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Gradient instability for $w < -1$

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Abstract

We show that in single scalar field models of the dark energy with equations of state satisfying $w \equiv p/\rho < -1$, the effective Lagrangian for fluctuations about the homogeneous background has a wrong sign spatial kinetic term. In most cases, spatial gradients are ruled out by microwave background observations. The sign of $w + 1$ is not connected to the sign of the time derivative kinetic term in the effective Lagrangian.

Matter whose equation of state satisfies $w \equiv p/\rho < -1$ violates a number of conditions, including the weak energy condition, generally assumed to apply to any reasonable model of physics [1]. However, the observational data do not exclude the possibility that the dark energy has $w < -1$ [2,3]. Current results [4] indicate $-1.67 < w < -0.61$ at 95% confidence level. The possibility of $w < -1$ has been explored by numerous authors (see, for instance, [5–17]). These models often contain a field with an unusual kinetic term, which is referred to as a phantom or ghost field. In this Letter we show that for $w < -1$, single scalar field models of the dark energy generally have a wrong sign gradient kinetic term for fluctuations about the homogeneous background. This result is not dependent on general relativistic effects and survives in the flat spacetime limit. Spatial inhomogeneities of the dark energy are tightly constrained by observations of the cosmic microwave background.

In our analysis we will assume a time-dependent but spatially homogeneous scalar background. Models with such backgrounds are treated in [18], whose notation we will adopt. We will show that for $w < -1$ spatial instabilities inevitably arise. Consider the low-energy effective theory of a scalar field coupled to gravity:

$$S = \int d^4x\, \sqrt{-g}\big[M_{\rm Pl}^2 R + P + U R + V R^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + \cdots\big], \qquad (1)$$

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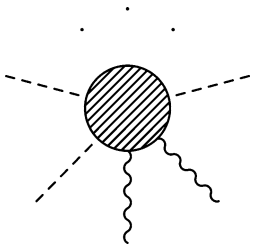

Fig. 1. The effective couplings of two gravitons to several quanta of the scalar field. The shaded region represents interactions involving only scalars.

where P, U and V are functions of the scalar field ϕ and its derivatives. (Because of the anti-symmetry of $R^{\mu\nu\rho\sigma}$ in its first two and also in its last two indices, no non-vanishing invariant can be formed from it using first derivatives of ϕ.) Naively we might expect that the higher-dimensional couplings of ϕ to the Ricci tensor would be suppressed by powers of the Planck mass $M_{\rm Pl}$, making them irrelevant for cosmology after the Planck epoch. However, such terms are generated by graphs such as that in Fig. 1. Writing the metric as $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}/M_{\rm Pl}$, we see that scalar-graviton interactions in Feynman diagrams are suppressed by the Planck mass, but when these interactions are reassembled into the Ricci tensor that suppression is absent. That is, the higher-dimensional terms in Eq. (1) will appear suppressed only by powers of the characteristic energy scale of the scalar field, M, which may be much smaller than $M_{\rm Pl}$.

We neglect terms in the action (1) which involve higher powers of the Ricci tensor. The terms we consider are ones that can generate contributions to the stress-energy tensor $T_{\mu\nu}$ which are not suppressed by powers of $M_{\rm Pl}$. Since $T_{\mu\nu}$ is obtained by varying the action with respect to the metric, terms with more than one power of $R^{\mu\nu\rho\sigma}$ yield contributions which are themselves proportional to the Ricci tensor and therefore vanish in the flat spacetime limit.

Assuming a spatially homogeneous background, only the time-derivatives of ϕ will be non-vanishing in Eq. (1). It may be shown that in the limit $M_{\rm Pl} \to \infty$, the term $R^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)V$ in the action contributes a term to the stress-energy tensor which can be reproduced by an appropriate change in the function U. Therefore we may restrict ourselves to $V = 0$ and consider the most general U in order to analyze the flat-space behavior of Eq. (1).

It is always possible to perform a rescaling of the metric in Eq. (1) $g_{\mu\nu} \to e^{2w} g_{\mu\nu}$, with $w = \log[1 + U/M_{\rm Pl}^2]$, so that the U term in Eq. (1) disappears, being absorbed into a redefinition of the P action for the ghost scalar field. (See, for example, Chapter 3, Section 7 of [19].) The action resulting from this rescaling, up to terms suppressed by powers of $1/M_{\rm Pl}$, is then

$$S = \int d^4x\, \sqrt{-g}\big[M_{\rm Pl}^2 R + P\big]. \tag{2}$$

The most general Lorentz invariant scalar Lagrangian without higher derivative terms (which we will consider later) is

$$\mathcal{L} = P(X,\phi), \tag{3}$$

where $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. (A potential term V would be the component of $P(X,\phi)$ that is independent of X.) Henceforth, $P'(X,\phi)$ will denote differentiation with respect to X. Since the scalar field ϕ is minimally coupled to gravity in Eq. (2), the stress-energy tensor is

$$T_{\mu\nu} = -\mathcal{L}g_{\mu\nu} + 2P'(X,\phi)\partial_\mu\phi\partial_\nu\phi, \tag{4}$$

and

$$\begin{aligned} w &= \frac{P(X,\phi)}{T_{00}} = \frac{P(X,\phi)}{-P(X,\phi) + 2\dot\phi^2 P'(X,\phi)} \\ &= -1 + \frac{2\dot\phi^2 P'(X,\phi)}{T_{00}}. \end{aligned} \tag{5}$$

