Mobility of Bodies in Contact—I: A New 2nd Order Mobility Index for Multiple-Finger Grasps

Elon Rimón (elon@robby.caltech.edu) Joel Burdick (jwb@robby.caltech.edu)
Dept. of Mechanical Engineering, CALTECH, Mail Code 104-44, Pasadena, CA 91125

Abstract

Using a configuration-space approach, this paper develops a coordinate invariant 2nd order mobility index for a body, $\mathcal{B}$, in frictionless contact with finger bodies $\mathcal{A}_1, \dots, \mathcal{A}_k$. The index captures the inherent mobility of $\mathcal{B}$ in an equilibrium grasp due to 2nd order, or surface curvature, effects. It differentiates between grasps which are deemed equivalent by the classical 1st order theories, but are physically different. In a companion paper we discuss applications and provide physical justification for using 2nd order immobility effects.

1 Introduction

We analyze the mobility of a body, $\mathcal{B}$, in frictionless contact with finger bodies $\mathcal{A}_1, \dots, \mathcal{A}_k$. Traditionally, the mobility of bodies in contact has been studied using 1st order theories that are based on instantaneous notions of force and velocity. For example, Ohwovoriole and Roth [6] describe the relative motions of bodies in contact via Screw Theory, which is a 1st order theory. Using 1st order notions, Reuleaux (1876) [7], Somoff (1900) [12], Mishra et. al (1987) [5], and Markenscoff et. al (1990) [3], derived bounds on the number of frictionless point-fingers required for force closure, which is one means to immobilize an object. However, the “maximal” and “minimal” 3-finger frictionless grasps shown in Figs 1(a) and 1(b) illustrate why 1st order theories are often inadequate in practice. 1st order theories indicate that both grasps possess one instantaneous degree of mobility, though intuition dictates that the object is immobilized in the minimal grasp.

The source of deficiency in the 1st order theories is that relative mobility is *not an infinitesimal notion, but a local one*. One must consider the local motions of the object, not only the tangential aspects of the motions, as employed by 1st order theories. In [9] we described a novel configuration-space based approach for analyzing the mobility of bodies in contact. The work in [9] also introduced the new concept of 2nd order mobility analysis, and a novel coordinate invariant 2nd order mobility index for 2-finger grasps. The index measures the effective mobility of a grasped object due to 2nd order, or surface curvature, effects.

This paper generalizes the 2nd order mobility index of [9] from 2 fingers to k fingers. To our knowledge, this is

Figure 1. (a) maximal 3-finger equilibrium grasp (b) minimal 3-finger grasp

the first complete second order mobility analysis to appear in the literature. The 2nd order analysis of rolling kinematics for an object undergoing roll-slide contact with two fingers has appeared in [10]. 1st and 2nd order considerations have appeared in work by Trinkle et. al [13, 14]. Their work, however, is concerned with stability criteria for contacting bodies under the influence of gravity, and is only tangentially related to our problem.

Importantly, the 2nd mobility order index developed below differentiates between alternative grasps having the same number of fingers. For example, it determines that the object in Fig. 1(a) is not immobilized, while it is completely immobilized in Fig. 1(b). This new analysis tool suggests that 2nd order effects can be used to immobilize a grasped object with fewer fingers than predicted by the 1st order theories. Applications of this insight for multi-fingered grasp planning and workpiece fixturing is discussed in the companion paper.

2 Configuration Space Terminology

A rigid object, $\mathcal{B}$, is located in an n -dimensional physical space, $\mathcal{W} = \mathbb{R}^n$ ($n = 2$ or 3), amidst rigid obstacles $\mathcal{A}_1, \dots, \mathcal{A}_k$. We interpret these obstacles as “finger bodies” and typically assume that they are stationary. The fingers contact $\mathcal{B}$ with *frictionless contact* and can deliver any force in the direction normal to the boundary of $\mathcal{B}$. Points on $\mathcal{B}$, when expressed in $\mathcal{B}$ ’s body coordinates, are denoted r . Points in $\mathcal{W}$ are denoted x . Every rigid placement of $\mathcal{B}$ in $\mathcal{W}$ is described by the *forward kinematics map*:

$$x = X(r, (d, R)) \triangleq Rr + d. \quad (1)$$

We use a configuration-space (c-space) based approach to analyze the mobility of $\mathcal{B}$ about its contact configuration with the fingers. For the prob-

lems addressed in this paper, one can focus on the c-space of $\mathcal{B}$, rather than the composite c-space of the $k+1$ rigid bodies. $\mathcal{B}$'s c-space is the smooth manifold $\mathcal{C} = \mathbb{R}^n \times SO(n)$, where $SO(n)$ is the group of rotations of $\mathbb{R}^n$. $SO(3)$ is parametrized by $\theta \in \mathbb{R}^3$, via the exponential map parametrization, where $\hat{\theta} = \theta/\|\theta\|$ is the axis of rotation, and $\|\theta\|$ is the angle of rotation [4]. We regard $SO(2)$ as a subgroup of $SO(3)$ with rotation axis $\hat{\theta}$ normal to the plane. The translation, d , is parametrized by $\mathbb{R}^3$. Thus $\mathcal{C}$ is parametrized by $\mathbb{R}^m$, where $m = \frac{1}{2}n(n+1)$. Points in $\mathbb{R}^m$ are denoted $q = (d, \theta)$, where d is the translation and θ is the parametrization of $SO(n)$. The forward kinematics map is thus written $X(r, q)$. $\mathcal{B}(q)$ denotes the subset of $\mathcal{W}$ occupied by $\mathcal{B}$ when its configuration is q .

