EIGENVALUE ESTIMATES FOR SCHRÖDINGER OPERATORS ON METRIC TREES

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Abstract. We consider Schrödinger operators on regular metric trees and prove Lieb-Thirring and Cwikel-Lieb-Rozenblum inequalities for their negative eigenvalues. The validity of these inequalities depends on the volume growth of the tree. We show that the bounds are valid in the endpoint case and reflect the correct order in the weak or strong coupling limit.

1. Introduction

It is well known that the moments of negative eigenvalues of the Schrödinger operator $-\Delta - V$ in $L_2(\mathbb{R}^d)$ can be estimated in terms of the classical phase space volume. Namely, the Lieb-Thirring inequality states that the bound

$$\operatorname{tr}(-\Delta - V)_-^\gamma \;\leq\; L_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}\, dx \tag{1.1}$$

holds true for any potential V if and only if

$$\gamma \geq \frac{1}{2} \;\text{ if }\; d=1, \quad \gamma > 0 \;\text{ if }\; d=2, \quad \gamma \geq 0 \;\text{ if }\; d \geq 3\,. \tag{1.2}$$

Here $x_\pm := \max\{0, \pm x\}$ denotes the positive and negative part of x. Inequality (1.1) is due to Lieb and Thirring [26] and, in the endpoint cases, to Cwikel [6], Lieb [24], Rozenblum [30] and Weidl [34]. We refer to [23] and [15] for recent reviews on this topic.

Our main objective is to establish the analog of (1.1) for Schrödinger operators on metric trees. A (rooted) metric tree Γ consists of a set of vertices and a set of edges, i.e., segments of the real axis which connect the vertices. We assume that Γ has infinite height, that is, it contains points at arbitrary large distance from the root. We define the Schrödinger operator formally as

$$-\Delta_{\mathcal{N}} - V \quad \text{in} \quad L_2(\Gamma)$$

with Kirchhoff matching conditions at the vertices and a Neumann boundary condition at the root of the tree.

Metric trees represent a special class of so called quantum graphs, which recently have attracted great interest; see, e.g., [3, 18, 20, 21] for extensive bibliographies about this subject. Many works devoted to quantum graphs concern questions about self-adjoint extensions, approximation by thin quantum wave guides and direct or inverse scattering properties of the Laplace operator on graphs, see the references above and also [11, 22]. Various functional inequalities for the Laplacian on metric trees have been established in [10, 27]. However, much less attention has been paid, with the exception of [28], to the classical question of finding appropriate estimates, similar to (1.1), on the discrete spectrum of Schrödinger operators on metric trees. As we shall see, the interplay between the spectral theory and the mixed dimensonality of a tree makes this a fascinating problem.

Key words and phrases. Schrödinger operator, metric tree, eigenvalue estimate, Lieb-Thirring inequality, Cwikel-Lieb-Rozenblum inequality.

Our main result concern *regular* metric trees, that is, trees which are symmetric with respect to the distance from the root; see Subsection 2.1 for a precise definition. We shall show that the validity of a suitable analog of (1.1) is characterized by the global branching of the tree Γ. The latter is expressed by the *branching function* $g_0(t) := \#\{x : |x| = t\}$ which counts the number of points of Γ as a function of the distance from the root. The function g_0 is clearly non-decreasing. Depending on its growth we may split the trees into two classes according to whether the integral

$$\int_0^\infty \frac{dt}{g_0(t)} \tag{1.3}$$

is finite (*transient trees*) or infinite (*recurrent trees*). It turns out that in the former case, the corresponding Lieb-Thirring inequality holds for all values $\gamma \geq 0$. For $\gamma = 0$ this is an estimate on the number of negative eigenvalues in terms of an integral of the potential, usually called a Cwikel-Lieb-Rozenblum inequality. On the other hand, if the integral (1.3) is infinite, then Lieb-Thirring inequalities do not hold for values of γ which are smaller than some critical value $\gamma_{\min} > 0$. In order to determine the value of $\gamma_{\min}$ we use the notion of the *global dimension* of a metric tree, see Definition 2.5. This dimension is equal to $d \geq 1$ if the branching function g_0 has a power-like growth at infinity with power $d-1$. We emphasize that in contrast to the Euclidean case, d need not be an integer.

For regular metric trees Γ with global dimension d and Schrödinger operators with symmetric potentials V we shall prove Lieb-Thirring inequalities of the form

$$\operatorname{tr}(-\Delta_{\mathcal{N}} - V)_-^\gamma \leq C \int_\Gamma V_+^{\gamma + \frac{1+a}{2}} g_0^{\frac{a}{d-1}} \, dx , \quad a \geq 0 . \tag{1.4}$$

The allowed values of γ are determined by the parameter a and by the global dimension d of Γ, see Theorem 2.7. For $a = 0$ the weight in the integral on the right hand side disappears and the inequality is very similar to its Euclidean version (1.1). Both sides then share the same growth in the strong coupling limit, see Remark 2.10 below. On the other hand, it requires the exponent $\gamma \geq 1/2$ and does not capture the fact that even smaller moments can be estimated for larger values of d. This motivates the inequality (1.4) with different choices of a. As a consequence of our result, the smallest value of γ such that (1.4) holds for some $a \geq 0$ (indeed, for $a = d-1$) is

$$\gamma_{\min} = \frac{2-d}{2} \quad 1 \leq d < 2 , \quad \gamma_{\min} = 0 \quad d > 2 . \tag{1.5}$$

We emphasize that we establish the inequality in these endpoint cases and that the resulting inequality for $1 \leq d < 2$ is *order-sharp* in the weak coupling limit, see Remark 2.11. As one may expect by analogy with the Euclidean situation, the case $d = 2$ is somewhat special, since the minimal value of γ is 0, but the inequality is not valid in the endpoint case.

We consider also the case of a homogeneous tree, i.e., a tree where all edges have equal length and all vertices are of the same degree. In this case, the function g_0 grows exponentially and the Laplacian $-\Delta_{\mathcal{N}}$ is positive definite. We prove Cwikel-Lieb-Rozenblum inequalities for the number of eigenvalues that a potential V generates below the bottom of the spectrum of $-\Delta_{\mathcal{N}}$.

An important ingredient in our proof of eigenvalue estimates are one-dimensional Sobolev inequalities with weights. In particular, if the integral (1.3) is finite, we combine them with a Sturm oscillation argument in order to deduce Cwikel-Lieb-Rozenblum inequalities. This yields remarkably good bounds on the constants. We believe that our technique, in particular the duality argument in Proposition 7.2, has applications beyond the context of this paper.

As we have pointed out, one of the main motivations for this work is to understand how the dimensionality of the underlying space is reflected in eigenvalue estimates. Several results in the literature can be viewed in this light. If the global dimension of the underlying space is, in contrast to our situation, *smaller* than the local dimension, then the eigenvalues are typically estimated by a sum of two terms. Lieb-Thirring inequalities of this form have been proved by Lieb, Solovej and Yngvason [25] for the Pauli operator. The second, non-standard term there corresponds to states in the lowest Landau level, which are localized in the plane orthogonal to the magnetic field. A two-term inequality of more obvious geometric nature was proved by Exner and Weidl [12] for Schrödinger operators in a waveguide $\omega \times \mathbb{R}$, $\omega \subset \mathbb{R}^{d-1}$. Here the second term corresponds to the global dimension, which is one, as opposed to the local dimension d. These two-term estimates are order-sharp both in the weak coupling regime (where the global dimension is dominant) and in the strong coupling regime (where the local dimension is dominant). In our situation, however, the global dimension is *larger* than the local dimension, and a two-term inequality would neither in the weak nor in the strong coupling regime be order-sharp. Therefore we propose *families* of inequalities, which are sharp in different coupling regimes. This is somewhat reminiscent of the family of inequalities proved by Hundertmark and Simon [17] for the discrete Laplacian on the lattice $\mathbb{Z}^d$, where the local dimension is 0 and the global dimension is d.

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2. Main results and discussions

2.1. **Preliminaries.** Let Γ be a rooted metric tree with root o. By $|x|$ we denote the unique distance between a point $x \in \Gamma$ and the root o. Throughout we assume that Γ is of infinite height, i.e., $\sup_{x\in\Gamma} |x| = \infty$. The branching number $b(x)$ of a vertex x is defined as the number of edges emanating from x. We assume the natural conditions that $b(x) > 1$ for any vertex $x \neq o$ and that $b(o) = 1$.

We define the Neumann Laplacian $-\Delta_{\mathcal{N}}$ as the self-adjoint operator in $L_2(\Gamma)$ associated with the closed quadratic form

$$\int_\Gamma |\varphi'(x)|^2\, dx, \quad \varphi \in H^1(\Gamma). \tag{2.1}$$

Here $H^1(\Gamma)$ consists of all continuous functions φ such that $\varphi \in H^1(e)$ on each edge e of Γ and

$$\int_\Gamma \left(|\varphi'(x)|^2 + |\varphi(x)|^2\right)\, dx < \infty.$$

The operator domain of $-\Delta_{\mathcal{N}}$ consists of all continuous functions φ such that $\varphi'(o) = 0$, $\varphi \in H^2(e)$ for each edge e of Γ and such that at each vertex $x \neq o$ of Γ the matching conditions

$$\varphi_-(x) = \varphi_1(x) = \cdots = \varphi_{b(x)}(x)\,, \quad \varphi'_-(x) = \varphi'_1(x) + \cdots + \varphi'_{b(x)}(x)$$

are satisfied. Here φ_- denotes the restriction of φ on the edge terminating in x and φ_j, $j = 1, \ldots, b(x)$, denote the restrictions of φ to the edges emanating from x, see, e.g., [28, 27] for details.

In this paper we are interested in Schrödinger operators $-\Delta_{\mathcal{N}} - V$ in $L_2(\Gamma)$. Throughout we assume that the potential V is a real-valued, sufficiently regular function on Γ, the positive part of which vanishes at infinity in a suitable sense. (We shall be more precise below.) In this case the negative spectrum of $-\Delta_{\mathcal{N}} - V$ consists of discrete eigenvalues of finite multiplicities. Our goal is to estimate the total number of these eigenvalues or, more generally, moments of these eigenvalues in terms of integrals of the potential V.

The starting point of our analysis is

Theorem 2.1. *Let $\gamma \geq 1/2$. Then there exists a constant L_γ such that for any rooted metric tree Γ and any V,*

$$\operatorname{tr}(-\Delta_{\mathcal{N}} - V)_-^\gamma \leq L_\gamma \int_\Gamma V(x)_+^{\gamma+\frac{1}{2}}\, dx. \tag{2.2}$$

We emphasize that the constant L_γ is independent of Γ. This result is clearly analogous to the standard one-dimensional Lieb-Thirring inequalities. An advantage is its universality. Moreover, we will see in Subsection 2.3 below, that the right hand side has the correct order of growth in the strong coupling limit when V is replaced by αV and $\alpha \to \infty$. On the other hand, it does not reflect the geometry of Γ at all and it does *not* display the correct behavior in the weak coupling limit when V is replaced by αV and $\alpha \to 0$.

