COLLAR LEMMA FOR HITCHIN REPRESENTATIONS.

GYE-SEON LEE†, TENGREN ZHANG‡

Abstract. In this article, we prove an analog of the classical collar lemma in the setting of Hitchin representations.

1. Introduction.

Let S be a closed oriented topological surface and let Γ be its fundamental group. The Teichmüller space of S, denoted $\mathcal{T}(S)$, is the space of hyperbolic structures on S. Via the holonomy representation, $\mathcal{T}(S)$ can be identified with a component of the space of conjugacy classes of representations from Γ to $PSL(2,\mathbb{R})$. One advantage of doing so is that it allows us to generalize $\mathcal{T}(S)$ in the following way. It is a standard fact in representation theory that for any $n \geq 2$, there is a unique (up to conjugation) irreducible representation $\iota_n : PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$. This gives, via post composition, an embedding

$$\mathcal{T}(S) \hookrightarrow \mathcal{X}_n(S) := Hom(\Gamma, PSL(n,\mathbb{R}))/PSL(n,\mathbb{R}).$$

The image of this embedding is known as the *Fuchsian locus* and the component of $\mathcal{X}_n(S)$ containing the Fuchsian locus is the *nth-Hitchin component*, denoted $Hit_n(S)$. By definition, $Hit_2(S) = \mathcal{T}(S)$, so Hitchin representations can be thought of as generalizations of Fuchsian representations.

For the hyperbolic structures in $\mathcal{T}(S)$, there is a classical result first due to Keen [16] known as the collar lemma. It gives an effective lower bound on the width of a collar neighborhood of a simple closed curve in a hyperbolic surface, which grows to ∞ as the length of the simple closed curve is shrunk to 0. A consequence of the collar lemma is that if two closed curves γ and η in a hyperbolic surface have non-vanishing geometric intersection number and γ is simple, then there is an explicit lower bound on the length of η in terms of the length of γ. This is a powerful tool that has been used to understand surfaces. For example, it was used to study the length spectrum of Riemann surfaces (see Buser [6]).

The goal of this paper is to generalize a version of the classical collar lemma to Hitchin representations. In this setting, the width of a collar neighborhood is not well defined since Hitchin representations in general do not give a metric on S. However, for every Hitchin representation ρ, we do still have a natural notion of length for homotopy classes of closed curves in S.

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By Labourie [17], we know that for any Hitchin representation ρ and any non-identity element X in Γ, $\rho(X)$ is diagonalizable over $\mathbb{R}$ with eigenvalues that have pairwise distinct moduli. Hence, given any representation ρ in $Hit_n(S)$ and any closed curve γ in S, we can define the *ρ-length* of γ to be

$$l_\rho(\gamma) = \log\left|\frac{\lambda_n}{\lambda_1}\right|,$$

where λ_n and λ_1 are the eigenvalues of $\rho(X)$ of largest and smallest modulus respectively, and $X \in \Gamma$ corresponds to the curve γ equipped with a choice of orientation. Observe that the ρ-length does not depend on the choice of orientation on γ, and is constant on each homotopy class of closed curves in S.

In the case when $\rho \in Hit_2(S)$, $l_\rho(\gamma)$ is exactly the hyperbolic length of the geodesic homotopic to γ, measured in the hyperbolic metric corresponding to ρ. Also, Choi-Goldman [8] proved that representations in $Hit_3(S)$ are exactly holonomies of convex $\mathbb{RP}^2$ structures on S. Moreover, each such convex $\mathbb{RP}^2$ structure also induces a natural Finsler metric, known as the Hilbert metric, on S. One can then verify that in the case when $\rho \in Hit_3(S)$, $l_\rho(\gamma)$ is the length of the geodesic homotopic to γ, measured in the Hilbert metric induced by the convex $\mathbb{RP}^2$ structure corresponding to ρ.

With this, we have the following theorem, which one can think of as a generalization of the collar lemma.

Theorem 1.1. *Let S be a surface of genus $g \geq 2$, and let γ, η be two non-contractible closed curves in S. Denote the geometric intersection number between γ and η by $i(\eta,\gamma)$. Then, for any $n \geq 2$ and any $\rho \in Hit_n(S)$, the following hold:*

(1) If $i(\eta,\gamma) \neq 0$, then
$$\frac{1}{\exp(l_\rho(\eta))} < 1 - \frac{1}{\exp(\frac{l_\rho(\gamma)}{n-1})}.$$
(2) If $i(\eta,\gamma) \neq 0$ and γ is simple, then there are non-negative integers u, v with $u \geq v$ and $u+v = i(\eta,\gamma)$ so that
$$\frac{1}{\exp(l_\rho(\eta))} < \left(1 - \frac{1}{\exp(\frac{l_\rho(\gamma)}{n-1})}\right)^u \left(1 - \frac{1}{\exp(l_\rho(\gamma))}\right)^v.$$
(3) Let $\delta_n > 0$ be the unique real solution to the equation $e^{-x} + e^{-x/(n-1)} = 1$. If η is a non-simple closed curve, then
$$l_\rho(\eta) > \delta_n.$$

We can say what the constants u and v are in (2) of Theorem 1.1. Choose orientations on γ and η, and let $\hat{i}(\gamma,\eta)$ be the algebraic intersection number between γ and η. Then

$$u = \frac{i(\gamma,\eta) + |\hat{i}(\gamma,\eta)|}{2} \text{ and } v = \frac{i(\gamma,\eta) - |\hat{i}(\gamma,\eta)|}{2}.$$

More geometrically, if we let w_0 and w_1 be number of times η crosses γ from left to right and from right to left respectively, then $u = \max\{w_0, w_1\}$ and $v = \min\{w_0, w_1\}$. Observe also that $\lim_{n\to\infty} \delta_n = \infty$ and $\delta_2 = \log(2)$.

Theorem 1.1 is in fact a consequence of a more general inequality between $l_\rho(\gamma)$ and the eigenvalues of the image of the group element corresponding to η under the representation ρ (see Proposition 2.12).

In the case of $\mathcal{T}(S)$, the first inequality in Theorem 1.1 can be rewritten as

$$(\exp(l_\rho(\eta)) - 1)(\exp(l_\rho(\gamma)) - 1) > 1.$$

This is weaker than a version of the classical collar lemma, which is the inequality

$$\sinh\left(\frac{l_\rho(\eta)}{2}\right)\sinh\left(\frac{l_\rho(\gamma)}{2}\right) > 1,$$

although in both inequalities, $l_\rho(\eta)$ grows logarithmically with $\frac{1}{l_\rho(\gamma)}$. Furthermore, while the classical collar lemma is sharp, we are unable to prove the same for Theorem 1.1. This led us to conjecture, in an earlier version of this paper, that for any $\rho \in Hit_n(S)$, there is some representation ρ' in the Fuchsian locus of $Hit_n(S)$ such that $l_\rho(\gamma) \geq l_{\rho'}(\gamma)$ for all $\gamma \in \Gamma$. If the conjecture holds, then we can obtain a sharp version of Theorem 1.1. Recently, Tholozan [20] proved that this conjecture is true for $n = 3$, but F. Labourie showed that it fails for all $n \geq 4$. See Section 3.3 for more details.

Choi [7] proved an analog of the Margulis lemma for convex $\mathbb{RP}^2$ surfaces. As a consequence, he showed the existence of a collar neighborhood in the convex $\mathbb{RP}^2$ surface about a simple closed curve of sufficiently short length, and found (non-explicit) lower bounds for the width of this collar neighborhood in terms of the length of the simple closed curve. This analog of the Margulis lemma was later extended by Cooper-Long-Tillman [9] to all convex real projective manifolds. Burger-Pozzetti [5] also recently proved a similar result for maximal representations into $Sp(2n, \mathbb{R})$.

Although the inequalities in Theorem 1.1 depend on n, we can obtain as a corollary the following "universal collar lemma" that is simultaneously true for all Hitchin representations.

Corollary 1.2 (Corollary 3.1)**.** *Let S be a surface of genus $g \geq 2$, and let γ, η be two non-contractible closed curves in S. Then for any $n \geq 2$ and any $\rho \in Hit_n(S)$, the following hold:*

(1) If $i(\eta, \gamma) \neq 0$, then

$$(\exp(l_\rho(\gamma)) - 1)(\exp(l_\rho(\eta)) - 1) > 1.$$

(2) If $i(\eta, \gamma) \neq 0$ and γ is simple, then

$$(\exp(l_\rho(\gamma)) - 1)\left(\exp\left(\frac{l_\rho(\eta)}{i(\eta,\gamma)}\right) - 1\right) > 1.$$

(3) If η is a non-simple closed curve, then

$$l_\rho(\eta) > \log(2).$$

Unfortunately, for $\rho \in Hit_n(S)$ when $n \geq 4$, it is not known whether there exists a metric on S that induces l_ρ as its length function. However, we can still interpret Theorem 1.1 and Corollary 1.2 geometrically by considering the $SL(n, \mathbb{R})$ symmetric space $\widetilde{M}$. Normalize the Riemannian metric on $\widetilde{M}$ so that for any $Z \in PSL(n, \mathbb{R})$ with real eigenvalues,

$$\inf\{d_{\widetilde{M}}(o, Z \cdot o) : o \in \widetilde{M}\} = \sqrt{2\sum_{i=1}^n (\log|\lambda_i|)^2},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of Z and $d_{\widetilde{M}}$ is the distance function on $\widetilde{M}$ induced by the normalized Riemannian metric. Let $M := \rho(\Gamma)\backslash\widetilde{M}$, and for any closed curve ω in M, let $l_M(\omega)$ be the length of ω measured in the Riemannian metric on M induced by the normalized Reimannian metric on $\widetilde{M}$. Then the following corollary of Theorem 1.1 and Corollary 1.2 holds.

Corollary 1.3 (Corollary 3.3). *Let γ, η be two non-contractible closed curves in S and let X, Y be elements in Γ corresponding to γ, η respectively. For any $\rho \in Hit_n(S)$, let γ', η' be two closed curves in M that correspond to $X, Y \in \Gamma$ respectively. Then the statements in Theorem 1.1 and Corollary 1.2 hold, with $l_\rho(\gamma)$ and $l_\rho(\eta)$ replaced with $l_M(\gamma')$ and $l_M(\eta')$ respectively.*

It is an important remark that this corollary (and hence Theorem 1.1) is not simply a quantitative version of the Margulis lemma on $PSL(n, \mathbb{R})$ because the closed curves γ' and η' do not need to intersect, even when $i(\gamma, \eta) \neq 0$. Theorem 1.1 is a property that is special to Hitchin representations. In fact, for any pair of simple closed curves in S, one can find a sequence of quasi-Fuchsian representations

$$\rho_i : \Gamma \to PSO(3,1)^+ \subset PSL(4, \mathbb{R})$$

so that the lengths of the geodesics in $\rho_i(\Gamma)\backslash\widetilde{M}$ corresponding to both of these two simple closed curves converge to 0 along this sequence. In particular, Theorem 1.1 does not hold on the space of quasi-Fuchsian representations. This is explained in greater detail in Section 3.2.

As a final consequence of Theorem 1.1, we have the following properness result.

Corollary 1.4 (Corollary 3.7). *Let $\mathcal{C} := \{\gamma_1, \dots, \gamma_k\}$ be a collection of closed curves in S that contains a pants decomposition, so that the complement of $\mathcal{C}$ in S is a union of discs. Then the map*

$$\begin{aligned} Hit_n(S) &\to \mathbb{R}^k \\ \rho &\mapsto (l_\rho(\gamma_1), \dots, l_\rho(\gamma_k)) \end{aligned}$$

is proper.

In other words, in order for a sequence $\{\rho_i\}_{i=1}^\infty$ in $Hit_n(S)$ to escape, the ρ_i-length of some curve in $\mathcal{C}$ must grow to ∞. Refer to Section 3.1 for more corollaries of Theorem 1.1.

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Contents

2. Proof of Theorem.

In this section, we give the proof of Theorem 1.1. We start by discussing some useful topological properties of Γ and its boundary in Section 2.1. Then for the sake of demonstrating the proof without too many technical details, we prove (1) of Theorem 1.1 for the special case of $Hit_3(S)$ in Section 2.2. Next, we develop the technical tools that we need in Section 2.3, and apply them in Section 2.4 to prove Theorem 1.1 in its full generality.

2.1. **Properties of the boundary of the group.** It is well-known that $\Gamma := \pi_1(S)$ is Gromov hyperbolic, so the Cayley graph of Γ has a natural boundary, denoted by $\partial_\infty\Gamma$, and the action of Γ on its Cayley graph extends to an action on $\partial_\infty\Gamma$. Moreover, if we choose $\rho \in \mathcal{T}(S)$, i.e. a hyperbolic structure on S, we get a ρ-equivariant identification of $\partial_\infty\Gamma$ with the boundary of the Poincaré disc $\partial\mathbb{D}$.

For any hyperbolic element $A \in PSL(2,\mathbb{R})$, the *axis* of A, denoted L_A, is the unique geodesic in $\mathbb{D}$ whose endpoints are the repelling and attracting fixed points of A in $\partial\mathbb{D}$. The proof of the main theorem relies crucially on an important property of the action of Γ on $\partial_\infty\Gamma$, which we state as Lemma 2.2. These are well-known facts about surface groups, but for want of a good reference, we will give the proof here.

Lemma 2.1. *Let B and B' be non-commuting elements in $PSL(2,\mathbb{R})$ that generate a subgroup consisting only of hyperbolic isometries. If the translation lengths of B and B' are the same and $L_{B'} \cap L_B = \emptyset$, then $(B \cdot L_{B'}) \cap L_{B'} = \emptyset$.*

Proof. Since B and B' do not commute, $L_B \neq L_{B'}$. Since the commutator $[B, B']$ is not parabolic, B and B' cannot share a fixed point. Hence, by changing coordinates and replacing B and B' with their inverses if necessary, we can assume that L_B and $L_{B'}$ are as in Figure 1 and B, B' translate along their axes in the directions drawn.

