FEYNMAN INTEGRALS AND PERIODS IN CONFIGURATION SPACES

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To Tanju Başarır Özkan, who saved Uzay

Abstract. We describe differential forms representing Feynman amplitudes in configuration spaces of Feynman graphs, and regularization and evaluation techniques, for suitable chains of integration, that give rise to periods of mixed Tate motives.

Contents

1. Introduction

This work is a continuation of our investigation [12] of algebro-geometric and motivic aspects of Feynman integrals in configuration spaces and their relation to periods of mixed Tate motives.

Section 2 introduces algebraic varieties $X^{\mathbf{V}_\Gamma}$ and $Z^{\mathbf{V}_\Gamma}$ that describe configuration spaces associated to a Feynman graph Γ, and smooth differential forms on these varieties, with singularities along diagonals, that extend the Feynman rules and Feynman propagators in configuration spaces to varieties containing a dense affine space that represents the "physical space" on which the Feynman diagram integration lives. We consider three possible variants of the geometry, and of the corresponding differential form, respectively given by the forms (2.2), (2.9) and (2.12), because this will allow us to present different methods of regularization and integration and show, in different ways, how the relation between Feynman integrals and periods of mixed Tate motives arises in the setting of configuration spaces.

The geometric setting builds upon our previous work [12], and we will be referring to that source for several of the algebro-geometric arguments we need to use here. In terms of the Feynman amplitudes themselves, the main difference between the approach followed in this paper and the one of [12] is that in our previous work we extended the Feynman propagator to the configuration space $X^{\mathbf{V}_\Gamma}$ as an algebraic differential form, which then had singularities not only along the diagonals, but also along a quadric Z_Γ (the configuration space analog of the graph hypersurfaces describing singularities of Feynman amplitudes in momentum space). In this paper, we extend the Feynman propagator to a $\mathcal{C}^\infty$ (non-algebraic) differential form on $X^{\mathbf{V}_\Gamma}$, which then has singularities only along the diagonals. In this way, we gain the fact that the motives involved are easy to control, and do not leave the class of Tate motives, while we move the difficulty in obtaining an interpretation of Feynman integrals as periods to the fact of working with a non-algebraic differential form. In the various sections of the paper we show different ways in which one can overcome this problem and end up with evaluations of a suitably regularized Feynman amplitude that gives periods of mixed Tate motives (multiple zeta values, by the recent result of [11]).

Section 3 can be read independently of the rest of the paper (or can be skipped by readers more directly interested in our main results on periods). Its main purpose is to explain the relation between the Feynman amplitudes (2.2), (2.9) and (2.12) and real and complex Green functions and the Bochner–Martinelli integral kernel operator. In particular, we show that the form of the Feynman amplitude leads naturally to a Bochner–Martinelli type theorem, adapted to the geometry of graph configuration spaces, which combines the usual Bochner–Martinelli theorem on complex manifolds [24] and the notion of harmonic functions and Laplacians on graphs.

In Section 4, we consider the first form (2.2) of the Feynman amplitude and its restriction to the affine space $\mathbb{A}^{D\mathbf{V}_\Gamma}(\mathbb{R})$ in the real locus in the graph configuration space $X^{\mathbf{V}_\Gamma}$, with $X = \mathbb{P}^D(\mathbb{C})$. We first describe a setting generalizing a recent construction, by Spencer Bloch, of cycles in the configuration spaces $X^{\mathbf{V}_\Gamma}$. Our setting assigns a middle-dimensional relative cycle, with boundary on the union of diagonals, to any acyclic orientation on the graph. We then consider the development of the propagator into Gegenbauer orthogonal polynomials. This is a technique for the calculation of Feynman diagrams in configuration spaces (known as x-space technique) that is well established in the physics literature since at least the early '80s, [13]. We split these x-space integrals into contributions parameterized by the chains described above. The decompositions of the chains of integration, induces a corresponding decomposition of the Feynman integral into an angular and a radial part. In the case of dimension $D = 4$, and for polygon graphs these simplify into integrals over simplexes of polylogarithm functions, which give rise to zeta values. For more general graphs, each star of a vertex contributes a certain combination of integrals of triple integrals of spherical harmonics. These can be expressed in terms of isoscalar coefficients. We compute explicitly, in the case $D = 4$ the top term of this expansion, and show that, when pairs of half edges are joined together to form an edge of the graph, one obtains a combination of nested sums that can be expressed in terms of Mordell–Tornheim and Apostol–Vu series.

