CONSIDERATIONS IN MULTIDIMENSIONAL FILTER BANK DESIGN

Tsuhuan Chen and P. P. Vaidyanathan

Department of Electrical Engineering, 116-81
California Institute of Technology
Pasadena, California 91125, USA

Abstract — One-dimensional (1D) and multidimensional (MD) analysis/synthesis filter banks have attracted much attention recently. While 1D filter banks have been well studied and various design approaches have been successfully developed, many problems in the MD case remain unsolved. In this paper, we exploit a number of issues in MD filter bank design, including: the choice of permissible passband supports, the cyclic property of decimation matrices that is required for filter bank design using transformations, etc. We also propose a class of useful tree-structure filter banks, which apply to the commonly-used hexagonal decimation.

1. INTRODUCTION

Analysis/synthesis filter banks have attracted much attention recently. One-dimensional (1D) filter banks have been well studied and various design approaches have been successfully developed [1]–[3]. Researchers have been trying to extend these 1D results to the multidimensional (MD) case, and a number of extensions have been made [4]–[11]. In particular, the authors of [10] proposed an MD filter bank design method using transforms. This method produces filter banks which achieve perfect reconstruction but does not provide explicit control over the filter response. On the other hand, the method mentioned in [8] has response control but there is limitation on the passband supports. In [11], 1D filter banks are used to derive MD filter banks. The design is very efficient but, with finite impulse response filters, the resulting filter banks can only achieve approximate reconstruction. With infinite response filters, perfect reconstruction can be achieved but time-reversal operators have to be used. Direct optimization of filter coefficients have also been tried in [5]–[7]. However, due to the large numbers of variables and constraints, such

optimization is very complex and often impossible.

In this paper, we will exploit a number of issues in MD filter bank design, including: the choice of permissible passband supports, the cyclic property of decimation matrices that is required for filter bank design using transformations, etc. We also propose a useful tree-structure filter bank for the commonly-used hexagonal decimation matrix.

Notations and Preliminaries: Notations of this paper are the same as those in [7], [12]. Refer to [7], [12] for other preliminaries of MD multirate systems. In particular, note that when an MD signal $x(\mathbf{n})$ is passed through an M -fold decimator followed by an M -fold expander to produce the signal $y(\mathbf{n})$, the corresponding frequency domain relation is

$$Y(\omega) = \frac{1}{M} \sum_{i=0}^{M-1} X(\omega - \omega_i), \quad (1)$$

where $M = |\det \mathbf{M}|$ and $\omega_i = 2\pi \mathbf{M}^{-T} \mathbf{k}_i$. The vectors $\mathbf{k}_i$ belong to $\mathcal{N}(\mathbf{M}^T)$, which is the set of all integer vectors of the form $\mathbf{M}^T \mathbf{x}$ for $\mathbf{x} \in [0, 1)^D$.

2. PERMISSIBLE PASSBAND SUPPORTS

Fig. 1 shows an MD analysis/synthesis filter bank. When $\hat{x}(\mathbf{n}) = x(\mathbf{n})$, we say that the system achieves perfect reconstruction. The Fourier transform of $\hat{x}(\mathbf{n})$, denoted as $\hat{X}(\omega)$, can be expressed in terms of the Fourier transforms of the input signal and the filters as

$$\hat{X}(\omega) = \frac{1}{M} \sum_{j=0}^{M-1} \sum_{i=0}^{M-1} X(\omega - \omega_i) H_j(\omega - \omega_i) F_j(\omega), \quad (2)$$

where ω_i are the aliasing offsets with respect to $\mathbf{M}$. The 0-th aliasing offset ω_0 is chosen to be the zero vector $\mathbf{0}$. All the terms in (2) containing $X(\omega - \omega_i)$ for $i \neq 0$ represent the aliasing. We want to choose the filters $H_j(\omega)$ and $F_j(\omega)$ properly such that the effect of aliasing is reduced or completely canceled.

Work supported in parts by the NSF grant MIP 8919196, and by matching funds from Tektronix, Inc., and Rockwell International.