For ϕ to account for the dark energy, we must have $T_{00} > 0$. Then, $w < -1$ requires that $P'(X,\phi) < 0$. Let $\phi_0 = \phi_0(t)$ be a solution to the equations of motion, and define $X_0 \equiv \dot\phi_0^2$. Then consider the fluctuations about this solution: $\phi = \phi_0 + \pi(x,t)$. When expanded in π, the effective Lagrangian will have the form

$$\begin{aligned} \mathcal{L} = {} & \big[P'(X_0,\phi_0) + 2\dot\phi_0^2 P''(X_0,\phi_0)\big]\dot\pi^2 \\ & - P'(X_0,\phi_0)|\nabla\pi|^2 + \cdots, \end{aligned} \tag{6}$$

which implies that for $P'(X_0,\phi_0) < 0$ there will exist field configurations with non-zero spatial gradients that have lower energy than the homogeneous configuration.[2]

[2] Here we mean energy associated with the Hamiltonian constructed from the Lagrangian for fluctuations about the background field configuration.

It is clear from Eq. (6) that there is no direct connection between the sign of $w+1$ and that of the $\dot{\pi}^2$ term in the effective Lagrangian, as long as $\dot{\phi}_0^2 P''(X_0,\phi_0)$ is not negligible with respect to $P'(X_0,\phi_0)$. However, in the usual quintessence models of the dark energy (see [20] and references therein), it is generally the case that the terms in P which are higher order in X can be neglected with respect to the leading term (i.e., $X/M^4 \ll 1$). In that case the $\dot{\pi}^2$ and $-(\nabla\pi)^2$ terms in the effective Lagrangian would both have a negative coefficient for $w<-1$.

If $P'(X_0,\phi_0)$ is negative, a finite expectation value for the gradients may be obtained if there are appropriate higher powers of $(\nabla\pi)^2$ in the effective Lagrangian, but this is problematic because it gives rise to a spatially inhomogeneous ground state for the dark energy and would lead to inhomogeneities far larger than the limit of 10^{-5} imposed by observations of the cosmic microwave background.[3] While a potential term such as $m^2\phi^2$ tends to confine the gradients to regions of size $1/m$, in most models of the dark energy $V''(\phi)$ must be small enough that these regions are of cosmological size.

In the $w<-1$ case, it is possible, by adding higher derivative terms to the Lagrangian, to avoid having finite spatial gradients lower the energy of the field. Consider, for example,

$$\mathcal{L} = P(X,\phi) + S(X,\phi)(\Box\phi)^2 \tag{7}$$

in which case

$$T_{00} = -\mathcal{L} + 2\big[P'(X,\phi)\dot{\phi}^2 + S'(X,\phi)\dot{\phi}^2\big(\partial^2\phi\big)^2 + 2S(X,\phi)\ddot{\phi}\big(\partial^2\phi\big) - \partial_0\big(\dot{\phi}S\partial^2\phi\big)\big]. \tag{8}$$

Setting the spatial gradients of ϕ to zero, we have that

$$\dot{\phi}^2 C_{\rm grad} - 2\partial_0(S\ddot{\phi})\dot{\phi} = \frac{(w+1)}{2}T_{00}, \tag{9}$$

where $C_{\rm grad}$ is the coefficient of $-(\nabla\pi)^2$ in the π Lagrangian. If $\partial_0(S\ddot{\phi})\dot{\phi} > 0$, then a model may have both $C_{\rm grad} > 0$ and $w<-1$. But for w significantly less than -1, this also requires $\ddot{\phi}^2$ to be at least of order $M^2\dot{\phi}^2$, unless $S(X,\phi)$ is made unnaturally large. It is not clear how to treat these higher derivative terms self-consistently beyond perturbation theory, so the dynamics of such models cannot be analyzed in a straightforward manner. The models we consider below have higher powers of first derivatives, but they satisfy the condition that $\ddot{\phi}^2 \ll (\dot{\phi}M)^2$.

