Robust Control for a Noncolocated Spring-Mass System

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Abstract

Robust control laws are presented for an undamped pair of coupled masses with a noncolocated sensor and actuator. This simple problem captures many of the features of more complex aircraft and space structure vibration control problems. The control problem is formulated in the structured singular value framework, which addresses the stability robustness to parameter variations directly. Controllers are designed by D-K iteration (commonly called μ-synthesis), and the resulting high-order controllers are reduced using Hankel model reduction. Design specifications such as settling time, actuator constraints, insensitivity to measurement noise, and parameter uncertainty are achieved by the resulting controllers. Design Problems #1 and #2 were considered in [2]. Design Problem #4 in [11] will be considered in this paper.

Introduction

Numerous researchers (as listed in [11]) have applied a variety of robust control design methodologies to the benchmark problem. Braatz and Morari [2] designed robust controllers for Design Problems #1 and #2 using the "DK iteration" method proposed by Doyle [6]. Though the design specifications cannot be described directly in the structured singular value framework, control, performance, disturbance, and measurement weights were chosen to meet the design specifications.

The paper is organized as follows. First the benchmark problem is briefly described. Then the structured singular value framework is reviewed. Design Problem #4 is then put into this framework. Due to lack of space, the state-space matrices for the controller, the gain and phase margins, the performance/stability robustness plots, and time simulations are not given here but will be presented at the *1992 ACC Conference* [3].

Benchmark Problem

Consider the two-mass/spring system in [11], which is a generic model of an uncertain dynamical system with noncolocated sensor and actuator. The system is represented in state-space form as

$$\dot{x} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k/m_1 & k/m_1 & 0 & 0\\ k/m_2 & -k/m_2 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1/m_1\\ 0\end{bmatrix}u + \begin{bmatrix}0\\ 0\\ 0\\ 1/m_2\end{bmatrix}w \quad (1)$$

$$y = x_2 + v \quad (2)$$

$$z = x_2 \quad (3)$$

where x_1 and x_2 are the positions of body 1 and body 2, x_3 and x_4 are the velocities of body 1 and body 2, u is the control input acting on body 1, y is the sensor measurement, w is the disturbance acting on body 2, v is sensor noise, and z is the output to be controlled. k is the spring constant, m_1 is the mass of body 1, and m_2 is the mass of body 2.

The coupled spring-mass system is assumed to have negligible damping. The spring constant and masses are assumed to be uncertain. The actuator is located on body 1 while the sensor is located on body 2, *i.e.* the sensor and actuator are noncolocated. This makes the system much harder to control than in the colocated case.

Design Problem #4 is described below. Specifications $(i - iv)$ are from [11]. Specifications $(v - vi)$ from [2] are additional practical constraints. We choose the measurement noise to be approximately the same as that for the laboratory flexible structure in [1]. Our actuator bandwidth limitation is more restrictive than that for the voice coil actuators in [1].

Design #4. Design a feedback/feedforward controller for a unit-step output command tracking problem for the controlled output, z, with the following properties:

(i) The control input $u(t)$ is limited to $|u| \le 1$.

(ii) Performance requirement: settling time and overshoot are both to be minimized.

(iii) Stability requirement: performance robustness and stability robustness with respect to the three uncertain parameters m_1, m_2, and k (with the nominal values $m_1 = m_2 = k = 1$) are both to maximized.

(iv) If there are conflicts between (2) and (3), then performance versus robustness trade-offs must be considered.

(v) The control system can tolerate Gaussian white noise with variance of $9 * 10^{-6}$.

(vi) Because of finite actuator response time, the bandwidth for the feedback controller must be ≤ 50 rad/s.

Structured Singular Value Framework

The goal of any controller design is that the overall system is stable and satisfies some minimum performance requirements. These requirements should be satisfied at least when the controller is applied to the *nominal* plant, that is, we require nominal stability and nominal performance.

In practice the real plant G_p is not equal to the model G. The term "robust" is used to indicate that some property holds for a set Π of possible plants G_p as defined by the uncertainty description. In particular, by robust stability we mean that the closed loop system is stable for all $G_p \in \Pi$. By robust performance we mean that the performance requirements are satisfied for all $G_p \in \Pi$. Performance is commonly defined in robust control theory using the H_∞-norm of some transfer function $\Sigma(G)$ of interest.

Definition 1 *The closed loop system exhibits nominal performance if*

$$\|\Sigma\|_\infty \equiv \sup_\omega \bar{\sigma}(\Sigma) \le 1. \quad (4)$$

Definition 2 *The closed loop system exhibits robust performance if*

$$\|\Sigma_p\|_\infty \equiv \sup_\omega \bar{\sigma}(\Sigma_p) \le 1, \quad \forall G_p \in \Pi. \quad (5)$$

For example, for rejection of disturbances at the plant output, Σ would be the weighted sensitivity

$$\begin{aligned}\Sigma &= W_1 S W_2, & S &= (I + GK)^{-1}\\ \Sigma_p &= W_1 S_p W_2, & S_p &= (I + G_p K)^{-1}.\end{aligned} \quad (6)$$

In this case, the input weight W_2 is usually chosen equal to the disturbance model. The output weight W_1 is used to specify the frequency range over which the sensitivity function should be small and to weight each output according to its importance. K is the transfer function of the controller.