Let L be the geodesic in $\mathbb{H}^2$ that is perpendicular to both $L_{B'}$ and L_B, and let R be the reflection about L. There is a unique geodesic K that is perpendicular to L_B and whose distances to L and $B \cdot L$ are equal. Let S be the reflection about K, and note that $B = SR$. Also, observe that the distance between K and L is realized

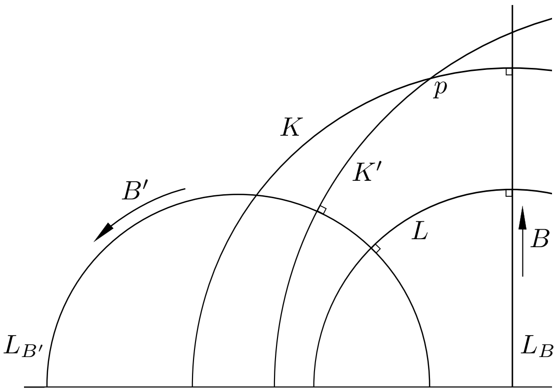

FIGURE 1. An impossible configuration of K and K' in Lemma 2.1

only by the points $K\cap L_B$ and $L\cap L_B$, and is half the translation length of B, which we denote by T. Furthermore, $(B\cdot L_{B'})\cap L_{B'}=(SR\cdot L_{B'})\cap L_{B'}=(S\cdot L_{B'})\cap L_{B'}$ is empty if and only if $K\cap L_{B'}$ is empty.

Thus, it is sufficient to show that $K\cap L_{B'}$ is empty. Suppose for contradiction that it is not. As before, there is a unique geodesic K' so that $B'=S'R$, where S' is the reflection about K'. Since the translation lengths of B and B' are the same, the symmetry between B and B' ensures that $K'\cap L_B$ is also nonempty.

Now, note that $K'\cap L_{B'}$ lies between $K\cap L_{B'}$ and $L\cap L_{B'}$ because

$$d(K\cap L_{B'},L\cap L_{B'})>d(K\cap L_B,L\cap L_B)=\frac{T}{2}=d(K'\cap L_{B'},L\cap L_{B'}).$$

Similarly, $K\cap L_B$ lies between $K'\cap L_B$ and $L\cap L_B$. This implies that K and K' have a common point of intersection, p (see Figure 1). Observe that $B'B^{-1}=S'RR^{-1}S^{-1}=S'S$ fixes p, but that is impossible because $B'B^{-1}$ is not elliptic. □

Lemma 2.2. *Let A, B, B' be pairwise non-commuting elements in Γ so that B and B' are conjugate. Let a^+, b^+, b'^+ be the attracting fixed points and a^-, b^-, b'^- be the repelling fixed points of A, B, B' respectively. If*

$$a^+,\ b'^+,\ b^+,\ a^-,\ b^-,\ b'^-$$

lie in $\partial_\infty\Gamma$ in that cyclic order, then

$$a^+,\ b'^+,\ B\cdot a^+,\ b^+,\ a^-,\ b^-,\ B^{-1}\cdot a^+,\ b'^-$$

lie in $\partial_\infty\Gamma$ in that cyclic order (see Figure 2).

Proof. Let s_0 be the open subsegment of $\partial_\infty\Gamma$ with endpoints b'^- and b^+ that does not contain b^-, and let s_1 be the open subsegment of $\partial_\infty\Gamma$ with endpoints b'^+ and b^+ that does not contain b^-. Observe that $B\cdot b'^-$ lies in s_0 and $B\cdot b'^+$ lies in s_1.

Choose a hyperbolic metric on S. This identifies $\partial_\infty\Gamma$ with $\partial\mathbb{D}$ and Γ with a discrete, torsion-free subgroup of $PSL(2,\mathbb{R})$. Since

$$a^+,\ b'^+,\ b^+,\ a^-,\ b^-,\ b'^-$$

lie in $\partial_\infty\Gamma$ in that cyclic order, L_B and $L_{B'}$ have to be disjoint. Moreover, B and B' have the same translation lengths and do not commute. Hence, we can apply Lemma 2.1 to conclude that $B\cdot L_{B'}$ and $L_{B'}$ are disjoint. This implies that both

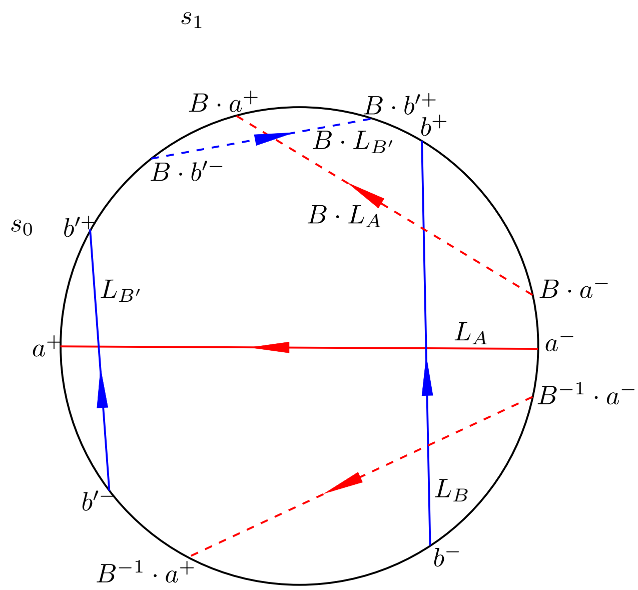

FIGURE 2. The cyclic order of the attracting and repelling fixed points of A, B, B' and BAB^{-1} along $\partial_\infty\Gamma$ in Lemma 2.2.

$B \cdot b'^-$ and $B \cdot b'^+$ have to lie in s_1. Since a^+ lies in s_0 between b'^- and b'^+, $B \cdot a^+$ must lie in s_1 between $B \cdot b'^-$ and $B \cdot b'^+$. In particular,

$$a^+,\ b'^+,\ B \cdot a^+,\ b^+,\ a^-$$

lie in $\partial_\infty\Gamma$ in that cyclic order (see Figure 2).

A similar argument, using B^{-1} instead of B, shows that

$$a^-,\ b^-,\ B^{-1} \cdot a^+,\ b'^-,\ a^+$$

lie in $\partial_\infty\Gamma$ in that cyclic order. This proves the lemma. $\square$

2.2. **Proof in the $PSL(3,\mathbb{R})$ case.** In order to demonstrate the main ideas of the proof without involving too many technicalities, we will first prove (1) of Theorem 1.1 in the special case when $n = 3$, i.e. $\rho : \Gamma \to PSL(3,\mathbb{R}) = SL(3,\mathbb{R})$ is a Hitchin representation.

By Choi-Goldman [8], we know that in this case, ρ is the holonomy of a convex $\mathbb{RP}^2$ structure on S. In other words, there is a strictly convex domain Ω_ρ in $\mathbb{RP}^2$ which is preserved by the Γ-action on $\mathbb{RP}^2$ induced by ρ, and on which the Γ-action is properly discontinuous and cocompact. Moreover, $\rho(X)$ is diagonalizable with positive pairwise distinct eigenvalues for any non-identity element $X \in \Gamma$, (see Theorem 3.2 of Goldman [13]) so $\rho(X)$ has an attracting and repelling fixed point in $\partial\Omega_\rho$. Since the Hilbert metric in Ω_ρ is invariant under projective transformations and the geodesics of the Hilbert metric are lines, one can use the Švarc-Milnor lemma (Proposition 8.19 of [3]) to construct a continuous map

$$\xi^{(1)} : \partial_\infty\Gamma \to \partial\Omega_\rho$$

which identifies the attracting fixed point of any $X \in \Gamma$ to the attracting fixed point of $\rho(X)$.

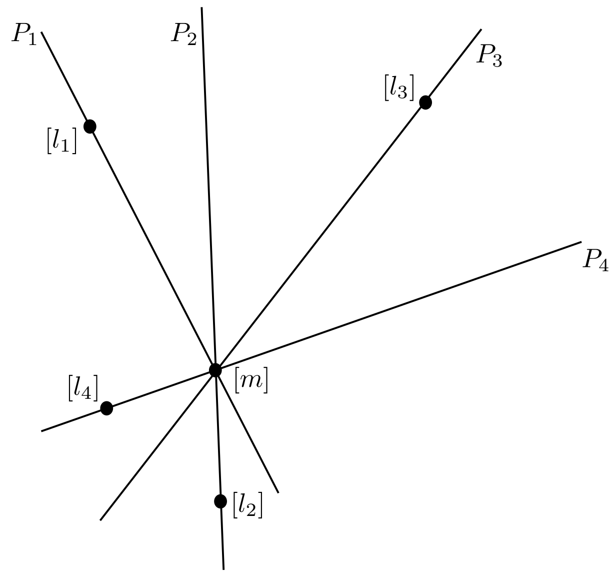

FIGURE 3. A choice of vectors l_i to compute the cross ratio (P_1, P_2, P_3, P_4).

Pick any four projective lines in $\mathbb{RP}^2$ that intersect at a common point, such that no three of the four agree. There is a classical projective invariant of these four projective lines, called the *cross ratio*, which can be defined as follows. Let the four projective lines be P_1, P_2, P_3, P_4 and let m be a vector in $\mathbb{R}^3$ so that $[m]$, the projective point corresponding to the $\mathbb{R}$-span of m, is the common point of intersection of the P_i. For each i, choose a vector $l_i \in \mathbb{R}^3$ so that $[l_i] \neq [m]$ and $[l_i]$ lies in P_i (see Figure 3). By choosing a linear identification

$$f : \bigwedge^3 \mathbb{R}^3 \to \mathbb{R},$$

we can evaluate the expression

$$(P_1, P_2, P_3, P_4) := \frac{m \wedge l_1 \wedge l_3}{m \wedge l_1 \wedge l_2} \cdot \frac{m \wedge l_4 \wedge l_2}{m \wedge l_4 \wedge l_3}$$

as an extended real number. One can then verify that the cross ratio (P_1, P_2, P_3, P_4) does not depend on the choice of m, l_1, l_2, l_3, l_4 or the choice of identification f.

This definition of the cross ratio agrees with the classical notion of the cross ratio of four points on a line in the following way. By taking the dual, the four lines $P_1, \ldots, P_4$ become four points $p_1, \ldots, p_4 \in (\mathbb{RP}^2)^*$, and they lie in the projective line in $(\mathbb{RP}^2)^*$ that corresponds to the point $[m]$ in $\mathbb{RP}^2$. One can then check that (P_1, P_2, P_3, P_4) is exactly the cross ratio of the four collinear points $p_1, \ldots, p_4$.

Proof of (1) of Theorem 1.1 when $n = 3$. Choose orientations on η and γ, and let A and B be elements in Γ that correspond to η and γ. Since $i(\eta, \gamma) \neq 0$, we can choose A and B so that if a^+, b^+ are the attracting fixed points and a^-, b^- are the repelling fixed points of A and B respectively, then

$$a^+,\ A \cdot b^+,\ b^+,\ a^-,\ b^-,\ A \cdot b^-$$

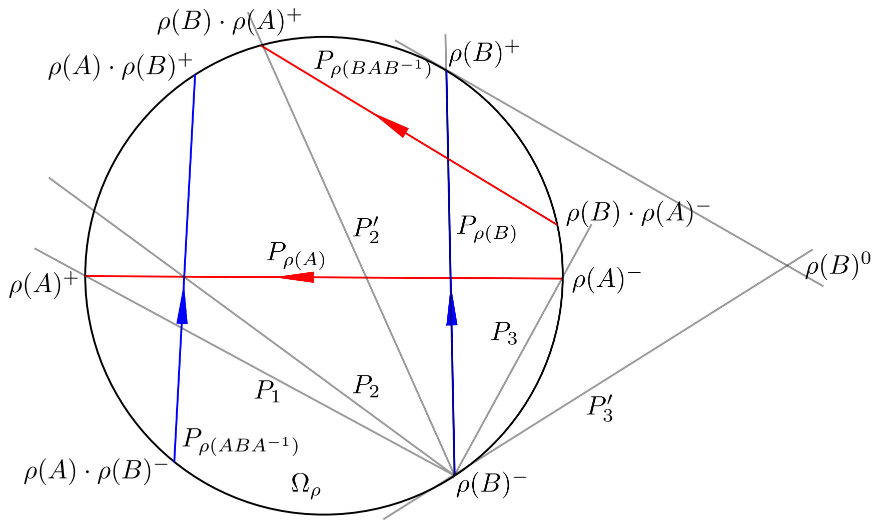

FIGURE 4. A schematic for the comparison between the cross ratios $(P_1, P_2, P_{\rho(B)}, P_3)$ and $(P_1, P_2', P_{\rho(B)}, P_3')$.

lie in $\partial_\infty\Gamma$ in that cyclic order. By Lemma 2.2, we see that

$$a^+,\ A\cdot b^+,\ B\cdot a^+,\ b^+,\ a^-,\ b^-,\ B^{-1}\cdot a^+,\ A\cdot b^-$$

lie in $\partial_\infty\Gamma$ in that cyclic order, because $A\cdot b^+$ and $A\cdot b^-$ are the attracting and repelling fixed points of ABA^{-1} respectively.

Choose any $\rho \in Hit_3(S)$. For any non-identity element $X \in \Gamma$, let $\rho(X)^+$, $\rho(X)^0$, $\rho(X)^-$ be the three fixed points for $\rho(X)$, where $\rho(X)^+$ is attracting and $\rho(X)^-$ is repelling. Denote by $P_{\rho(X)}$ the line segment in Ω_ρ with endpoints $\rho(X)^+$ and $\rho(X)^-$.

Now, let

- P_1 be the line through $\rho(B)^-$ and $\rho(A)^+$,
- P_2 be the line through $\rho(B)^-$ and $P_{\rho(A)} \cap P_{\rho(ABA^{-1})}$,
- P_3 be the line through $\rho(B)^-$ and $\rho(A)^-$,
- P_2' be the line through $\rho(B)^-$ and $\rho(B)\cdot\rho(A)^+$,
- P_3' be the line through $\rho(B)^-$ and $\rho(B)^0$.

