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All higher curvature gravities can be bootstrapped from their linearizations

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Abstract

We show that the full covariant versions of higher curvature order gravities, like that of GR itself, can be derived by self-coupling from their linear, flat space, versions. Separately, we comment on the initial version of the bootstrap.

1 Introduction

One physically illuminating way of understanding General Relativity (GR) is as a free spin 2 $m = 0$ gauge field in flat space that must become self-coupled if it is to interact consistently with matter via its stress tensors. That is, unlike the $s = 1$ Maxwell theory, that can remain neutral when coupled to charges, but more like its Yang-Mills generalization, GR is intrinsically nonlinear because it must be self-interacting, i.e., "charged". This was shown [1] in closed form using first order, Palatini, formulation. Further, the requirement is applicable starting from any, not just flat, background, thereby allowing for a cosmological term [2] as well. Recently the question was raised whether other models, notably actions including quadratic curvatures, could also be bootstrapped [3]. This note shows that indeed they can. In the Appendix, we reconsider the first proposed realization of the bootstrap, and some of its problems.

2 Bootstrap

We begin with a brief resume of the relevant parts of [1]. The free $s = 2$ gauge field action in flat space has the schematic form

$$A_L[\eta, \Gamma, h] = \int \left[\eta^{\mu\nu}(\Gamma\Gamma)_{\mu\nu} + \Gamma^{\alpha}_{\mu\nu}\partial_\alpha h^{\mu\nu}\right]. \tag{1}$$

Indices are only shown to indicate the tensorial ranks of the fundamental variables, and will be omitted henceforth. Here η is the flat metric, Γ the affinity/connection, treated as an independent variable (first order, Palatini form) and $h^{\mu\nu}$ the (symmetric) contravariant tensor (density) field. [It should be recalled that although η is the flat metric, it has both tensor index positions and density weight (it need not be cartesian), whose character is then shared by the added h-field, so as to keep overall invariance.] Thus in (1), its stress tensor (or density) T becomes the source of the field itself (we omit matter for ease of notation). This quantity is given by the usual (Belinfante-Rosenfeld) definition as the variation of A with respect to the (background) metric η, so $T \sim \Gamma\Gamma$. At this point it bears repeating the often forgotten fact that the stress tensors of $s > 1$ gauge fields are NOT invariant under their abelian gauge transformations (not even on-shell). This is especially true of the spin 2 field, where (linear) elevators can annihilate Γ – hence T – at any point; T is only a Poincare tensor; fortunately, those facts in no way invalidate the bootstrap, whose final (total) action is invariant under the full diffeo group. As a first step, we include T's effect by coupling it to h, by the addition of a term $h\Gamma\Gamma$ to the integrand of (1), setting its – dimensionally required – coefficient to unity. At this stage, then, (1) has changed "slightly" to the cubic form

$$A_{NL}[g = \eta + h, \Gamma] = \int g^{\cdot\cdot}[\Gamma\Gamma + \partial\Gamma] = \int g^{\cdot\cdot}\mathrm{Ricci}_{\cdot\cdot}(\Gamma), \qquad g = \eta + h \tag{2}$$

simply by adding zero, namely $\Gamma\partial\eta$, to the second term and integrating by parts. [Throughout, the tensor and density character of the added h is the same as that of the η to which it is added, of course.] But now, only the combination g, rather than the separate tensors η and h, appears – and the bootstrap stops – as it should, because (2) is the full Palatini GR action! Actually, we never had to specify whether Γ was an independent or (linearized) metric connection: read either way, (2) and its derivation are correct, when the above Palatini method is slightly generalized. To see this, consider the second order form of (1), where Γ is now taken to be $\Gamma(h) \sim \eta^{\cdot\cdot}\partial_{\cdot}h_{\cdot\cdot}$, so that it becomes

$$A_L[\eta, h] = \int \eta_{\cdot\cdot}\partial_{\cdot}h^{\cdot\cdot}\partial_{\cdot}h^{\cdot\cdot}. \tag{3}$$

To bootstrap the resulting η variation, $\partial_{\cdot}h^{\cdot\cdot}\partial_{\cdot}h^{\cdot\cdot}$, requires using $h_{\cdot\cdot}$, the covariant anti-density, inverse, highly nonlinear function of $h^{\cdot\cdot}$, to contract the variation's indices. This is why Palatini form is simply cubic in its variables, while traditional second order form is an infinite series in the metric (and its first derivatives), whether co- or contra-variant of any density order (again we added zero as $\partial\eta$):

$$A_{NL}[g = \eta + h] = \int g^{\cdot\cdot}\Gamma(g)\Gamma(g) = \int g_{\cdot\cdot}\partial g^{\cdot\cdot}\partial g^{\cdot\cdot}, \tag{4}$$

perhaps the most compact form of the EH action, one that shows that GR simply modifies the free field kinematics by a (metric) form factor. Indeed, a recent reformulation of GR [4] arrived at similar forms by demanding this property. [As a point of history, expressions like (4) have actually been known for (at least) eight decades [5], if not in our modern context.]