Doyle [4] derived the *structured singular value*, μ, to test for robust performance. To use μ we must model the uncertainty (the set Π of possible plants G_p) as norm bounded perturbations (Δ_i) on the nominal system. Through weights each perturbation is normalized to be of size one:

$$\bar{\sigma}(\Delta_i) \le 1, \quad \forall \omega. \quad (7)$$

The perturbations, which may occur at different locations in the system, are collected in the diagonal matrix $\Delta_U = \mathrm{diag}[\Delta_1, \cdots, \Delta_n]$ (the U denotes uncertainty) and the system is arranged to match the block diagrams in Fig. 1. The interconnection matrix M in Fig. 1 is determined by the nominal model (G), the size and nature of the uncertainty, the performance specifications, and the controller (K). The definition of μ is:

Definition 3 *Let M be a square complex matrix and the set $\Delta = \{\mathrm{diag}[\Delta_1, \cdots, \Delta_n]\}$. Then $\mu_\Delta(M)$ is defined such that $\mu_\Delta^{-1}(M)$ is equal to the smallest $\bar{\sigma}(\Delta)$ for Δ making $(I + \Delta M)$ singular, i.e.*

$$\mu_\Delta^{-1}(M) = \min_{\Delta \in \Delta} \{\bar{\sigma}(\Delta) : \det(I + \Delta M) = 0\}. \quad (8)$$

For Fig. 1, robust stability and robust performance can be tested by

Theorem 1 *The closed loop system exhibits robust stability if and only if the closed loop system is nominally stable and*

$$\sup_\omega \mu_{\Delta_U}(M_{11}) < 1. \quad (9)$$

Theorem 2 *The closed loop system exhibits robust performance if and only if the closed loop system is nominally stable and*

$$\sup_\omega \mu_\Delta(M) < 1. \quad (10)$$

$\mu_\Delta(M)$ depends on both the elements of the matrix M and the structure of the perturbation matrix $\Delta = \mathrm{diag}[\Delta_U, \Delta_P]$. Δ_P is often chosen to be a full square matrix with dimension equal to the number of outputs (the subscript P denotes performance). Note that the issue of robust stability is simply a special case of robust performance. Also note that robust performance implies robust stability, *i.e.* $\sup_\omega \mu_\Delta(M) \ge \sup_\omega \mu_{\Delta_U}(M_{11})$.

It is a key idea that μ is a *general* analysis tool for determining robust performance. Any system with uncertainty adequately modeled as in (7) can be put into $M - \Delta$ form, and robust performance can be tested using (10). Standard programs calculate M and Δ [7], given the transfer functions describing the system components and the location of the uncertainty blocks Δ_i.

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Upper Bound for μ with Complex Δ Because calculating μ exactly is fairly difficult, its well-known upper bound is used instead. Define

$$\mathbf{D} = \{\operatorname{diag}[d_i I_i] : \dim(I_i) = \dim(\Delta_i), d_i \text{ positive real scalar}\}, \quad (11)$$

then [4]

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathbf{D}} \overline{\sigma}(DMD^{-1}). \quad (12)$$

The upper bound is almost always within a percent or so of μ for real problems [10], so for engineering purposes μ never has to be calculated exactly.

Controller Synthesis M is a function of the controller K. The H_∞-optimal control problem is to find a stabilizing K which minimizes $\sup_\omega \overline{\sigma}(M(K))$. The state-space approach for solving the H_∞ control problem is described in [9].

The D-K iteration method (often called μ-synthesis) is an *ad hoc* method which attempts to minimize the tight upper bound of μ in (12), *i.e.* it attempts to solve

$$\min_K \inf_{D \in \mathbf{D}} \sup_\omega \overline{\sigma}(DM(K)D^{-1}). \quad (13)$$

The approach in D-K iteration is to alternatively minimize $\sup_\omega \overline{\sigma}(DM(K)D^{-1})$ for either K or D while holding the other constant. For fixed D, the controller synthesis is solved via H_∞-optimization. For fixed K, the quantity is minimised as a convex optimization. The resulting D as a function of frequency is fitted with an invertible stable minimum-phase transfer function and wrapped back into the nominal interconnection structure. This increases the number of states of the scaled G, which leads the next H_∞-synthesis step to give a higher order controller. The iterations stop after $\sup_\omega \overline{\sigma}(DM(K)D^{-1})$ is no longer diminished. The resulting high-order controller is reduced using Hankel model reduction [8]. Though this method is not guaranteed to converge to a global minimum, it has been used extensively to design robust controllers and seems to work well [5].

Application of μ to Design Problem #4

The spring constant and the two masses are assumed to be uncertain and are given by

$$k = k_0 + w_k\delta_k, \quad m_1 = m_{10} + w_1\delta_1, \quad m_2 = m_{20} + w_2\delta_2, \quad (14)$$

where k_0, m_{10}, and m_{20} are the nominal values and the weights w_k, w_1, and w_2 are used to normalize the uncertainties δ_i so that $|\delta_i| \leq 1$. Simultaneous perturbations in the δ_i are allowed, as long as $|\delta_i| \leq 1$ for each uncertainty i.