By using $\xi^{(1)}$ to identify $\partial_\infty\Gamma$ with $\partial\Omega_\rho$, we have that

$$\rho(A)^+,\ \rho(A)\cdot\rho(B)^+,\ \rho(B)\cdot\rho(A)^+,\ \rho(B)^+,\ \rho(A)^-,\ \rho(B)^-$$

lie in $\partial\Omega_\rho$ in that cyclic order (see Figure 4.) It is a classically known property of the cross ratio (see Proposition 2.10) that

$$(P_1, P_2, P_{\rho(B)}, P_3) > (P_1, P_2', P_{\rho(B)}, P_3').$$

Let $0 < \alpha_1 < \alpha_2 < \alpha_3$ be the eigenvalues of $\rho(A)$ and $0 < \beta_1 < \beta_2 < \beta_3$ be the eigenvalues of $\rho(B)$. It is an easy cross ratio computation (see Lemma 2.8 and Lemma 2.9) that

$$(P_1, P_2', P_{\rho(B)}, P_3') = \frac{\beta_3}{\beta_3 - \beta_2}.$$

$$(P_1, P_2, P_{\rho(B)}, P_3) = \frac{\alpha_3}{\alpha_1}.$$

Hence, we have

$$\frac{\alpha_3}{\alpha_1} > \frac{\beta_3}{\beta_3 - \beta_2},$$

which implies that

$$\frac{\beta_2}{\beta_3} < 1 - \frac{\alpha_1}{\alpha_3}.$$

Similarly, by reversing the roles of $\rho(B)^-$ and $\rho(B)^+$, and using $\rho(B)^{-1}$ in place of $\rho(B)$, we can also show that

$$\frac{\beta_1}{\beta_2} < 1 - \frac{\alpha_1}{\alpha_3}.$$

Combining these inequalities gives

$$\frac{\beta_1}{\beta_3} = \frac{\beta_1}{\beta_2} \cdot \frac{\beta_2}{\beta_3} < \left(1 - \frac{\alpha_1}{\alpha_3}\right)^2,$$

which is equivalent to the inequality

$$\frac{\alpha_1}{\alpha_3} + \left(\frac{\beta_1}{\beta_3}\right)^{\frac{1}{2}} < 1.$$

Since $l_\rho(\eta) = \log(\frac{\alpha_3}{\alpha_1})$ and $l_\rho(\gamma) = \log(\frac{\beta_3}{\beta_1})$, (1) of Theorem 1.1 in the case when $n = 3$ follows immediately. □

2.3. **Properties of Frenet curves of Hitchin representations.** Next, we want to generalize the proof given in Section 2.2 to any Hitchin representation. We will devote this section to developing the tools needed to do so. In the rest of the paper, we use the same notation for points in $\mathbb{RP}^{n-1}$ and for lines in $\mathbb{R}^n$. It should be clear from the context which we are referring to.

Denote by $\mathcal{F}(\mathbb{R}^n)$ the space of complete flags in $\mathbb{R}^n$. Labourie [17] and Guichard [14] gave a beautiful characterization of representations in $Hit_n(S)$ as representations that admit an equivariant Frenet curve $\partial_\infty\Gamma \to \mathcal{F}(\mathbb{R}^n)$. When $n = 3$, the Frenet curve, post-composed with the projection from $\mathcal{F}(\mathbb{R}^3)$ to $\mathbb{RP}^2$, is exactly the map $\xi^{(1)} : \partial_\infty\Gamma \to \partial\Omega_\rho$ described in Section 2.2. This characterization will be the main tool we use to extend our proof in Section 2.2 to the general case.

We will start by first defining the Frenet property.

Notation 2.3. let $\xi : S^1 \to \mathcal{F}(\mathbb{R}^n)$ be a continuous closed curve. For any $k = 1, \dots, n-1$ and any point $x \in S^1$, let $\xi(x)^{(k)} := \pi_k(\xi(x))$, where $\pi_k : \mathcal{F}(\mathbb{R}^n) \to Gr(k, n)$ is the obvious projection.

Definition 2.4. A closed curve $\xi : S^1 \to \mathcal{F}(\mathbb{R}^n)$ is *Frenet* if the following hold:

(1) Let $x_1, \dots, x_k$ be pairwise distinct points in S^1 and let $n_1, \dots, n_k$ be positive integers so that $\sum_{i=1}^k n_i = n$. Then

$$\sum_{i=1}^{k} \xi(x_i)^{(n_i)} = \mathbb{R}^n.$$

(2) Let $x_1, \dots, x_k$ be pairwise distinct points in S^1 and let $n_1, \dots, n_k$ be positive integers so that $m := \sum_{i=1}^k n_i \le n$. Then for any $x \in S^1$,

$$\lim_{\substack{x_i \to x, \forall i \\ x_i \neq x_j, \forall i \neq j}} \sum_{i=1}^k \xi(x_i)^{(n_i)} = \xi(x)^{(m)}.$$

The Frenet property ensures ξ has good continuity properties and is "maximally transverse". Combining the work of Labourie (Theorem 1.4 of [17]) and Guichard (Théorème 1 of [14]), one can characterize the representations in the $Hit_n(S)$ as those that preserve an equivariant Frenet curve.

Theorem 2.5 (Guichard, Labourie)**.** *A representation ρ in the character variety*

$$Hom(\Gamma, PSL(n,\mathbb{R}))/PSL(n,\mathbb{R})$$

lies in $Hit_n(S)$ if and only if there exists a ρ-equivariant Frenet curve $\xi : \partial_\infty\Gamma \to \mathcal{F}(\mathbb{R}^n)$. If ξ exists, then it is unique.

We will now prove several properties of these Frenet curves that will be needed. These are special cases of more general properties that appear in Section 2 of [21]. However, for the sake of completeness, we will reproduce the proofs.

Lemma 2.6. *Let a, m_0, b, m_1, m_2, m_3 be pairwise distinct points on $\partial_\infty\Gamma$ in that cyclic order and let $\rho \in Hit_n(S)$ with corresponding Frenet curve ξ. Also, let $P := \mathbb{P}(\xi(a)^{(1)} + \xi(b)^{(1)})$. Then the following hold:*

(1) Let k_0, k_1, k_2, k_3, be non-negative integers that sum to $n-2$ and let $M := \sum_{i=0}^3 \xi(m_i)^{(k_i)}$. The map

$$f_M : \partial_\infty\Gamma \to P$$

given by

$$f_M : x \mapsto \begin{cases} \mathbb{P}(\xi(x)^{(1)} + \sum_{i=0}^3 \xi(m_i)^{(k_i)}) \cap P & \text{if } x \neq m_j \\ \mathbb{P}(\xi(m_j)^{(k_j+1)} + \sum_{i\neq j} \xi(m_i)^{(k_i)}) \cap P & \text{if } x = m_j \end{cases}$$

is a homeomorphism with $f_M(a) = \xi(a)^{(1)}$ and $f_M(b) = \xi(b)^{(1)}$.

(2) Let k_0, k_1, k_2 be non-negative integers that sum to $n-1$, and let s be the closed subsegment of $\partial_\infty\Gamma$ with endpoints a, b that does not contain m_0. Also, let $M := \xi(m_0)^{(k_0)}$. Then there is some closed subsegment ω of P with endpoints $\xi(a)^{(1)}$, $\xi(b)^{(1)}$ so that the map

$$g_M : s \to \omega$$

given by

$$g_M : x \mapsto \begin{cases} \mathbb{P}(\xi(x)^{(k_2)} + \xi(m_1)^{(k_1)} + \xi(m_0)^{(k_0)}) \cap P & \text{if } x \neq m_1 \\ \mathbb{P}(\xi(m_1)^{(k_1+k_2)} + \xi(m_0)^{(k_0)}) \cap P & \text{if } x = m_1 \end{cases}$$

is a homeomorphism with $g_M(a) = \xi(a)^{(1)}$ and $g_M(b) = \xi(b)^{(1)}$.

Proof. Before we start the proof, observe that for any non-negative integers $t_0, \dots, t_4$ so that $\sum_{i=0}^4 t_i = n-1$, the intersection $\mathbb{P}(\xi(x)^{(t_4)} + \sum_{i=0}^3 \xi(m_i)^{(t_i)}) \cap P$ is a single point, otherwise $\mathbb{P}(\xi(a)^{(1)} + \xi(b)^{(1)}) \subset \mathbb{P}(\xi(x)^{(t_4)} + \sum_{i=0}^3 \xi(m_i)^{(t_i)})$, which contradicts the Frenet property of ξ.

Proof of (1). Since ξ is Frenet, f_M is continuous. Moreover, because the domain and target of f_M are both topologically a circle, it is sufficient to show that f_M

is injective. Suppose for contradiction that there exist $x \neq x'$ such that $f_M(x) = f_M(x')$. We will assume that $x, x' \neq m_i$ for all $i = 0, 1, 2, 3$ as the other cases are similar. Then

$$\begin{aligned} \sum_{i=0}^{3} \xi(m_i)^{(k_i)} + \xi(x)^{(1)} &= \sum_{i=0}^{3} \xi(m_i)^{(k_i)} + f_M(x) \\ &= \sum_{i=0}^{3} \xi(m_i)^{(k_i)} + f_M(x') \\ &= \sum_{i=0}^{3} \xi(m_i)^{(k_i)} + \xi(x')^{(1)}, \end{aligned}$$

which is impossible because ξ is Frenet. The fact that $f_M(a) = \xi(a)^{(1)}$ and $f_M(b) = \xi(b)^{(1)}$ is easily verified.

Proof of (2). First, observe that g_M viewed as a map from s to P is continuous. Also, for any x in s, $g_M(x) = \xi(a)^{(1)}$ if and only if $x = a$ and $g_M(x) = \xi(b)^{(1)}$ if and only if $x = b$. This proves that the image of g_M is a subsegment ω of P with endpoints $\xi(a)^{(1)}$, $\xi(b)^{(1)}$.

To finish the proof, we only need to show that g_M is injective. Choose x, x' in the interior of s with $x \neq x'$, and assume without loss of generality that a, x', x, b lie along s in that order. Again, we assume that $x, x' \neq m_1$ as the other cases are similar. For any positive integer $i \leq k_2$, let

$$M_i := \xi(x)^{(i-1)} + \xi(x')^{(k_2-i)} + \xi(m_1)^{(k_1)} + \xi(m_0)^{(k_0)}.$$

By (1), we know that $f_{M_i}(x)$ lies on ω strictly between $f_{M_i}(x')$ and $f_{M_i}(b) = \xi(b)^{(1)}$. Also, observe that $f_{M_i}(x) = f_{M_{i+1}}(x')$. This implies that $f_{M_{k_2}}(x)$ lies on ω strictly between $f_{M_1}(x')$ and $\xi(b)^{(1)}$. In particular, $g_M(x) = f_{M_{k_2}}(x) \neq f_{M_1}(x') = g_M(x')$, so g_M is injective. □

In the proof of the $n = 3$ case given in Section 2.2, the classical cross ratio in $\mathbb{RP}^2$ was the main computational tool used to obtain our estimates. We will now define a generalization of the cross ratio for $\mathbb{RP}^{n-1}$.

Definition 2.7. Let $P_1, \dots, P_4$ be four hyperplanes in $\mathbb{R}^n$ that intersect along a $(n-2)$-dimensional subspace $M = Span\{m_1, \dots, m_{n-2}\} \subset \mathbb{R}^n$, so that no three of the four P_i agree. For $i = 1, \dots, 4$, let $L_i = [l_i]$ be a line through the origin in P_i that does not lie in M. Define the *cross ratio* by

$$(P_1, P_2, P_3, P_4) := \frac{m_1 \wedge \cdots \wedge m_{n-2} \wedge l_1 \wedge l_3 \cdot m_1 \wedge \cdots \wedge m_{n-2} \wedge l_4 \wedge l_2}{m_1 \wedge \cdots \wedge m_{n-2} \wedge l_1 \wedge l_2 \cdot m_1 \wedge \cdots \wedge m_{n-2} \wedge l_4 \wedge l_3}.$$

In the above definition, choose an identification between $\bigwedge^n(\mathbb{R}^n)$ and $\mathbb{R}$ to evaluate the fraction on the right as a real number. One can check that this number does not depend on the identification chosen, the choice of basis $\{m_1, \dots, m_{n-2}\}$ for M, the choice of L_i in P_i, or the choice of representatives l_i for L_i. When convenient, we sometimes use the notation

$$(L_1, L_2, L_3, L_4)_M := (P_1, P_2, P_3, P_4).$$

Also, at times, in our notation for the cross ratio, we replace the subspaces L_i, P_i and M of $\mathbb{R}^n$ with their projectivizations. As with the $n = 3$ case, this definition of the cross ratio agrees with the classical cross ratio of four points along a projective line in $(\mathbb{RP}^{n-1})^*$.

The following two lemmas summarizes some basic properties of this cross ratio.

Lemma 2.8. *Let $L_1,\dots,L_5$ be pairwise distinct lines in $\mathbb{R}^n$ through 0 and let M, M' be $(n-2)$-dimensional subspaces of $\mathbb{R}^n$ not containing L_i for any $i=1,\dots,5$, so that no three of the five $M+L_i$ agree and no three of the five $M'+L_i$ agree.*

(1) For any $X\in PSL(n,\mathbb{R})$, $(X\cdot L_1,\dots,X\cdot L_4)_{X\cdot M}=(L_1,\dots L_4)_M$.
(2) Suppose L_1,L_2,L_3,L_4 lie in a plane. Then
$$(L_1,L_2,L_3,L_4)_M=(L_1,L_2,L_3,L_4)_{M'}.$$
(3) $(L_1,L_2,L_3,L_4)_M=(L_4,L_3,L_2,L_1)_M$.
(4) $(L_1,L_2,L_3,L_5)_M\cdot(L_1,L_3,L_4,L_5)_M=(L_1,L_2,L_4,L_5)_M$.
(5) $(L_1,L_2,L_3,L_4)_M\cdot(L_1,L_3,L_2,L_4)_M=1$.
(6) $(L_1,L_2,L_3,L_4)_M=1-(L_1,L_2,L_4,L_3)_M$.