Starting with §5, we consider the form (2.9) of the Feynman amplitude, which corresponds to a complexification, and a doubling of the dimension of the relevant configuration spaces $Z^{\mathbf{V}_\Gamma} \simeq (X \times X)^{\mathbf{V}_\Gamma}$. The advantage of passing to this formulation is that one is then dealing with a closed form, unlike the case of (2.2) considered in the previous section, hence cohomological arguments become available. We first describe the simple modifications to the geometry of configuration spaces, with respect to the results of our previous work [12], which are needed in order to deal with this doubling of dimension. We introduce the wonderful compactification $F(X, \Gamma)$ of the configuration space $Z^{\mathbf{V}_\Gamma}$, which works pretty much as in the case of the wonderful compactification $\overline{\mathrm{Conf}}_\Gamma(X)$ of $X^{\mathbf{V}_\Gamma}$ described in [12], [30], [31], with suitable modifications. The purpose of passing to the wonderful compactification is to realize the complement of the diagonals in the configuration space explicitly as the complement of a union of divisors intersecting transversely, so that we can describe de Rham cohomology classes in terms of representatives that are algebraic forms with logarithmic poles along the divisors. We also discuss, in this section, the properties of the motive of the wonderful compactification and we verify that, if the underlying variety X is a Tate motive, the wonderful compactification also is, and so are the unions and intersections of the divisors obtained in the construction. We also describe iterated Poincaré residues of these forms along the intersections of divisors corresponding to $\mathcal{G}_\Gamma$-nests of biconnected induced subgraphs of the Feynman graph, according to the general theory of iterated Poincaré residues of forms with logarithmic poles, [1] , [2], [15], [23].

In §6, we consider a first regularization method for the Feynman integral based on the amplitude (2.9), which is essentially a cutoff regularization and is expressed in terms of principal-value currents and the theory of Coleff–Herrera residue currents, [8], [14]. We show that, when we regularize the Feynman integral as a principal value current, the ambiguity is expressed by residue currents supported on the intersections of the divisors of the wonderful compactification, associated to the $\mathcal{G}_\Gamma$-nests, which are related to the iterated Poincaré residues described in §5. In particular, when evaluated on algebraic test differential forms on these intersections, the currents describing the ambiguities take values that are periods of mixed Tate motives.

In §7, we introduce a more directly algebro-geometric method of regularization of both the Feynman amplitude form (2.9) and of the chain of integration, which is based on the *deformation to the normal cone* [18], which we use to separate the chain of integration from the locus of divergences of the form. We check again that the motive of the deformation space constructed using the deformation to the normal cone remains mixed Tate, and we show once again that the regularized Feynman integral obtained in this way gives rise to a period.

2. Feynman amplitudes in configuration space

In the following we let X be a D-dimensional smooth projective variety, which contains a dense subset isomorphic to an affine space $\mathbb{A}^D$, whose set of real points $\mathbb{A}^D(\mathbb{R})$ we identify with Euclidean D-dimensional spacetime. For instance, we can take $X = \mathbb{P}^D$. We assume that D is even and we write $D = 2\lambda + 2$.

2.1. Feynman graphs.

A *graph* Γ is a one-dimensional finite CW-complex. We denote the set of vertices of Γ by $\mathbf{V}_\Gamma$, the set of edges by $\mathbf{E}_\Gamma$, and the boundary map by $\partial_\Gamma : \mathbf{E}_\Gamma \to (\mathbf{V}_\Gamma)^2$.

