Multivariable Anti-Windup and Bumpless Transfer: A General Theory

P. J. Campo * M. Morari † C. N. Nett ‡

Abstract

A general theory is developed to address the anti-windup/bumpless transfer (AWBT) problem. Analysis results applicable to any linear time invariant system subject to plant input limitations and substitutions are presented. Quantitative performance objectives for AWBT compensation are outlined and several proposed AWBT methods are evaluated in light of these objectives. A synthesis procedure which highlights the performance trade-offs for AWBT compensation design is outlined.

1 Introduction

Actuator saturation limits the input of all physical systems and in many practical situations override or selector schemes are implemented to use the available plant inputs to keep several outputs within a specified range. This involves switching between any of several linear controllers, each designed to achieve different closed-loop characteristics. Common practice is to design the linear controller (or controllers in the case of overrides) ignoring the effects of limitations and substitutions. Then "anti-windup" or "bumpless transfer" (AWBT) compensation is added to the control system in order to minimize the adverse effects of limitations and substitutions on closed loop performance. The idea is that the control system performance should "degrade gracefully" when the inevitable limitations and substitutions occur. Given their practical importance, a wide variety of problem specific AWBT techniques have been developed. While many of these schemes are successful (at least in specific single-input-single-output situations) they are largely intuitively based and have little theoretical foundation.

In this paper we formalize these techniques and advance a general AWBT analysis and synthesis theory applicable to any linear time invariant (LTI) system subject to plant input limitations and substitutions. This theory is based on a minimum number of simple assumptions and provides a framework for the consideration of a number of possible AWBT objectives.

A major void in the existing AWBT literature is a clear exposition of the objectives (and associated engineering trade-offs) which lead to "graceful performance degradation" in any reasonably general setting. The abstract framework we develop allows us to study most of the proposed schemes in an effort to shed light on this issue.

2 Theoretical Framework

We consider the system shown in Figure 1, where P is an LTI interconnection structure, K an LTI controller, and N a memoryless nonlinearity which models the plant input limitation/substitution. These limitations and substitutions cause the actual plant input, u', to be different from the controller output, u. It is assumed that in the absence of limitations or substitutions K stabilizes P and provides acceptable performance, *i.e.*, for $N = I$, the (weighted) closed loop transfer function from external input, w, to controlled output, e, is small in some sense (*e.g.*, H^2, H^∞, etc).

The interconnection structure of Figure 1 provides a very general framework for studying linear systems which include feedback. Indeed, any feedforward/feedback interconnection of linear system elements can be brought into this form. It is important to emphasize that w represents all exogenous inputs (including commands, disturbances, and sensor noises), e includes all signals for which performance specifications are provided (typically tracking errors and filtered actuator signals), and y represents all signals available to the controller K (including sensor outputs, commands, and possibly measurements of actuator positions).

The nonlinearity N is included to account for plant input limitations and substitutions and is assumed to be memoryless and conic sector bounded. At this point we develop a number of general stability results based on conic sector approximations of N. In the next section we will outline the development of these approximations for a number of common limitation/substitution mechanisms.

The approach taken here originated with the work of Zames in the early 1960s [13]. The basic approach is to approximate the nonlinear system components with linear ones and obtain norm bounds on the error involved in this approximation. The linear system is then studied subject to nonlinear perturbations within the specified norm bounds. If it can be shown that the linear system has certain properties (*e.g.* stability) for all perturbations within the norm bounds, then it is certain that the original nonlinear system has these properties as well.

We will be concerned with signals which remain finite for all finite values of time. A mathematical characterisation of the set of such functions is given by [4]

Definition 1 *L_{2e} is the extended space of vector valued functions, $x(t)$, with the property*

$$\|x(t)\|_T \equiv \left[\int_0^T x^*(t)x(t)dt\right]^{1/2} < \infty \tag{1}$$

for all $T \geq 0$.

The notion of a conic sector bounded nonlinearity is captured by

Definition 2 *Given $N : L_{2e} \to L_{2e}$ and the LTI operators C and R, N is said to be inside $Cone(C, R)$ if*

$$\|N(x) - Cx\|_T \leq \|Rx\|_T \tag{2}$$

for all $T \geq 0$ and for all $x \in L_{2e}$.

A conic sector provides an LTI approximation to the input-output behavior of N. The cone center, C, provides an approximate output, Cx, for any input x. The cone radius, R, provides a measure of the error inherent in this approximation. For example the SISO saturation nonlinearity $N : x(t) \to sat(x(t))$ where

$$sat(x(t)) = \begin{cases} x(t) & |x(t)| \leq 1 \\ sign(x(t)) & |x(t)| > 1 \end{cases} \tag{3}$$

is inside $Cone(\frac{1}{2}, \frac{1}{2})$. The operator $C : x(t) \to \frac{1}{2}x(t)$ is our linear approximation to N, and $R : x(t) \to \frac{1}{2}x(t)$ gives us a measure of the error in this approximation (as much as 100% in this case).