Proof. (1), (3), (4) and (5) follow immediately from the definition of the cross ratio. To prove (2), observe that there is a projective transformation X that fixes L_1, L_2, L_3 and maps M to M'. Since L_4 lies in the plane containing L_1, L_2 and L_3, X must also fix L_4. This allows us to use (1) to get (2).

To prove (6), assume that $M+L_1,\dots,M+L_4$ are pairwise distinct; the other cases are similar. Choose a basis $e_1,\dots,e_n$ for $\mathbb{R}^n$ so that
$$M=Span\{e_1,\dots,e_{n-2}\},\ L_1=[e_{n-1}],\ L_4=[e_n],\ L_2=[\sum_{i=1}^n e_i],\ L_3=[\sum_{i=1}^n\alpha_ie_i]$$
for some real numbers $\alpha_1,\dots,\alpha_n$. The assumption that $M+L_1,\dots,M+L_4$ are pairwise distinct implies that α_{n-1} and α_n are non-zero real numbers. One can then easily compute that
$$(L_1,L_2,L_3,L_4)_M=\frac{\alpha_n}{\alpha_{n-1}}\text{ and }(L_1,L_2,L_4,L_3)_M=\frac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}}.$$
□

In view of (2) of Lemma 2.8, we will denote $(L_1,L_2,L_3,L_4)_M$ by (L_1,L_2,L_3,L_4) in the case when L_1, L_2, L_3, L_4 lie in the same plane.

Lemma 2.9. *Let $X\in PSL(n,\mathbb{R})$ be diagonalizable with n real eigenvalues $\lambda_1,\dots,\lambda_n$ of pairwise distinct moduli, so that $|\lambda_1|<\cdots<|\lambda_n|$. Let L_i and L_j be fixed lines through the origin in $\mathbb{R}^n$ corresponding to the eigenvalues λ_i and λ_j respectively, with $i<j$, and let L be a line through the origin in the plane L_i+L_j so that $L_i\neq L\neq L_j$. Then*
$$(L_i,L,X\cdot L,L_j)=\frac{\lambda_j}{\lambda_i}.$$

Proof. Choose a basis $e_1,\dots,e_n$ for $\mathbb{R}^n$ so that $[e_k]$ is a fixed line through the origin of $\rho(X)$ corresponding to the eigenvalue λ_k. In this basis, $\rho(X)$ is the diagonal matrix $[x_{u,v}]$, where
$$x_{u,v}=\left\{\begin{array}{ll}0 & \text{if}\quad u\neq v\\ \lambda_u & \text{if}\quad u=v.\end{array}\right.$$
Let M be the $n-2$ dimensional subspace $Span\{e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_n\}$ of $\mathbb{R}^n$. Via a projective transformation that fixes $e_1,\dots,e_n$, we can assume $L=[e_i+e_j]$. The lemma follows from an easy computation using the cross ratio definition. □

The next task is to understand how the cross ratio interacts with Frenet curves.

Proposition 2.10. *Let $\rho \in Hit_n(S)$ and let ξ be the corresponding Frenet curve. Also, let a, b, c, m_0, d, m_1 be pairwise distinct points along $\partial_\infty \Gamma$, in that cyclic order, and let k_0, k_1 be non-negative integers that sum to $n-2$. For any $x \in \partial_\infty \Gamma$, define*

$$P_x = \begin{cases} \xi(x)^{(1)} + \xi(m_0)^{(k_0)} + \xi(m_1)^{(k_1)} & \text{if} \quad x \neq m_0, m_1 \\ \xi(m_i)^{(k_i+1)} + \xi(m_{1-i})^{(k_{1-i})} & \text{if} \quad x = m_i \end{cases}$$

Then the following hold:

(1) $(P_a, P_b, P_{m_0}, P_d) > (P_a, P_b, P_{m_0}, P_{m_1})$.
(2) $(P_a, P_b, P_{m_0}, P_d) > (P_a, P_c, P_{m_0}, P_d)$.

Proof. We will only show the proof of (1); the same proof together with the Lemma 2.8 gives (2). Let

$$\begin{aligned} L_{m_0} &= P_{m_0} \cap \Big(\xi(a)^{(1)} + \xi(b)^{(1)} \Big), \\ L_{m_1} &= P_{m_1} \cap \Big(\xi(a)^{(1)} + \xi(b)^{(1)} \Big), \\ L_d &= P_d \cap \Big(\xi(a)^{(1)} + \xi(b)^{(1)} \Big). \end{aligned}$$

Choose vectors $l_{m_0}, l_{m_1}, l_a, l_b, l_d$ in $\mathbb{R}^n$ such that

$$[l_{m_0}] = L_{m_0},\ [l_{m_1}] = L_{m_1},\ [l_a] = \xi(a)^{(1)},\ [l_b] = \xi(b)^{(1)},\ [l_d] = L_d.$$

By (1) of Lemma 2.6, we can ensure, by replacing each l_i with $-l_i$ if necessary, that

$$\begin{aligned} l_{m_0} &= \alpha l_a + (1-\alpha) l_b, \\ l_d &= \beta l_a + (1-\beta) l_b, \\ l_{m_1} &= \gamma l_a + (1-\gamma) l_b \end{aligned}$$

for $0 < \alpha < \beta < \gamma < 1$. Then one can compute

$$\begin{aligned} (P_a, P_b, P_{m_0}, P_d) &= \frac{1-\alpha}{1-\frac{\alpha}{\beta}} \\ &> \frac{1-\alpha}{1-\frac{\alpha}{\gamma}} \\ &= (P_a, P_b, P_{m_0}, P_{m_1}) \end{aligned}$$

□

2.4. **Proof in the $PSL(n,\mathbb{R})$ case.** We will now use the technical facts established in Section 2.3 to prove Theorem 1.1. For the rest of this section, fix $\rho \in Hit_n(S)$ and let ξ be its corresponding Frenet curve. By a theorem of Labourie (Theorem 1.5 of [17]), we know that for every non-identity element $X \in \Gamma$, $\rho(X) \in PSL(n,\mathbb{R})$ has a lift to $SL(n,\mathbb{R})$ that is diagonalizable with positive pairwise distinct eigenvalues. These eigenvalues will also be referred to as the eigenvalues of $\rho(X)$.

The next lemma is the main computation in the proof of Theorem 1.1.

Lemma 2.11. *Let B be a non-identity element in Γ and let b^-, b^+ be the repelling and attracting fixed points of B respectively. Pick $k = 0, \dots, n-2$, and for any*

$x \in \partial_\infty \Gamma$, define

$$P_x = P_x^{(k)} := \begin{cases} \xi(b^-)^{(k)} + \xi(b^+)^{(n-k-2)} + \xi(x)^{(1)} & \text{if } \ x \neq b^-, b^+ \\ \xi(b^-)^{(k+1)} + \xi(b^+)^{(n-k-2)} & \text{if } \ x = b^- \\ \xi(b^-)^{(k)} + \xi(b^+)^{(n-k-1)} & \text{if } \ x = b^+ \end{cases}$$

Suppose that x_1, x_2, x_3 are points in $\partial_\infty \Gamma$ so that

$$x_1, \ x_2, \ B \cdot x_1, \ b^+, \ x_3, \ b^-$$

lie on $\partial_\infty \Gamma$, in that cyclic order. Then

$$(P_{x_1}, P_{x_2}, P_{b^+}, P_{x_3}) > \frac{\beta_{k+2}}{\beta_{k+2} - \beta_{k+1}},$$

where $0 < \beta_1 < \cdots < \beta_n$ are the eigenvalues of $\rho(B)$.

Proof. By Proposition 2.10 and (5), (6) of Lemma 2.8, we have

$$\begin{aligned} (P_{x_1}, P_{x_2}, P_{b^+}, P_{x_3}) &> (P_{x_1}, P_{B\cdot x_1}, P_{b^+}, P_{b^-}) \\ &= \frac{1}{(P_{x_1}, P_{b^+}, P_{B\cdot x_1}, P_{b^-})} \\ &= \frac{1}{1 - (P_{b^+}, P_{x_1}, P_{B\cdot x_1}, P_{b^-})} \\ &= \frac{(P_{b^+}, P_{B\cdot x_1}, P_{x_1}, P_{b^-})}{(P_{b^+}, P_{B\cdot x_1}, P_{x_1}, P_{b^-}) - 1} \end{aligned} \tag{2.1}$$

Note that for all $j = 1, \ldots, n$,

$$L_j := \xi(b^-)^{(j)} \cap \xi(b^+)^{(n-j+1)}$$

is the fixed line through the origin in $\mathbb{R}^n$ of $\rho(B)$ corresponding to the eigenvalue β_j. Also, observe that P_{b^+} and P_{b^-} intersect the plane $\xi(b^-)^{(k+2)} \cap \xi(b^+)^{(n-k)}$ at L_{k+2} and L_{k+1} respectively. Let

$$L := P_{x_1} \cap (\xi(b^-)^{(k+2)} \cap \xi(b^+)^{(n-k)}),$$

and it is clear that $P_{B\cdot x_1} \cap (\xi(b^-)^{(k+2)} \cap \xi(b^+)^{(n-k)}) = \rho(B) \cdot L$. Thus, we can use Lemma 2.9, to conclude that

$$(P_{b^+}, P_{B\cdot x_1}, P_{x_1}, P_{b^-}) = (L_{k+2}, \rho(B) \cdot L, L, L_{k+1}) = \frac{\beta_{k+2}}{\beta_{k+1}}.$$

Combining this with inequality (2.1) proves the lemma. □

Applying Lemma 2.11 to our setting, we obtain the following proposition.

Proposition 2.12. *Let A, B be elements in Γ so that a^+, b^+, a^-, b^- lie in $\partial_\infty \Gamma$ in that cyclic order. Here, a^+, b^+ are the attracting fixed points and a^-, b^- are the repelling fixed points for A, B respectively. Also, let $\alpha_1 < \cdots < \alpha_n$, $\beta_1 < \cdots < \beta_n$ be the eigenvalues of $\rho(A)$, $\rho(B)$ respectively. For every $k = 0, \ldots, n-2$, the following hold:*

(1)

$$\frac{\alpha_n}{\alpha_1} > \frac{\beta_{k+2}}{\beta_{k+2} - \beta_{k+1}}.$$

(2) Let η, γ be closed curves in S corresponding to A and B respectively. If γ is simple, then

$$\frac{\alpha_n}{\alpha_1} > \left(\frac{\beta_{k+2}}{\beta_{k+2}-\beta_{k+1}}\right)^u \cdot \left(\frac{\beta_{n-k}}{\beta_{n-k}-\beta_{n-k-1}}\right)^{i(\eta,\gamma)-u}$$

for some non-negative integer $u \leq i(\eta,\gamma)$ that is independent of k.

Proof. Proof of (1). Let ω be the subsegment of $\mathbb{P}(\xi(a^+)^{(1)} + \xi(a^-)^{(1)})$ with endpoints $\xi(a^+)^{(1)}$, $\xi(a^-)^{(1)}$ that has non-empty intersection with $\mathbb{P}(\xi(b^-)^{(k)} + \xi(b^+)^{(n-k-1)})$. Define

$$\begin{aligned} p &:= \mathbb{P}(\xi(b^-)^{(k)} + \xi(b^+)^{(n-k-1)}) \cap \omega, \\ q &:= \mathbb{P}(\xi(b^-)^{(k)} + \xi(b^+)^{(n-k-2)} + \xi(A \cdot b^+)^{(1)}) \cap \omega, \end{aligned}$$

and note that q exists by (1) of Lemma 2.6. Also, observe that

$$\rho(A) \cdot p = \mathbb{P}(\xi(A \cdot b^-)^{(k)} + \xi(A \cdot b^+)^{(n-k-1)}) \cap \omega,$$

so (2) of Lemma 2.6 implies that $\rho(A) \cdot p$ lies between $\xi(a^+)^{(1)}$ and q in ω. Lemma 2.8, Lemma 2.9 and Proposition 2.10 together then allow us to conclude that

$$\begin{aligned} \frac{\alpha_n}{\alpha_1} &= (\xi(a^+)^{(1)}, \rho(A) \cdot p, p, \xi(a^-)^{(1)}) \\ &> (\xi(a^+)^{(1)}, q, p, \xi(a^-)^{(1)}) \\ &= (P_{a^+}, P_{A \cdot b^+}, P_{b^+}, P_{a^-}) \end{aligned}$$

where

$$P_x = P_x^{(k)} := \begin{cases} \xi(x)^{(1)} + \xi(b^-)^{(k)} + \xi(b^+)^{(n-k-2)} & \text{if} \quad x \neq b^-, b^+ \\ \xi(b^-)^{(k+1)} + \xi(b^+)^{(n-k-2)} & \text{if} \quad x = b^- \\ \xi(b^-)^{(k)} + \xi(b^+)^{(n-k-1)} & \text{if} \quad x = b^+. \end{cases}$$

By Lemma 2.2, we know that a^+, $A \cdot b^+$, $B \cdot a^+$, b^+, a^-, b^- lie along $\partial_\infty \Gamma$ in that cyclic order. This allows us to apply Lemma 2.11 with x_1, x_2, x_3 as a^+, $A \cdot b^+$, a^- respectively to obtain the desired inequality.