Given a finger body $\mathcal{A}_i \subset \mathbb{R}^n$, its corresponding c-space obstacle, denoted $\mathcal{CA}_i$, is the set of all configurations q such that $\mathcal{B}(q)$ intersects $\mathcal{A}_i$: $\mathcal{CA}_i \triangleq \{q \in \mathbb{R}^m : \mathcal{B}(q) \cap \mathcal{A}_i \neq \emptyset\}$. The boundary of $\mathcal{CA}_i$, denoted $\mathcal{S}_i$, consists of those configurations q where the surfaces of $\mathcal{B}(q)$ and $\mathcal{A}_i$ touch each other, while their interiors are disjoint. We assume that each $\mathcal{S}_i$ is smooth. The free configuration space (or freespace), $\mathcal{F}$, is the space that remains after removing all of the c-obstacle interiors from $\mathcal{C}$. Admissible motions of $\mathcal{B}$ are thus curves in $\mathcal{F}$. If there are k fingers holding $\mathcal{B}$, the object configuration, q_0 , lies on the boundary of $\mathcal{F}$, at the intersection of the c-obstacle boundaries, $q_0 \in \bigcap_{i=1}^k \mathcal{S}_i$.

3 Review of Rigid Body Mobility Theory

The mobility theory introduced in [9] is based on the concept of the *free motions* of $\mathcal{B}$. Let $\mathcal{B}$ be held by k stationary and frictionless fingers $\mathcal{A}_1, \dots, \mathcal{A}_k$. The free motions of $\mathcal{B}$ are those local motions along which $\mathcal{B}$ either breaks away from or roll-slides on the surface of the fingers. More precisely, let q_0 be $\mathcal{B}$'s configuration, and let $\mathcal{D}$ be a small m -dimensional ball centered at q_0 . The free motions are c-space paths that emanate from q_0 and lie in $\mathcal{D} \cap \mathcal{F}$. The 1st order (i.e. tangents and tangent hyperplanes) and 2nd order (i.e. curvatures and curvature forms) properties of these paths and the c-obstacle boundaries can be directly related to the mobility of $\mathcal{B}$ at q_0 . The 1st order properties of these curves can be related to other well known 1st order theories, such as Screw Theory. We cast these notions in a c-space framework, as this new interpretation is the basis for the novel consideration of 2nd and higher order aspects of mobility.

To make this *key concept* precise and to aid in the physical understanding of this mobility theory, we introduce the following signed distance function. It measures the distance of a configuration point q from $\mathcal{S}_i$, and is a convenient representation of rigidity in c-space,

$$d_i(q) \triangleq \begin{cases} \text{dst}(q, \mathcal{S}_i) & \text{if } q \text{ is outside of } \mathcal{CA}_i \\ 0 & \text{if } q \text{ is on } \mathcal{S}_i \\ -\text{dst}(q, \mathcal{S}_i) & \text{if } q \text{ is in the interior of } \mathcal{CA}_i \end{cases}$$

Figure 2. The 1st order approximation to the free motions of $\mathcal{B}$ at q_0 . $\alpha(0)$ and $\beta(0)$ are 1st order roll-slide motions. $\alpha(t)$ locally lies in $\mathcal{F}$, $\beta(t)$ locally lies in the c-obstacle.

where $\text{dst}(q, \mathcal{S}_i)$ is the Euclidean distance of q from $\mathcal{S}_i$ in c-space. Note that $d_i(q_0) = 0$ when $\mathcal{B}$ contacts $\mathcal{A}_i$. The unit normal to $\mathcal{S}_i$ at q_0 , pointing outward with respect to $\mathcal{CA}_i$, is denoted $\hat{n}_i(q_0)$.

3.1 1st Order Free Motions

Let $\alpha(t)$ be a c-space path such that $\alpha(0) = q_0$, $q_0 \in \mathcal{S}_i$, and let $\dot{\alpha}(0) = \dot{q}$. The 1st order free motions are related to the following first-order Taylor expansion of d_i along $\alpha(t)$:

$$(d_i \circ \alpha)(t) = d_i(q_0) + (\nabla d_i(q_0) \cdot \dot{q})t + O(t^2).$$

First note that $d_i(q_0) = 0$, since $q_0 \in \mathcal{S}_i$. Second, it can be shown that $\|\nabla d_i(q_0)\| = 1$ for the Euclidean distance [1] [p. 66]. It can also be verified that $\nabla d_i(q_0)$ is equal to the unit normal $\hat{n}_i(q_0)$ (Figure 2). Thus:

$$(d_i \circ \alpha)(t) = (\hat{n}_i(q_0) \cdot \dot{q})t + O(t^2). \quad (2)$$

(A formula for $\hat{n}_i(q_0)$ is provided by Theorem 1 below.)

Definition 1 ([9]) Let $\mathcal{B}$ be in contact with $\mathcal{A}_i$ at configuration q_0 . The 1st order free motions of $\mathcal{B}$ at q_0 is the set (halfspace of $T_{q_0} \mathbb{R}^m$)

$$M_{q_0}^1(\mathcal{CA}_i) \triangleq \{\dot{q} \in T_{q_0} \mathbb{R}^m : \hat{n}_i(q_0) \cdot \dot{q} \geq 0\}.$$

The halfspace's boundary, $\{\dot{q} \in T_{q_0} \mathbb{R}^m : \hat{n}_i(q_0) \cdot \dot{q} = 0\}$, is called the set of 1st order roll-slide motions. Its interior, $\{\dot{q} \in T_{q_0} \mathbb{R}^m : \hat{n}_i(q_0) \cdot \dot{q} > 0\}$, is termed the set of 1st order escape motions. For k fingers,

$$M_{q_0}^1(\mathcal{CA}_1, \dots, \mathcal{CA}_k) \triangleq \bigcap_{i=1}^k M_{q_0}^1(\mathcal{CA}_i)$$

(we shall use $M_{q_0}^1$ for $M_{q_0}^1(\mathcal{CA}_1, \dots, \mathcal{CA}_k)$).