The main goal of this paper is to obtain eigenvalue estimates which take the global structure of Γ into account. We shall consider trees which possess certain additional symmetry properties. Namely, we impose

Assumption 2.2. The tree Γ is *regular*, i.e., all the vertices at the same distance from the root have equal branching numbers and all the edges emanating from these vertices have equal length.

Let x be a vertex such that there are $k+1$ vertices on the (unique) path between o and x including the endpoints. We denote by t_k the distance $|x|$ and by b_k the branching number of x. Moreover, we put $t_0 := 0$ and $b_0 := 1$. Note that t_k and b_k are only well-defined for regular trees and that these numbers, in the regular case, uniquely determine the tree.

We define the *(first) branching function* $g_0 : \mathbb{R}_+ \to \mathbb{N}$ by

$$g_0(t) := b_0\, b_1 \cdots b_k, \quad \text{if } t_k < t \leq t_{k+1}, \quad k \in \mathbb{N}_0.$$

Here $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Note that g_0 is a non-decreasing function and that $g_0(t)$ coincides with the number of points $x \in \Gamma$ such that $|x| = t$. The rate of growth of g_0 reflects the rate of growth of the tree Γ. More precisely, g_0 measures how the surface of the 'ball' $\{x \in \Gamma : |x| < t\}$ grows with t. Of great importance in our analysis will be the fact whether the *reduced height* of Γ,

$$\ell_\Gamma := \int_0^\infty \frac{dt}{g_0(t)} \tag{2.3}$$

is finite or not.

In addition to Assumption 2.2 we shall impose

Assumption 2.3. The function V is *symmetric*, i.e., for any $x \in \Gamma$ the value $V(x)$ depends only on the distance $|x|$ between x and the root o.

With slight abuse of notation we shall write sometimes V instead of $V(|\cdot|)$.

2.2. **Eigenvalue estimates on trees.** In this subsection we present our main results. We denote by $N(T)$ the number of negative eigenvalues (counting multiplicities) of a self-adjoint, lower bounded operator T. We begin with the case where the reduced height (2.3) is finite. In this case we shall prove

Theorem 2.4 (**CLR bounds for trees of finite reduced height**). *Let Γ be a regular metric tree with $\ell_\Gamma < \infty$ and let $w : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function such that for some $2 < q \leq \infty$*

$$M := \sup_{t \geq 0} \left(\int_0^t g_0(s)^{\frac{q}{2}} w(s)^{-\frac{q-2}{2}}\, ds \right)^{2/q} \int_t^\infty \frac{ds}{g_0(s)} < \infty \ . \tag{2.4}$$

Let $p := q/(q-2)$. Then there exists a constant $N_p(\Gamma, w)$ such that

$$N(-\Delta_{\mathcal{N}} - V) \leq N_p(\Gamma, w) \int_\Gamma V(|x|)_+^p w(|x|)\, dx \tag{2.5}$$

for all symmetric V. Moreover, the sharp constant in (2.5) *satisfies*

$$N_p(\Gamma, w) \leq (1 + p')^{p-1} \left(1 + \frac{1}{p'}\right)^p M^p.$$

By definition, if $q = \infty$ condition (2.4) is understood as

$$\sup_{t \geq 0} \left(\sup_{0 \leq s \leq t} \frac{g_0(s)}{w(s)} \right) \int_t^\infty \frac{ds}{g_0(s)} < \infty \ ,$$

and one has $N_1(\Gamma, w) \leq M$.

In order to give more explicit estimates we assume that the growth of the branching function is sufficiently regular in the sense of

Definition 2.5. A regular metric tree Γ has *global dimension* $d \geq 1$ if its branching function satisfies

$$0 < c_1 := \inf_{t \geq 0} \frac{g_0(t)}{(1+t)^{d-1}} \leq \sup_{t \geq 0} \frac{g_0(t)}{(1+t)^{d-1}} =: c_2 < \infty \, . \tag{2.6}$$

Obviously, if Γ has global dimension d, then it has finite reduced height if and only if $d > 2$. In this case Theorem 2.4 implies

Corollary 2.6. *Assume that Γ has global dimension $d > 2$. Then for any $a \geq 1$ there exists a constant $C(a, \Gamma)$ such that for any symmetric V*

$$N(-\Delta_{\mathcal{N}} - V) \leq C(a, \Gamma) \int_\Gamma V(|x|)^{\frac{1+a}{2}} g_0(|x|)^{\frac{a}{d-1}}\, dx \, .$$

Next we turn to the case of infinite reduced height $\ell_\Gamma = \infty$. It is easy to see that Schrödinger operators $-\Delta_{\mathcal{N}} - V$ on such trees with non-trivial $V \geq 0$ have at least one negative eigenvalue, no matter how small V is. Hence it is impossible to estimate the number of eigenvalues from above by a weighted integral norm of the potential. However, under the assumption that the tree has a global dimension we can prove estimates for the *moments* of negative eigenvalues of $-\Delta_{\mathcal{N}} - V$. Moreover, we can treat the case $0 \leq a < 1$ which was left open in Corollary 2.6. Our result is

Theorem 2.7 (**LT bounds for trees**). *Let Γ be a regular metric tree with global dimension $d \geq 1$.*

(1) *Assume that either $1 \leq d < 2$ and $0 \leq a \leq d-1$, or else that $d \geq 2$ and $0 \leq a < 1$. Then for any $\gamma \geq \frac{1-a}{2}$ there exists a constant $C(\gamma, a, \Gamma)$ such that for any symmetric V*

$$\operatorname{tr}(-\Delta_{\mathcal{N}} - V)_-^\gamma \leq C(\gamma, a, \Gamma) \int_\Gamma V(|x|)_+^{\gamma + \frac{1+a}{2}} g_0(|x|)^{\frac{a}{d-1}}\, dx. \tag{2.7}$$

(2) *Assume that either $1 \leq d < 2$ and $a > d-1$, or else that $d = 2$ and $a \geq 1$. Then for any $\gamma > (1+a)\,\frac{2-d}{2d}$ there exists $C(\gamma, a, \Gamma)$ such that* (2.7) *holds for any symmetric V.*

(3) *Assume that $d > 2$ and that $a \geq 1$. Then for any $\gamma \geq 0$ there exists $C(\gamma, a, \Gamma)$ such that (2.7) holds for any symmetric V.*

One can prove that our conditions on γ are not only sufficient but (except for the limiting case in Part (2)) also necessary for the validity of (2.7). This is further discussed in Subsection 2.3. Part (3) is in fact an immediate consequence of Corollary 2.6 and an argument by Aizenman and Lieb [2]. It is stated here for the sake of completeness.

If the branching function g_0 grows 'very' fast, the Laplacian $-\Delta_{\mathcal{N}}$ is positive definite. In this case it is reasonable not only to estimate the number of negative eigenvalues of $-\Delta_{\mathcal{N}} - V$, but also the number of eigenvalues less then the bottom of the spectrum of $-\Delta_{\mathcal{N}}$. We carry through this analysis for a special class of trees.

A regular metric tree is called *homogeneous* if all the edges have the same length τ and if the branching number $b_k = b > 1$ is independent of k. Homogeneous trees correspond intuitively to trees of infinitely large global dimension. By scaling it is no loss of generality to assume that $\tau = 1$. The branching function g_0 then reads

$$g_0(t) = b^j, \qquad j < t \leq j+1, \quad j \in \mathbb{N}_0 .$$

The Laplacian $-\Delta_{\mathcal{N}}$ (or rather its Dirichlet version) on a homogeneous tree was studied in [32]. It follows from the analysis there that $-\Delta_{\mathcal{N}}$ is positive definite and its essential spectrum starts at

$$\lambda_b = \left(\arccos \frac{1}{R_b} \right)^2, \qquad R_b = \frac{b^{\frac12} + b^{-\frac12}}{2} .$$

We shall prove

Theorem 2.8 (**CLR bounds for homogeneous trees**). *Let Γ be a homogeneous tree with edge length 1 and branching number $b > 1$ and let $w : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function such that for some $2 < q \leq \infty$*

$$M := \sup_{t \geq 0} (1+t)^{-1} \left(\int_0^t (1+s)^q w^{-\frac{q-2}{2}} \, ds \right)^{2/q} .$$

Let $p = q/(q-2)$. Then there exists a constant $N_p(b, w)$ such that

$$N(-\Delta_{\mathcal{N}} - V - \lambda_b) \leq N_p(b, w) \int_\Gamma V(|x|)_+^p \, w(|x|) \, dx \tag{2.8}$$

for all symmetric V. Moreover, the sharp constant in (2.8) satisfies

$$N_p(b, w) \leq C(b) \, (1+p')^{p-1} \left(1 + \frac{1}{p'} \right)^p M^p \tag{2.9}$$

with some constant $C(b)$ depending only on b.

Choosing $w(t) = (1+t)^a$ we obtain the following strengthening of Corollary 2.6.

Corollary 2.9. *Let Γ be a homogeneous tree with edge length 1 and branching number $b > 1$. Then for any $a \geq 1$ there exists a constant $C(a, b)$ such that for any symmetric V*

$$N(-\Delta_{\mathcal{N}} - V - \lambda_b) \leq C(a, b) \int_\Gamma V(|x|)_+^{\frac{1+a}{2}} \, (1+|x|)^a \, dx .$$

2.3. **Discussion.** In this subsection we discuss the inequality (2.7) and the conditions for its validity given in Theorem 2.7.

Remark 2.10 (**Strong coupling limit**). Inequality (2.7) with $a = 0$ coincides with (2.2),

$$\operatorname{tr}(-\Delta_{\mathcal{N}} - V)_-^\gamma \leq L_\gamma \int_\Gamma V(|x|)_+^{\gamma+\frac{1}{2}}\, dx, \quad \gamma \geq \frac{1}{2}.$$

This inequality reflects the correct behavior in the strong coupling limit. Indeed, if V is, say, continuous and of compact support then standard Dirichlet-Neumann bracketing [31, Thm. XIII.80] leads to the Weyl-type asymptotic formula

$$\lim_{\alpha\to\infty} \alpha^{-\gamma-\frac{1}{2}} \operatorname{tr}\left(-\Delta_{\mathcal{N}} - \alpha V\right)_-^\gamma = L_{\gamma,1}^{\mathrm{cl}} \int_\Gamma V(|x|)_+^{\gamma+\frac{1}{2}}\, dx, \quad \gamma \geq 0, \tag{2.10}$$

with

$$L_{\gamma,1}^{\mathrm{cl}} := \frac{\Gamma(\gamma+1)}{2\sqrt{\pi}\,\Gamma(\gamma+3/2)}\,. \tag{2.11}$$

This shows in particular that (2.7) can *not* hold for $a < 0$.