Proof of (2). Let r^-, r^+ be the closed subsegments of $\partial_\infty \Gamma$ with endpoints a^- and a^+, so that b^- and $A \cdot b^-$ lie in r^-, while b^+ and $A \cdot b^+$ lie in r^+. Orient both r^- and r^+ from a^- to a^+. Define $\mathcal{B}$ to be the set of unordered pairs $\{b'^+, b'^-\}$ in the Γ-orbit of $\{b^+, b^-\}$ so that b'^+ lies in r^+ between b^+ and $A \cdot b^+$, while b'^- lies in r^- between b^- and $A \cdot b^-$.

Every pair in $\mathcal{B}$ is the set of attracting and repelling fixed points for some B' in Γ that is conjugate to B. Since η is simple, we know that for every $\{b'^+, b'^-\}$ and $\{b''^+, b''^-\}$ in $\mathcal{B}$, b'^+ precedes b''^+ (in the orientation on r^+) if and only if b'^- precedes b''^- (in the orientation of r^-). The orientation on r^- and r^+ thus induce an ordering on $\mathcal{B}$. Also, observe that $|\mathcal{B}| = i(\eta, \gamma) + 1$, so we can label the pairs in $\mathcal{B}$ according to the order, i.e.

$$\mathcal{B} = \left\{ \{b_1^+, b_1^-\}, \ldots, \{b_{m+1}^+, b_{m+1}^-\} \right\},$$

where $b_1^+ = b^+$, $b_1^- = b^-$, $b_{m+1}^+ = A \cdot b^+$, $b_{m+1}^- = A \cdot b^-$, and $m = i(\eta, \gamma)$.

For each i, let B_i be the element in Γ that is conjugate to either B or B^{-1} so that its attracting and repelling fixed points are b_i^+ and b_i^- respectively. By Lemma

2.2, a^+, b^+_{i+1}, $B_i\cdot a^+$, b^+_i, a^-, b^-_i lie along $\partial_\infty\Gamma$ in that cyclic order, so we can apply Proposition 2.11 with x_1, x_2, x_3 as a^+, b^+_{i+1}, a^- respectively to conclude that

$$(P_{a^+,i}, P_{b^+_{i+1},i}, P_{b^+_i,i}, P_{a^-,i}) > \frac{\beta_{k+2}}{\beta_{k+2}-\beta_{k+1}} \tag{2.2}$$

if B_i is conjugate to B, and

$$(P_{a^+,i}, P_{b^+_{i+1},i}, P_{b^+_i,i}, P_{a^-,i}) > \frac{\beta_{n-k}}{\beta_{n-k}-\beta_{n-k-1}} \tag{2.3}$$

if B_i is conjugate to B^{-1}, where

$$P_{x,i} = P^{(k)}_{x_i} := \begin{cases} \xi(x)^{(1)} + \xi(b^-_i)^{(k)} + \xi(b^+_i)^{(n-k-2)} & \text{if} \quad x \neq b^-_i, b^+_i \\ \xi(b^-_i)^{(k+1)} + \xi(b^+_i)^{(n-k-2)} & \text{if} \quad x = b^-_i \\ \xi(b^-_i)^{(k)} + \xi(b^+_i)^{(n-k-1)} & \text{if} \quad x = b^+_i. \end{cases}$$

Fix any $k = 0, \dots, n-2$, and let ω be the subsegment of $\mathbb{P}(\xi(a^+)^{(1)} + \xi(a^-)^{(1)})$ with endpoints $\xi(a^+)^{(1)}$, $\xi(a^-)^{(1)}$ that has non-empty intersection with $\mathbb{P}(\xi(b^-_1)^{(k)} + \xi(b^+_1)^{(n-k-1)})$. For $i = 1, \dots, m+1$, define

$$p_i := \mathbb{P}(\xi(b^-_i)^{(k)} + \xi(b^+_i)^{(n-k-1)}) \cap \omega,$$

and for $i = 2, \dots, m+1$, define

$$q_i := \mathbb{P}(\xi(b^-_{i-1})^{(k)} + \xi(b^+_{i-1})^{(n-k-2)} + \xi(b^+_i)^{(1)}) \cap \omega.$$

Observe that (2) of Lemma 2.6 implies that p_i and q_i are well-defined, and that $\xi(a^-)^{(1)}, p_1, q_1, p_2, q_2, \dots, p_m, q_m, p_{m+1}, \xi(a^+)^{(1)}$ lie in ω in that order. Hence, by similar arguments as those used in the proof of (1), we have

$$\begin{aligned} (\xi(a^+)^{(1)}, p_{i+1}, p_i, \xi(a^-)^{(1)}) &> (\xi(a^+)^{(1)}, q_{i+1}, p_i, \xi(a^-)^{(1)}) \\ &= (P_{a^+,i}, P_{b^+_{i+1},i}, P_{b^+_i,i}, P_{a^-,i}) \end{aligned}$$

We can then use Lemma 2.9 and Lemma 2.8 to obtain

$$\begin{aligned} \frac{\alpha_n}{\alpha_1} &= (\xi(a^+)^{(1)}, p_{m+1}, p_1, \xi(a^-)^{(1)}) \\ &= \prod_{i=1}^{m} (\xi(a^+)^{(1)}, p_{i+1}, p_i, \xi(a^-)^{(1)}) \\ &> \prod_{i=1}^{m} (P_{a^+,i}, P_{b^+_{i+1},i}, P_{b^+_i,i}, P_{a^-,i}). \end{aligned} \tag{2.4}$$

Let $\mathcal{B}_+ := \{i : B_i \text{ is conjugate to } B\}$, $\mathcal{B}_- := \{i : B_i \text{ is conjugate to } B^{-1}\}$ and let $u := |\mathcal{B}_+|$. Then combining the inequalities (2.2), (2.3) and (2.4) yields

$$\begin{aligned} \frac{\alpha_n}{\alpha_1} &> \prod_{i\in\mathcal{B}_+} (P_{a^+,i}, P_{b^+_{i+1},i}, P_{b^+_i,i}, P_{a^-,i}) \cdot \prod_{i\in\mathcal{B}_-} (P_{a^+,i}, P_{b^+_{i+1},i}, P_{b^+_i,i}, P_{a^-,i}) \\ &> \left(\frac{\beta_{k+2}}{\beta_{k+2}-\beta_{k+1}}\right)^u \cdot \left(\frac{\beta_{n-k}}{\beta_{n-k}-\beta_{n-k-1}}\right)^{i(\eta,\gamma)-u}. \end{aligned}$$

□

With Proposition 2.12, we can now prove Theorem 1.1.

Proof of Theorem 1.1. In this proof, we will use the same notation as we used in Proposition 2.12.

Proof of (1). Choose orientations on γ and η. The hypothesis on γ and η imply that there are group elements A, B in Γ corresponding to η, γ respectively, so that

$$a^+,\ b^+,\ a^-,\ b^-$$

lie along $\partial_\infty\Gamma$ in that cyclic order. Let $0 < \alpha_1 < \cdots < \alpha_n$ and $0 < \beta_1 < \cdots < \beta_n$ be the eigenvalues of $\rho(A)$ and $\rho(B)$ respectively. By (1) of Proposition 2.12, we know that for all $k = 0, \dots, n-2$,

$$\frac{\alpha_n}{\alpha_1} > \frac{\beta_{k+2}}{\beta_{k+2} - \beta_{k+1}},$$

which implies that

$$\frac{\beta_{k+1}}{\beta_{k+2}} < 1 - \frac{\alpha_1}{\alpha_n}.$$

Taking the product over all $k = 0, \dots, n-2$, we get

$$\frac{\alpha_1}{\alpha_n} + \left(\frac{\beta_1}{\beta_n}\right)^{\frac{1}{n-1}} < 1.$$

Since $l_\rho(\eta) = \log(\frac{\alpha_n}{\alpha_1})$ and $l_\rho(\gamma) = \log(\frac{\beta_n}{\beta_1})$, the above inequality gives us (1).

Proof of (2). By Proposition 2.12, we know that there is some non-negative integer $u \leq i(\eta,\gamma)$ so that for any $k = 0, \dots, n-2$, we have

$$\frac{\alpha_1}{\alpha_n} < \left(1 - \frac{\beta_{k+1}}{\beta_{k+2}}\right)^u \left(1 - \frac{\beta_{n-k-1}}{\beta_{n-k}}\right)^{i(\eta,\gamma)-u}.$$

In particular, we also have that for any $k = 0, \dots, n-2$,

$$\frac{\alpha_1}{\alpha_n} < \left(1 - \frac{\beta_{k+1}}{\beta_{k+2}}\right)^{i(\eta,\gamma)-u} \left(1 - \frac{\beta_{n-k-1}}{\beta_{n-k}}\right)^u,$$

so we can assume that

$$\frac{\alpha_1}{\alpha_n} < \left(1 - \frac{\beta_{k+1}}{\beta_{k+2}}\right)^u \left(1 - \frac{\beta_{n-k-1}}{\beta_{n-k}}\right)^v.$$

for some non-negative integers u, v so that $u \geq v$ and $u + v = i(\eta,\gamma)$. This implies that

$$\frac{\alpha_1}{\alpha_n} < \left(1 - \frac{\beta_{k+1}}{\beta_{k+2}}\right)^u \left(1 - \frac{\beta_1}{\beta_n}\right)^v,$$

which we can rewrite as

$$\frac{\beta_{k+1}}{\beta_{k+2}} < 1 - \frac{(\frac{\alpha_1}{\alpha_n})^{\frac{1}{u}}}{(1 - \frac{\beta_1}{\beta_n})^{\frac{v}{u}}}.$$

By taking the product of the above inequality over $k = 0, \dots, n-2$, we have

$$\left(\frac{\beta_1}{\beta_n}\right)^{\frac{1}{n-1}} < 1 - \frac{(\frac{\alpha_1}{\alpha_n})^{\frac{1}{u}}}{(1 - \frac{\beta_1}{\beta_n})^{\frac{v}{u}}},$$

which can be rewritten as

$$\frac{\alpha_1}{\alpha_n} < \left(1 - \left(\frac{\beta_1}{\beta_n}\right)^{\frac{1}{n-1}}\right)^u \left(1 - \frac{\beta_1}{\beta_n}\right)^v,$$

from which (2) follows.

Proof of (3). Choose an orientation on η. Since η is non-simple, there are group elements A, B corresponding to η so that

$$a^+,\ b^+,\ a^-,\ b^-$$

lie along $\partial_\infty \Gamma$ in that cyclic order. Let $0 < \alpha_1 < \cdots < \alpha_n$ and $0 < \beta_1 < \cdots < \beta_n$ be the eigenvalues of $\rho(A)$ and $\rho(B)$ respectively. Note that $\rho(B)$ is either conjugate to $\rho(A)$ or $\rho(A)^{-1}$, so $\frac{\beta_n}{\beta_1} = \frac{\alpha_n}{\alpha_1}$. Hence, the same computation as the proof of (1) then yields the inequality

$$\left(\frac{\alpha_1}{\alpha_n}\right)^{\frac{1}{n-1}} + \frac{\alpha_1}{\alpha_n} < 1,$$

which is equivalent to

$$\left(1 - \frac{\alpha_1}{\alpha_n}\right)^{n-1} - \frac{\alpha_1}{\alpha_n} > 0. \tag{2.5}$$

Consider the polynomial $P_n(x) = x^{n-1} + x - 1$. Note that for $n \geq 2$, $P_n(x)$ is strictly increasing on the interval $[0,1]$, $P_n(0) = -1$ and $P_n(1) = 1$. Hence, P_n has a unique zero in the interval $(0,1)$, which we denote by x_n. It then follows that

$$\{x \in [0,1] : P_n(x) > 0\} = (x_n, 1].$$

Also, observe that

$$P_n\left(1 - \frac{\alpha_1}{\alpha_n}\right) = \left(1 - \frac{\alpha_1}{\alpha_n}\right)^{n-1} - \frac{\alpha_1}{\alpha_n}$$

and $0 < 1 - \frac{\alpha_1}{\alpha_n} < 1$. Since $\frac{\alpha_1}{\alpha_n}$ satisfies the inequality (2.5), we have

$$x_n < 1 - \frac{\alpha_1}{\alpha_n} < 1.$$

This implies that

$$l_\rho(\eta) = \log\left(\frac{\alpha_n}{\alpha_1}\right) > \delta_n := -\log(1 - x_n).$$

□

3. Further remarks.

In this section, we prove some corollaries to Theorem 1.1, show that it does not hold for quasi-Fuchsian representations, and perform a comparison with the classical collar lemma.

3.1. **Corollaries.**

In this subsection, we will mention some consequences to Theorem 1.1. The first is a universal collar lemma that holds simultaneously for all Hitchin representations.

Corollary 3.1. *Let S be a surface of genus $g \geq 2$, and let γ, η be two non-contractible closed curves in S. Then for any $n \geq 2$ and any $\rho \in Hit_n(S)$, the following hold:*

(1) If $i(\eta, \gamma) \neq 0$, then

$$(\exp(l_\rho(\gamma)) - 1)(\exp(l_\rho(\eta)) - 1) > 1.$$

(2) If $i(\eta,\gamma)\neq 0$ and γ is simple, then

$$(\exp(l_\rho(\gamma))-1)\left(\exp\left(\frac{l_\rho(\eta)}{i(\eta,\gamma)}\right)-1\right)>1.$$

(3) If η is a non-simple closed curve, then

$$l_\rho(\eta)>\log(2).$$

Proof. Proof of (1). For all $n\geq 2$, (1) of Theorem 1.1 tells us that

$$\frac{1}{\exp(l_\rho(\eta))}<1-\frac{1}{\exp(\frac{l_\rho(\gamma)}{n-1})}\leq 1-\frac{1}{\exp(l_\rho(\gamma))}.$$

By rearranging the terms in this inequality, we then have

$$(\exp(l_\rho(\eta))-1)(\exp(l_\rho(\gamma))-1)>1.$$

The proof of (2) is very similar to the proof of (1), and (3) is obvious since $\{\delta_i\}_{i=2}^\infty$ is an increasing sequence and $\delta_2=\log(2)$. □

An easy corollary to Theorem 1.1 is a generalization of a direct consequence of the classical collar lemma in the case of $\mathcal{T}(S)$.