When an orientation is chosen on Γ, we write $\partial_\Gamma(e) = \{s(e), t(e)\}$, the source and target vertices of the oriented edge e.

A *looping* edge is an edge for which the two boundary vertices coincide and *multiple edges* are edges between the same pair of vertices. We assume that our graphs have no looping edges.

2.1.1. *Induced subgraphs and quotient graphs.* A subgraph $\gamma \subseteq \Gamma$ is called an *induced subgraph* if its set of vertices $\mathbf{E}_\gamma$ is equal to $\partial_\Gamma^{-1}((\mathbf{V}_\gamma)^2)$, that is, γ has all the edges of Γ on the same set of vertices. We will denote by $\mathbf{SG}(\Gamma)$ the set of all induced subgraphs of Γ and by

$$\mathbf{SG}_k(\Gamma) = \{\gamma \in \mathbf{SG}(\Gamma) \mid |\mathbf{V}_\gamma| = k\}, \tag{2.1}$$

the subset $\mathbf{SG}_k(\Gamma) \subseteq \mathbf{SG}(\Gamma)$ of all induced subgraphs with k vertices.

For $\gamma \in \mathbf{SG}(\Gamma)$, we denote by $\Gamma//\gamma$ the graph obtained from Γ by shrinking each connected component of γ to a separate vertex. The quotient graph $\Gamma//\gamma$ does not have looping edges since we consider only induced subgraphs.

2.2. **Feynman amplitude.**

When computing Feynman integrals in configuration space, one considers singular differential forms on $X^{\mathbf{V}_\Gamma}$, integrated on the real locus of this space.

Definition 2.1. *The Euclidean massless Feynman amplitude in configuration space is given by the differential form*

$$\omega_\Gamma = \prod_{e\in\mathbf{E}_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \bigwedge_{v\in\mathbf{V}_\Gamma} dx_v. \tag{2.2}$$

The form (2.2) defines a $\mathcal{C}^\infty$-differential form on the configuration space

$$\mathrm{Conf}_\Gamma(X) = X^{\mathbf{V}_\Gamma} \smallsetminus \cup_{e\in\mathbf{E}_\Gamma}\Delta_e, \tag{2.3}$$

with singularities along the diagonals

$$\Delta_e = \{(x_v \mid v \in \mathbf{V}_\Gamma) \mid x_{s(e)} = x_{t(e)}\}. \tag{2.4}$$

Remark 2.2. The diagonals (2.4) corresponding to edges between the same pair of vertices are the same, consistently with the fact that the notion of degeneration that defines the diagonals is based on "collisions of points" and not on contraction of the edges connecting them. This suggests that we may want to exclude graphs with multiple edges. However, multiple edges play a role in the definition of Feynman amplitudes (see Definition 2.1 above and §4 of [12]) hence we allow this possibility. On the other hand, the definition of configuration space is void in the presence of looping edges, since the diagonal Δ_e associated to a looping edge is the whole space $X^{\mathbf{V}_\Gamma}$, and its complement is empty. Thus, our assumption that graphs have no looping edges is aimed at excluding this geometrically trivial case.

Remark 2.3. Our choice of $\|x_{s(e)} - x_{t(e)}\|^2$ in the Feynman propagator differs from the customary choice of propagator (see for instance [12]) where $(x_{s(e)} - x_{t(e)})^2$ is used instead, but these two expressions agree on the locus

$X^{\mathbf{V}_\Gamma}(\mathbb{R})$ of real points, which is the chain of integration of the Feynman amplitude. The latter choice gives a manifestly algebraic differential form, but at the cost of introducing a hypersurface $\mathcal{Z}_\Gamma$ in $X^{\mathbf{V}_\Gamma}$ where the singularities of the form occur, which makes it difficult to control explicitly the nature of the motive. Our choice here only gives a $\mathcal{C}^\infty$ differential form, but the domain of definition is now simply $\mathrm{Conf}_\Gamma(X)$, whose motivic nature is much easier to understand, see [12].