Remark 2.11 (**Weak coupling limit**). Assume that Γ has global dimension $d \in [1, 2)$. Inequality (2.7) with $a = d-1$, $\gamma = (2-d)/2$ reads

$$\operatorname{tr}(-\Delta_{\mathcal{N}} - V)_-^{\frac{2-d}{2}} \leq C\left(\frac{2-d}{2}, d-1, \Gamma\right) \int_\Gamma V(|x|)_+ g_0(|x|)\, dx.$$

This inequality reflects the correct behavior in the weak coupling limit. Indeed, it is shown in [19] that $-\Delta_{\mathcal{N}} - \alpha V$ has at least one negative eigenvalue whenever $\int_\Gamma V(|x|)\, dx > 0$, and that for α sufficiently small this eigenvalue, say $\lambda_1(\alpha)$, is unique and satisfies

$$-\, a_1\, \alpha^{\frac{2}{2-d}} \;\leq\; \lambda_1(\alpha) \;\leq\; -a_2\, \alpha^{\frac{2}{2-d}}, \quad \alpha \to 0, \tag{2.12}$$

for suitable constants $a_1 \geq a_2 > 0$ depending on V. This fact shows also that (2.7) does *not* hold for $1 \leq d < 2$, $a \geq 0$ and $\gamma < (1+a)\frac{2-d}{2d}$. We do not know whether (2.7) holds in the endpoint case $\gamma = (1+a)\frac{2-d}{2d}$ when $1 \leq d < 2$ and $a > d-1$.

Similarly, when Γ has global dimension $d = 2$, one can show that $-\Delta_{\mathcal{N}} - \alpha V$ has at least one negative eigenvalue whenever $\int_\Gamma V(|x|)\, dx > 0$. Hence (2.7) does *not* hold for $d = 2$, $a \geq 0$ and $\gamma = 0$.

Remark 2.12 (**Dirac-potential limit**). As we have seen in the previous remark, the condition $\gamma > (1+a)(2-d)/(2d)$ in Part (2) of Theorem 2.7 comes from the weak coupling limit. Now we explain that the condition $\gamma \geq (1-a)/2$ in Part (1) comes from what may be called the Dirac-potential limit. Consider the sequence of potentials $V_n = n\chi_{(0,n^{-1})}$. Using a trial function supported near the root o one easily proves that $\operatorname{tr}(-\Delta_{\mathcal{N}} - V_n)_-^\gamma$ is bounded away from zero uniformly in n. On the other hand, $\int V_n^{\gamma+\frac{a+1}{2}} g_0^{\frac{a}{(d-1)}}\, dx$ tends to zero if $\gamma < (1-a)/2$. This shows that the condition $\gamma \geq \frac{1-a}{2}$ is necessary for the validity of (2.7).

Remark 2.13 (**Slowly decaying potentials**). Assume that V is a symmetric function which is locally sufficiently regular and obtains the asymptotics $V(t) \sim \alpha t^{-s}$ as $t \to \infty$ for some $s > 0$, $\alpha > 0$. By standard methods (see, e.g., [31, Thm. XIII.6]) one shows that the operator $-\Delta_{\mathcal{N}} - V$ has only a finite number of negative eigenvalues provided $s > 2$. However, the semi-classical expression for the number of negative eigenvalues, i.e. the right hand side of (2.10) with $\gamma = 0$, is only finite under the more restrictive condition $s > 2d$. Our Corollary 2.6 with sufficiently large a gives a quantitative estimate on the number of negative eigenvalues for the whole range of exponents $s > 2$ if $d > 2$. Similarly, in the case $1 \leq d \leq 2$ we obtain quantitative information about the magnitude of the eigenvalues, which goes beyond semi-classics.

Remark 2.14 (**Dirichlet boundary conditions**). The reader might wonder how our main theorems change, if a Dirichlet instead of a Neumann boundary condition is imposed at the root. Let $-\Delta_{\mathscr{D}}$ be the self-adjoint operator in $L_2(\Gamma)$ generated by the quadratic form (2.1) with form domain $H_0^1(\Gamma) := \{\phi \in H^1(\Gamma) : \phi(0) = 0\}$. By the variational principle, any bound for $-\Delta_{\mathcal{N}} - V$ implies a bound for $-\Delta_{\mathscr{D}} - V$. However, it turns out that inequalities for the latter operator hold for a strictly larger range of parameters. Indeed, the analog of Theorems 2.4 states that the inequality

$$\operatorname{tr}(-\Delta_{\mathscr{D}} - V)_-^\gamma \le C(\gamma, a, \Gamma) \int_\Gamma V(|x|)_+^{\gamma + \frac{1+a}{2}} g_0(|x|)^{\frac{a}{d-1}}\, dx.$$

holds provided either $0 \le a < 1$ and $\gamma \ge (1-a)/2$, or else $a \ge 1$ and $\gamma \ge 0$ and $d \ne 2$, or else $a \ge 1$ and $\gamma > 0$ and $d = 2$. This follows (except for the statement for $\gamma = 0$, $1 \le d < 2$) from Theorem 7.4. There is also an analog of Theorem 2.4 for $-\Delta_{\mathscr{D}}$ which is obtained by simply interchanging the two intervals of integration in the assumption (2.4). We omit the details. For spectral asymptotics of the operator $-\Delta_{\mathscr{D}} - V$ we refer to [27].

2.4. **One-dimensional Schrödinger operators with metric.** Our symmetry assumptions will allow us to reduce the spectral analysis of the operator $-\Delta_{\mathcal{N}} - V$ to the spectral analysis of a family of one-dimensional Schrödinger-type operators. The main ingredient in the proof of Theorem 2.7 will be an inequality for such operators, which is of independent interest.

We consider a positive, measurable and locally bounded function g on $[0,\infty)$ and denote by $H^1(\mathbb{R}_+, g)$ the space of all functions $f \in H^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$\int_0^\infty \left(|f'(t)|^2 + |f(t)|^2\right) g(t)\, dt < \infty.$$

The quadratic form

$$\int_0^\infty |f'(t)|^2 g(t)\, dt \tag{2.13}$$

with form domain $H^1(\mathbb{R}_+, g)$ defines a self-adjoint operator A_g in $L_2(\mathbb{R}_+, g)$. Note that this operator corresponds to the differential expression

$$A_g = -g^{-1} \frac{d}{dt} g \frac{d}{dt}\,,$$

and that functions f in its domain satisfy Neumann boundary conditions $f'(0) = 0$ at the origin (at least when g is sufficiently regular near 0).

For our first results we assume that g grows sufficiently fast in the sense that

$$\int_t^\infty \frac{ds}{g(s)} < \infty \qquad \forall\, t > 0. \tag{2.14}$$

We shall prove that under this condition the number of negative eigenvalues of the Schrödinger operators $A_g - V$ can be estimated in terms of weighted L_p-norms of V. More precisely, one has

Theorem 2.15. *Assume* (2.14) *and let* $w : \mathbb{R}_+ \to \mathbb{R}_+$ *be a positive function such that for some* $2 < q \le \infty$

$$M := \sup_{t \ge 0} \left(\int_0^t g(s)^{\frac{q}{2}} w(s)^{-\frac{q-2}{2}}\, ds \right)^{2/q} \int_t^\infty \frac{ds}{g(s)} < \infty\ . \tag{2.15}$$

Let $p := q/(q-2)$. Then the inequality

$$N(A_g - V) \le C_p(w,g) \int_0^\infty V_+^p w\,dt \tag{2.16}$$

holds for all V, and the sharp constant $C_p(w,g)$ in (2.16) satisfies

$$M^p \le C_p(w,g) \le \left(1+p'\right)^{p-1}\left(1+\frac{1}{p'}\right)^p M^p.$$

Moreover, if $M = \infty$ then there is no constant $C_p(w,g)$ such that (2.16) holds for all V.

By definition, if $q = \infty$ condition (2.15) is understood as

$$M := \sup_{t\ge 0}\left(\sup_{0\le s\le t}\frac{g(s)}{w(s)}\right)\int_t^\infty \frac{ds}{g(s)} < \infty,$$

and the *sharp* constant is $C_1(w,g) = M$. This leads to the following beautiful estimate.

Example 2.16. Taking $w(t) = g(t)\int_t^\infty g^{-1}(s)\,ds$ and $q = \infty$ one obtains

$$N(A_g - V) \le \int_0^\infty V(t)_+\, g(t)\left(\int_t^\infty \frac{ds}{g(s)}\right)dt\,, \tag{2.17}$$

which is sharp (meaning that the estimate is no longer true for all g and all V if the right hand side is multiplied by a constant less than one). As a consequence one also finds

$$N(A_g - V) \le \int_0^\infty \frac{dt}{g}\int_0^\infty V_+ g\,dt\,.$$

Theorem 2.15 gives a complete characterization of weights for which the number of negative eigenvalues can be estimated by a weighted norm of the potential. When g grows very fast, the operator A_g will be positive definite and in this case one may not only ask for the number of eigenvalues of $A_g - V$ below 0 but also below the bottom of the spectrum of A_g. We turn to this question next. We assume, in addition to (2.14), that

$$\sup_{t>0}\int_0^t g(s)\,ds\int_t^\infty \frac{ds}{g(s)} < \infty. \tag{2.18}$$

This condition is necessary and sufficient for the operator A_g to be positive definite, see Proposition 5.1 below or [33, Thm. 5.2]. We denote the bottom of its spectrum by $\lambda(A_g) > 0$ and assume that $\lambda(A_g)$ is *not* an eigenvalue of A_g. Let ω be the unique (up to a constant) distributional solution of the differential equation

$$-\left(g\omega'\right)' = \lambda(A_g)\,g\,\omega \qquad \text{on } \mathbb{R}_+ \tag{2.19}$$

satisfying the boundary condition $\omega'(0) = 0$. Since $\lambda(A_g)$ is not an eigenvalue, the function ω is not square-integrable with respect to the weight g. We quantify the growth of $\omega^2 g$ by assuming that

$$\int_0^\infty \omega^{-2}g^{-1}\,ds < \infty. \tag{2.20}$$

Under these conditions one has

Theorem 2.17. *Assume* (2.14), (2.18) *and* (2.20). *Let $w : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function such that for some $2 < q \le \infty$*

$$M := \sup_{t>0}\left(\int_0^t \omega^q g^{\frac{q}{2}} w^{-\frac{q-2}{2}}\,ds\right)^{2/q}\int_t^\infty \omega^{-2}g^{-1}\,ds < \infty,$$

and put $p := \frac{q}{q-2}$. Then the inequality

$$N(A_g - V - \lambda(A_g)) \le C_p(w,g,\omega)\int_0^\infty V_+^p w\,dt \tag{2.21}$$

holds for all V, and the sharp constant $C_p(w,g,\omega)$ satisfies

$$M^p \le C_p(w,g,\omega) \le \big(1+p'\big)^{p-1}\left(1+\frac{1}{p'}\right)^p M^p. \tag{2.22}$$

Finally, we present some estimates without imposing the condition (2.14). It is easy to see that if the integral in (2.14) is infinite, then $A_g - V$ will have a negative eigenvalue for any non-negative $V \not\equiv 0$, hence no estimate on the number of eigenvalues in terms of norms of V can hold. Below we shall prove that estimates on *moments* of eigenvalues do hold. For the sake of simplicity we restrict ourselves to the case where g has power-like growth, i.e.,

$$0 < c_1 := \inf_{t>0} \frac{g(t)}{(1+t)^{d-1}} \le \sup_{t>0} \frac{g(t)}{(1+t)^{d-1}} =: c_2 < \infty \tag{2.23}$$

for some $d \ge 1$. Note that (2.14) holds iff $d > 2$. We shall consider inequalities of the form

$$\operatorname{tr}(A_g - V)_-^\gamma \le L \int_0^\infty V(t)_+^{\gamma+\frac{a+1}{2}} (1+t)^a\, dt, \qquad L = L(\gamma, a, d, c_1, c_2). \tag{2.24}$$

In Remark 7.3 below we show that the relation between the exponent of V and that of the weight $(1+t)$ can not be improved. Our result is

Theorem 2.18. *Assume* (2.23) *for some* $d \ge 1$.