Under the assumption that all filters have large stopband attenuation, the choice of passband supports of filters is very important for alias cancellation. In the 1D case, it has been shown that if the analysis filters $H_j(\omega)$ are chosen as in Fig. 2(a), e.g., as in a uniform discrete Fourier transform (DFT) filter bank, and $|F_j(\omega)| = |H_j(\omega)|$, then the aliasing can not be canceled unless $M = 2$ [3]. To see this, consider the case $M = 3$. The aliasing offsets are 0, $2\pi/3$ and $4\pi/3$. Fig. 3 shows the aliasing terms in each of the three channels. Around $\omega = \pi/3$, there are two aliasing terms. One comes from $X(\omega - 2\pi/3)$, and the other comes from $X(\omega - 4\pi/3)$. Therefore, there is no way that these two aliasing terms can be canceled. On the other hand, if the analysis and synthesis filters are chosen as in Fig. 2(b), e.g., as in a cosine modulated system [12], aliasing can be canceled for any M . We say that the passband supports in Fig. 2(a) are nonpermissible (except for $M = 2$) and those in Fig. 2(b) are permissible.

A similar issue also exists in the MD case. Assume that all the analysis and synthesis filters are 'good', i.e., they have large stopband attenuation. Also assume that in each channel, the analysis filter and the synthesis filter have the same passband support. Using the same argument as in the 1D case, one can verify that the passband supports shown in Fig. 4(a) are permissible for the analysis/synthesis filter bank with decimation factor

$$\mathbf{M} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}. \quad (3)$$

On the other hand, Fig. 4(b) shows an example of 'nonpermissible' supports, although the filters in both Fig. 4(a) and Fig. 4(b) all have real coefficients. Note that this is indeed the choice in [5], where the filters are structurally forced to cancel aliasing. This explains why the filter banks in [5] do not have good stopband attenuation.

One typical choice of passband supports for MD multirate signal processing is the parallelepiped shape. For a given decimation matrix $\mathbf{M}$, one defines the corresponding symmetric parallelepiped as $SPD(\pi\mathbf{M}^{-T})$, which is the set of real vectors that can be written as $\pi\mathbf{M}^{-T}\mathbf{x}$, for $\mathbf{x} \in [-1, 1]^D$. It has been shown that prefiltering a signal by a filter having passband support in $SPD(\pi\mathbf{M}^{-T})$ can prevent aliasing due to $\mathbf{M}$ -fold decimation [7], [12]. For the special case that all the filters in a filter bank have parallelepiped-shaped passband supports, we can prove the following. Suppose both $H_j(\omega)$ and $F_j(\omega)$ have passband supports in $SPD(\pi\mathbf{M}^{-T}) + \omega_j$, where ω_j are the aliasing offsets. Assume that all filters have large stopband attenuation.

Then, the aliasing can be canceled if and only if

$$-\mathbf{k}_j \bmod \mathbf{M}^T = \mathbf{k}_j, \quad \forall \mathbf{k}_j \in \mathcal{N}(\mathbf{M}^T). \quad (4)$$

In other words, these passband supports are permissible if and only if (4) is satisfied.

For example, consider the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad (5)$$

which defines the well-known quincunx decimation. It can be verified that (4) is satisfied, so the supports as shown in Fig. 5 are permissible. In fact, we can conclude from (4) that for any $\mathbf{M}$ with $|\det \mathbf{M}| = 2$, the supports $SPD(\pi\mathbf{M}^{-T}) + \omega_j$ are always permissible.

For the 1D case, the condition in (4) requires that $M = 2$. This is consistent with the results in [3]. If we want to cancel aliasing with nonpermissible supports, these filters can not be 'good'. This indeed explains why the alias-free DFT filter banks given in [13] can not have both good analysis filters and good synthesis filters, except for the case $M = 2$. See Fig. 6 which shows the frequency response of a synthesis filter in the three-channel DFT filter bank of [13]. Notice the 'bump' in the stopband around π .

In [14], filters with allpass polyphase components were designed. These filters were used in [13], [15] together with DFT matrices to build filter banks that achieve perfect reconstruction. This approach results in filters similar to Fig. 2(a). Therefore, we conclude that the filters in these perfect reconstruction filter banks can not be good unless $M = 2$. This gives another explanation why there are 'spikes' in the stopband of filters in [14].