*Chemical Engineering, 210-41, California Institute of Technology, Pasadena, CA 91125.

†Author to whom correspondence should be addressed. Chemical Engineering, 210-41, California Institute of Technology, Pasadena, CA 91125, MM@IMC.CALTECH.EDU.

‡Control Systems Laboratory, GE Corporate Research and Development, Bldg. KW, Room D219, P.O. Box 8, Schenectady, NY 12301.

Any representation of all nonlinearities in $Cone(C, R)$ can be replaced by an equivalent representation in terms of all nonlinearities in $Cone(0, I)$. Specifically $y = N(x)$ with $N \in Cone(C, R)$ if and only if $y = Cx + R\hat{N}(x)$ for some $\hat{N} \in Cone(0, I)$. Thus the set of all nonlinearities in $Cone(C, R)$ can be replaced by the LTI blocks C and R, and the set of all cone bounded nonlinearities in $Cone(0, I)$. This allows us to state all nonlinear stability results in terms of the $Cone(0, I)$ and thereby simplifies the notation.

Extracting the cone center and radius associated with N of Figure 1, we arrive at the feedback interconnection of Figure 2 where M is an LTI operator with transfer function $M(s)$ and Δ is a (possibly nonlinear) block diagonal operator in the set $\mathbf{\Delta}$

$$\mathbf{\Delta} \equiv \{\Delta \mid \Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_n) \;\; \Delta_i \in Cone(0, I)\} \qquad (4)$$

With these preliminaries we present the main stability result, a version of the multiloop circle criterion (see for example [11]).

Theorem 1 *The system in Figure 2 is stable*[1] *for all $\Delta \in \mathbf{\Delta}$ if*

1. $M(s)$ is stable

2. $\inf_{T \in \mathcal{T}} \|TM_{11}(s)T^{-1}\|_\infty \leq 1$

where

$$\mathcal{T} \equiv \{T \mid T = \mathrm{diag}(T_1, \ldots, T_n),\; T\Delta T^{-1} \in \mathbf{\Delta},\; \forall \Delta \in \mathbf{\Delta}\} \qquad (5)$$

Since a simple parametrization of the set $\mathcal{T}$ is not available, the optimization problem implied in 2. is not tractable. We note however that the set

$$\mathcal{T}' \equiv \{T \mid T \in \mathcal{T} \text{ and } T \in \mathcal{C}^{n \times n}\} \qquad (6)$$

is characterized only by the structure of T. Specifically, $\mathcal{T}'$ consists of all block diagonal constant matrices whose block structure is compatible with $\mathbf{\Delta}$ in the sense that for each diagonal block in $\mathbf{\Delta}$ the corresponding block in $\mathcal{T}'$ is diagonal, and for each full block in $\mathbf{\Delta}$ the corresponding block in $\mathcal{T}'$ is a scalar times identity. This simplification motivates

Corollary 1 *The system in Figure 2 is stable for all $\Delta \in \mathbf{\Delta}$ if*

1. $M(s)$ is stable

2. $\inf_{T \in \mathcal{T}'} \|TM_{11}(s)T^{-1}\|_\infty \leq 1$

This simplification is significant since a complete solution to 2. is available from state space structured singular value theory [5]. In addition, this framework can be extended in a straightforward manner to assess (nonlinear) robust stability with respect to uncertainties in the linear plant model.

It should be noted that Theorem 1 provides only sufficient conditions for nonlinear stability, unlike the necessary and sufficient conditions provided by linear structured singular value theory. In addition, because these results guarantee stability for all nonlinearities in the specified cone, the sufficient conditions may be very conservative when tight conic sector bounds are not available.

3 Limitations and Substitutions as Cone Bounded Nonlinear Operators

We model the effects of MIMO actuator saturations, $u' = Nu$, with a diagonal operator defined by

$$N = \mathrm{diag}(n_1, \ldots, n_n) \qquad (7)$$

where $n_i(u) = sat(u_i)$.

A conic sector representation of this nonlinearity is provided by $C = \frac{1}{2}I, R = \frac{1}{2}I$, and a diagonal nonlinear operator $\Delta \in \mathbf{\Delta}$, where

$$\mathbf{\Delta} \equiv \{\Delta \mid \Delta = \mathrm{diag}(\delta_1, \ldots \delta_n) \quad \delta_i \in Cone(0, 1)\} \qquad (8)$$

We note that the zero operator, $u' = 0(u) = 0$ is contained in this cone. This is required since in principle the elements of u could be arbitrarily large while those of u' are bounded by ± 1. If from physical arguments, the controller output can be assumed to be bounded (for example, by bounding the magnitude of the exogenous inputs and system initial conditions) then a tighter conic sector approximation can be derived. An immediate consequence is that in order to guarantee global stability of the system of Figure 1 for all $N \in (\frac{1}{2}I, \frac{1}{2}I)$ we must have (when $N = 0$) P and K stable. More formally we have

Lemma 1 *The system of Figure 1 is stable for all $N \in Cone(\frac{1}{2}I, \frac{1}{2}I)$ only if*