The objective of Design Problem #4 is for the controlled variable z to follow the desired trajectory r, *i.e.* we want $\|z - r\|_\infty$ small. Weighted versions of the noise, disturbance, control input, and performance variable are given by

$$r = w_r r', \quad w = w_w w', \quad u' = w_u u, \quad (z - r)' = w_{zr}(z - r), \quad (15)$$

where in general the input weights w_r and w_w weigh the frequencies of interest and determine the relative importance of the noise and disturbance. w_{zr} is the performance weight and w_u is used to limit the magnitude of the control input. It is suspected that the high frequency roll-off required by specification (*vi*) will automatically force specification (*v*) to be satisfied, so (*v*) is not directly accounted for in the controller synthesis. Specification (*v*) will be checked in the time simulations.

k, m_1, and m_2 from (14) and r, w, u, and z from (15) are substituted into the state-space equations (1-3) and written in block diagram form in Fig. 2. The block diagram has x, r', w', u as inputs and $\dot{x}$, u', $(z - r)'$, r, and y as outputs.

By inspection, the block diagram in Fig. 2 is rearranged to form the block diagram in Fig. 3, where

$$N = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 & -\frac{1}{m_1} & \frac{1}{m_1} & 0 & 0 & 0 & \frac{1}{m_1} \\ \frac{k}{m_1} & -\frac{k}{m_1} & 0 & 0 & \frac{1}{m_2} & 0 & \frac{1}{m_2} & 0 & \frac{w_w}{m_2} & 0 \\ w_k & -w_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{kw_1}{m_1} & -\frac{kw_1}{m_1} & 0 & 0 & \frac{w_1}{m_1} & -\frac{w_1}{m_1} & 0 & 0 & 0 & -\frac{w_1}{m_1} \\ -\frac{kw_2}{m_1} & \frac{kw_2}{m_1} & 0 & 0 & -\frac{w_2}{m_2} & 0 & -\frac{w_2}{m_2} & 0 & -\frac{w_2 w_w}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_u \\ 0 & w_{zr} & 0 & 0 & 0 & 0 & 0 & -w_{zr}w_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_r & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (16)$$

and the normalized performance variable $\hat{e}$, the normalized disturbance $\hat{d}$, and the uncertainty block Δ_U are given by

$$\hat{e} = \begin{pmatrix} u' \\ (z - r)' \end{pmatrix}, \quad \hat{d} = \begin{pmatrix} r' \\ w' \end{pmatrix}, \quad \Delta_u = \begin{pmatrix} \delta_k & & \\ & \delta_1 & \\ & & \delta_2 \end{pmatrix}. \quad (17)$$

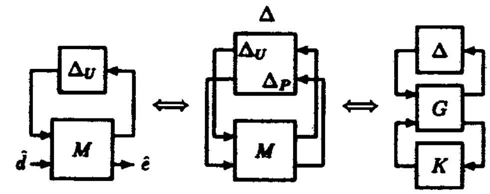

Figure 1: General Interconnection Structures

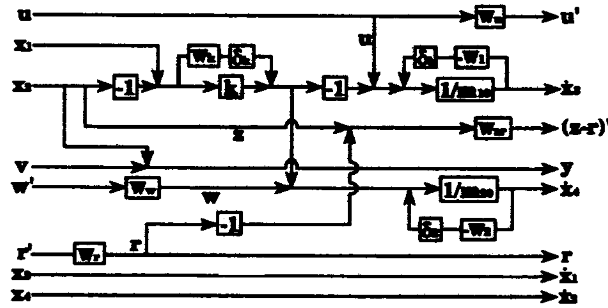

Figure 2: Block Diagram for Coupled Mass-Spring System

Δ_P is the performance Δ-block which relates the outputs to inputs, K is the controller transfer function, and I_4 is the 4×4 identity matrix. Closing the integrator loop in Fig. 3 gives the system interconnection structures in Fig. 1. Note that K has two inputs—the reference trajectory r and the measured variable y. $K = [K_{FF}\ K_{FB}]$, where K_{FF} is the feedforward and K_{FB} is the feedback controller.

We will use the D-K iteration method by allowing the uncertainties in k, m_1, and m_2 to be complex. The D-K iteration method described in Section 3 approximately maximizes the performance for the worst-case plant described by the nominal plant plus the complex uncertainties. As such, the method will give a controller whose performance is insensitive to the complex uncertainties. Thus the performance of the controller will also be insensitive to the corresponding real uncertainties.

Due to lack of space, the choice of weights, the state-space matrices for the controller, the gain and phase margins, the performance/stability robustness plots, and time simulations are not given here but will be presented at the *1992 ACC Conference* [3].

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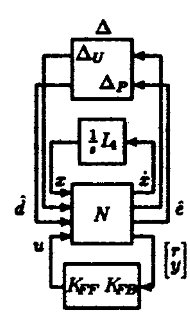

Figure 3: Simplified Block Diagram for Coupled Mass-Spring System