It can be shown [9] that $M_{q_0}^1$ is coordinate invariant. Given two parametrizations of the same c-space, q and

$\bar{q}$, let $\bar{q}_0$ be the parametrization of q_0 in the $\bar{q}$ coordinates, and let $(\bar{\cdot})$ denote objects in the $\bar{q}$ -space. Then $\dot{q} \in M_{q_0}^1(\mathcal{CA}_1, \dots, \mathcal{CA}_k)$ iff $\dot{\bar{q}} \in M_{\bar{q}_0}^1(\bar{\mathcal{CA}}_1, \dots, \bar{\mathcal{CA}}_k)$. The 1st order free motions also admit the following geometrical interpretation. If $\dot{q} \in M_{q_0}^1(\mathcal{CA}_i)$ is a 1st order escape motion, its corresponding path, $\alpha(t)$ with $\alpha(0) = q_0$ and $\dot{\alpha}(0) = \dot{q}$, locally lies in the freespace, for all $t \in [0, \epsilon]$, for some $\epsilon > 0$. That is, $\mathcal{B}(\alpha(t))$ locally breaks away from $\mathcal{A}_i$, no matter the value of the higher derivatives of $\alpha(t)$. If $\dot{q} \in M_{q_0}^1(\mathcal{CA}_i)$ is a 1st order roll-slide motion, it is *not possible* to determine from (2) if $\alpha(t)$ locally lies in the freespace or if it enters $\mathcal{CA}_i$. For example, the curves $\alpha(t)$ and $\beta(t)$ in Fig. 2 have the same 1st order roll-slide motion, yet $\alpha(t)$ locally lies in the freespace, while $\beta(t)$ does not.

3.2 2nd Order Free Motions

We now consider the 2nd order characteristics of the free-motion curves, as these lead to our novel 2nd order mobility theory. Let $\alpha(t)$ be a smooth c-space path such that $\alpha(0) = q_0$, $\dot{\alpha}(0) = \dot{q}$, and $\ddot{\alpha}(0) = \ddot{q}$. The 2nd order free motions are related to the following second-order Taylor expansion of d_i along $\alpha(t)$:

$$\begin{aligned} (d_i \circ \alpha)(t) &= d_i(q_0) + (\nabla d_i(q_0) \cdot \dot{q})t \\ &+ \frac{1}{2}(\dot{q}^T D^2 d_i(q_0) \dot{q} + \nabla d_i(q_0) \cdot \ddot{q})t^2 + O(t^3) = \\ &(\hat{n}_i(q_0) \cdot \dot{q})t + \frac{1}{2}(\dot{q}^T D^2 d_i(q_0) \dot{q} + \hat{n}_i(q_0) \cdot \ddot{q})t^2 + O(t^3), \end{aligned} \quad (3)$$

since $d_i(q_0) = 0$ and $\nabla d_i(q_0) = \hat{n}_i(q_0)$. If $\hat{n}_i(q_0) \cdot \dot{q} > 0$ i.e., when $\dot{q}$ is a 1st order escape motion, the linear term in (3) locally determines the sign of $(d_i \circ \alpha)(t)$. The 2nd order term affects the sign of $(d_i \circ \alpha)(t)$ only when $\dot{\alpha}(0) = \dot{q}$ lies in $T_{q_0} \mathcal{S}_i$ i.e., when $\dot{q}$ is a 1st order roll-slide motion. Since $\nabla d_i(q) = \hat{n}_i(q)$ for all $q \in \mathcal{S}_i$, $\dot{q}^T D^2 d_i(q_0) \dot{q} = \dot{q}^T [D\hat{n}_i(q_0)]\dot{q}$ for all $\dot{q} \in T_{q_0} \mathcal{S}_i$. The quadratic form $\dot{q}^T [D\hat{n}_i(q_0)]\dot{q}$ in (3) is the *curvature form* of $\mathcal{S}_i$ at q_0 . An expression for the curvature form was given in [9].

It is clear from (3) that the free-motion curves are determined to 2nd order by their *velocity and acceleration* at q_0 . The collection $(q_0, \dot{q}, \ddot{q})$ of all velocities and accelerations of paths $\alpha(t)$ such that $\alpha(0) = q_0$ is called the 2nd jet space of $\mathcal{C} = \mathbb{R}^m$ at q_0 , and is denoted $J_{q_0}^2 \mathcal{C}$.

Definition 2 ([9]) The 2nd order free motions of $\mathcal{B}$ at q_0 is the subset of $(\dot{q}, \ddot{q})$ in $J_{q_0}^2 \mathcal{C}$ satisfying

$$M_{q_0}^2(\mathcal{CA}_i) \triangleq \{(\dot{q}, \ddot{q}) \in J_{q_0}^2 \mathcal{C} : \hat{n}_i(q_0) \cdot \dot{q} = 0 \text{ and } \dot{q}^T [D\hat{n}_i(q_0)]\dot{q} + \hat{n}_i(q_0) \cdot \ddot{q} \geq 0\}.$$

Analogous to the 1st order case, pairs $(\dot{q}, \ddot{q})$ that satisfy $\hat{n}_i(q_0) \cdot \dot{q} = 0$ and $\dot{q}^T [D\hat{n}_i(q_0)]\dot{q} + \hat{n}_i(q_0) \cdot \ddot{q} = 0$ are called

¹ $J^2 \mathcal{C} = \bigcup_q J_q^2 \mathcal{C}$, called the 2nd jet bundle of $\mathcal{C}$, is a “two-tiered” vector bundle: every q is a base point for a fiber of points $(\dot{q}, \ddot{q}) \in \mathbb{R}^m \times \mathbb{R}^m$, and every $(q, \dot{q})$ is a base point for the fiber of $\ddot{q} \in \mathbb{R}^m$.