Corollary 3.2. *For any $n\geq 2$ and any $\rho\in Hit_n(S)$, there are at most $3g-3$ closed curves in S of ρ-length at most δ_n.*

In the case of $\mathcal{T}(S)$, one can replace the number $\delta_2=\log(2)$ with $4\cdot\sinh^{-1}(1)$ (see Theorem 4.2.2 of [6]).

Proof. By (1) of Theorem 1.1, if η and γ are closed curves in S such that $i(\eta,\gamma)\neq 0$, then $l_\rho(\eta)$ and $l_\rho(\gamma)$ cannot both be smaller than δ_n. Moreover, (3) of Theorem 1.1 tells us that any closed curve of ρ-length less than δ_n has to be simple. Thus, the set of closed curves of ρ-length less than δ_n has to be a pairwise disjoint collection of simple closed curves, so the size of this collection is at most $3g-3$. □

Let $\widetilde{M}$ be the $SL(n,\mathbb{R})$ symmetric space, and let $d_{\widetilde{M}}$ be the distance function given by the Riemannian metric on $\widetilde{M}$. It is a well-known that for any $Z\in\Gamma$, the translation length of $\rho(Z)$, $\inf\{d_{\widetilde{M}}(o,\rho(Z)\cdot o):o\in\widetilde{M}\}$, is

$$c_n\sqrt{\sum_{i=1}^n(\log\lambda_i)^2},$$

for some positive constant c_n depending only on n. Here, $0<\lambda_1<\cdots<\lambda_n$ are the eigenvalues of $\rho(Z)$. (See II.10 of Bridson-Haefliger [3].) For our purposes, we normalize the metric on $\widetilde{M}$ so that $c_n=\sqrt{2}$, i.e. so that the image of the totally geodesic embedding of $\mathbb{H}^2$ in $\widetilde{M}$ induced by $\iota_n:PSL(2,\mathbb{R})\to PSL(n,\mathbb{R})$ has sectional curvature $-\frac{6}{n(n-1)(n+1)}$. Then for any discrete, faithful representation $\rho:\Gamma\to PSL(n,\mathbb{R})$, and for any rectifiable closed curve ω in $M:=\rho(\Gamma)\backslash\widetilde{M}$, let $l_M(\omega)$ be the length of ω in the Riemannian metric on M.

In the case when $\rho\in Hit_n(S)$, we can use Theorem 1.1, to obtain a relationship between the lengths of curves in the quotient locally symmetric space M.

Corollary 3.3. *Let γ, η be two non-contractible closed curves in S and let X, Y be elements in Γ corresponding to γ, η respectively. For any $\rho \in Hit_n(S)$, let γ', η' be two closed curves in $\rho(\Gamma)\backslash\widetilde{M} =: M$ that correspond to $X, Y \in \Gamma = \pi_1(M)$ respectively. Then the statements in Theorem 1.1 and Corollary 1.2 hold, with $l_\rho(\gamma)$ and $l_\rho(\eta)$ replaced with $l_M(\gamma')$ and $l_M(\eta')$ respectively.*

Proof. Pick any $Z \in \Gamma \setminus \{\mathrm{id}\}$, and let ω in S and ω' in M be closed curves corresponding to Z. Observe then that the translation length of $\rho(Z)$ in $\widetilde{M}$ is a lower bound for $l_M(\omega')$.

Also, since

$$2\sum_{i=1}^{n} x_i^2 - (x_n - x_1)^2 = (x_1 + x_n)^2 + 2(x_2^2 + \cdots + x_{n-1}^2) \geq 0,$$

we have that

$$(x_n - x_1)^2 \leq 2\sum_{i=1}^{n} x_i^2.$$

This allows us to compute

$$l_\rho(\omega) = \log\left(\frac{\lambda_n}{\lambda_1}\right) \leq \sqrt{2\sum_{i=1}^{n}(\log\lambda_i)^2} \leq l_M(\omega'),$$

where $0 < \lambda_1 < \cdots < \lambda_n$ are the eigenvalues of $\rho(Z)$. □

Note that in Corollary 3.3, the closed curves γ' and η' in the locally symmetric space do not necessarily intersect, even when $i(\gamma, \eta) \neq 0$. In particular, Corollary 3.3 is not simply a quantitative version of the Margulis lemma on $PSL(n, \mathbb{R})$. However, it does imply a quantitative version of the Margulis lemma in the following way. Let $f : S \to M := \rho(\Gamma)\backslash\widetilde{M}$ be a π_1-injective map such that $f(\gamma)$ and $f(\eta)$ are rectifiable curves in the Riemannian metric on M. Then the statements in Theorem 1.1 and Corollary 1.2 hold, with $l_\rho(\gamma)$ and $l_\rho(\eta)$ replaced with $l_M(f(\gamma))$ and $l_M(f(\eta))$ respectively. In particular, we have a collar lemma for the image of the harmonic maps corresponding to Hitchin representations that were given by Corlette [10], Labourie [18] or Eells-Sampson [11].

Theorem 1.1 and Corollary 1.2 also allow us to deduce consequence that are similar to Corollary 3.3, but with the Hilbert metric on the symmetric space instead of the Riemannian one. The symmetric space $\widetilde{M}$ can be given a Hilbert metric in the following way. Let $S(n, \mathbb{R})$ be the space of symmetric n by n matrices with real entries and let $P(n, \mathbb{R})$ be the set of positive-definite matrices in $S(n, \mathbb{R})$. Let $\mathbb{P}(P)$ and $\mathbb{P}(S)$ be the projectivizations of $P(n, \mathbb{R})$ and $S(n, \mathbb{R})$ respectively, and observe that $\mathbb{P}(P)$ is a properly convex domain in $\mathbb{P}(S) \simeq \mathbb{RP}^{N-1}$, where $N = \frac{n(n+1)}{2}$. This allows us to equip $\mathbb{P}(P)$ with a Hilbert metric.

Moreover, we can define a $PSL(n, \mathbb{R})$-action on $\mathbb{P}(S)$ by $g \cdot A := gAg^T$ for any $g \in PSL(n, \mathbb{R})$ and any $A \in \mathbb{P}(S)$. Note that this action preserves the projective structure on $\mathbb{P}(S)$, and also preserves $\mathbb{P}(P)$. In fact, $PSL(n, \mathbb{R})$ acts transitively on $\mathbb{P}(P)$, and the stabilizer of the projective class of the identity matrix in $\mathbb{P}(P)$ is $PSO(n)$, so the symmetric space $\widetilde{M}$ can be identified with $\mathbb{P}(P)$. This equips $\widetilde{M}$ with a Hilbert metric. Denote $\widetilde{M}$ equipped with the Hilbert metric by $\widetilde{M}'$, and for any discrete, faithful representation $\rho : \Gamma \to PSL(n, \mathbb{R})$, let $l_{M'}$ be the length function on $M' := \rho(\Gamma)\backslash\widetilde{M}'$ induced by the Hilbert metric. Theorem 1.1 and Corollary 1.2 then also implies the following corollary.

Corollary 3.4. *Let γ, η be two non-contractible closed curves in S and let X, Y be elements in Γ corresponding to γ, η respectively. For any $\rho \in Hit_n(S)$, let γ', η' be two closed curves in M' that correspond to $X, Y \in \Gamma = \pi_1(M')$ respectively. Then the statements in Theorem 1.1 and Corollary 1.2 hold, with $l_\rho(\gamma)$ and $l_\rho(\eta)$ replaced with $\frac{1}{2}l_{M'}(\gamma')$ and $\frac{1}{2}l_{M'}(\eta')$ respectively.*

Proof. For any $Z \in \Gamma \setminus \{\mathrm{id}\}$, let $0 < \lambda_1 < \cdots < \lambda_n$ be the eigenvalues of $\rho(Z)$. We can assume without loss of generality that $\rho(Z)$ is a diagonal. Let E_{ij} be the (n,n)–matrix with 1 at position (i,j) and zero everywhere else, and let $B_{ij} = E_{ij} + E_{ji}$. Obviously, $\{B_{ij}\}_{i\leq j}$ is a basis of $S(n,\mathbb{R}) = \mathbb{R}^N$, and it is easy to verify that $\rho(Z)\cdot B_{ij} = \lambda_i\lambda_j B_{ij}$. That means B_{ij} is an eigenvector of $\rho(Z)$ with eigenvalue $\lambda_i\lambda_j$. Consequently, the translation length of $\rho(Z)$ is

$$\log\left(\frac{\lambda_n^2}{\lambda_1^2}\right) = 2\log\left(\frac{\lambda_n}{\lambda_1}\right)$$

(see Proposition 2.1 in [9]). The corollary then follows easily. □

The image of the irreducible representation $\iota_n : PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$ lies in a conjugate of the subgroup $PSO(k,k+1) \subset PSL(2k+1,\mathbb{R})$ when $n = 2k+1$, and a conjugate of $PSp(2k,\mathbb{R}) \subset PSL(2k,\mathbb{R})$ when $n = 2k$. Hence, we can define Hitchin components of the character varieties

$$Hom(\Gamma, PSO(k,k+1))/PSO(k,k+1),\ Hom(\Gamma, PSp(2k,\mathbb{R}))/PSp(2k,\mathbb{R})$$

in the same way as we did for $PSL(n,\mathbb{R})$. Denote these Hitchin components by $Hit_n(S)'$. Note that $Hit_n(S)'$ can be naturally identified with a subset of $Hit_n(S)$. In the case when $\rho \in Hit_n(S)$ happens to be an element of $Hit_n(S)'$, we can strengthen (2) of Proposition 2.12, which we state as the following corollary.

Corollary 3.5. *Let A, B be elements in Γ so that a^+, b^+, a^-, b^- lie in $\partial_\infty\Gamma$ in that cyclic order. Here, a^+, b^+ are the attracting fixed points and a^-, b^- are the repelling fixed points for A, B respectively. Let $\rho \in Hit_n(S)'$ and let $\alpha_1 < \cdots < \alpha_n$, $\beta_1 < \cdots < \beta_n$ be the eigenvalues of $\rho(A)$, $\rho(B)$ respectively. Finally, let η and γ be closed curves on S corresponding to A and B respectively. If γ is a simple closed curve in S, then for every $k = 0,\dots,n-2$,*

$$\frac{\alpha_n}{\alpha_1} > \left(\frac{\beta_{k+2}}{\beta_{k+2}-\beta_{k+1}}\right)^{i(\eta,\gamma)}$$

Proof. Since $\rho(B)$ is a diagonalizable element in $PSO(k,k+1)$ or $PSp(2k,\mathbb{R})$, we see that $\beta_{k+1} = \frac{1}{\beta_{n-k}}$ for $k = 0,\dots,n-1$. Apply this to (2) of Proposition 2.12. □

From this corollary, the same proof as (2) of Theorem 1.1 allows us to obtain the following stronger inequality in the case when $\rho \in Hit_n(S)'$.

Corollary 3.6. *Let γ, η be two non-contractible closed curves in S so that γ is simple and $i(\eta,\gamma) \neq 0$. Then for any $\rho \in Hit_n(S)'$,*

$$\frac{1}{\exp(l_\rho(\eta))} < \left(1 - \frac{1}{\exp(\frac{l_\rho(\gamma)}{n-1})}\right)^{i(\eta,\gamma)}.$$

Finally, combining (1) of Proposition 2.12 with some results proven in [21], one obtains the following properness result.

Corollary 3.7. *Let $\mathcal{C} := \{\gamma_1, \dots, \gamma_k\}$ be a collection of closed curves in S that contains a pants decomposition, so that the complement of $\mathcal{C}$ in S is a union of discs. Then the map*

$$\begin{array}{rcl} \Psi : Hit_n(S) & \to & \mathbb{R}^k \\ \rho & \mapsto & (l_\rho(\gamma_1), \dots, l_\rho(\gamma_k)) \end{array}$$

is proper.

In other words, whether or not a sequence $\{\rho_i\}$ divergences in the Hitchin component can be detected by the ρ_i-lengths of the curves in $\mathcal{C}$. We postpone the proof of this corollary to Appendix A because it uses some technical results from [21].

3.2. **Counter example for non-Hitchin representations.** Note that in our proof, we used very strongly that the representations we consider are in $Hit_n(S)$ because we used properties of the Frenet curve to obtain our estimates. In fact, the collar lemma is special to Hitchin representations, and does not hold even on the space of discrete and faithful representations from Γ to $PSL(n, \mathbb{R})$.

To see this, consider the space of conjugacy classes of quasi-Fuchsian representations from Γ to $PSL(2, \mathbb{C}) = PSO(3, 1)^+ \subset PSL(4, \mathbb{R})$, which is the group of orientation preserving isometries of $\mathbb{H}^3$. These are discrete and faithful representations whose limit set in the Riemann sphere is a Jordan curve. It is well-known that each quasi-Fuchsian representation ρ induces a convex cocompact hyperbolic structure on the three-manifold $S \times I$. Also, for any non-identity element X in Γ, the closed geodesic γ in $S \times I$ (equipped with the hyperbolic metric induced by ρ) corresponding to X has ρ-length

$$l_\rho(\gamma) = \log \left| \frac{\lambda_4}{\lambda_1} \right|,$$

where λ_4 and λ_1 are the eigenvalues of $\rho(X)$ with largest and smallest modulus respectively (see Proposition 2.1 of Cooper-Long-Tillman [9]).