1. K stabilizes P.

2. P is stable.

3. K is stable.

Another common limitation/substitution mechanism arises from the use of selectors to achieve multiple control objectives with the available plant inputs (see *e.g.* [7] [8]). In general the actual plant input u', is chosen from among the outputs of several parallel controllers each providing different closed loop characteristics (corresponding for example to different plant operating modes). In this case K of Figure 1 is given by $K = \begin{bmatrix} K_1 \\ \vdots \\ K_k \end{bmatrix}$ so that $u = \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$ and at any time $u'(t) = u_i(t)$ for some $i = 1, \ldots, k$, where k is the number of parallel controllers. With no further assumptions on the switching mechanism or signal u, we can obtain conic sector bounds for the case $k = 2$. Specifically we have $C = \begin{bmatrix} \frac{1}{2}I_n & \frac{1}{2}I_n \end{bmatrix}, R = \begin{bmatrix} \frac{1}{2}I_n & -\frac{1}{2}I_n \end{bmatrix}$ where n is the number of plant inputs. The obvious role of an AWBT mechanism in this case is to insure that the controllers which are "switched out", or off-line, are properly updated so that they are ready to be "switched in" at any time.

Partitioning the exogenous input w as $w = \begin{bmatrix} r \\ d \end{bmatrix}$ where r is an externally supplied command, and d represents all other exogenous disturbances and noises, and similarly the input to K as $y = \begin{bmatrix} r \\ y_m \end{bmatrix}$ where y_m represents the measured plant outputs, and defining $K_k \equiv [I \quad 0]$ we include the possibility of switching off all feedback control, em i.e., $u' = u_k$ will correspond to manual operation with $u' = r$. As in the case of saturations we state an obvious necessary condition for stability when arbitrary substitutions can occur

Lemma 2 *The system of Figure 1 with $K = \begin{bmatrix} K_1 \\ \vdots \\ K_k \end{bmatrix}$ and N a selector such that $\forall\, t\; \exists\, i \in \{1, \ldots, k\}$ such that $u'(t) = u_i(t)$, is stable only if K_i stabilizes $P\; \forall\, i \in \{1, \ldots, k\}$.*

This implies of course that P be stable in the case that a viable selector option is manual control.

4 A General AWBT Scheme

We now turn out attention to the development of a general AWBT compensation scheme. We begin with Figure 1, introduce a minimal realization for K as

$$K(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad (9)$$

and reiterate our assumption that $K(s)$ provides acceptable nominal performance (*i.e.*, when $N = I$). We are interested in replacing this implementation of K with one which exhibits graceful performance degradation when limitations and substitutions occur. In order to develop an alternate implementation, we assume that u' can be measured or estimated. Estimation of u' requires a nonlinear model of the limitation/substitution process, which may be simple to obtain as in the case of a well defined actuator saturation, or nearly impossible, as for selectors whose action depends upon events external to the control system. The measured (or estimated) value of u' will be denoted u_m.

We now consider the generalized controller implementation of Figure 3. Here we have provided u_m to the controller block and added

[1] Here and throughout the paper we refer to L_{2e} stability [4]. For L_{2e} stable systems, inputs of bounded energy give rise to outputs of bounded energy.

an AWBT mechanism, Λ, which operates on information provided by the controller, v, to generate an anti-windup action, z. The possibility of including non-trivial measurement dynamics and measurement noise associated with u_m is provided by the specification of P_{31} and P_{32} of Figure 3. The case that u' can be measured perfectly, *i.e.*, $u_m = u'$, corresponds to $P_{31} = 0$ (no noise) and $P_{32} = I$ (no measurement dynamics). In order to maintain complete generality, we will provide the AWBT operator, Λ, with all information available to the controller including its state, x, and all inputs. Partitioning the AWBT action as $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ we allow it to act on the states of the controller via z_1 and the output of the controller via z_2. This gives rise to the following realization

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \left[\begin{array}{c|cccc} A & B & 0 & I & 0 \\ \hline C & D & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right] \quad \text{where } v = \begin{bmatrix} x \\ y \\ u_m \\ z_1 \\ z_2 \end{bmatrix} \tag{10}$$

We define the AWBT operator Λ to be ***admissible*** if it satisfies the following properties:

A1). $\Lambda : v \rightarrow z$ is linear and time invariant.

A2). $u - u_m = 0 \Rightarrow z = 0 \quad \forall t$.

The first condition simply requires that our AWBT compensation can be realized as a linear system. While this may seem arbitrary, most proposed AWBT schemes satisfy this condition. Furthermore if we are to consider nonlinear design problems, it makes little sense to require the initial controller design, K_{11}, to be linear. The second condition enforces the notion that we do not want the AWBT block, Λ, to effect the nominal controller, K_{11}, when there is no limitation or substitution and our measurement of the plant input is not corrupted by noise.