2nd order roll-slide motions, and the other pairs in $M_{q_0}^2(\mathcal{CA}_i)$ are termed 2nd order escape motions. For k fingers,

$$M_{q_0}^2(\mathcal{CA}_1, \dots, \mathcal{CA}_k) \triangleq \bigcap_{i=1}^k M_{q_0}^2(\mathcal{CA}_i).$$

Any 1st order escape motion is automatically free to 2nd order. Thus, our definition of 2nd order free motions focuses on those free-motion curves that are 1st order roll-slide motions. The 2nd order free motions possess the following two important properties. First, they are *coordinate invariant*. Given two parametrizations of the same c-space, q and $\bar{q}$ as mentioned above, we have that

$$(\dot{q}, \ddot{q}) \in M_{q_0}^2(\mathcal{CA}_1, \dots, \mathcal{CA}_k) \text{ iff } (\dot{\bar{q}}, \ddot{\bar{q}}) \in M_{\bar{q}_0}^2(\bar{\mathcal{CA}}_1, \dots, \bar{\mathcal{CA}}_k).$$

$M_{q_0}^2$ has the following geometrical interpretation. If $(\dot{q}, \ddot{q}) \in M_{q_0}^2(\mathcal{CA}_i)$ is a 2nd order escape motion, its corresponding path, $\alpha(t)$ with $\alpha(0) = q_0$, $\dot{\alpha}(0) = \dot{q}$, $\ddot{\alpha}(0) = \ddot{q}$, locally lies in the freespace, for all $t \in [0, \epsilon]$, for some $\epsilon > 0$. If $(\dot{q}, \ddot{q}) \in M_{q_0}^2(\mathcal{CA}_i)$ is a 2nd order roll-slide motion it is *not possible* to determine from (3) if $\alpha(t)$ locally lies in $\mathcal{F}$ or if it enters $\mathcal{CA}_i$.

4 Relevant Rigid Body Dynamics

The real-world forces acting on $\mathcal{B}$ due to finger contacts are denoted $F_1(x_1), \dots, F_k(x_k)$, where x_i is the contact point between $\mathcal{A}_i$ and $\mathcal{B}$. These forces give rise to a net generalized wrench acting in $\mathcal{B}$'s c-space, which we denote as $w(q; F_1, \dots, F_k)$. The change in $\mathcal{B}$'s kinetic energy due to such forces is [9]:

$$\frac{d}{dt} K(q, \dot{q}) = w(t) \cdot \dot{q}. \quad (4)$$

where the kinetic energy, K , is given by $K(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q}$, where $M(q)$ is the $m \times m$ inertia matrix of $\mathcal{B}$. We neglect potential energy effects, such as gravity, and assume that the wrench w is generated solely by finger forces. Eq. (4) leads to the interpretation of wrenches as *covectors* i.e., linear functions $w : T_q \mathbb{R}^m \rightarrow \mathbb{R}$ mapping velocity vectors $\dot{q}$ to scalars $\dot{K} = w \cdot \dot{q}$, representing the *work* done by the force.

Theorem 1 below relates the net wrench on $\mathcal{B}$ to the geometry of the fingers' c-obstacles. It is well known, and is based on the following *virtual work principal* [11]. Let $X_r(q) = X(r, q)$ denote the forward kinematics map when r is kept fixed on $\mathcal{B}$, while only q varies. Then

$$w(q, F_i) \cdot \dot{q} = F_i(x_i) \cdot X_{r_i}(q), \quad (5)$$

where r_i is the description of the contact point x_i in $\mathcal{B}$'s body coordinates.

Theorem 1 ([9]) The wrench due to a single-finger contact force $F_i(x_i)$ acting on $\mathcal{B}(q)$ is:

$$w(q, F_i) = [DX_{r_i}(q)]^T F_i(x_i), \quad (6)$$

where $DX_{r_i}(q) = \frac{d}{dq} X_{r_i}(q)$, and $w(q, F_i)$ is written as a column vector. If $F_i(x_i)$ is normal to the surface of

$B(q)$ at x_i and is pushing into $B(q)$, then w is normal to S_i , pointing outward with respect to CA_i (Figure 2):

$$w(q, F_i) = \lambda_i \hat{n}_i(q) \quad \text{for } \lambda_i \geq 0.$$

More generally, if $k \geq 1$ fingers push on $B(q)$ with normal forces $F_1(x_1), \dots, F_k(x_k)$, the net wrench w is orthogonal to the subspace generated by intersection of the tangent hyperplanes to the individual finger c-obstacles, and is given by

$$w(q; F_1, \dots, F_k) = \sum_{i=1}^k w(q, F_i) = \sum_{i=1}^k \lambda_i \hat{n}_i(q) \quad \lambda_i \geq 0.$$

At a k -fingered equilibrium grasp the net wrench on B must be zero. From Theorem 1 we get:

Corollary 4.1 Let k fingers push on B with normal forces $F_1(x_1), \dots, F_k(x_k)$, while B is at configuration q_0 . q_0 can be made an equilibrium grasp by suitable choice of the finger force magnitudes iff zero lies in the convex hull of $\hat{n}_1(q_0), \dots, \hat{n}_k(q_0)$,

$$0 = \lambda_1 \hat{n}_1(q_0) + \dots + \lambda_k \hat{n}_k(q_0), \quad (7)$$

for $\lambda_1, \dots, \lambda_k$ such that $\lambda_i \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$.