1. *Let either* $1 \le d < 2$ *and* $0 \le a \le d-1$, *or else* $d \ge 2$ *and* $0 \le a < 1$. *Then* (2.24) *holds iff* $\gamma \ge (1+a)/2$.
2. *Let either* $1 \le d < 2$ *and* $a > d-1$, *or else* $d = 2$ *and* $a \ge 1$. *Then* (2.24) *holds iff* $\gamma > (1+a)(2-d)/(2d)$.
3. *Let* $d > 2$ *and* $a \ge 1$. *Then* (2.24) *holds for any* $\gamma \ge 0$.

Part (3) is of course a consequence of Theorem 2.15 (for $\gamma = 0$) and of an argument by Aizenman and Lieb [2] (for $\gamma > 0$). Note carefully that for small a (Part (1)) the inequality (2.24) holds in the endpoint case, while it does *not* for large a (Part (2)). This is a phenomenon due to the Neumann boundary conditions which is not present when Dirichlet boundary conditions are imposed instead, see Theorem 7.4.

2.5. **Outline of the paper.** This paper is organized as follows. In Section 3 we prove Theorem 2.1 and a weighted version of it about arbitrary, not necessarily regular, metric trees. In Section 4 we show how our main results, Theorems 2.4, 2.7 and 2.8, follow from the results about one-dimensional Schrödinger operators in Subsection 2.4. In Section 5 we give the proofs of Theorems 2.15 and 2.17. Section 6 is of auxiliary character and contains the proof of a family of Sobolev interpolation inequalities which will be useful in the proof of Theorem 2.18. Finally, in Section 7 we will use a duality argument and estimates for Dirichlet eigenvalues in order to obtain the statements of Theorem 2.18.

3. Eigenvalue estimates on general metric trees

This section is devoted to the proof of Theorem 2.1. Moreover, we shall also prove the following weighted analog.

Theorem 3.1. *Let $a > 0$ and $\gamma > (1+a)/2$. Then there exists a constant $C_a(\gamma)$ such that*

$$\operatorname{tr}(-\Delta_{\mathcal N} - V)_-^\gamma \le C_a(\gamma) \int_\Gamma V(x)_+^{\gamma+\frac{1+a}{2}} |x|^a\, dx. \tag{3.1}$$

We emphasize that the constant in (3.1) can be chosen independently of the tree. For the proofs of Theorems 2.1 and 3.1 we use the following results about half-line operators.

Proposition 3.2. *Let $\Gamma = \mathbb{R}_+$ and $a \geq 0$. Let $\gamma > (1+a)/2$ if $a > 0$ and $\gamma \geq 1/2$ if $a = 0$. Then there exists a constant $L^{EK}_{\gamma,a}$ such that*

$$\operatorname{tr}(-\Delta_{\mathcal{N}} - V)^{\gamma}_{-} \leq L^{EK}_{\gamma,a} \int_0^\infty V(t)_+^{\gamma+\frac{1+a}{2}} t^a \, dt \tag{3.2}$$

for all V.

To prove (3.2) we extend V to an even function W on $\mathbb{R}$. Then the left hand side of (3.2) can be estimated from above by the corresponding moments of the whole-line operator $-d^2/dx^2 - W$, and the claimed inequality for that operator follows from [7] and [34]. Using in addition the sharp constants from [16] and [2] one obtains for $a = 0$ the following bounds on the constants,

$$L^{EK}_{\gamma,0} \leq 4\, L^{\mathrm{cl}}_{\gamma,1} \text{ if } \gamma \geq \frac{1}{2}, \qquad L^{EK}_{\gamma,0} \leq 2\, L^{\mathrm{cl}}_{\gamma,1} \text{ if } \gamma \geq \frac{3}{2} \tag{3.3}$$

with $L^{\mathrm{cl}}_{\gamma,1}$ from (2.11). Note that the inequality (3.2) with this constant for $\gamma = 1/2$ and $a = 0$ is sharp, and therefore so is (2.2) for $\gamma = 1/2$. Now we turn to the

Proof of Theorems 2.1 and 3.1. The idea is to impose Neumann boundary condition at all but one emanating edges of all vertices. This decreases the operator $-\Delta_{\mathcal{N}} - V$. The resulting operator can be identified with a direct sum of half-line operators for which one can use Proposition 3.2.

To be more precise, we decompose the graph $\Gamma = \bigcup_j \Gamma_j$ into a disjoint union of infinite halflines Γ_j. Then $L_2(\Gamma) = \bigoplus_j L_2(\Gamma_j)$ and $H^1(\Gamma) \subset \sum_j H^1(\Gamma_j)$. By the variational principle, this implies

$$-\Delta_{\mathcal{N}} - V \geq \bigoplus_j \left(-\Delta^{\Gamma_j}_{\mathcal{N}} - V_j \right),$$

where $-\Delta^{\Gamma_j}_{\mathcal{N}}$ is the Neumann Laplacian on Γ_j and V_j is the restriction of V to Γ_j. Hence Proposition 3.2 yields

$$\begin{aligned}
\operatorname{tr}(-\Delta_{\mathcal{N}} - V)^{\gamma}_{-} &\leq \sum_j \operatorname{tr}_{L_2(\Gamma_j)} \left(-\Delta^{\Gamma_j}_{\mathcal{N}} - V_j \right)^{\gamma}_{-} \\
&\leq L^{EK}_{\gamma,\alpha} \sum_j \int_{\Gamma_j} V_j(x)_+^{\gamma+\frac{1+a}{2}} \operatorname{dist}(x, \partial\Gamma_j)^a \, dx \\
&\leq L^{EK}_{\gamma,\alpha} \int_{\Gamma} V(x)_+^{\gamma+\frac{1+a}{2}} |x|^a \, dx,
\end{aligned}$$

as claimed. □

4. Eigenvalue estimates on regular trees

In this section we show how our main results, Theorems 2.4, 2.7 and 2.8, can be deduced from the results about one-dimensional Schrödinger operators in Subsection 2.4. To do so, we exploit the symmetry of the tree and the potential, which allows us to decompose $-\Delta_{\mathcal{N}} - V$ into a direct sum of half-line Schrödinger operators in weighted L_2-spaces. We recall this construction next.

4.1. **Orthogonal decomposition.** In this subsection we recall the results of Carlson [5] and of Naimark and Solomyak [27, 28]. We need some notation. For each $k \in \mathbb{N}$ we define the higher order branching functions $g_k : \mathbb{R}_+ \to \mathbb{N}_0$ by

$$g_k(t) := \begin{cases} 0, & t < t_k \,, \\ 1, & t_k \leq t < t_{k+1} \,, \\ b_{k+1} b_{k+2} \cdots b_n, & t_n \leq t < t_{n+1},\ k < n \,, \end{cases}$$

and introduce the weighted Sobolev space $H_0^1((t_k,\infty), g_k)$ as the closure of $C_0^\infty(t_k,\infty)$ in the norm

$$\left[\int_{t_k}^\infty \left(|f'(t)|^2 + |f(t)|^2 \right) g_k(t)\, dt \right]^{\frac12} .$$

Let $\mathcal{A}_k$ be the self-adjoint operator in $L_2((t_k,\infty),\, g_k)$ given by the quadratic form

$$\mathfrak{a}_k[f] := \int_{t_k}^\infty |f'(t)|^2 g_k(t)\, dt$$

with form domain $H_0^1((t_k,\infty), g_k)$. Notice that the operators $\mathcal{A}_k$ with $k \geq 1$ satisfy *Dirichlet* boundary condition at t_k, while the operator $\mathcal{A}_0$ satisfies *Neumann* boundary condition at $t_0 = 0$.

The following statement is taken from [28] and [33].

Proposition 4.1. *Let $V \in L_\infty(\Gamma)$ be symmetric. Then $-\Delta_{\mathcal{N}} - V$ is unitarily equivalent to the orthogonal sum of operators*

$$-\Delta_{\mathcal{N}} - V \simeq (\mathcal{A}_0 - V) \oplus \sum_{k=1}^\infty \oplus \big(\mathcal{A}_k - V_k\big)^{[b_1 \ldots b_{k-1}(b_k - 1)]}. \tag{4.1}$$

Here the symbol $[b_1 ... b_{k-1}(b_k-1)]$ means that the operator $\mathcal{A}_k - V_k$ appears $b_1...b_{k-1}(b_k-1)$ times in the orthogonal sum, and V_k denotes the restriction of V to the interval (t_k,∞).

4.2. **Proof of Theorems 2.4 and 2.7.** Let us compare the operators $\mathcal{A}_k$ with each other. From the definition of the function g_k it follows that

$$\frac{\int_{t_k}^\infty \left(|f'|^2 - V_k|f|^2\right) g_k\, dt}{\int_{t_k}^\infty |f|^2 g_k\, dt} = \frac{\int_{t_k}^\infty \left(|f'|^2 - V_k|f|^2\right) g_0\, dt}{\int_{t_k}^\infty |f|^2 g_0\, dt}.$$

Since every function $f \in H_0^1((t_k,\infty), g_k)$ can be extended by zero to a function in $H^1(\mathbb{R}_+, g_0)$, the variational principle shows that

$$\operatorname{tr}(\mathcal{A}_k - V_k)_-^\gamma \leq \operatorname{tr}(\mathcal{A}_0 - \chi_{(t_k,\infty)} V)_-^\gamma \tag{4.2}$$

for any $k \in \mathbb{N}$ and $\gamma \geq 0$.