It is straightforward to determine that any admissible Λ must be a memoryless linear transformation — equivalently a constant matrix — which has a representation as

$$z = \Lambda v = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} \begin{bmatrix} -C & -D & I & 0 & -I \end{bmatrix} v \tag{11}$$

or, more simply

$$z = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} (u_m - u) \tag{12}$$

Combining the realization (10) and Equation (11) we arrive at Figure 4 where $e_u \equiv u_m - u$. Every AWBT scheme which satisfies **A1).** and **A2).** can be obtained by proper selection of Λ_1 and Λ_2 in Figure 4.

Incorporating the AWBT block, Λ, into the controller we obtain Figure 5 where explicit realizations for U and V are given by

$$V(s) = \left[\begin{array}{c|c} A - H_1C & H_1 \\ \hline H_2C & H_2 \end{array}\right] \tag{13}$$

$$U(s) = \left[\begin{array}{c|c} A - H_1C & B - H_1D \\ \hline H_2C & H_2D \end{array}\right] \tag{14}$$

with $H_1 = \Lambda_1(I + \Lambda_2)^{-1}, H_2 = (I + \Lambda_2)^{-1}$. We require $\Lambda_2 \neq -I$ for well-posedness of the AWBT feedback loop, but Λ_1 and Λ_2 are otherwise arbitrary constant matrices. With $H_2 = I$ selection of H_1 can be interpreted as a "tracking mode" implementation as suggested by Åström and Wittenmark [1].

Since the realization of K_{11} was assumed to be minimal, we can arbitrarily assign the eigenvalues of $A - H_1C$ by proper selection of H_1. It should also be noted that in the case that $A - H_1C$ is stable, U and V correspond to stable right coprime factors of K_{11}. Indeed it is easy to verify that $K_{11} = V^{-1}U$. With the implementation of Figure 5 we generate the V^{-1} factor of K_{11} with feedback around the nonlinearity N (via u_m). Thus when $u = u' = u_m$, Figure 5 is equivalent to Figure 1.

The advantage of the implementation of Figure 5 is that we have the degrees of freedom represented in H_1 and H_2 with which to improve the performance of the closed loop when limitations and substitutions occur. The development of meaningful objectives to guide the selection of H_1 and H_2 is the subject of the next section.

5 Design Objectives for AWBT

5.1 Stabilization

Our primary concern must be that the system remain stable when limitations and substitutions occur. Although necessary and sufficient conditions for nonlinear stability are not available, we can apply the sufficient conditions of Section 2 to obtain an AWBT compensated system which is guaranteed to be stable. In general we will be interested in selecting H_1 and H_2 to satisfy the conditions of Corollary 1. Usually the the zero operator will be included in the sector description of N. In this case we have the necessary condition (Lemma 1) that H_1 must stabilize $A - H_1C$.

5.2 State Positioning

Beyond stabilization we require that the AWBT compensated system should avoid windup and excessive transients when limitations and substitutions occur. In general, windup results in the implementation of Figure 1 because the controller states are not correctly updated when $u' \neq u$ (*i.e.*, for a given controller input y, the controller states, x, achieve values different than they would in the absence of a limitation or substitution). This improper state update results in incorrect controller outputs. In the case of limitations this often causes the system to remain in saturation too long resulting in the output overshoot characteristic of windup. In the case of substitutions, incorrect controller states cause undesirable transients (bumps) when the substitution is removed. It is immediately clear from this discussion why controllers with fast dynamics (and in particular static, or proportional, controllers) do not exhibit windup problems.

There are two basic mechanisms by which the AWBT compensated system (Figure 5) can minimize the state positioning problem. First, if the correct state update under limitation or substitution is known, for example if K is an observer based compensator, this update procedure can be applied whenever $u_m \neq u$.

If the "correct" update procedure is unknown, for example if K is provided in transfer function form, the AWBT compensated controller, $[U \quad I - V]$ should be made (nearly) memoryless so that the effects of incorrect states will be minimized. By nearly memoryless we mean that the dynamics of $[U \quad I - V]$ should be fast relative to the closed loop dynamics. This can be achieved by selecting H_1 so that the poles of $A - H_1C$ are far in the left half plane. This implies that H_1 should be large.

5.3 Noise Sensitivity

Another concern in the AWBT design is that measurement noise associated with u_m should not have a significant effect on nominal ($N = I$) performance. (Since the implementation of Figure 1 does not use u_m, its performance is not affected by these measurement noises). To clarify the effect of this measurement noise, n, we consider a special case of Figure 5 with $w = n$, $N = I$, $P_{11} = P_{21} = 0$ (we assume n does not affect e or y directly), and $P_{31} = I$. Evaluating the transfer function from n to e we find

$$T_{en} = P_{12}[I - UP_{22} - (I - V)P_{32}]^{-1}(I - V)P_{31} \tag{15}$$

In order to minimize the effect of measurement noise on the nominal closed loop we require $\bar{\sigma}(T_{en}(j\omega))$ to be small for frequencies at which the noise n is significant. This objective requires that H_1 and H_2 be chosen such that $V \approx I$ in this range of frequencies. (This would be the case for $H_1 \rightarrow 0$ and $H_2 \rightarrow I$). Note that this objective is in conflict with making the dynamics of $[U \quad I - V]$ fast.