Penultimately, a word about the meaning of "linearized". In the above, GR, case, Γ did not count as a "power" of the h-field, being an independent variable. Only η and h are relevant here, and the rule is to keep their lowest non-vanishing order, i.e., $\eta \times \Gamma\Gamma$ and $\Gamma\partial h$. For the general R^2 (and higher) models below, it will be obvious that we only keep η to contract their indices, but linearize the Lagrange multiplier term as shown. Also where there are several η contracting, each is to be separately bootstrapped of course, as is done implicitly. Finally, to dispel one last possible cloud, we have minimized formalism by making the h"T" addition to η"T" according to the co/contra and density character of the initial η. One might worry that the true Belinfante-Rosenfeld (say) "T" is, instead, the contra-tensor density of the coefficient of the covariant tensor η in the action, so that this use of "T" is cheating. Of course the beauty of tensor notation is that it precludes cheating. But let's verify pedantically: If we have a "wrong" (say contravariant) η in the action, then we can express it as an infinite series in the right one and then bootstrap each term with the correct "T", to reach . . . exactly the shortcut answer, as a moment's reflection shows – same infinite series, but now in $(\eta + h)$, rather than just in η – now resum: Trees saved!

3 Higher order Gravities

Only gravity actions quadratic in the curvature have flat space linearizations, unlike say Ricci^3 models that start as $\int \eta R^3 \sim h^3$, but our results will also apply there. There are only two quadratic ones in $D = 4$, where the Gauss-Bonnet identity leaves only the $a\text{Ricci}^2 + bR^2$ combination. As is well-known (and obvious), Palatini and second-order formulations here describe very different models, whether or not we keep the linear GR term. The Palatini forms are easy to bootstrap, but at the cost of being (even) less physical than their more popular second order variants. For our purposes, since the GR and higher curvature parts would bootstrap separately, we treat the latter by themselves, getting the final form just adding GR at the end, but this is only to save space. The linear actions are – a sum of – terms

$$I_L(\eta, h) = \int \eta^{\cdot\cdot} \dots \text{Riem}^n(\Gamma), \tag{5}$$

where $\text{Riem}(\Gamma) \sim \Gamma\Gamma + \partial\Gamma$ is the Riemann tensor, with 4 open indices; any lower, Ricci or scalar R, contributions are included simply by different index contractions with the etas. For GR, we saw there is a 1-line completion via (the single) $\eta \to g$ substitution, one that is not available for $n > 1$ due to the larger number of η's, and because as stated, the Γ are now explicit functions of h. To reduce (5) to a GR-like form, we use a Lagrange multiplier trick to keep the ostensible Palatini form of (5), adding

$$\Delta I_L = \int \lambda[\Gamma - \Gamma_L(\eta, h)], \qquad \Gamma_L \sim \eta^{\cdot\cdot}\partial_{\cdot} h_{\cdot\cdot}, \tag{6}$$

an expression that enforces the metric nature of the (linearized) affinity. This η-dependence must also be bootstrapped, to reach the full nonlinear affinity $\Gamma_{NL} \sim (\eta + h)^{\cdot\cdot}\partial_{\cdot}(\eta + h)_{\cdot\cdot}$:

$$\Delta I_{NL} = \int \lambda[\Gamma - \Gamma_{NL}(g = \eta + h)], \tag{7}$$

along with the obvious required bootstrapping the explicit etas in (5) to $\eta + h$. [To see the Γ bootstrap in detail, write $\Gamma^\alpha_{\beta\gamma} = 1/2\eta^{\alpha\delta}[\partial_\delta h_{\beta\gamma} + \text{perm}]$ and replace $\eta^{\cdot\cdot}$ by $\eta + h = g$ while adding zero to $\partial_{\cdot} h$ via $\partial_{\cdot}\eta$.] But this is exactly the desired nonlinear result, that, using the lambda constraint $\Gamma = \Gamma(g)$, the final action is (the sum of) invariant terms,

$$I_{NL}(g) = \int g^{\cdot\cdot} \dots \text{Riem}^n[\Gamma(g)]. \tag{8}$$

As claimed, this method covers any invariant functionals of Riem, (hence also Ricci and R). What we have not covered are functionals involving (covariant) derivatives of the R's, because these terms require their own bootstrap, a straightforward, but separate, task. Our conclusion at this stage is either encouraging, showing that any possible invariant action is the product of the same humble linearized beginnings, or discouraging in that GR is not uniquely defined by this simple physical demand. Whatever one's views of this, it is pretty clear that it is the correct *a priori* outcome, since matter coupling remains the same whatever the gravity model, so the field equations,

$$H^{\mu\nu}(g) = T_M^{\mu\nu}(\psi, g), \qquad D_\nu T_M^{\mu\nu} = 0 \tag{9}$$

polarize into the linear plus nonlinear $H_L = -H_{NL} + T_m$, with $\nabla \cdot H_L = 0$ identically. Hence the universal identification of $-H_{NL}$ with T(grav).

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Appendix. Another derivation of GR?

Long before [1], the first attempt at a bootstrap derivation of GR was given in [6], using a different, functional equation, technique to obtain "$\eta \to \eta + h$" from self-coupling, leading to the requirement that the action depend only on the latter combination. This raises the question whether it might have been used here as well. A re-reading of [6], however, prompts some queries about that method. Firstly, the linear, $\sim \Box h$, term in the field equation was obtained as if it were part of a stress tensor, from varying an action $\sim \int h\text{Ricci}(\eta)$, a somewhat puzzling choice. Secondly it was explicitly assumed that the final action, because it only depended on g, was necessarily a diffeo invariant, which is of course a stretch. Indeed, this assumption is manifestly wrong since the linear GR kinetic, "$\Box h$" term can only be part of one invariant, namely GR itself, rather than any invariant as is stated. Nevertheless, the underlying idea that self-coupling leads to a functional that depends only on $\eta + h$ is of course correct. We conclude that a proper version of [6], one that would better justify its functional $\eta \longrightarrow \eta + h$ buildup, would indeed also apply to the present problem.

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