Assuming the validity of Theorems 2.15 and 2.18 we now give the

Proof of Theorems 2.4 and 2.7. In the case of Theorem 2.4 put $\gamma = 0$ and let q and w be such that (2.4) holds. Moreover, put $p = q/(q-2)$. In the case of Theorem 2.7 let γ be as indicated there and put $p = \gamma + (1+a)/2$ and $w(t) := g_0(t)^{a/(d-1)}$. It follows from Theorems 2.15 and 2.18, respectively, that in both cases there exists a constant C such that

$$\operatorname{tr}(\mathcal{A}_0 - V)_-^\gamma \leq C \int_0^\infty V(t)_+^p w(t)\, dt$$

for all V. Combining this with the orthogonal decomposition (4.1) and inequality (4.2) we obtain

$$\begin{aligned}
\operatorname{tr}(-\Delta_{\mathcal{N}} - V)_-^\gamma &= \operatorname{tr}(\mathcal{A}_0 - V)_-^\gamma + \sum_{k=1}^\infty b_1 \cdots b_{k-1}(b_k - 1) \operatorname{tr}(\mathcal{A}_k - \chi_{(t_k,\infty)} V)_-^\gamma \\
&\le C \int_0^\infty V(t)_+^p w(t)\, dt \\
&\quad + C \sum_{k=1}^\infty \left(b_1 \cdots b_{k-1}(b_k - 1) \int_{t_k}^\infty V(t)_+^p w(t)\, dt \right) \\
&= C \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} (b_0 \cdots b_k) V(t)_+^p w(t)\, dt \\
&= C \int_\Gamma V(|x|)_+^p \, w(|x|)\, dx,
\end{aligned}$$

as claimed. ☐

4.3. **Proof of Theorem 2.8.** In this subsection we assume that g_0 is the first branching function of a homogeneous metric tree with edge length 1 and branching number $b > 1$. Denote by λ_b the bottom of its essential spectrum and by ω the function on $\mathbb{R}_+$ satisfying in distributional sense

$$-(g_0 \omega')' = \lambda_b g_0 \, \omega \,,$$

$$\omega'(0) = 0, \quad \omega(j+) = \omega(j-), \quad \omega'(j-) = b\omega'(j+), \quad j \in \mathbb{N}\,.$$

In the proof of Theorem 2.8 we need the following technical result.

Lemma 4.2. *There exist constants* $0 < C_1 < C_2 < \infty$ *such that*

$$C_1 \frac{1+t}{\sqrt{g_0(t)}} \le \omega(t) \le C_2 \frac{1+t}{\sqrt{g_0(t)}}\,, \quad t \ge 0\,. \tag{4.3}$$

Assuming this for the moment we give the

Proof of Theorem 2.8. Proceeding in the same way as in the proof of Theorems 2.4 and 2.7 one sees that it suffices to prove that

$$N(\mathcal{A}_0 - V - \lambda_b) \le C \int_0^\infty V(t)_+^p w(t)\, dt\,. \tag{4.4}$$

We shall deduce this from Theorem 2.17 with $g = g_0$. By the explicit form of g_0 we see that (2.14) and (2.18) are satisfied. Moreover, $\lambda_b = \lambda(\mathcal{A}_0)$ and ω is the generalized ground state of $\mathcal{A}_0$ in the sense of (2.19). It follows from Lemma 4.2 that the assumption (2.20) is satisfied and that one has

$$\begin{aligned}
&\left(\int_0^t \omega^q g_0^{\frac{q}{2}} w^{-\frac{q-2}{2}}\, ds \right)^{2/q} \int_t^\infty \omega^{-2} g^{-1}\, ds \\
&\qquad \le \left(\frac{C_2}{C_1} \right)^2 \left(\int_0^t (1+s)^q w^{-\frac{q-2}{2}}\, ds \right)^{2/q} \frac{1}{1+t}.
\end{aligned}$$

Hence (4.4) follows from Theorem 2.17. ☐

We are left with the

Proof of Lemma 4.2. A direct calculation shows that

$$\omega(t) = \alpha_j \, \cos(\mu(t-j)) + \beta_j \, \cos(\mu(j+1-t)), \quad j < t < j+1,$$

with $\mu := \sqrt{\lambda_b}$, $\alpha_0 := 1$, $\beta_0 := 0$ and

$$\alpha_{j-1} \cos\mu + \beta_{j-1} = \alpha_j + \beta_j \cos\mu \,, \quad -\alpha_{j-1} = b\, \beta_j \,.$$

This can be rewritten as

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} b^{-\frac12} \begin{pmatrix} 2 & b^{\frac12} \\ -b^{-\frac12} & 0 \end{pmatrix} \begin{pmatrix} \alpha_{j-1} \\ \beta_{j-1} \end{pmatrix},$$

and by induction one easily finds that

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} b^{-\frac{j}{2}} \begin{pmatrix} j+1 & j\, b^{\frac12} \\ -j\, b^{-\frac12} & -j+1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}.$$

This implies

$$\omega(t) = g_0(t)^{-\frac12} (j+1) \left(\cos(\mu(t-j)) - \frac{j}{j+1} b^{-\frac12} \cos(\mu(j+1-t)) \right)$$

if $j < t < j+1$, and hence

$$\omega(t) \; \sim \; g_0(t)^{-\frac12} \, (1+t) \, \varphi(t), \quad t \to \infty, \tag{4.5}$$

where φ is periodic with period 1 and

$$\varphi(t) = \cos\mu t - b^{-\frac12} \cos(\mu(1-t)), \quad 0 < t < 1.$$

The estimates

$$\frac{b^{\frac12} - b^{-\frac12}}{b^{\frac12} + b^{-\frac12}} \; \geq \; \varphi(t) \; \geq \; b^{-\frac12} \, \frac{b^{\frac12} - b^{-\frac12}}{b^{\frac12} + b^{-\frac12}} > 0, \quad 0 < t < 1\,,$$

and the asymptotics (4.5) imply that (4.3) holds for all sufficiently large t. On the other hand, by the Sturm oscillation theorem (or by direct calculation) ω is bounded and bounded away from zero on compacts. This proves the lemma. □

5. Estimates on the number of eigenvalues

5.1. **Proof of Theorem 2.15.** Our goal in this section is to prove the statements of Theorem 2.15. An important ingredient will be weighted Hardy-Sobolev inequalities. The characterization of all admissible weights is independently due to Bradley, Maz'ya and Kokilashvili. The constant in (5.3) below is due to Opic. We refer to [29, Thm. 6.2] for the proof and further historical remarks.

Proposition 5.1. *Let $2 \leq q \leq \infty$. The inequality*

$$\left(\int_0^\infty |w(r) u(r)|^q \, dr \right)^{2/q} \leq S^2 \int_0^\infty |v(r) u'(r)|^2 \, dr \tag{5.1}$$

holds for all absolutely continuous functions u on $[0,\infty)$ with $\lim_{r\to\infty} u(r) = 0$ if and only if

$$T := \sup_{r>0} \left(\int_0^r |w(s)|^q \, ds \right)^{1/q} \left(\int_r^\infty |v(s)|^{-2} \, ds \right)^{1/2} < \infty. \tag{5.2}$$

In this case, the sharp constant S in (5.1) satisfies

$$T \leq S \leq \left(1 + \frac{q}{2}\right)^{1/q} \left(1 + \frac{2}{q}\right)^{1/2} T. \tag{5.3}$$

If $q = \infty$, then (5.2) means

$$T := \sup_{r>0} \left(\sup_{0\le s\le r} |w(s)| \right) \left(\int_r^\infty |v(s)|^{-2}\, ds \right)^{1/2} < \infty,$$

and in (5.3) one has $T = S$. Now everything is in place to give the

Proof of Theorem 2.15. Let $w \ge 0$ such that M defined in (2.15) is finite. Then Proposition 5.1 yields for all $u \in H^1(\mathbb{R}_+, g)$,

$$\left(\int_0^\infty |u|^q g^{\frac{q}{2}} w^{-\frac{q-2}{2}}\, dt \right)^{2/q} \le S^2 \int_0^\infty |u'|^2 g\, dt, \tag{5.4}$$

where

$$M \le S^2 \le \left(1 + \frac{q}{2}\right)^{2/q} \left(1 + \frac{2}{q}\right) M.$$

We now use an argument in the spirit of [14] to deduce (2.16) from (5.4). Let ω be the solution of $-(g\omega')' - V\omega g = 0$ that satisfies the boundary condition $\omega'(0) = 0$. By Sturm-Liouville theory (see, e.g., [35, Thm. 14.2]) the number of zeros of ω coincides with the number N of negative eigenvalues of $A_g - V$. Denote these zeros by $0 < a_1 < a_2 < \ldots < a_N < \infty$ and apply (5.4) to $u\omega\chi_{(a_j,a_{j+1})}$. Integrating by parts and using Hölder's inequality (noting that $1/p + 2/q = 1$) we obtain

$$\begin{aligned}\left(\int_{a_j}^{a_{j+1}} |\omega|^q g^{\frac{q}{2}} w^{-\frac{q-2}{2}}\, dt \right)^{2/q} &\le S^2 \int_{a_j}^{a_{j+1}} |\omega'|^2 g\, dt = S^2 \int_{a_j}^{a_{j+1}} V|\omega|^2 g\, dt \\ &\le S^2 \left(\int_{a_j}^{a_{j+1}} V^p w\, dt \right)^{1/p} \left(\int_{a_j}^{a_{j+1}} |\omega|^q\, g^{\frac{q}{2}} w^{-\frac{q-2}{2}}\, dt \right)^{2/q}.\end{aligned}$$

This implies that

$$1 \le S^{2p} \int_{a_j}^{a_{j+1}} V^p w\, dt\,, \quad \forall\, j = 1, \ldots N\,.$$

Summing this inequality over all intervals (a_j, a_{j+1}) we obtain

$$N(A_g - V) \le S^{2p} \int_0^\infty V_+^p w\, dt.$$

This proves (2.16) and shows that the sharp constant satisfies $C(w) \le S^{2p}$. The lower bound $C(w) \ge S^{2p}$ follows from Theorem 7.1 below. This implies also that (2.16) does not hold if $M = \infty$ and completes the proof. □

For later reference we include

Example 5.2. Assume that g satisfies (2.23) for some $d > 2$. Then for any $1 \le a < \infty$

$$N(A_g - V) \le C_a \int_0^\infty V_+^{\frac{1+a}{2}} (1+t)^a\, dt$$

where

$$\left(\frac{c_1}{c_2}\right)^{\frac{1+a}{2}} M_a^{\frac{1+a}{2}} \le C_a \le \frac{(2a)^a}{(a+1)^{\frac{a+1}{2}} (a-1)^{\frac{a-1}{2}}} \left(\frac{c_2}{c_1}\right)^{\frac{1+a}{2}} M_a^{\frac{1+a}{2}}\,.$$

and

$$\begin{aligned}M_a &:= \sup_{t>0} \left(\int_0^t (1+s)^{\frac{(d-1)(a+1)-2a}{a-1}}\, ds \right)^{\frac{a-1}{a+1}} \int_t^\infty (1+s)^{-d+1}\, ds \\ &= \left(\frac{a-1}{a+1}\right)^{\frac{a-1}{a+1}} (d-2)^{-\frac{2a}{a+1}}.\end{aligned}$$

(For $a = 1$ one has $(c_1/c_2)M_1 \le C_1 \le (c_2/c_1)M_1$ and $M_1 := (d-2)^{-1}$.) This follows by choosing $w(t) = (1+t)^a$ and $q = 2(a+1)/(a-1)$ after elementary calculations.