5.4 Directional Sensitivity

A final objective, originally pointed out by Doyle *et al.* [6], is that the AWBT compensated system should provide robust performance with respect to diagonal input uncertainty. For MIMO systems, a saturation nonlinearity can cause the direction of the actual plant input u', to be different from the controller output, u. It is well known that some systems can be very sensitive to diagonal perturbations which change the direction of the plant input. We can evaluate robust performance of the AWBT compensated system by computing

$\mu_D \equiv \frac{1}{\beta}$ where β is the scaling which provides

$$\inf_{\substack{T_1 \in \mathcal{T}' \\ T_2 = t_2 I_n}} \left\| \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ \beta M_{21} & \beta M_{22} \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{bmatrix} \right\|_\infty = 1 \tag{16}$$

and $M(s)$ is obtained by rearranging Figure 5 to that arrive at Figure 2.

As defined μ_D is an upper bound on $\|T_{ew}\|_{L_{2e}-L_{2e}}$ of Figure 5 for all $N \in Cone(C, R)$. If the saturation nonlinearity bounds include the zero operator then we cannot expect worst case nonlinear performance to be any better than open loop performance (for which $\|T_{ew}\|_{L_{2e}-L_{2e}} = \|P_{11}\|_\infty$).

It may be that no H_1 and H_2 exist for which μ_D is acceptable. In this case, a simple nonlinear modification to the AWBT compensated system (in the spirit of MAW of [6]) has been shown to improve performance [2]. The controller output, u, is operated on by S defined by

$$S(u) = \begin{cases} u & \|u\|_\infty \le 1 \\ \frac{u}{\|u\|_\infty} & \|u\|_\infty > 1 \end{cases} \tag{17}$$

This additional block scales the controller output, without affecting its direction, so that its largest element has magnitude one. The saturation will have no effect on $S(u)$ since by construction $\|S(u)\|_\infty \le 1$. This effectively replaces the diagonal saturation operator with a scalar times identity operator in the same cone.

Robust performance with respect to the scalar times identity perturbation can be evaluated with $\mu_S \equiv \frac{1}{\beta}$ where β is such that

$$\inf_{\substack{T_1 \in \mathcal{C}^{n\times n} \\ T_2 = t_2 I_n}} \left\| \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ \beta M_{21} & \beta M_{22} \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{bmatrix} \right\|_\infty = 1 \tag{18}$$

In general $\mu_S \le \mu_D$ since the set over which T_1 varies is smaller in (16) than in (18) (*i.e.* $\mathcal{T}' \subseteq \mathcal{C}^{n\times n}$).

6 Proposed AWBT Schemes in the General Framework

6.1 Observer Based Compensator

We first consider observer based compensator (OBC) designs. The controller states in these designs have a physical interpretation as estimates of the states of the plant. This special structure makes clear the design of an AWBT mechanism which insures that the controller states assume their correct values (namely estimates of the plant states) regardless of plant input limitations and substitutions.

If the controller has access to the full plant state, *i.e.*, $y = \begin{bmatrix} x_P \\ y_m \end{bmatrix}$, then K will be a static state feedback. Since the controller is memoryless it has no states to "windup" (or attain "incorrect values") when limitations or substitutions occur. If K is such that the conditions of Corollary 1 are satisfied and μ_D is acceptable, then no AWBT compensation is required. If μ_D is excessive but μ_S is acceptable then the simple directionality compensation (17) is sufficient.

In the case that the full state is not available, an observer is constructed to supply state estimates which are used to provide feedback. Defining $w = \begin{bmatrix} r \\ d \end{bmatrix}$ and $y = \begin{bmatrix} r \\ y_m \end{bmatrix}$, we introduce a realization of the interconnection structure P of Figure 1:

$$P(s) = \left[\begin{array}{c|ccc} A_P & B_{1P} & B_{2P} & B_{3P} \\ \hline C_{1P} & D_{11P} & D_{12P} & D_{13P} \\ 0 & I & 0 & 0 \\ C_{3P} & D_{31P} & D_{32P} & 0 \end{array}\right] \tag{19}$$

Implicit in this realization is the assumption that r is not corrupted by noise and that P_{33} is strictly proper. The corresponding observer based compensator is of the form:

$$K(s) = \left[\begin{array}{c|cc} A_P - LC_{3P} + B_{3P}F & B_{1P} - LD_{31P} & L \\ \hline F & 0 & 0 \end{array}\right] \tag{20}$$

where L the observer gain and F is the state feedback gain. The observer error, $e_{obs} \equiv x - \hat{x}$, can be shown to obey the relation (when there is no model error)

$$\dot{e}_{obs} = (A_P - LC_{3P})e + (B_{2P} - LD_{32P})d + B_{3P}(u' - u) \tag{21}$$

The last term driving the estimator error results from plant input limitations and substitutions. We see that limitations and substitutions result in an incorrect state update in the controller resulting in a poor estimate of the true plant state. This windup effect is clearly observed in practice and simulation for OBC controllers implemented in this form.