Tree-structured Filter Banks

Consider the following matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}, \quad (6)$$

which defines the so-called hexagonal decimation. We now present a choice of permissible supports for hexagonal decimation. The above $\mathbf{M}$ can be factorized as

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}}_{\mathbf{M}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{\mathbf{Q}}, \quad (7)$$

where $\mathbf{D}$ is a diagonal matrix and $\mathbf{Q}$ defines the quincunx decimation. Therefore, we can use the tree structure shown in Fig. 7 to obtain the analysis filter bank for hexagonal decimation. We choose $H_0(\omega)$ and $H_1(\omega)$ to be lowpass and highpass (in the vertical-frequency

direction) filters, and choose $G_0(\omega)$ and $G_1(\omega)$ as in Fig. 5. Since the filters in each stage are permissible, the equivalent four analysis filters, which have supports as in Fig. 8, are also permissible. The synthesis filter bank can be obtained similarly. Also note that both the analysis and synthesis filters in the lowest frequency channel have passband supports in $SPD(\pi M^{-T})$, which is often desired in practice.

This idea can be easily generalized to any M which can be factorized as the product of diagonal matrices and quincunx matrices Q (in any order), to obtain permissible supports. Moreover, both the analysis and synthesis filters corresponding to the lowest frequency band have passband supports in $SPD(\pi M^{-T})$.

3. CYCLIC PROPERTY OF DECIMATION MATRICES

In [10], a design method which transforms 1D filter banks to MD filter banks was proposed. Suppose we have a 1D M -channel alias-free filter bank in which all the filters are zero-phase. This method transforms these 1D filters to obtain MD filters which are then used to construct an MD analysis/synthesis filter bank with M -fold decimators, where $|\det M| = M$. Among other constraints, this method requires that $\mathcal{N}(M)$ forms a cyclic group under addition modulo M , i.e., there exists some m_1 in $\mathcal{N}(M)$ such that

$$\{m_1, 2m_1, \dots, Mm_1\} \bmod M = \mathcal{N}(M). \quad (8)$$

We say this m_1 is a primitive element of $\mathcal{N}(M)$. We now present an easy way to test whether $\mathcal{N}(M)$ is cyclic, without searching through all elements in $\mathcal{N}(M)$. This test uses the Smith form $M = U\Lambda V$ where U and V are unimodular integer matrices, and Λ is a diagonal matrix with integer diagonal elements $\lambda_0, \dots, \lambda_{D-1}$ where $\lambda_i | \lambda_{i+1}$ for all i [16]. This test is summarized in the following theorem:

Theorem 1. The set $\mathcal{N}(M)$ is a cyclic group under addition modulo M if and only if $|\lambda_{D-1}| = |\det M|$. $\diamond$

Sketch of Proof: First, it can be shown that $\mathcal{N}(M)$ is cyclic if and only if $\mathcal{N}(\Lambda)$ is cyclic. Now, since Λ is diagonal, it is easy to show that $\mathcal{N}(\Lambda)$ is cyclic if and only if λ_i and λ_j are relatively prime for any $i \neq j$. And since $\lambda_i | \lambda_{i+1}$, the result follows. $\triangle\triangle\triangle$

Remark: It is not necessary to compute the complete Smith form to test the cyclic property of $\mathcal{N}(M)$. The quantity λ_{D-1} can be computed as

$$\lambda_{D-1} = \frac{\Delta_D}{\Delta_{D-1}}, \quad (9)$$

where Δ_i is the greatest common divisor of all $i \times i$ minors of M [12]. Since $\Delta_D = \det M$, we can conclude from Theorem 1 that $\mathcal{N}(M)$ is cyclic if and only if $\Delta_{D-1} = \pm 1$. For the two-dimensional (2D) case, the 1×1 minors of a 2×2 matrix are simply the four elements. In this case, $\mathcal{N}(M)$ is cyclic if and only if all these four elements do not have a nontrivial common factor.

For example, the Smith form of quincunx matrix Q [8] has $\Lambda = \text{diag}[1 \ 2]$, so $\mathcal{N}(Q)$ is cyclic with $[1 \ 0]^T$ as a primitive element. In fact, we can conclude from previous facts that whenever $|\det M|$ is a prime number, $\mathcal{N}(M)$ is cyclic. As another example, the Smith form of the hexagonal matrix M [8] has $\Lambda = \text{diag}[1 \ 4]$. So, $\mathcal{N}(M)$ is cyclic. Either $[0 \ 1]^T$ or $[0 \ -1]^T$ can be a primitive element in this case. On the other hand, $\mathcal{N}(M)$ with $M = \text{diag}[2 \ 2]$ is not a cyclic group, so the approach in [10] can not be used to design a 2D filter bank with this M . However, since M is diagonal, *separable* approaches can be used instead (i.e., separate operations in each dimension).