It is also illustrative to include another proof of estimate (2.17) in Example 2.16: The Birman-Schwinger principle implies

$$N(A_g - V) \le \operatorname{tr}_{L^2(\mathbb{R}_+,gdt)} \left(V_+^{\frac12} A_g^{-1} V_+^{\frac12} \right). \tag{5.5}$$

Since the operator $V_+^{\frac12} A_g^{-1} V_+^{\frac12}$ is non-negative, we have

$$\operatorname{tr}_{L^2(\mathbb{R}_+,gdt)} \left(V_+^{\frac12} A^{-1} V_+^{\frac12} \right) = \int_0^\infty G(t,t)\, V(t)_+\, g(t)\, dt, \tag{5.6}$$

where $G(t,t)$ is the diagonal of the Green function of the operator A. It follows from Sturm-Liouville theory (see, e.g., [35, Thm. 7.8]) that

$$G(t,t) = \frac{u_1(t)\, u_2(t)}{g(t) W(t)}\,,$$

where u_1, u_2 are two linearly independent solutiuons of $-(gu')' = 0$ and $W = u_1' u_2 - u_1 u_2'$ is their Wronskian. A direct calculation gives

$$u_1(t) = 1, \quad u_2(t) = \int_t^\infty \frac{ds}{g(s)}\,, \quad W(t) = \frac{1}{g(t)}\,.$$

In view of (5.5) and (5.6) this yields estimate (2.17).

5.2. **Proof of Theorem 2.17.** In this subsection we are working under the assumptions (2.14), (2.18) and (2.20) of Theorem 2.17. Recall that ω is the 'ground state' of the operator A. Since g may be non-smooth (it is a step function in the case of the tree) the differential equation (2.19) has to be understood in quadratic form sense, i.e.,

$$\int_0^\infty \omega' f' g\, dt = \lambda(A) \int_0^\infty \omega f g\, dt \tag{5.7}$$

for all $f \in H^1(\mathbb{R}_+, g)$ with compact support in $[0,\infty)$. The following identity is usually called ground state representation.

Lemma 5.3. *For any* $h = \omega^{-1} f \in \omega^{-1} H^1(\mathbb{R}_+, g)$,

$$\int_0^\infty |f'|^2 g\, dt - \lambda(A) \int_0^\infty |f|^2 g\, dt = \int_0^\infty |h'|^2 \omega^2 g\, dt. \tag{5.8}$$

We include a sketch of the proof for the sake of completeness.

Proof. It suffices to consider $h \in C_0^\infty(\overline{\mathbb{R}_+})$. Then

$$|(\omega h)'|^2 = \omega^2 |h'|^2 + \omega' (\omega |h|^2)'$$

and (5.8) follows from (5.7) with $f = \omega |h|^2$. $\square$

With (5.8) at hand we can proceed to the

Proof of Theorem 2.17. We denote by B the operator in $L_2(\mathbb{R}_+, \omega^2 g)$ corresponding to the quadratic form

$$\int_0^\infty |h'|^2 \omega^2 g\, dt$$

with form domain $H^1(\mathbb{R}_+, \omega^2 g)$. Then by the ground state representation (5.3) and Glazman's lemma (see e.g. [4, Thm. 10.2.3])

$$N(A - V - \lambda(A)) = N(B - V),$$

and the result follows from Theorem 2.15. $\square$

6. Sobolev interpolation inequalities

In this section we fix a parameter $d \geq 1$ and study inequalities of the form

$$\left(\int |u|^q (1+t)^{\beta q-1}\,dt\right)^{2/q} \leq K(q,\beta,d)\left(\int |u'|^2(1+t)^{d-1}\,dt\right)^{\theta} \left(\int |u|^2(1+t)^{d-1}\,dt\right)^{1-\theta} \tag{6.1}$$

for all $u \in H^1(\mathbb{R}_+,(1+t)^{d-1})$. We are interested in the values of β and q for which this inequality holds. We always fix

$$\theta := \frac{d-2\beta}{2}. \tag{6.2}$$

In the endpoint case $q=\infty$ we use the convention that (6.1) means

$$\sup |u|^2(1+t)^{2\beta} \leq K(\infty,\beta,d)\left(\int |u'|^2(1+t)^{d-1}\,dt\right)^{\theta}\left(\int |u|^2(1+t)^{d-1}\,dt\right)^{1-\theta}$$

for all $u \in H^1(\mathbb{R}_+,(1+t)^{d-1})$. Note that this makes sense even in the special case $\beta = 0$ (where the product βq in (6.1) is not well-defined).

Theorem 6.1. *Let $d \geq 1$ and $\frac{d-2}{2} \leq \beta \leq \frac{d}{2}$.*

1. *If $1 < d \leq 2$ and $0 < \beta \leq \frac{d-1}{2}$, or if $d > 2$ and $\frac{d-2}{2} \leq \beta \leq \frac{d-1}{2}$, then (6.1) holds for all $2 \leq q \leq \infty$.*
2. *If $d \geq 1$ and $\frac{d-1}{2} < \beta \leq \frac{d}{2}$, then (6.1) holds for all $2 \leq q \leq \left(\beta - \frac{d-1}{2}\right)^{-1}$.*
3. *If $1 \leq d < 2$ and $\beta = 0$, then (6.1) holds for $q = \infty$.*
4. *If $1 \leq d \leq 2$ and $-\frac{2-d}{2} \leq \beta \leq 0$, then (6.1) does* not *hold for $2 \leq q < \infty$.*
5. *If $1 \leq d < 2$ and $-\frac{2-d}{2} \leq \beta < 0$, or if $d = 2$ and $\beta = 0$, then (6.1) does* not *hold for $q = \infty$.*
6. *If $d \geq 1$ and $\frac{d-1}{2} < \beta \leq \frac{d}{2}$, then (6.1) does* not *hold for $\left(\beta - \frac{d-1}{2}\right)^{-1} < q \leq \infty$.*

We refer to Figure 1 below for the region of allowed parameters.

Remark 6.2. In (6.1) the exponent $\beta q - 1$ of the weight on the left hand side is coupled to the interpolation exponent θ in (6.2). This is in a certain sense optimal. Indeed, if the inequality

$$\left(\int |u|^q(1+t)^{\sigma-1}\,dt\right)^{2/q} \leq K\left(\int |u'|^2(1+t)^{d-1}\,dt\right)^{\theta}\left(\int |u|^2(1+t)^{d-1}\,dt\right)^{1-\theta}$$

holds for some $\sigma > 0$ and all $u \in H^1(\mathbb{R}_+,(1+t)^{d-1})$, then necessarily $\sigma \leq q(d-2\theta)/2$. (To see this put $u(t) = v(lt)$ and let $l \to 0$.) Note that with the value (6.2) of θ one has $q(d-2\theta)/2 = \beta q$.

We break the proof into several lemmas which prove inequality (6.1) in the endpoint cases.

Lemma 6.3. *If $1 < d \leq 2$ and $0 < \beta \leq \frac{d-1}{2}$, or if $d > 2$ and $\frac{d-2}{2} \leq \beta \leq \frac{d-1}{2}$, then (6.1) holds for $q = 2$ with the constant*

$$K(2,\beta,d) = \beta^{-d+2\beta}.$$

Proof. Integration by parts shows

$$\begin{aligned}\int |u|^2(1+t)^{2\beta-1}\,dt &= (-\beta)^{-1}\Re\int \overline{u}u'\left((1+t)^{2\beta}-1\right)\,dt\\ &\leq \beta^{-1}\int |u||u'|(1+t)^{2\beta}\,dt.\end{aligned}$$

We shall assume now that $\beta<\frac{d-1}{2}$. The proof in the case of equality follows along the same lines. Then $p:=\frac{d-2\beta}{d-1-2\beta}$ satisfies $1<p<\infty$, and by Hölder we can continue to estimate

$$\begin{aligned}\int |u|^2(1+t)^{2\beta-1}\,dt \leq \beta^{-1}&\left(\int |u|^2(1+t)^{2\beta-1}\,dt\right)^{1/p}\\ &\times\left(\int |u|^{\frac{p-2}{p-1}}|u'|^{\frac{p}{p-1}}(1+t)^{\frac{2\beta(p-1)+1}{p-1}}\,dt\right)^{\frac{p-1}{p}}.\end{aligned}$$

By the definition of p one has

$$\frac{2\beta(p-1)+1}{p-1}=\frac{(d-1)(p-2)}{2(p-1)}+\frac{(d-1)p}{2(p-1)},$$

and hence again by Hölder,

$$\begin{aligned}&\int |u|^{\frac{p-2}{p-1}}|u'|^{\frac{p}{p-1}}(1+t)^{\frac{2\beta(p-1)+1}{p-1}}\,dt\\ &\quad\leq\left(\int |u|^2(1+t)^{(d-1)}\,dt\right)^{\frac{p-2}{2(p-1)}}\left(\int |u'|^2(1+t)^{(d-1)}\,dt\right)^{\frac{p}{2(p-1)}}.\end{aligned}$$

This proves the inequality with the claimed constant. □

Lemma 6.4. *If $1<d\leq 2$ and $0<\beta\leq\frac{d-1}{2}$, or if $d>2$ and $\frac{d-2}{2}\leq\beta\leq\frac{d-1}{2}$, then* (6.1) *holds for $q=\infty$ with the constant*

$$K(\infty,\beta,d)=\left(\frac{2}{d-2\beta}\right)^{d-2\beta}\left(\frac{d-1-2\beta}{2\beta}\right)^{d-1-2\beta}.$$

Here we use the convention that $0^0=1$. Hence for $\beta=\frac{d-1}{2}$ one has $K(\infty,\frac{d-1}{2},d)=2$.

Proof. Let $p:=\frac{2}{d-2\beta}$. Our assumptions imply that $\frac{2}{d}<p\leq 2$ if $1<d\leq 2$ and $1\leq p\leq 2$ if $d>2$. By Schwarz we estimate

$$\begin{aligned}|u(t)|^p &\leq p\int_t^\infty |u|^{p-1}|u'|\,ds\\ &\leq p\left(\int_0^\infty |u'|^2(1+s)^{d-1}\,ds\right)^{1/2}\left(\int_t^\infty |u|^{2(p-1)}(1+s)^{-d+1}\,ds\right)^{1/2}\end{aligned}$$

This proves the assertion if $p=1$, i.e., $\beta=\frac{d-2}{2}$ and $d>2$. If $p=2$ the assertion follows from the estimate

$$\int_t^\infty |u|^{2(p-1)}(1+s)^{-d+1}\,ds\leq(1+t)^{-2(d-1)}\int_0^\infty |u|^{2(p-1)}(1+s)^{d-1}\,ds.$$

In the remaining case $1<p<2$ we use Hölder to obtain

$$\begin{aligned}&\int_t^\infty |u|^{2(p-1)}(1+s)^{-d+1}\,ds\\ &\quad\leq\left(\int_t^\infty (1+s)^{-\frac{(d-1)p}{2-p}}\,ds\right)^{2-p}\left(\int_0^\infty |u|^2(1+s)^{d-1}\,ds\right)^{p-1}\\ &\quad=\left(\frac{2-p}{dp-2}\right)^{2-p}(1+t)^{-dp+2}\left(\int_0^\infty |u|^2(1+s)^{d-1}\,ds\right)^{p-1}.\end{aligned}$$

This proves the inequality with the claimed constant. □

Lemma 6.5. *If $1 \le d < 2$ and $\beta = 0$, then (6.1) holds for $q = \infty$ with the constant*
$$K(\infty, 0, d) = (2d)^d (2(d-1))^{-2(d-1)} (2-d)^{-1}.$$