If instead of using the controller output to drive the state estimator, as is implicit in the realization (20), we use the measured or estimated value of the plant input (as would be the case for an extended Kalman filter implementation utilizing a nonlinear model of N) we obtain a realization corresponding to Figure 5 of

$$[U(s) \quad I - V(s)] = \left[\begin{array}{c|ccc} A_P - LC_{3P} & B_{1P} - LD_{31P} & L & B_{3P} \\ \hline F & 0 & 0 & 0 \end{array}\right] \tag{22}$$

This is equivalent to the general AWBT implementation of Figure 5 when $H_1 \equiv B_{3P}$ and $H_2 \equiv I$. To see this define

$$\begin{aligned} A &= A_P - LC_{3P} + B_{3P}F \\ B &= [B_{1P} - LD_{31P} \quad L] \\ C &= F \\ D &= [0 \quad 0] \end{aligned}$$

corresponding to the realization of K in (20), and substitute A, B, C, D, and $H_1 = B_{3P}, H_2 = I$ into equations (13) and (14) to arrive at (22). With this implementation the observer error obeys

$$\dot{e}_{obs} = (A_P - LC_{3P})e + (B_{2P} - LD_{32P})d + B_{3P}(u' - u_m) \tag{23}$$

If the measurement of the plant input is exact ($u' = u_m$), the observer error is not affected by limitations or substitutions. With this implementation the observer states (and hence the controller states) remain correct and we do not see state positioning problems when limitations or substitutions occur.

The nonlinear stability, noise sensitivity, and directional sensitivity of this scheme are determined solely be the initial observer and state feedback designs. If these issues are not considered in the nominal linear design, we have no guarantee that simply employing the explicit observer implementation will provide adequate performance when limitations and substitutions occur.

6.2 Hanus' Conditioned Controller

We next consider an AWBT technique proposed by Hanus [9] applicable when $w = \begin{bmatrix} r \\ d \end{bmatrix}$, $y = \begin{bmatrix} r \\ y_m \end{bmatrix}$ and $[K_1 \quad K_2] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D_1 & D_2 \end{array}\right]$. In Hanus' scheme a "realizable reference", r^r, is used in the controller state equation in place of r when $u_m \neq u$. This "realizable reference" is the the reference signal that would make $u_m = u$ if applied to the controller state *and* output equations in place of r. As Hanus points out, such a back calculation is unnecessary since it is equivalent to implementation of a "conditioned controller" which has the realization

$$u = \left[\begin{array}{c|ccc} A - B_1 D_1^{-1} C & 0 & B_2 - B_1 D_1^{-1} D_2 & B_1 D_1^{-1} \\ \hline C & D_1 & D_2 & 0 \end{array}\right] \begin{bmatrix} r \\ y_m \\ u_m \end{bmatrix} \tag{24}$$

We note immediately that stability for $N = 0$ implies that $A - B_1 D_1^{-1} C$ must be stable (Lemma 1) which in turn implies that $K_1 = \left[\begin{array}{c|c} A & B_1 \\ \hline C & D_1 \end{array}\right]$ must be minimum phase. This is generally not restrictive for 2-degree-of-freedom (2 DOF) designs since making K_1 non-minimum phase introduces non-minimum phase characteristics in T_{er} which are not required for stabilization, and is therefore always undesirable. It can, however, be restrictive for 1-degree-of-freedom (or error feedback) controllers for which $K_1 = -K_2$.

An additional limitation of this technique is the requirement that a left inverse of D_1 must exist. This implies that D_1 must have full column rank which precludes application to control designs in which K_1 is strictly proper.

From the realization of the "conditioned controller" (24) we see that this implementation corresponds to the choice, $H_1 = B_1 D_1^{-1}$, $H_2 = I$ in the general AWBT formulation (Equations (13) and (14)). Furthermore this choice makes the states of the controller uncontrollable from r (and y_m as well in the 1 DOF case). This has the advantage that slow modes in the controller are not driven by r (and y_m for 1 DOF controllers) and therefore do not windup. However, if $K_1 \neq -K_2$ and the conditioned controller contains slow poles (eigenvalues of $A - B_1 D_1^{-1} C$ near the $j\omega$ axis) we may expect to see poor anti-windup performance. Furthermore if BD_1^{-1} is large and $A - B_1 D_1^{-1} C$ is fast, we can expect that $\|T_{en}\|_\infty$ will be large at high frequencies and that measurement noise associated with u_m will negatively impact nominal performance.