4. CONCLUSIONS

Summarizing, we have addressed a number of issues in MD filter bank design. We have discussed the idea of 'permissible' passband supports. This idea can explain many problems in [13], [14], and [5] that have not been noticed before. We also presented a simple test to check whether $\mathcal{N}(M)$ is cyclic, which is a crucial condition for designing MD filter banks using transformations.

REFERENCES

- [1] M. J. T. Smith and T. P. Barnwell III, "A new filter-bank theory for time-frequency representation," *IEEE Trans. on ASSP*, pp. 314-327, March 1987.
- [2] M. Vetterli, "A theory of multirate filter banks," *IEEE Trans. on ASSP*, pp. 356-372, March 1987.
- [3] P. P. Vaidyanathan, "Theory and design of M -channel maximally decimated quadrature mirror filters with arbitrary M , having the perfect-reconstruction property," *IEEE Trans. on ASSP*, pp. 476-492, April 1987.
- [4] R. Ansari and C.-L. Lau, "Two-dimensional IIR filters for exact reconstruction in tree structured sub-band decomposition," *Electronics Letters*, vol. 23, no. 12, pp. 633-634, June 1987.
- [5] E. Viscito, and J. P. Allebach, "Design of perfect reconstruction multi-dimensional filter banks using cascaded Smith form matrices," *IEEE ISCAS*, Finland, pp. 831-834, June 1988.
- [6] G. Karlsson and M. Vetterli, "Theory of two-dimensional multirate filter banks," *IEEE Trans. on ASSP*, pp. 925-937, June 1990.

- [7] E. Viscito, and J. P. Allebach, "The analysis and design of multidimensional FIR perfect reconstruction filter banks for arbitrary sampling lattices," *IEEE Trans. on CAS*, vol. 38, no. 1, pp. 29-41, January 1991.
- [8] P. P. Vaidyanathan, "New results in multidimensional multirate systems," *Proc. IEEE Int. Symp. on Circuits and Systems*, pp. 468-471, Singapore, June 1991.
- [9] J. Kovačević and M. Vetterli, "Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathcal{R}^n$," *IEEE Trans. on Information Theory*, vol. 38, no. 2, pp. 533-555, March 1992.
- [10] I. A. Shah and A. A. C. Kalker, "Generalized theory of multidimensional M -band filter bank design," *Proc. of the sixth EUSIPCO*, pp. 969-972, August 1992.
- [11] T. Chen and P. P. Vaidyanathan, "Multidimensional multirate filters and filter banks derived from one dimensional filters," *IEEE Trans. on SP*, May 1993.
- [12] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.
- [13] K. Swaminathan, and P. P. Vaidyanathan, "Theory and design of uniform DFT, parallel, quadrature mirror filter banks," *IEEE Trans. on Circuits and Systems*, vol. 33, pp. 1170-1191, December 1986.
- [14] M. Renfors and T. Saramäki, "Recursive N th-band digital filters—Part I: design and properties," *IEEE Trans. on CAS*, vol. 34, no. 1, pp. 24-39, January 1987.
- [15] S. K. Mitra, C. D. Creusere, and H. Babic, "A novel implementation of perfect reconstruction QMF banks using IIR filters for infinite length signals," *IEEE IS-CAS*, pp. 2312-2315, San Diego, May 1992.
- [16] H. J. S. Smith, "On systems of linear indeterminate equations and congruences", *Philos. Trans. Royal Society*, London, vol. 151, pp. 293-326, 1861.

Fig. 1. Analysis/synthesis filter bank.

Fig. 2. Frequency responses of analysis filters.

Fig. 3. Aliasing terms in a three-channel filter bank.

Fig. 4. Passband supports in an MD filter bank: (a) permissible, (b) nonpermissible.

Fig. 5. Passband supports for quincunx decimation.

Fig. 6. A synthesis filter in a three-channel DFT filter bank.

Fig. 7. Tree-structured analysis filter bank.

Fig. 8. Passband supports for hexagonal decimation.