It is a theorem of Bers (Theorem 1 of [1]) that the space of quasi-Fuchsian representations can be naturally identified with $\mathcal{T}(S) \times \mathcal{T}(\overline{S})$, where $\overline{S}$ is S with the opposite orientation. For any quasi-Fuchsian representation ρ let (ρ^+, ρ^-) denote the pair of Fuchsian representations that corresponds to ρ, so that $\rho^+ \in \mathcal{T}(S)$ and $\rho^- \in \mathcal{T}(\overline{S})$. Then for any closed non-contractible curve γ in S, let γ_ρ be the geodesic representative of γ in the hyperbolic metric on $S \times I$ corresponding to ρ, and let γ_{ρ^+} and γ_{ρ^-} be the geodesic representatives of γ in the hyperbolic metrics on S and $\overline{S}$ corresponding to ρ^+ and ρ^- respectively. By Theorem 3.1 of Epstein-Marden-Markovic [12] we know that

$$l_\rho(\gamma_\rho) \le \min\{2 \cdot l_{\rho^+}(\gamma_{\rho^+}), 2 \cdot l_{\rho^-}(\gamma_{\rho^-})\}.$$

For any pair of simple closed curves η and γ and for any $\epsilon > 0$, let ρ be a quasi-Fuchsian representation so that

$$l_{\rho^+}(\gamma_{\rho^+}) < \frac{\epsilon}{2} \quad \text{and} \quad l_{\rho^-}(\eta_{\rho^-}) < \frac{\epsilon}{2}.$$

Hence, $l_\rho(\gamma_\rho)$ and $l_\rho(\eta_\rho)$ are both smaller than ϵ. This implies that the analog of Theorem 1.1 does not hold on the space of discrete and faithful, or even Anosov, representations from Γ to $PSL(4,\mathbb{R})$. (See Guichard-Wienhard [15] for more background on Anosov representations.)

3.3. **Comparison with the classical collar lemma.** Let ρ be a representation in the Fuchsian locus of $Hit_n(S)$ and let h be the corresponding Fuchsian representation in $\mathcal{T}(S)$. Also, let X be a non-identity element in Γ and let γ be a curve in S corresponding to X. If λ^{-1} and λ are the two eigenvalues of $h(X)$, then λ^{-n+1}, λ^{-n+3}, $\dots$, λ^{n-3}, λ^{n-1} are the n eigenvalues of $\rho(X)$. Hence we can get the lengths

$$l_h(\gamma) = 2\log(\lambda) \quad \text{and} \quad l_\rho(\gamma) = 2(n-1)\log(\lambda).$$

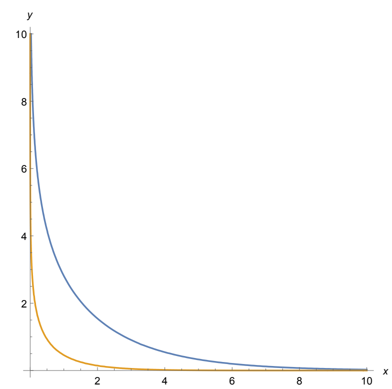

FIGURE 5. The blue curve $\sinh(\frac{x}{2})\sinh(\frac{y}{2}) = 1$ and the orange curve $(e^x-1)(e^y-1) = 1$

Since $h \in \mathcal{T}(S)$, the classical collar lemma holds. In other words, for any pair of curves γ and η in S such that η is simple and $i(\eta,\gamma) > 0$, we have (Corollary 4.1.2 of Buser [6])

$$I_{\gamma,\eta}(h) := \sinh\left(\frac{l_h(\gamma)}{2}\right)\sinh\left(\frac{l_h(\eta)}{2}\right) > 1. \tag{3.1}$$

This inequality is sharp, in the sense that for any S, there are curves η and γ in S and a sequence of Fuchsian representations $\{h_i\}$ such that $I_{\gamma,\eta}(h_i)$ converges to 1. For more details, refer to Section 6 of Matelski [19].

On the other hand, (1) of Corollary 1.2, specialized to the $n = 2$ case, is the inequality

$$(e^{l_h(\gamma)} - 1)(e^{l_h(\eta)} - 1) > 1.$$

This is weaker than the inequality (3.1), because

$$e^x - 1 > \frac{e^{-\frac{1}{2}x}}{2}(e^x - 1) = \sinh\left(\frac{x}{2}\right)$$

for every $x > 0$ (see Figure 5). Moreover, we are unable to show that the inequality (3.1) fails in $Hit_n(S)$ for any $n > 2$. This led us to make the following conjecture in an earlier version of this paper.

Conjecture 3.8. *Let ρ be a representation in $Hit_n(S)$. Then there is some representation ρ' in the Fuchsian locus of $Hit_n(S)$ such that*

$$l_\rho(\gamma) \geq l_{\rho'}(\gamma), \text{ for any } \gamma \in \Gamma.$$

This conjecture implies that

$$\sinh\Big(\frac{l_\rho(\gamma)}{2(n-1)}\Big)\sinh\Big(\frac{l_\rho(\eta)}{2(n-1)}\Big)>1$$

for any $\rho\in Hit_n(S)$, which is sharp on every $Hit_n(S)$ because it is sharp when restricted to the Fuchsian locus. Recently, Tholozan proved (Section 0.4 of [20]) that the conjecture holds in the case when $n=3$. Furthermore, F. Labourie disproved our conjecture in the case when $n\geq 4$. We will give his argument here.

For any closed curve γ in S, let $L_\gamma : Hit_n(S)\to\mathbb{R}$ denote the map given by $L_\gamma(\rho)=l_\rho(\gamma)$. As before, let $Hit_n(S)'$ be the $PSp(2k,\mathbb{R})$ or $PSO(k,k+1)$ Hitchin components when $n=2k$ and $n=2k+1$ respectively, and recall that for all $\rho\in Hit_n(S)'$, $l_\rho(\gamma)=2\log|\lambda_n(\rho(X))|$, where $X\in\Gamma$ is a group element corresponding to γ and $\lambda_n(\rho(X))$ is the eigenvalue of $\rho(X)$ with largest modulus. Proposition 10.3 of Bridgeman-Canary-Labourie-Sambarino [4] then implies that for any $\rho\in Hit_n(S)'$, the set of differentials $\{dL_\gamma : \gamma \text{ a closed curve in } S\}$ generates the entire cotangent space of $Hit_n(S)'$ at ρ.

Observe that when $n\geq 4$, $Hit_n(S)'\subset Hit_n(S)$ properly contains the Fuchsian locus. Thus, to show that the conjecture is false, it is sufficient to show that it fails on $Hit_n(S)'$. Suppose for contradiction that the conjecture holds on $Hit_n(S)'$. Choose a point ρ_0 in the Fuchsian locus, and take a smooth path ρ_t, $t\in(-\epsilon,\epsilon)$ with $\epsilon>0$, whose non-zero tangent vector $U\in T_{\rho_0}Hit_n(S)'$ is not tangential to the Fuchsian locus. Along the path, choose a sequence of representations $\{\rho_{t_i}\}_{i=1}^\infty$ which converges to ρ_0 as $i\to\infty$ so that $t_i>0$ for all i.

Since the conjecture holds, there exists the corresponding sequence of Fuchsian representations ρ'_{t_i} such that $L_\gamma(\rho_{t_i})\geq L_\gamma(\rho'_{t_i})$, for any $\gamma\in\Gamma$. Also, since ρ_{t_i} converges to ρ_0, we see that $L_\gamma(\rho'_{t_i})$ is bounded for all $\gamma\in\Gamma$, so the sequence $\{\rho'_{t_i}\}_{i=1}^\infty$ lie in a compact subset of the Fuchsian locus. By picking subsequence, we can assume without loss of generality that $\{\rho'_{t_i}\}_{i=1}^\infty$ converges to some ρ'_0 in the Fuchsian locus. The continuity of L_γ then implies that $L_\gamma(\rho_0)\geq L_\gamma(\rho'_0)$ for all $\gamma\in\Gamma$, so $\rho_0=\rho'_0$ because both ρ_0 and ρ'_0 lie in the Fuchsian locus.

Thus, the sequence $\{\rho'_{t_i}\}_{i=1}^\infty$ converges to ρ_0 as well. Choose a Riemannian metric on a neighborhood of ρ_0 in $Hit_n(S)$. By taking a further subsequence of $\{\rho_{t_i}\}_{i=1}^\infty$, we can also assume that either one of the following cases hold:

(i) $\rho'_{t_i}=\rho_0$ for all i.
(ii) the unit vectors at ρ_0 that are tangential to the geodesic between ρ'_{t_i} and ρ_0 converge to some unit vector $V\neq 0$ in $T_{\rho_0}Hit_n(S)'$ that is tangential to the Fuchsian locus.

If (i) holds, then we have that $dL_\gamma(U)\geq 0$ for all $\gamma\in\Gamma$. On the other hand, if (ii) holds, then

$$\begin{aligned} dL_\gamma(U) &= \frac{d}{dt}L_\gamma(\rho_t)|_{t=0}\\ &= \lim_{i\to\infty}\frac{L_\gamma(\rho_{t_i})-L_\gamma(\rho_0)}{t_i}\\ &\geq \lim_{i\to\infty}\frac{L_\gamma(\rho'_{t_i})-L_\gamma(\rho'_0)}{t_i}\\ &= \Big(\lim_{i\to\infty}s_i\Big)\cdot dL_\gamma(V)\end{aligned}$$

for some sequence of positive numbers $\{s_i\}_{i=1}^{\infty}$. Note that if $\lim_{i\to\infty} s_i = \infty$ then, $dL_\gamma(V) \leq 0$, which is impossible since $V \neq 0$ is tangential to the Fuchsian locus and all hyperbolic metrics on S have the same area. Hence, $dL_\gamma(U+sV) \geq 0$ for some $s \leq 0$.

In either case, there is some vector $W \in T_{\rho_0}Hit_n(S)'$ (possibly the 0 vector) that is tangential to the Fuchsian locus so that $dL_\gamma(U+W) \geq 0$ for all γ. Furthermore, since $U+W \neq 0$, the fact that the differentials dL_γ generate the cotangent space of $Hit_n(S)'$ at ρ_0 implies that $dL_\gamma(U+W) > 0$ for some γ. By a similar argument, we can also show that there is some vector $W' \in T_{\rho_0}Hit_n(S)'$ that is tangential to the Fuchsian locus so that $dL_\gamma(-U+W') \geq 0$ for all γ, and this inequality holds strictly for some γ.

Adding these two inequalities together gives $dL_\gamma(W+W') \geq 0$ for all γ, and $dL_\gamma(W+W') > 0$ for some γ. However, this is impossible since $W+W'$ is tangential to the Fuchsian locus.

Appendix A. Proof of Corollary 3.7

In this appendix, we will prove the properness result stated as Corollary 3.7. We begin by recalling some results from [21] that we will need.

Let $\mathcal{P} := \{\gamma_1, \dots, \gamma_{3g-3}\}$ be an oriented pants decomposition of S, i.e. a maximal collection of pairwise non-intersecting, pairwise non-homotopic, homotopically non-trivial, oriented simple closed curves on S. These curves cut S into $2g-2$ pairs of pants, which we label by $P_1, \dots, P_{2g-2}$, and also gives us a real analytic diffeomorphism

$$Hit_n(S) \to (\mathbb{R}^+)^{(3g-3)(n-1)} \times \mathbb{R}^{(3g-3)(n-1)} \times \mathbb{R}^{(2g-2)(n-1)(n-2)},$$

which one should think of as a generalization of the Fenchel-Nielsen coordinates on the Teichmüller space $\mathcal{T}(S)$ (see Proposition 3.5 of [21]).

The first $(3g-3)(n-1)$ positive numbers are called the *boundary invariants*. For any $\rho \in Hit_n(S)$, these are the numbers

$$\beta_{\gamma_j,k} := \log\left|\frac{\lambda_{k+1}(\rho(X_j))}{\lambda_k(\rho(X_j))}\right|,$$

where $k = 1, \dots, n-1$ and $j = 1, \dots, 3g-3$. Here, $X_j \in \Gamma$ is a group element that corresponds to γ_j and $\lambda_1(\rho(X_j)), \dots, \lambda_n(\rho(X_j))$ are the eigenvalues of $\rho(X_j)$ arranged in increasing order of moduli. Note that each of the $3g-3$ curves in $\mathcal{P}$ are associated $n-1$ of these numbers. They capture the eigenvalue data of the holonomy about each of the curves in $\mathcal{P}$, and are a generalization of the Fenchel-Nielsen length coordinates.

The next $(3g-3)(n-1)$ real numbers are called the *gluing parameters*, and these are also associated to the curves in $\mathcal{P}$. Informally, the $n-1$ gluing parameters associated to each curve in $\mathcal{P}$ is the data specifying how one should "glue" the representations on adjacent pair of pants together along a common boundary component. Hence, these generalize the Fenchel-Nielsen twist coordinates. Just like the Fenchel-Nielsen twist coordinates, to specify these gluing parameters formally, we need to make additional topological choices to define what is "zero gluing". In this case, this additional topological choice we make is a pair of distinct points $a_j, b_j \in \partial_\infty\Gamma$ so that X_j^-, a_j, X_j^+, b_j lie in $\partial_\infty\Gamma$ in that cyclic order.

For simplicity, we will fix such a choice once and for all in the following way. Let P_1, P_2 be the two pairs of pants that have γ_j as a common boundary component (it is possible for $P_1 = P_2$). For $i = 1, 2$, choose A_i, B_i and C_i be elements in Γ corresponding the boundary components of P_i so that $C_i \cdot B_i \cdot A_i = \mathrm{id}$ and $A_1 = A_2^{-1} = X_j$. Let a_j be the repelling fixed point of B_1 and b_j be the repelling fixed point of C_2. The gluing parameters are then

$$g_{\gamma_j,k} := \log\big(-(P_{k,1}, P_{k,2}, P_{k,4}, P_{k,3})\big)$$

for $k = 1, \dots, n-1$, where $\xi : \partial_\infty\Gamma \to \mathcal{F}(\mathbb{R}^n)$ is the Frenet curve corresponding to ρ and

$$\begin{array}{rcl} P_{k,1} & := & \xi(X_j^-)^{(k)} + \xi(X_j^+)^{(n-k-1)} \\ P_{k,2} & := & \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k-1)} + \xi(a_j)^{(1)} \\ P_{k,3} & := & \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k)} \\ P_{k,4} & := & \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k-1)} + \xi(b_j)^{(1)} \end{array}$$

are four hyperplanes in $\mathbb{R}^n$ that intersect at $M_k := \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k-1)}$.