6.3 Internal Model Control

The internal model control (IMC) structure [10] has been suggested as a way of implementing controllers in order to avoid AWBT problems for open loop stable systems. Although IMC was not developed as an AWBT method, two equivalent schemes, proposed in the CHANCE project [3] and by Irving, have been introduced specifically to achieve AWBT. In the IMC framework, the "classical controller" (K of Figure 1) is given by

$$K = -(I - Q\hat{G})^{-1} Q \tag{25}$$

where Q is the IMC controller and $\hat{G}$ is a model of the plant. Note that stability of Q is necessary and sufficient for K to stabilize $\hat{G}$. Introducing realizations of $\hat{G}$ and Q

$$Q(s) = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right] \quad \hat{G}(s) = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right]$$

we compute a realization for K using (25)

$$K(s) = \left[\begin{array}{cc|c} A_Q & B_Q C_G & B_Q \\ B_G C_Q & A_G + B_G D_Q C_G & B_G D_Q \\ \hline -C_Q & -D_Q C_Q & -D_Q \end{array}\right]$$

The IMC implementation corresponds to Figure 5 with $U = [Q \; -Q]$, $V = I - Q\hat{G}$, $e = r - y_m$,

$$w = \begin{bmatrix} r \\ d \end{bmatrix}, \quad y = \begin{bmatrix} r \\ y_m \\ u_m \end{bmatrix}, \text{ and } P(s) = \begin{bmatrix} I & -I & -G \\ I & 0 & 0 \\ 0 & I & G \\ 0 & 0 & I \end{bmatrix}$$

In terms of the realizations for $\hat{G}$ and Q we have

$$U(s) = \left[\begin{array}{c|cc} A_Q & B_Q & -B_Q \\ \hline C_Q & D_Q & -D_Q \end{array}\right] \quad V(s) = \left[\begin{array}{cc|c} A_Q & B_Q C_G & 0 \\ 0 & A_G & B_G \\ \hline -C_Q & -D_Q C_Q & I \end{array}\right]$$

With these realizations it can be verified that the IMC structure implementation of K corresponds to $H_1 = \begin{bmatrix} 0 \\ B_G \end{bmatrix}$, $H_2 = I$ in the general AWBT formulation. We see from the realization of $V(s)$ that if the open loop plant has dynamics much slower than the closed loop dynamics, windup (in these sense of state positioning errors) will occur. The distinguishing feature of IMC is that, for any stabilizing K, $M(s)$ of Figure 2 is stable and $M_{11}(s)$ is identically zero. Thus the conditions of Corollary 1 are satisfied trivially. In the absence of plant model mismatch, implementation in the IMC structure yields a stable closed loop for *any* N.

For the IMC implementation we have (with $N = I$ and no model error)

$$T_{en} = -GQG \tag{26}$$

If the plant rolls off sufficiently fast or the noise, n, is not significant inside the closed loop bandwidth, the effect of n on e should be modest. There are no inherent properties of IMC which provide robustness with respect to diagonal input uncertainty. Design guidelines for the initial linear controller design (Q or K) to provide good robustness with respect to this uncertainty can be found in [12].

6.4 High Gain Conventional Anti-windup

The final AWBT technique we will examine is referred to as high gain conventional anti-windup (CAW). High gain CAW is applicable to one degree of freedom error feedback structures. The AWBT action is provided by feeding $u_m - u$ back through a high gain, X, to the controller input, e, as shown in Figure 6. Typically $X = \alpha I$ with $\alpha \gg 1$. Rearranging Figure 6 to the standard AWBT diagram of Figure 5 provides, $w = \begin{bmatrix} r \\ d \end{bmatrix}$, $y = \begin{bmatrix} r - y_m \\ u_m \end{bmatrix}$, $e = r - y_m$, $U = [I + KX]^{-1} K$, and $V = [I + KX]^{-1}$. In terms of the realization of K, (9), we have

$$U(s) = \left[\begin{array}{c|c} A - BX[I+DX]^{-1}C & B[I+DX]^{-1} \\ \hline [I+DX]^{-1}C & [I+DX]^{-1}D \end{array}\right] \tag{27}$$

$$V(s) = \left[\begin{array}{c|c} A - BX[I+DX]^{-1}C & BX[I+DX]^{-1} \\ \hline -[I+DX]^{-1}C & [I+DX]^{-1} \end{array}\right] \tag{28}$$

This is equivalent to $\Lambda_1 = BX$ and $\Lambda_2 = DX$ in Figure 4 (or $H_1 = BX[I+DX]^{-1}, H_2 = [I+DX]^{-1}$). Thus high gain CAW can be seen to be a special case of the general AWBT compensation formulation. Rather than acting on the controller states and outputs, the AWBT action, z, acts on the controller input.

With X large and D invertible it is easy to see that the poles of U and V approach the zeros of K (the eigenvalues of $A - BX[I + DX]^{-1}C$ approach the eigenvalues of $A - BD^{-1}C$). Thus for non-minimum phase controllers this scheme will not result in stable U and V. If the zero operator is included in the conic sector description of N this precludes satisfaction of the conditions of Corollary 1.

It is also clear that for X large and D non-zero, $U \approx 0$ and $V \approx 0$ so that from (15) we see that measurement noise associated with u_m will not be attenuated unless $\bar{\sigma}(P_{12}P_{31}(j\omega))$ is small in the frequency range where n is significant.