5 The Mobility Index of k -Fingered Grasps

A mobility index [9] is an integer-valued function that measures the mobility, or effective number of degrees of freedom, of B when held in an equilibrium grasp. First, in Section 5.1, we limit our analysis to non-redundant finger arrangements (though the ensuing results can be extended to include redundant finger arrangements). In Section 5.2 we review the fact that the 1st mobility index is identical for all k -fingered equilibrium grasps. The inability of 1st order theories to differentiate between the mobility of grasps which are intuitively different motivates our development of the 2nd order mobility index in Section 5.3. This index captures the mobility of B , taking the bodies' surface curvature into consideration. Throughout this section B is held at a frictionless equilibrium grasp by fingers $A_1, \dots, A_k$, at configuration q_0 .

5.1 Essential Finger Arrangements

It will be convenient to restrict attention only to the following generic equilibrium grasps, termed *essential*.

Definition 3 A finger A_i is essential to the grasp if its force is necessary to maintain the equilibrium. I.e., if its c-obstacle normal $\hat{n}_i(q_0)$ is necessary for spanning the origin in (7).

The next lemma guarantees the uniqueness of the λ_i 's. The proofs of the next two lemmas are given in [8].

Lemma 5.1 $\lambda_1, \dots, \lambda_k$ in (7) are non-zero and unique iff the fingers $A_1, \dots, A_k$ are all essential.

Figure 3. (a) non-essential equilibrium arrangement (b) an essential one

Thus, once the position of the fingers relative to B is fixed in an essential equilibrium arrangement, the λ_i 's depend only on the finger c-obstacle normals $\{\hat{n}_1(q_0), \dots, \hat{n}_k(q_0)\}$, which are fixed for a given contact arrangement. We will also need the following lemma.

Lemma 5.2 If all the k fingers participating in the equilibrium grasp are essential, then any collection of $k-1$ finger c-obstacle normals from $\{\hat{n}_1(q_0), \dots, \hat{n}_k(q_0)\}$ is linearly independent.

Remark: It follows from the lemma that k fingers are all essential only for $k \leq m+1$ fingers i.e., $k \leq 4$ for planar grasps and $k \leq 7$ for 3D grasps. There is no problem with this upper bound, as we shall see that generic finger arrangements involving $k \geq m+1$ fingers completely immobilize B due to 1st order effects.

The equilibrium grasp shown in Fig. 3(a) is an example of a non-essential finger arrangement, since the equilibrium can be maintained with a single antipodal pair. Fig. 3(b) illustrates the notion that essential finger arrangements are generic. That is, for a non-essential equilibrium arrangement with $k \leq m+1$ fingers, almost any equilibrium arrangement obtained by small movement of the fingers about B becomes essential. We henceforth consider only equilibrium grasps with essential finger arrangements.

5.2 All k -Fingered Grasps have the Same 1st Order Mobility Index

According to Definition 1, B 's 1st order free motions lie in the intersection of the individual 1st order free motion halfspaces that are associated with the k contacting finger bodies.

Proposition 5.3 ([9]) The 1st order free motions of B at an equilibrium, $M_{q_0}^1$, form a subspace of $T_{q_0}R^m$, given by the intersection of the tangent hyperplanes to the finger c-obstacles:

$$M_{q_0}^1 = \bigcap_{i=1}^k T_{q_0} S_i.$$

If all the fingers are essential for the equilibrium, the dimension of $M_{q_0}^1$ is $m-k+1$, where $m=3$ or 6 . By definition, $M_{q_0}^1 = \{0\}$ when $m-k+1 < 0$.

The only 1st order free motions of B at an equilibrium grasp are 1st order roll-slide motions with respect to every finger. The 1st order mobility index of an equilibrium grasp is defined as the dimension of $M_{q_0}^1$:

$$m_{q_0}^1 \triangleq \max\{0, m - k + 1\}. \quad (8)$$

$m_{q_0}^1$ is *coordinate invariant*, a fact that follows from the coordinate invariance of the 1st order free motions. A key fact expressed by (8) is that $m_{q_0}^1$ is *identical for all k -fingered grasps*. For example, the 3-finger grasps in Fig. 1 have the same 1st order mobility index of unity (since $m = k = 3$). Only the 2nd order index introduced in the next section differentiates between grasps involving the same number of fingers.

5.3 The 2nd Order Mobility Index

At the equilibrium configuration q_0 , the c-obstacle boundaries intersect each other, $q_0 \in \bigcap_{i=1}^k \mathcal{S}_i$. Moreover, the 1st order free motions of $\mathcal{B}$ are $M_{q_0}^1 = \bigcap_{i=1}^k T_{q_0} \mathcal{S}_i$. The second-order geometry of each $\mathcal{S}_i$ at q_0 is captured by its curvature form:

$$\kappa_i(q_0, \dot{q}) = \dot{q}^T D^2 d_i(q_0) \dot{q} = \dot{q}^T D \hat{n}_i(q_0) \dot{q} \quad \dot{q} \in T_{q_0} \mathcal{S}_i,$$

We now use the λ_i 's in the equilibrium condition (7) to define a weighted sum of the fingers' c-space curvature forms.

Definition 4 Let $\lambda_1, \dots, \lambda_k$ be the coefficients of the equilibrium equation (7). The **c-space relative distance** of an equilibrium grasp is the real-valued function d_{rel} defined by

$$d_{\text{rel}}(q) \triangleq \lambda_1 d_1(q) + \dots + \lambda_k d_k(q).$$

The **c-space relative curvature form** for the equilibrium grasp is the quadratic form:

$$\kappa_{\text{rel}}(q_0, \dot{q}) \triangleq \dot{q}^T D^2 d_{\text{rel}}(q_0) \dot{q} = \sum_{i=1}^k \lambda_i \kappa_i(q_0, \dot{q}),$$

such that $\dot{q} \in M_{q_0}^1$.

d_{rel} and κ_{rel} are well defined since the λ_i 's depend only on the geometrical arrangement of the fingers with respect to $\mathcal{B}$ (by the essential grasp assumption). Since $\hat{n}_i(q_0) = \nabla d_i(q_0)$ for all i , we have from (7) that

$$\nabla d_{\text{rel}}(q_0) = \lambda_1 \nabla d_1(q_0) + \dots + \lambda_k \nabla d_k(q_0) = 0.$$

Hence d_{rel} has a *critical point* at q_0 . We shall see below that this guarantees the desired coordinate invariance. But first let us define the 2nd order mobility index.