Proof. If $d = 1$ one has
$$|u(t)|^2 \le 2 \int_t^\infty |u||u'|\, ds \le 2 \left(\int_0^\infty |u|^2\, ds \right)^{1/2} \left(\int_0^\infty |u'|^2\, ds \right)^{1/2}, \tag{6.3}$$
as claimed. If $1 < d < 2$ then we estimate for any $R > 0$
$$\begin{aligned} |u(t)|^2 &\le 2 \left(\int_0^R |u||u'|\, ds + \int_R^\infty |u||u'|\, ds \right) \\ &\le 2 \left(\left(\int_0^\infty |u'|^2 s^{d-1}\, ds \right)^{1/2} \|u\|_\infty \left(\int_0^R s^{-d+1}\, ds \right)^{1/2} \right. \\ &\qquad \left. + \left(\int_0^\infty |u'|^2 s^{d-1}\, ds \right)^{1/2} \left(\int_0^\infty |u|^2 s^{d-1}\, ds \right)^{1/2} R^{-d+1} \right) \\ &= 2 \left(\int_0^\infty |u'|^2 s^{d-1}\, ds \right)^{1/2} \left[\|u\|_\infty (2-d)^{-1/2} R^{(2-d)/2} \right. \\ &\qquad \left. + \left(\int_0^\infty |u|^2 s^{d-1}\, ds \right)^{1/2} R^{-d+1} \right] .v \end{aligned}$$
Choosing t such that $u(t) = \|u\|_\infty$ and optimizing with respect to R we find that
$$\|u\|_\infty^2 \le K \left(\int |u'|^2 s^{d-1}\, ds \right)^{d/2} \left(\int |u|^2 s^{d-1}\, ds \right)^{(2-d)/2}$$
with the constant as claimed. This implies (and, by a scaling argument, is actually equivalent to) the assertion. □

Lemma 6.6. *If $d = 1$ and $0 < \beta \le \frac{1}{2}$, then (6.1) holds for $q = 2$ with the constant*
$$K(2, \beta, 1) = 2^{-2\beta} (1 - 2\beta)^{2\beta - 1} \beta^{-1}.$$

Proof. It suffices to prove the inequality
$$\int |v|^2 s^{-1+2\beta}\, ds \le K \left(\int |v'|^2\, ds \right)^{(1-2\beta)/2} \left(\int |v|^2 s^{d-1}\, ds \right)^{(1+2\beta)/2}.$$
(Actually, a scaling argument as in the proof of Theorem 6.1 below shows that this inequality is equivalent – with the same constant – to the inequality (6.1).) Using (6.3) we estimate for any $R > 0$
$$\begin{aligned} \int |v|^2 s^{-1+2\beta}\, ds &\le \|v\|_\infty^2 \int_0^R s^{-1+2\beta}\, ds + \|v\|_2^2 R^{-1+2\beta} \\ &\le \beta^{-1} \|v\| \|v'\| R^{2\beta} + \|v\|_2^2 R^{-1+2\beta}, \end{aligned}$$
and the claim follows by optimizing with respect to R. □

Proof of Theorem 6.1. First assume that $1 < d \le 2$ and $0 < \beta \le \frac{d-1}{2}$, or $d > 2$ and $\frac{d-2}{2} \le \beta \le \frac{d-1}{2}$. The assertion (1) has been proved in the endpoint cases $q = 2$ and $q = \infty$ in Lemmas 6.3 and 6.4. Estimating
$$\int |u|^q (1+t)^{\beta q - 1}\, dt \le \sup \left(|u|^{q-2} (1+t)^{\beta(q-2)} \right) \int |u|^2 (1+t)^{\beta 2 - 1}\, dt$$

we obtain the assertion (1) also in the case $2 < q < \infty$.

Next we prove the assertion (2). Let $d \geq 1$, $\frac{d-1}{2} < \beta \leq \frac{d}{2}$. First assume that $q = 2$. If $d = 1$, the inequality holds by Lemma 6.6. If $d > 1$ we put $p : (2\beta - d + 1)^{-1}$ and apply Hölder's inequality to find

$$\int |u|^2 (1+t)^{2\beta-1}\, dt \leq \left(\int |u|^2 (1+t)^{d-2}\, dt \right)^{\frac{p-1}{p}} \leq \left(\int |u|^2 (1+t)^{d-1}\, dt \right)^{\frac{1}{p}}.$$

Estimating the first factor on the right side using Lemma 6.3 with $\beta \leq \frac{d-1}{2}$ we obtain the assertion in the case $q = 2$. Now let $q = \left(\beta - \frac{d-1}{2}\right)^{-1}$. We estimate

$$\int |u|^q (1+t)^{2\beta-1}\, dt \leq \left(\sup |u|^2 (1+t)^{d-1} \right)^{\frac{d-2\beta}{2\beta-d+1}} \left(\int |u|^2 (1+t)^{d-1}\, dt \right).$$

The first factor on the right side is estimated using (6.3) if $d = 1$ and using Lemma 6.4 with $\beta \leq \frac{d-1}{2}$ if $d > 1$. This proves the assertion in the case $q \left(\beta - \frac{d-1}{2}\right)^{-1}$. By Hölder's inequality we obtain (2) for arbitrary $2 < q < \left(\beta - \frac{d-1}{2}\right)^{-1}$.

The assertion (3) was proved in Lemma 6.5.

To prove the negative results let $1 \leq d \leq 2$ and assume that (6.1) holds for some β and some $2 \leq q \leq \infty$. We apply the inequality to the function $u(t) = v(t/l)$, where v is a smooth function with bounded support. Letting $l \to \infty$ we obtain

$$\left(\int |v|^q s^{\beta q - 1}\, ds \right)^{2/q} \leq K(q, \beta, d) \left(\int |v'|^2 s^{d-1}\, ds \right)^{\theta} \left(\int |v|^2 s^{d-1}\, ds \right)^{1-\theta}. \tag{6.4}$$

Note that v can be chosen non-zero in a neighborhood of the origin. We deduce that the inequality can not hold for $\beta < 0$, and if $q < \infty$ then it can not hold for $\beta = 0$ either. This proves assertion (4) and the first part of (5). It remains to prove that (6.1) or equivalently (6.4) does not hold if $d = 2$, $\beta = 0$ and $q = \infty$. This follows by considering the sequence of trial functions $v_n(s) := \min\{1, (\log n - \log s)/\log n\}$ if $s \leq n$ and $v_n(s) = 0$ for $s > n$.

Finally, to prove (6) let $d \geq 1$ and $\frac{d-1}{2} < \beta \leq \frac{d}{2}$. Again we apply the inequality to the function $u(t) = v(t/l)$, where v is a smooth function with bounded support. As $l \to 0$, the left hand side decays like $l^{2/q}$ (resp. becomes constant when $q = \infty$) whereas the right hand side decays like $l^{2\beta - d + 1}$. We conclude that the condition $q \leq \left(\beta - \frac{d-1}{2}\right)^{-1}$ is necessary for (6.1) to hold. $\square$

7. Estimates for moments of eigenvalues

Our goal in this section will be to prove the Lieb-Thirring bounds in Theorem 2.18. Throughout we will assume that g has power-like growth in the sense of (2.23) for some $d \geq 1$.

7.1. **One-bound-state inequalities and duality.**

A first step towards Theorem 2.18 is to prove that the lowest eigenvalue of the operator $A_g - V$ can be estimated from below by a weighted L_p-norm of the potential.

Theorem 7.1. *Assume* (2.23) *for some* $d \geq 1$ *and let* $a, \gamma \geq 0$. *Then the inequality*

$$\sup \operatorname{spec} \left((A_g - V)_-^{\gamma} \right) \leq C \int_{\mathbb{R}_+} V(t)_+^{\gamma + \frac{a+1}{2}} g(t)^{\frac{a}{d-1}}\, dt, \qquad C = C(\gamma, a, d, c_1, c_2), \tag{7.1}$$

holds for all V if and only if a and γ satisfy the assumptions of Theorem 2.18.

In the case $\gamma = 0$, inequality (7.1) means that if $\int_{\mathbb{R}_+} V(t)_+^{\frac{a+1}{2}} (1+t)^a\, dt < C^{-1}$ then $\inf \operatorname{spec}(\mathcal{A}_g - V) \geq 0$.

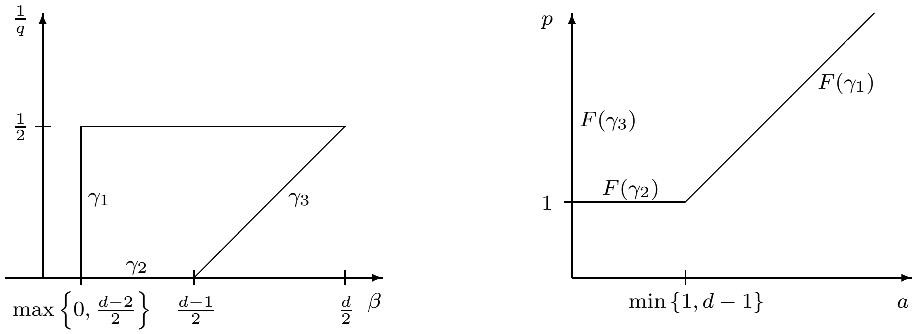

FIGURE 1. Parameter range of the Sobolev interpolation inequalities. Here $F(1/q, \beta) = (q, (d-1-2\beta)q+2)/(q-2)$ and $F(\gamma_1) = \{(p,a) : p = (a+1)/\min\{2,d\}\}$.

The proof of Theorem 7.1 is based on the following abstract duality result, which does not use the explicit form of g.

Proposition 7.2. *Assume that the parameters $a > -1$, $\gamma \geq 0$ and $p := \gamma + \frac{1+a}{2}$ are related to the parameters $2 < q \leq \infty$, $\frac{d-2}{2} \leq \beta < \frac{d}{2}$ and $\theta := \frac{d-2\beta}{2}$ by*

$$p = \frac{q}{q-2}, \qquad q = \frac{2p}{p-1}, \qquad a = \frac{(d-1-2\beta)q+2}{q-2}, \qquad \beta = \frac{dp-1-a}{2p}, \tag{7.2}$$

see Figure 1. Then the inequality (7.1) *holds if and only if*

$$\left(\int |u|^q g^{\frac{\beta q-1}{d-1}}\,dt\right)^{2/q} \leq K(q,\beta,g)\left(\int |u'|^2 g\,dt\right)^{\theta}\left(\int |u|^2 g\,dt\right)^{1-\theta}. \tag{7.3}$$

for all $u \in H^1(\mathbb{R}_+, g)$. In this case, the constants are related by

$$K(q,\beta,g) = L^{\frac{q-2}{q}}\theta^{-\theta}(1-\theta)^{\theta-1} \tag{7.4}$$

In the case $q = \infty$, (7.3) means

$$\sup |u|^2 g^{\frac{2\beta}{d-1}} \leq L\theta^{-\theta}(1-\theta)^{\theta-1}\left(\int |u'|^2 g\,dt\right)^{\theta}\left(\int |u|^2 g\,dt\right)^{1-\theta}.$$

for all $u \in H^1(\mathbb{R}_+, g)$.