7 An Approach to Synthesis

Each of the proposed AWBT schemes reviewed in Section 6 considers only a subset of the AWBT objectives of Section 5. The analysis tools and quantitative performance objectives presented suggest an approach to the synthesis problem which recognizes all of these objectives.

The basis for the design of H_1 and H_2 (and therefore any admissible AWBT scheme) are:

1. $M(s)$ must be stable.
2. $\inf_{T \in \mathcal{T}'} \|TM_{11}(s)T^{-1}\|_\infty \leq 1$.
3. $\inf_{H_1, H_2} \mu_D$
4. $Re\{\lambda_i(A - H_1 C)\} \ll 0$.

Objectives 1. and 2. correspond to the conditions of Corollary 1 (stabilization), 3. encompasses the directionality and (by including measurement noise, n, in w) noise sensitivity objectives, and 4. insures that the poles of U and V are fast so that state positioning errors are minimized.

In the general case considering 1.—3. simultaneously amounts to a μ-optimal (or H^∞) static controller design subject to a μ or H^∞ constraint. Solutions to such a problem are not at hand.

A topic of current research is to examine limiting case solutions and to use the analysis tests available for each objective to obtain insight into the performance trade-offs involved in the selection of H_1 and H_2. This insight should allow us to formulate simplified performance objectives which are more amenable to the available synthesis techniques, while preserving the essence of the original objectives. For example it seems feasible to select H_1 and H_2 neglecting the directionality issue and use a simple (if ad hoc) nonlinearity (20) to deal with the directionality problem.

An additional area of interest is to obtain an understanding of the impact of the initial design of K on achievable AWBT performance. In certain situations we may need to modify (detune) the initial design in order to guarantee nonlinear stability and provide acceptable AWBT performance.

8 Conclusions

We have developed a general theoretical framework for studying the performance of control systems subject to plant input limitations and substitutions. This includes the development of analysis tools applicable to a broad class of limitation and substitution mechanisms. Quantitative performance objectives, applicable in the general case, which result in graceful performance degradation of the nominally linear system are presented. These objectives and analysis tools lead to the development of a general synthesis problem which is the subject of on-going research.

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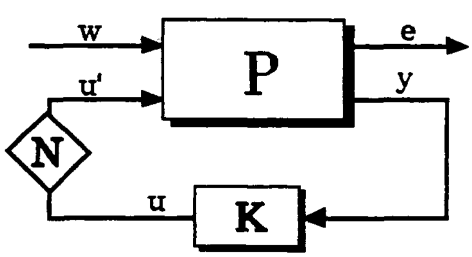

Figure 1: The general interconnection structure.

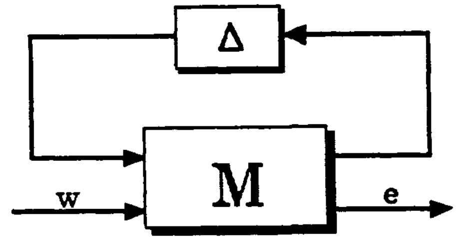

Figure 2: The general analysis structure.

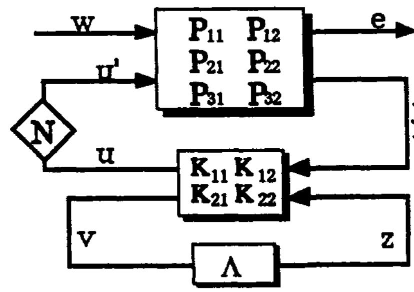

Figure 3: The AWBT structure.

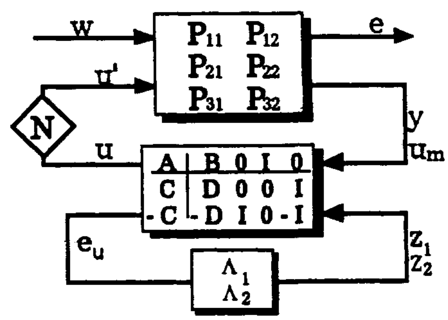

Figure 4: The AWBT structure simplified.

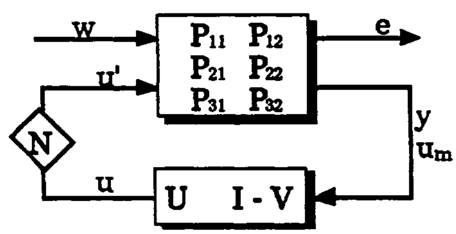

Figure 5: The general AWBT formulation.

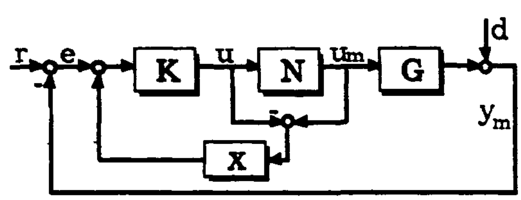

Figure 6: High Gain CAW.