Definition 5 The 2nd order mobility index of an equilibrium, denoted $m_{q_0}^2$, is the number of non-negative eigenvalues of the c-space relative curvature, $D^2 d_{\text{rel}}(q_0)$, at q_0 .

The relative curvature is defined on $M_{q_0}^1$. Hence $m_{q_0}^1$ is an upper bound on the possible values of $m_{q_0}^2$ i.e., $0 \leq m_{q_0}^2 \leq m_{q_0}^1$. If $m_{q_0}^1 = 0$, the body is completely immobilized to 1st order, and the 2nd order index carries no useful information. The 2nd order index is always useful for planar grasps involving $k = 2, 3$ fingers, and

for 3D grasps involving $k = 2, \dots, 6$ fingers. In these cases, $\mathcal{B}$ is not immobilized to 1st order ($m_{q_0}^1 > 0$), but may be immobilized to 2nd order.

The following two propositions discuss important properties of $m_{q_0}^2$. The first is its coordinate invariance. Let $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}$ be a smooth function with a critical point at q_0 , $\nabla \sigma(q_0) = 0$. The *Morse index* of σ at q_0 is the number of negative eigenvalues of $D^2 \sigma(q_0)$. It can be shown that the Morse index is coordinate invariant, as following lemma states:

Lemma 5.4 Let $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}$ be a smooth map, let $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a smooth coordinate transformation (a diffeomorphism) such that $q = f(\bar{q})$, and let $\bar{\sigma}(\bar{q}) = (\sigma \circ f)(\bar{q})$. There is one-to-one correspondence between the critical points of σ and $\bar{\sigma}$,

$$\nabla \bar{\sigma}(\bar{q}_0) = 0 \quad \text{iff} \quad \nabla \sigma(q_0) = 0.$$

In addition, the sign of the eigenvalues of $D^2 \sigma$, when evaluated at a critical point, is **preserved**. That is, for every $\bar{q} \in T_{\bar{q}_0} \mathbb{R}^m$ there exists $\dot{q} \in T_{q_0} \mathbb{R}^m$, $\dot{q} = Df(\bar{q}_0) \bar{\dot{q}}$, such that

$$\dot{q}^T [D^2 \sigma(q_0)] \dot{q} = \bar{\dot{q}}^T [D^2 \bar{\sigma}(\bar{q}_0)] \bar{\dot{q}}. \quad (9)$$

In our case $\sigma(q) = d_{\text{rel}}(q)$ is the relative distance function, and $\nabla d_{\text{rel}}(q_0) = 0$ at an equilibrium grasp q_0 . The lemma, together with the coordinate invariance of $M_{q_0}^1$ (Section 3.1), implies the following invariance property:

Proposition 5.5 Let q and $\bar{q}$ be two parametrizations of the same c-space, related by $q = f(\bar{q})$. Let q_0 and $\bar{q}_0$ be the equilibrium configuration in the respective parametrizations. Then the 2nd order mobility index is **preserved** under the coordinate transformation i.e., $m_{q_0}^2 = m_{\bar{q}_0}^2$.

The following proposition provides a *key interpretation* for the 2nd order index. It is a main contribution of this paper.

Proposition 5.6 Let $\dot{q} \in M_{q_0}^1$ be a 1st order free motion of $\mathcal{B}$ at the equilibrium. If $\kappa_{\text{rel}}(q_0, \dot{q}) > 0$, there exists $\bar{q}$ such that the path $\alpha(t)$ with $\alpha(0) = q_0$, $\dot{\alpha}(0) = \dot{q}$, $\ddot{\alpha}(0) = \dot{q}$ locally lies in the **freespace**,

$$\alpha(t) \in \mathcal{F} \quad \text{for all } t \in [0, \epsilon], \text{ for some } \epsilon > 0.$$

If $\kappa(q_0, \dot{q}) < 0$, any $\bar{q}$ yields a path $\alpha(t)$ that locally lies outside the freespace,

$$\alpha(t) \in \mathbb{R}^m - \mathcal{F} \quad \text{for all } t \in (0, \epsilon], \text{ for some } \epsilon > 0.$$

Proof: By hypothesis, $q_0 \in \bigcap_{i=1}^k \mathcal{S}_i$ and $\dot{q} \in M_{q_0}^1 = \bigcap_{i=1}^k T_{q_0} \mathcal{S}_i$. Hence $d_i(q_0) = 0$ and $\dot{q} \perp \nabla d_i(q_0)$ for all i . The 2nd order expansion of $(d_i \circ \alpha)(t)$ is thus:

$$(d_i \circ \alpha)(t) = \frac{1}{2} \{ \dot{q}^T [D^2 d_i(q_0)] \dot{q} + \nabla d_i(q_0) \cdot \ddot{q} \} t^2,$$

for $i = 1, \dots, k$. For convenience, let $a_i = \dot{q}^T [D^2 d_i(q_0)] \dot{q}$.