Our analysis shows that $w<-1$ scalar models typically require a wrong sign $(\nabla\pi)^2$ term in the effective Lagrangian. Previous analyses of ghost models [1,21] have focused on the problems associated with negative energy, particularly with a kinetic term $\mathcal{L} = -(\partial_\mu\phi)^2$ that has the wrong sign for *both* the time- and space-derivatives. The classical equations of motion for such models do not exhibit growing modes of non-zero spatial gradients, although the energy of the field is unbounded from below. Models with $w<-1$ that do not have a wrong sign time-derivative kinetic term in the effective Lagrangian can result from a Lorentz invariant action, as we demonstrate below. However, both Lorentz invariance and time translation invariance are spontaneously broken by a time-dependent condensate.

In [18] a model with $\mathcal{L} = P(X)$ was proposed in which a ghost field has a time-dependent condensate (from now on we take the Lagrangian to be a function of X only, and therefore invariant under the shift $\phi\to\phi+c$). We use units in which the dimensional scale M of the model is unity ($M\sim 10^{-3}$ eV if the ghost comprises the dark energy). The flat spacetime equation of motion is

$$\partial_\mu\big[P'(X)\partial^\mu\phi\big] = 0. \tag{10}$$

Homogeneous solutions of the equations of motion with $\dot{\phi}^2 = c^2$ were considered in [18]. In general, the existence of a $\dot{\phi}$ condensate allows for exotic equations of state, including $w<-1$. In what follows we let

$$P(X) = -1 + 4(X-1)^2 + 3(X-1)^3, \tag{11}$$

which leads to $w<-1$ with $T_{00}>0$, for X in a left neighborhood of 1.[4]

[3] A condensate of gradients with a preferred magnitude, determined by the higher order terms that stabilize Eq. (6), will spontaneously break the $O(3)$ rotational symmetry down to $O(2)$. The homotopy group $\pi_2[O(3)/O(2)]$ is non-trivial, which leads to the formation of global monopole (hedgehog) configurations.

[4] Notice that the model in Eq. (11) has a positive leading-order kinetic term (i.e., the term linear in X). It is possible to construct models with the desired properties in which this sign is either positive or negative.

The energy density is given by

$$T_{00} = \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L} = 2\dot{\phi}^2 P'(X) - \mathcal{L}, \tag{12}$$

which is not necessarily minimized by a particular ghost condensate $\phi = ct$, although it is a solution to the flat spacetime equations of motion for any value of c. This is possible because there is a conserved charge associated with the shift symmetry,

$$Q = \int d^3x\, P'(X)\dot{\phi}, \tag{13}$$

so configurations which do not extremize T_{00} can still be stable. In fact, the Lagrangian describing small fluctuations has the correct sign of $\dot{\pi}^2$ if $P'(X) + 2XP''(X) > 0$. This condition is satisfied in a neighborhood of $X = 1$ by (11) given above. For $c^2 < 1$ there is then a local instability to the formation of gradients, as required by our earlier results.

Ghost models of the dark energy which approach $w = -1$ asymptotically make potentially interesting predictions for the evolution of the equation of state for the dark energy. In a FRW universe, the equation of motion for the ghost field is

$$\partial_\mu\left[a^3(t)P'(X)\partial^\mu\phi\right] = 0, \tag{14}$$

where $a(t)$ is the FRW scale factor. If there is a value $c_*^2 = \dot{\phi}^2 = X$ such that $P'(c_*^2) = 0$, then Eq. (5) implies that $w = -1$ when $X = c_*^2$. The model described by Eq. (11) has $c_*^2 = 1$, and if we apply Eq. (14) to it, we see that if we start from $X = c_i^2$ with c_i close to c_*, then we are driven asymptotically towards $X = c_*^2$ and $w = -1$.

In the model described by Eq. (11), we may be driven towards $w = -1$ either from above or from below, depending on whether we chose to start from $c_i^2 > 1$ or from $c_i^2 < 1$. We have argued that $w < -1$ is problematic because of spatial gradient instabilities, so that the case in which we are driven to $w = -1$ from above is more interesting.

Near the asymptotic value $c_* = 1$ we have

$$\dot{\pi} = \frac{P'(c_i^2)c_i}{2P''(c_*^2)c_*^2}\left(\frac{a_i}{a}\right)^3, \tag{15}$$

where higher order terms in a_i/a have been neglected. Thus, in this regime,

$$\begin{aligned} w &= -1 - \frac{4P''(c_*^2)c_*^3\dot{\pi}}{P(c_*^2)} \\ &= -1 - \frac{2P'(c_i^2)c_*c_i}{P(c_*^2)}\left(\frac{1+z}{1+z_i}\right)^3. \end{aligned} \tag{16}$$

Eq. (16) offers a prediction for the w parameter of the dark energy as a function of the redshift z, which could be tested by cosmological observation [22].

In summary, from Eqs. (5) and (6) we find that in single scalar field models of the dark energy with $w < -1$, the kinetic term for fluctuations about the homogeneous background has a wrong sign gradient term. On the other hand, there is no direct connection between the sign of the $\dot{\pi}^2$ kinetic term in the effective Hamiltonian and the sign of $w + 1$.

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