Proof of Proposition 7.2. Below we will only consider $u \in H^1(\mathbb{R}_+, g)$ and $V \geq 0$ such that the right hand side of (7.1) is finite.

Equation (7.1) holds for all V if and only if

$$\frac{\int |u'|^2 g\,dt - \int V|u|^2 g\,dt}{\int |u|^2 g\,dt} \geq -\left(L\int V^p g^{\frac{a}{d-1}}\,dt\right)^{2/(2p-1-a)} \tag{7.5}$$

holds for all u and V. Write $V = \alpha W$ with α such that

$$\int W^p g^{\frac{a}{d-1}}\,dt = 1. \tag{7.6}$$

Thus (7.5) holds for all u and V if and only if

$$\sup_{\alpha>0}\left(\alpha\int W|u|^2 g\,dt - \alpha^{\frac{1}{1-\theta}} L^{\frac{q-2}{q(1-\theta)}}\int |u|^2 g\,dt\right) \leq \int |u'|^2 g\,dt \tag{7.7}$$

holds for all u and all W obeying (7.6). By calculating the supremum we find that (7.8) holds for all u and all W obeying (7.6) if and only if

$$\sup\left\{\int W|u|^2 g\,dt : \int W^p g^{\frac{a}{d-1}}\,dt = 1\right\} \leq K\left(\int |u'|^2 g\,dt\right)^{\theta}\left(\int |u|^2 g\,dt\right)^{1-\theta} \tag{7.8}$$

for all u. By duality

$$\sup\left\{\int W|u|^2 g\,dt : \int W^p g^{\frac{a}{d-1}}\,dt = 1\right\} = \left(\int |u|^q g^{\frac{\beta q-1}{d-1}}\,dt\right)^{2/q}.$$

Hence (7.8) holds for all u if and only if (7.3) holds for all u. $\square$

Proof of Theorem 7.1. Assumption (2.23) implies that Theorem 6.1 holds (with another constant) if $(1+t)^{d-1}$ is replaced by g. Simple arithmetic shows that if (q,β) and (p,a) are related as in (7.2), then the allowed values (q,β) in Theorem 6.1 correspond to the allowed values (p,a) in Theorem 2.18. In view of Proposition 7.2 we obtain the assertion of Theorem 7.1. $\square$

Remark 7.3. We claim that if the inequality

$$\sup\operatorname{spec}\left((A_g - V)_-^{\gamma}\right) \leq C\int_{\mathbb{R}_+} V(t)_+^{\gamma+\frac{1+a}{2}} g(t)^b\,dt \tag{7.9}$$

holds for some $\gamma \geq 0$, $a \geq 0$, $b \geq 0$ and all V, then one has necessarily $b \geq a/(d-1)$. Obviously, the inequality becomes weaker as b increases. This motivates why we restrict ourselves to the case $b = a/(d-1)$ when considering the inequalities (2.24).

To prove the claim we apply a similar duality argument as in the proof of Proposition 7.2 and find that (7.9) is equivalent to

$$\left(\int |u|^q g^{\frac{p-b}{p-1}}\,dt\right)^{2/q} \leq K\left(\int |u'|^2 g\,dt\right)^{\theta}\left(\int |u|^2 g\,dt\right)^{1-\theta}, \quad u \in H^1(\mathbb{R}_+, g),$$

where p and q are as in that proposition and $\theta = (p-\gamma)/p$. It follows from Remark 6.2 that $(d-1)(p-b)/(p-1)+1 \leq q(d-2\theta)/2$. This means $b \geq a/(d-1)$, as claimed.

7.2. **Estimates in the case of a Dirichlet boundary condition.** Here we will establish the analog of Theorem 2.18 when a Dirichlet instead of a Neumann boundary condition is imposed at the origin. More precisely we denote by $A_{\mathscr{D}}$ the self-adjoint operator in $L_2(\mathbb{R}_+, g)$ corresponding to the quadratic form (2.13) with form domain $H_0^1(\mathbb{R}_+, g) := \{f \in H^1(\mathbb{R}_+, g) : f(0) = 0\}$. In this case the conditions for the validity of a Lieb-Thirring inequality become much simpler than in Theorem 2.18.

Theorem 7.4. *Assume* (2.23) *for some* $d \geq 1$ *and let* $a \geq 0$, $\gamma > 0$. *Then the inequality*

$$\operatorname{tr}(A_{\mathscr{D}} - V)_-^{\gamma} \leq L\int_{\mathbb{R}_+} V(t)_+^{\gamma+\frac{a+1}{2}}(1+t)^a\,dt, \qquad L = L(\gamma, a, d, c_1, c_2), \tag{7.10}$$

holds for all V *if and only if* a, γ *satisfy*

$$\begin{aligned} \gamma &\geq \frac{1-a}{2} \qquad &&\text{if } 0 \leq a < 1,\\ \gamma &> 0 &&\text{if } a \geq 1. \end{aligned}$$

We emphasize that we did not discuss the case $\gamma = 0$ in Theorem 7.4.

When proving Theorem 7.4 we will use a result from [8] and [9] concerning the operator $-\frac{d^2}{dr^2} - \frac{1}{4r^2} - W$ in $L_2(\mathbb{R}_+)$ with a Dirichlet boundary condition at the origin.

Proposition 7.5. *Let $0 \le a < 1$ and $\gamma \ge \frac{1-a}{2}$ or $a \ge 1$ and $\gamma > 0$, then*

$$\operatorname{tr}\left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - W\right)_-^\gamma \le C_{\gamma,a} \int_{\mathbb{R}_+} W(r)_+^{\gamma+\frac{1+a}{2}} r^a \, dr \tag{7.11}$$

with a constant $C_{\gamma,a}$ independent of W.

Before we can apply this estimate we have to replace the (possibly non-smooth) function g by a smooth function with the same behavior at infinity. To this end we consider the self-adjoint operator $\mathfrak{B}_{\mathscr{D}}$ in $L_2(\mathbb{R}_+)$ corresponding to the quadratic form

$$\begin{aligned} \mathfrak{b}_{\mathscr{D}}[u] &= \int_{\mathbb{R}_+} \left| \left(\frac{u(t)}{(1+t)^{(d-1)/2}} \right)' \right|^2 (1+t)^{d-1} \, dt \\ &= \int_{\mathbb{R}_+} \left(|u'|^2 + \frac{(d-1)(d-3)|u|^2}{4(1+t)^2} \right) dt \end{aligned} \tag{7.12}$$

defined on $H_0^1(\mathbb{R}_+)$. We prove now that the eigenvalues of $A_{\mathscr{D}} - V$ can be estimated – modulo a change in the coupling constant – from above and below by those of $\mathfrak{B}_{\mathscr{D}} - V$. A similar idea was used in [13] to obtain Lieb-Thirring inequalities for Schrödinger operators with background potentials.

Lemma 7.6. *Assume* (2.23) *for some $d \ge 1$ and put $\beta := c_2/c_1$. Then for any $V \ge 0$ and $\gamma \ge 0$ we have*

$$\operatorname{tr}(\mathfrak{B}_{\mathscr{D}} - \beta^{-1} V)_-^\gamma \le \operatorname{tr}(A_{\mathscr{D}} - V)_-^\gamma \le \operatorname{tr}(\mathfrak{B}_{\mathscr{D}} - \beta V)_-^\gamma. \tag{7.13}$$

Proof. We shall prove that for any $\tau > 0$

$$N(\mathfrak{B}_{\mathscr{D}} - \beta^{-1} V + \tau) \le N(A_{\mathscr{D}} - V + \tau) \le N(\mathfrak{B}_{\mathscr{D}} - \beta V + \tau). \tag{7.14}$$

This will imply the statement since

$$\operatorname{tr} T_-^\gamma = \gamma \int_0^\infty \tau^{\gamma-1} N(T+\tau) \, d\tau.$$

To prove the second inequality in (7.14) suppose that

$$\int_{\mathbb{R}_+} \left(|f'|^2 - V|f|^2 \right) g \, dt < -\tau \int_{\mathbb{R}_+} |f|^2 g \, dt$$

for some $f \in H_0^1(\mathbb{R}_+, g)$. Using (2.23) we conclude that

$$\begin{aligned} c_1 \int_{\mathbb{R}_+} \left(|f'|^2 - \beta V |f|^2 \right) (1+t)^{d-1} \, dt &\le \int_{\mathbb{R}_+} \left(|f'|^2 - V|f|^2 \right) g \, dt \\ &\le -\tau \int_{\mathbb{R}_+} |f|^2 g \, dt \\ &\le -\tau c_1 \int_{\mathbb{R}_+} |f|^2 (1+t)^{d-1} \, dt \, . \end{aligned}$$

It follows from Glazman's lemma (see, e.g., [4, Thm. 10.2.3]) that

$$N(A_{\mathscr{D}} - V + \tau) \le N(\tilde{A}_{\mathscr{D}} - \beta V + \tau),$$

where $\tilde{A}_{\mathscr{D}}$ denotes the operator $L^2(\mathbb{R}_+, (1+t)^{d-1})$ corresponding to the quadratic form $\int |f'|^2 (1+t)^{d-1} \, dt$ with a Dirichlet boundary condition. Since $\tilde{A}_{\mathscr{D}} - \beta V$ in $L^2(\mathbb{R}_+, (1+t)^{d-1})$ is unitarily equivalent to $\mathfrak{B}_{\mathscr{D}} - \beta V$ in $L^2(\mathbb{R}_+)$, we obtain the second inequality in (7.14). The first one is proved similarly. $\square$

Proof of Theorem 7.4. We may assume that $V \geq 0$. We use the operator inequality

$$-\frac{d^2}{dr^2} - \frac{1}{4r^2} \leq -\frac{d^2}{dr^2} + \frac{(d-1)(d-3)}{4r^2}\,.$$

(Note also that the form domain of the operator on the LHS is strictly larger than $H_0^1(\mathbb{R}_+)$.) It follows that

$$\operatorname{tr}(\mathfrak{B}_{\mathscr{D}} - \beta V)_-^\gamma \;\leq\; \operatorname{tr}\left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - \beta V\right)_-^\gamma\,.$$

The result now follows from Proposition 7.5 and Lemma 7.6. □

7.3. **Putting it all together.** Finally we give the

Proof of Theorem 2.18. The variational principle implies that the eigenvalues of the Dirichlet and the Neumann problems interlace (see, e.g., [4, Thm. 10.2.5]). Hence

$$\operatorname{tr}(A-V)_-^\gamma \leq \sup\operatorname{spec}\left((A-V)_-^\gamma\right) + \operatorname{tr}(A_{\mathscr{D}} - V)_-^\gamma.$$

We estimate the first term on the right hand side via Theorem 7.1 (recall (2.23)) and the second one via Theorem 7.4. This completes the proof of the 'if' part of the statement. The 'only if' statement follows from the 'only if' part of Theorem 7.1. □

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