In the first case $\lambda_1 a_1 + \dots + \lambda_k a_k > 0$, and we have to show that there exists $\vec{q}$ such that $a_i + \nabla d_i(q_0) \cdot \vec{q} > 0$ for all i . Consider the following set of linear equations,

$$a_2 + \nabla d_2 \cdot \vec{q} = \epsilon_2, \quad \dots, \quad a_k + \nabla d_k \cdot \vec{q} = \epsilon_k,$$

in which $\vec{q}$ is the variable and $\epsilon_2, \dots, \epsilon_k$ are parameters. Writing the equations in matrix form gives

$$[\nabla d_2 \dots \nabla d_k]^T \vec{q} = \vec{\epsilon} - \vec{a}, \quad (10)$$

where $\vec{\epsilon} = (\epsilon_2, \dots, \epsilon_k)^T$ and $\vec{a} = (a_2, \dots, a_k)^T$. Let $\vec{q}$ be chosen as a linear combination of $\nabla d_2, \dots, \nabla d_k$:

$$\vec{q} = \sigma_2 \nabla d_2 + \dots + \sigma_k \nabla d_k = [\nabla d_2 \dots \nabla d_k] \vec{\sigma},$$

where $\vec{\sigma} = (\sigma_2, \dots, \sigma_k)^T$ are parameters yet to be determined. Substituting for $\vec{q}$ in Eq. (10) gives

$$[\nabla d_2 \dots \nabla d_k]^T [\nabla d_2 \dots \nabla d_k] \vec{\sigma} = \vec{\epsilon} - \vec{a}.$$

All the fingers are essential, hence, according to Lemma 5.2, $\{\nabla d_2, \dots, \nabla d_k\}$ are linearly independent. This implies that the matrix $[\nabla d_2 \dots \nabla d_k]^T [\nabla d_2 \dots \nabla d_k]$ is invertible. Thus we may choose $\vec{\sigma}$ (and therefore $\vec{q}$) such that $\epsilon_2, \dots, \epsilon_k$ have any desired value. In particular, they may have any desired positive value, which implies that a $\vec{q}$ exists such that

$$a_i + \nabla d_i \cdot \vec{q} = \epsilon_i > 0 \quad \text{for } i = 2, \dots, k. \quad (11)$$

Last, we show that for ϵ_i sufficiently small, $a_1 + \nabla d_1 \cdot \vec{q}$ is positive as well. Since $\lambda_1 > 0$, we may equivalently show that $\lambda_1(a_1 + \nabla d_1 \cdot \vec{q}) > 0$. Using Eq. (7), we substitute $\sum_{i=2}^k -\lambda_i \nabla d_i$ for $\lambda_1 \nabla d_1$ in $\lambda_1(a_1 + \nabla d_1 \cdot \vec{q}) > 0$,

$$\lambda_1(a_1 + \nabla d_1 \cdot \vec{q}) = \lambda_1 a_1 - \lambda_2 \nabla d_2 \cdot \vec{q} - \dots - \lambda_k \nabla d_k \cdot \vec{q}.$$

Substituting for the terms $\lambda_i \nabla d_i \cdot \vec{q}$ for $i = 2, \dots, k$ according to (11) gives

$$\begin{aligned} \lambda_1(a_1 + \nabla d_1 \cdot \vec{q}) &= \lambda_1 a_1 - \sum_{i=2}^k \lambda_i(\epsilon_i - a_i) \\ &= \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_k a_k - \sum_{i=2}^k \lambda_i \epsilon_i > 0 \end{aligned}$$

for $\epsilon_2, \dots, \epsilon_k$ sufficiently small, since $\lambda_1 a_1 + \dots + \lambda_k a_k > 0$ by hypothesis.

Consider now the case where $\lambda_1 a_1 + \dots + \lambda_k a_k < 0$. We have to show that for all $\vec{q}$, there exists an i , $1 \leq i \leq k$, such that $a_i + \nabla d_i \cdot \vec{q} < 0$. Suppose to the contrary, that a $\vec{q}$ exists such that $a_i + \nabla d_i \cdot \vec{q} \geq 0$ for all i . Premultiplying each inequality by λ_i and then adding the resulting inequalities gives $\lambda_1 a_1 + \dots + \lambda_k a_k \geq 0$, contradicting the hypothesis. $\square$

Prop. 5.6 has the following interpretation. If $m_{q_0}^2 > 0$, then there exists accelerations of $\mathcal{B}$ such that it can break away from the k fingers. If $m_{q_0}^2 = 0$ (even though $m_{q_0}^1 > 0$), then there exists NO accelerations of $\mathcal{B}$ which allow it to break away from the fingers. The object is in fact immobile.

6 Conclusion

This paper introduced a configuration-space based methodology for analyzing the mobility of bodies in contact. Mobility can be characterized in this framework by the properties of curves lying in the free configuration space. Based on the analysis of these curves,

we introduced coordinate invariant first and second order *mobility indices*, which are integer value functions that capture the intrinsic mobility of an object held in and equilibrium grasp due to 1st and 2nd order effects.

We showed that the 1st order index (which is analogous to the definition of mobility traditionally applied to closed loop linkages) is determined solely by the number of fingers, k . Thus, all generic k -fingered equilibrium grasps look alike to 1st order. However, Fig. 1 clearly shows that this is not true. Consequently, *reasoning about the mobility of an object at an equilibrium grasp using forces and velocities implies too crude an approximation*. This deficiency motivated the 2nd order mobility theory developed in this work. We consider one ramification of this work here, while others are discussed in the companion paper.