Finally, the remaining $(2g-2)(n-1)(n-2)$ real numbers are called the *internal parameters*, and are associated to the pairs of pants $P_1, \dots, P_{2g-2}$. To each P_j, we associate $(n-1)(n-2)$ internal parameters, and they parameterize the Hitchin representations on a pair of pants after the boundary invariants are fixed. These are defined in great detail in Section 3 of [21]. For our purposes though, we do not need to know what these parameters are, but only the following proposition.

Proposition A.1. *Fix a pair of pants P_{j_0} given by $\mathcal{P}$. Let $\{\rho_i\}$ be a sequence in $Hit_n(S)$ so that*

- *The boundary invariants corresponding to ∂P_{j_0} remain bounded away from 0 and ∞ along $\{\rho_i\}$.*
- *Some internal parameter corresponding to P_{j_0} grows to ∞ or $-\infty$ along $\{\rho_i\}$.*

Let γ be a closed curve in S with the property that any closed curve homotopic to γ has non-empty intersection with P_{j_0}. Then $\lim_{i\to\infty} l_{\rho_i}(\gamma) = \infty$.

Proof. The proof of this proposition is a slight modification of the proof of the main theorem given in Section 5.1 of [21]. In Section 3.2 of [21], there is a description of a particular way to cut each P_j into two ideal triangles that share all three edges. Doing this over all P_j gives us $6g-6$ edges. Here, we view each of these edges $e = [a, b]$ as a Γ-orbit of a pair of distinct points $a, b \in \partial_\infty\Gamma$.

Let $\rho \in Hit_n(S)$ and ξ the corresponding Frenet curve. As was done in Section 4.4 of [21], one can associate a particular positive number $K[a, b]$ to each of these $6g-6$ edges $[a, b]$. Using this, define

$$K(\rho, j_0) := \min\{K[a, b] : [a, b] \subset P_{j_0}\}.$$

The same argument as given in Section 5.1 of [21] proves that

$$\lim_{i\to\infty} K(\rho_i, j_0) = \infty.$$

Let $X \in \Gamma$ be a group element corresponding to γ and let X^+ and X^- be the attracting and repelling fixed points of X in $\partial_\infty\Gamma$. Let $e = [a, b]$ be an edge in P_{j_0} so that there is a lift $\widetilde{e} = \{a, b\}$ with the property that X^-, a, X^+, b lie in $\partial_\infty\Gamma$ in that cyclic order. Such an edge exists by the hypothesis we imposed on γ. For any $p = 0, \dots, n-1$, one can define subsegments $c_p(\widetilde{e})$ of the projective line

$\mathbb{P}(\xi(X^-)^{(1)} + \xi(X^+)^{(1)}) \subset \mathbb{RP}^{n-1}$ associated to each lift $\widetilde{e} = \{a, b\}$ of $e = [a, b]$. These are called the *crossing* (p)*-subsegments* (see Definition 4.7 of [21]). Using the cross ratio, we can define a notion of length for these subsegments, which we denote by $l(c_p(\widetilde{e}))$ (see Definition 4.8 of [21]).

By the proof of Proposition 4.16 of [21], we see that

$$\frac{1}{n}\sum_{p=0}^{n-1} l(c_p(\widetilde{e})) \geq K(\rho, j_0).$$

Furthermore, by Lemma 4.9 and Lemma 4.10 of [21], we have

$$l_\rho(\gamma) \geq l(c_p(\widetilde{e})),$$

for all $p = 0, \dots, n-1$, which allows us to conclude that

$$l_\rho(\gamma) \geq K(\rho, j_0).$$

Combining this with the fact that $\lim_{i\to\infty} K(\rho_i, j_0) = \infty$ gives the proposition. □

With the above proposition, we are ready to prove Corollary 3.7. Let $\{\rho_i\}$ be a sequence in $Hit_n(S)$, let $\mathcal{C} := \{\gamma_1, \dots, \gamma_k\}$ satisfy the hypothesis of Corollary 3.7 and let $\mathcal{P} := \{\gamma_1, \dots, \gamma_{3g-3}\} \subset \mathcal{C}$ be a pants decomposition. Observe that the hypothesis on $\mathcal{C}$ ensures the following:

- For any $\gamma \in \mathcal{P}$, there is some $\gamma' \in \mathcal{C}$ that intersects γ transversely.
- For each pair of pants P given by $\mathcal{P}$, there is some $\gamma \in \mathcal{C}$ so that any closed curve homotopic to γ has non-empty intersection with P.

The pants decomposition $\mathcal{P}$ then gives us a parameterization of $Hit_n(S)$ as described above. We will prove Corollary 3.7 in the following steps.

(1) If there is some boundary invariant $\beta_{\gamma_j,k}$ so that $\lim_{i\to\infty} \beta_{\gamma_j,k}(\rho_i) = \infty$, then $\lim_{i\to\infty} l_{\rho_i}(\gamma_j) = \infty$.
(2) If there is some boundary invariant $\beta_{\gamma_j,k}$ so that $\lim_{i\to\infty} \beta_{\gamma_j,k}(\rho_i) = 0$, then $\lim_{i\to\infty} l_{\rho_i}(\gamma) = \infty$ for any closed curve γ that intersects γ_j transversely.
(3) If all the boundary invariants remain bounded away from 0 and ∞ and some internal parameter associated to a pair of pants P grows to $\pm\infty$, then $\lim_{i\to\infty} l_{\rho_i}(\gamma) = \infty$ for any closed curve γ with the property that any closed curve homotopic to γ has non-empty intersection with P.
(4) If all the boundary invariants remain bounded away from 0 and ∞ and there is some gluing parameter $g_{\gamma_j,k}$ so that $\lim_{i\to\infty} g_{\gamma_j,k}(\rho_i) = \pm\infty$, then $\lim_{i\to\infty} l_{\rho_i}(\gamma) = \infty$ for any γ that intersects γ_j transversely.

Note that together, the four statements above prove Corollary 3.7. Statement (1) is obvious because

$$l_\rho(\gamma_j) = \sum_{k=1}^{n-1} \beta_{\gamma_j,k}(\rho)$$

and all the boundary invariants are positive. Also, Statement (3) is a restatement of Proposition A.1, and Statement (2) is an immediate consequence of (1) of Proposition 2.12, which is a main result in this paper. The rest of this appendix will be the proof of Statement (4).

Let $X_j, X \in \Gamma$ correspond to γ_j and γ respectively so that X_j^-, X^-, X_j^+, X^+ lie in $\partial_\infty\Gamma$ in that cyclic order. We previously chose a pair of points $a_j, b_j \in \partial_\infty\Gamma$ so that X_j^-, a_j, X_j^+, b_j lie in $\partial_\infty\Gamma$ in that cyclic order in order to define the gluing parameters $g_{\gamma_j,k}$ associated to γ_j. If we choose $X_j^l \cdot a_j$ and $X_j^m \cdot b_j$ in place of a_j

and b_j, we get another collection of gluing parameters, which we denote by $g^{l,m}_{\gamma_j,k}$. The next lemma explains the relationship between $g_{\gamma_j,k} = g^{0,0}_{\gamma_j,k}$ and $g^{l,m}_{\gamma_j,k}$. Its proof is an easy computation which we omit.

Lemma A.2. *Let $\rho \in Hit_n(S)$ and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\rho(X_j)$ arranged in increasing order of their moduli. For any integers l, m, we have*

$$g^{l,m}_{\gamma_j,k}(\rho) = (m-l)\log\left(\frac{\lambda_{k+1}}{\lambda_k}\right) + g_{\gamma_j,k}(\rho).$$

In particular, when the boundary invariants corresponding to γ_j are bounded away from 0 and ∞ along a sequence of representations $\{\rho_i\}$ in $Hit_n(S)$, then $\lim_{i\to\infty} g_{\gamma_j,k}(\rho_i) = \pm\infty$ if and only if $\lim_{i\to\infty} g^{l,m}_{\gamma_j,k}(\rho_i) = \pm\infty$. Statement (4) then follows immediately from this observation and the following proposition.

Proposition A.3. *Let $\rho \in Hit_n(S)$ and let γ_j, γ, X_j^-, X_j^+, X^-, X^+, a_j, b_j be as above. Let l and m be integers such that X_j^-, $X_j^{l-1}\cdot a_j$, X^-, $X_j^l \cdot a_j$, X_j^+, $X_j^{m+1}\cdot b_j$, X^+, $X_j^m \cdot b_j$ lie in $\partial_\infty\Gamma$ in that cyclic order. Then*

$$3l_\rho(\gamma) \geq g^{l,m}_{\gamma_j,k}(\rho) \quad \textit{and} \quad 3l_\rho(\gamma) \geq -g^{l-1,m+1}_{\gamma_j,k}(\rho)$$

for all $k = 1, \dots, n-1$.

Proof. The technique used in this proof is the same as that used in the proofs of Lemma 4.18 of [21]. For any $k = 1, \dots, n-1$, let

$$\begin{aligned}
P_{k,0} &:= \xi(X_j^-)^{(k)} + \xi(X_j^+)^{(n-k-1)},\\
P_{k,1} &:= \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k-1)} + \xi(X^-)^{(1)},\\
P'_{k,2} &:= \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k-1)} + \xi(X_j^{l-1}\cdot a_j)^{(1)},\\
P_{k,2} &:= \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k-1)} + \xi(X_j^{l}\cdot a_j)^{(1)},\\
P_{k,3} &:= \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k)},\\
P'_{k,4} &:= \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k-1)} + \xi(X_j^{m}\cdot b_j)^{(1)},\\
P_{k,4} &:= \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k-1)} + \xi(X_j^{m+1}\cdot b_j)^{(1)},\\
P_{k,5} &:= \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k-1)} + \xi(X^+)^{(1)}.
\end{aligned}$$

Also, for all $i = 0, \dots, 5$, let

$$\begin{aligned}
L'_{k,i} &:= P'_{k,i} \cap \big(\xi(X^-)^{(1)} + \xi(X^+)^{(1)}\big),\\
L_{k,i} &:= P_{k,i} \cap \big(\xi(X^-)^{(1)} + \xi(X^+)^{(1)}\big)
\end{aligned}$$

and let

$$\begin{aligned}
L_{k,a_j} &:= \big(\xi(X_j^{l-1}\cdot a_j)^{(k-1)} + \xi(X_j^{l}\cdot a_j)^{(n-k)}\big) \cap \big(\xi(X^-)^{(1)} + \xi(X^+)^{(1)}\big),\\
L_{k,b_j} &:= \big(\xi(X_j^{m+1}\cdot b_j)^{(k-1)} + \xi(X_j^{m}\cdot b_j)^{(n-k)}\big) \cap \big(\xi(X^-)^{(1)} + \xi(X^+)^{(1)}\big).
\end{aligned}$$

It follows from Lemma 2.5 of [21] that

$$\xi(X^-)^{(1)}\,,\ L_{k,a_j}\,,\ L_{k,2}\,,\ L_{k,3}\,,\ L_{k,4}\,,\ L_{k,b_j}\,,\ \xi(X^+)^{(1)}$$

lie in the projective line $\xi(X^-)^{(1)} + \xi(X^+)^{(1)}$ in that cyclic order. Also, by Lemma 4.11 of [21], we know

$$3l_\rho(\gamma) \geq \log\big(\xi(X^-)^{(1)}, L_{k,a_j}, L_{k,b_j}, \xi(X^+)^{(1)}\big),$$

which implies that

$$3l_\rho(\gamma) \geq \log\left(\xi(X^-)^{(1)}, L_{k,2}, L_{k,3}, \xi(X^+)^{(1)}\right)$$

and

$$3l_\rho(\gamma) \geq \log\left(\xi(X^-)^{(1)}, L_{k,3}, L_{k,4}, \xi(X^+)^{(1)}\right)$$

by Lemma 2.9. Using Lemma 2.8 and Lemma 2.6, we can also deduce that

$$\begin{aligned}
\left(\xi(X^-)^{(1)}, L_{k,2}, L_{k,3}, \xi(X^+)^{(1)}\right) &= \left(\xi(X^-)^{(1)}, L_{k,2}, L_{k,3}, \xi(X^+)^{(1)}\right)_{M_k} \\
&= (P_{k,1}, P_{k,2}, P_{k,3}, P_{k,5}) \\
&\geq (P_{k,0}, P_{k,2}, P_{k,3}, P'_{k,4}) \\
&= 1 - (P_{k,0}, P_{k,2}, P'_{k,4}, P_{k,3}) \\
&= 1 + e^{g^{l,m}_{\gamma_j,k}} \\
&\geq e^{g^{l,m}_{\gamma_j,k}}
\end{aligned}$$

and

$$\begin{aligned}
\left(\xi(X^-)^{(1)}, L_{k,3}, L_{k,4}, \xi(X^+)^{(1)}\right) &= \left(\xi(X^-)^{(1)}, L_{k,3}, L_{k,4}, \xi(X^+)^{(1)}\right)_{M_k} \\
&= (P_{k,1}, P_{k,3}, P_{k,4}, P_{k,5}) \\
&\geq (P'_{k,2}, P_{k,3}, P_{k,4}, P_{k,0}) \\
&= 1 - \frac{1}{(P_{k,0}, P'_{k,2}, P_{k,4}, P_{k,3})} \\
&= 1 + e^{-g^{l-1,m+1}_{\gamma_j,k}} \\
&\geq e^{-g^{l-1,m+1}_{\gamma_j,k}}
\end{aligned}$$

where $M_k := \xi(X_j^-)^{(k-1)} + \xi(X_j^+)^{(n-k-1)}$. □

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Gye-Seon Lee
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
D-69120 Heidelberg, Germany
lee@mathi.uni-heidelberg.de

Tengren Zhang
Mathematics Department
University of Michigan
530 Church St, Ann Arbor, MI 48109
tengren@math.umich.edu