Fig. 4 schematically illustrates the c-space obstacles corresponding to the 3-finger grasps of Fig. 1. In both cases $M_{q_0}^1$ is a 1-dimensional subspace, so that $0 \leq m_{q_0}^2 \leq 1$. However, it can be shown that $m_{q_0}^2 = 1$ for the maximal 3-finger grasp, while $m_{q_0}^2 = 0$ for the minimal grasp. Indeed, Fig. 4(a) depicts a path $\alpha(t)$ that start at q_0 and locally lies in the freespace. No such path emanating from q_0 exists in Fig. 4(b) (the exact c-space picture of Fig. 1(b) is shown in Figs 5 and 6). The discriminating nature $m_{q_0}^2$, coupled with the fact that 2nd order effects can effectively lower the mobility (as predicted by 1st order theories) of the object in an equilibrium grasp, leads to important applications. Consider, for example, exploitation of 2nd order effects for object immobilization.

We say that $\mathcal{B}$ is *completely immobile* if its configuration q_0 is completely isolated from the freespace by c-obstacles. $\mathcal{B}$ is *immobile to 1st order* if $m_{q_0}^1 = 0$. 1st order immobility is sufficient for complete immobility of $\mathcal{B}$. Almost all 2D or 3D objects can be held in frictionless force-closure grasp (which, it can be shown, is equivalent to 1st order immobility) by $m+1$ contact points, where m is the c-space dimension. Thus, 4 contact points suffice for 2D grasps and 7 for 3D grasps [3, 7, 12]. Moreover, $m+1$ is the smallest possible number when only the point-contact aspect of the fingers is considered.

However, our 2nd order mobility theory suggests that objects can be immobilized with less than $m+1$ frictionless fingers if curvature effects are taken into account. We say that an object held in equilibrium grasp is *immobile to 2nd order* if $m_{q_0}^2 = 0$. Czyzowicz et. al. [2] have shown that $n+1$ frictionless contact points suffice to completely immobilize almost any n -dimensional *polygonal* shape. In our language, their result can be interpreted to say that because of the flatness of the polygonal faces, almost any 2D or 3D polygonal object can be immobilized to 2nd order by $n+1$ frictionless point fingers. We can extend this result to a larger class of objects using the theory outlined in this paper.

Research in progress supports the following two conjectures. *If one is free to choose the fingers' point of contact, then generic smooth n -dimensional objects can be completely immobilized by $n+1$ frictionless fingers—*

Figure 4. (a) 3-finger grasp with $m_{q_0}^2 = 1$ (b) 3-finger grasp with $m_{q_0}^2 = 0$

Figure 5. A portion of the c-space of the minimal 3-fingered grasp (Fig. 1(b)). The dark lines indicate the location of the constant orientation slices shown in the figure below

i.e., 3 fingers (instead of 4 for force closure) for 2D grasps and 4 fingers (instead of 7 for force closure) for 3D grasps. Furthermore, in many fixturing applications it is conceivable that the surface geometry of the work holding fixtures can be chosen. In that case our theory indicates that: *If one is allowed to choose suitably concave finger tips, then generic smooth n -dimensional objects can be completely immobilized by n frictionless fingers.* That is, generic 2D objects can be immobilized with 2 frictionless fingers, while generic 3D bodies can be immobilized with 3 frictionless fingers. This possibility, to the best of our knowledge, has never appeared in the literature before.

References

- [1] F. H. Clarke. *Optimization and Nonsmooth Analysis*. SIAM Publication, 1990.
- [2] J. Urrutia J. Czyzowicz, I. Stojmenovic. Immobilizing a polytope. In *Lecture Notes in Computer Science*, volume 519, pages 214–227. 1991.
- [3] X. Markenscoff, L. Ni, and C. H. Papadimitriou. The geometry of grasping. *The International Journal of Robotics Research*, 9(1):61–74, Feb 1990.
- [4] J. M. McCarthy. *An Introduction to Theoretical Kinematics*. MIT Press, Cambridge, MA, 1990.
- [5] B. Mishra, J. T. Schwartz, and M. Sharir. On the existence and synthesis of multifinger positive grips. *Algorithmica*, 2:541–558, 1987.
- [6] M. S. Ohwovoriole and B. Roth. An extension of screw theory. *J. of Mechanical Design*, 103:725–735, 1981.
- [7] F. Reuleaux. *The Kinematics of Machinery*. Macmillan 1876, republished by Dover, NY, 1963.
- [8] E. Rimon and J. W. Burdick. Curvature effects of bodies in contact: Their impact on mobility and their forces of restraint. Tech report, ME Department, California Institute of Technology, Sept 1993.
- [9] E. Rimon and J. W. Burdick. Towards planning with force constraints: On the mobility of bodies in contact. In *Proc. IEEE Int. Conf. on Robotics and Automation*, pages 1:994–1000, Atlanta, GA, May 1993.
- [10] N. Sarkar, X. Yun, and V. Kumar. Dynamic control of 3-d rolling contacts in two-arm manipulation. In *Proc. IEEE Int. Conf. on Robotics and Automation*, pages 3:978–983, Atlanta, May 1993.
- [11] R. Segev and A. Ailon. A geometrical setting for the newtonian mechanics of robots. *J. of the Franklin Institute*, 322(3):173–183, Sept 1986.
- [12] P. Somoff. Über gebiete von schraubengeschwindigkeiten eines starren körpers bei verschiebener zahl von stützflächen. *Zeitschrift für Mathematik und Physik*, 45:245–306, 1900.
- [13] J. C. Trinkle, A. O. Farahat, and P. F. Stiller. First-order stability cells of frictionless rigid body systems. Tech report, CS Dept., Texas A&M, Jan. 1993.
- [14] J. C. Trinkle, A. O. Farahat, and P. F. Stiller. Second-order stability cells of frictionless rigid body systems. Tech report, CS Dept., Texas A&M, May 1993.

Figure 6. Constant orientation c-space slices of the minimal 3-fingered grasp. The slices are taken at $\theta = -25^\circ, 0^\circ, 25^\circ$.