Formally (before considering the issue of divergences), the Feynman integral is obtained by integrating the form (2.2) on the locus of real points of the configuration space.

Definition 2.4. *The chain of integration for the Feynman amplitude is taken to be the set of real points of this configuration space,*

$$\sigma_\Gamma = X(\mathbb{R})^{\mathbf{V}_\Gamma}. \tag{2.5}$$

The form (2.2) defines a top form on σ_Γ. There are two sources of divergences in integrating the form (2.2) on σ_Γ: the intersection between the chain of integration and the locus of divergences of ω_Γ,

$$\sigma_\Gamma \cap \cup_e \Delta_e = \cup_e \Delta_e(\mathbb{R})$$

and the behavior at infinity, on $\Delta_\infty := X \smallsetminus \mathbb{A}^D$. In physics terminology, these correspond, respectively, to the *ultraviolet* and *infrared* divergences.

We will address these issues in §4 below.

2.3. **Variations upon a theme.** In addition to the form (2.2) considered above, we will also consider other variants, which will allow us to discuss different possible methods to address the question of periods and in particular the occurrence of multiple zeta values in Feynman integrals in configuration spaces.

2.3.1. *Complexified case.* In this setting, instead of the configuration space $X^{\mathbf{V}_\Gamma}$ and its locus of real points $\sigma = X^{\mathbf{V}_\Gamma}(\mathbb{R})$, we will work with a slightly different space, obtained as follows.

As above, let X be a smooth projective variety, and Z denote the product $X \times X$. Let $p : Z \to X$, $p : z = (x, y) \mapsto x$ be the projection.

Given a graph Γ, the *configuration space* $F(X, \Gamma)$ of Γ in Z is the complement

$$Z^{\mathbf{V}_\Gamma} \setminus \bigcup_{e \in \mathbf{E}_\Gamma} \Delta_e^{(Z)} \cong (X \times X)^{\mathbf{V}_\Gamma} \setminus \bigcup_{e \in \mathbf{E}_\Gamma} \Delta_e^{(Z)}, \tag{2.6}$$

in the cartesian product $Z^{\mathbf{V}_\Gamma} = \{(z_v \mid v \in \mathbf{V}_\Gamma)\}$ of the *diagonals associated to the edges* of Γ,

$$\Delta_e^{(Z)} \cong \{(z_v \mid v \in \mathbf{V}_\Gamma) \in Z^{\mathbf{V}_\Gamma} \mid p(z_{s(e)}) = p(z_{t(e)})\}. \tag{2.7}$$

The relation between the configuration space $F(X, \Gamma)$ and the previously considered $\mathrm{Conf}_\Gamma(X)$ of (2.3) is described by the following.

Lemma 2.5. *The configuration space* $F(X,\Gamma)$ *is isomorphic to*

$$F(X,\Gamma) \simeq \mathrm{Conf}_\Gamma(X) \times X^{\mathbf{V}_\Gamma}, \tag{2.8}$$

and the diagonals (2.7) *and related to those of* (2.4) *by* $\Delta_e^{(Z)} \cong \Delta_e \times X^{\mathbf{V}_\Gamma}$.

2.3.2. *Feynman amplitudes in the complexified case.* We assume that the smooth projective variety Z contains a dense open set isomorphic to affine space $\mathbb{A}^{2D}$, with coordinates $z = (x_1, \dots, x_D, y_1, \dots, y_D)$.

Definition 2.6. *Given a Feynman* Γ *with no looping edges, we define the corresponding Feynman amplitude (weight) as*

$$\omega_\Gamma^{(Z)} = \prod_{e \in \mathbf{E}_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2D-2}} \bigwedge_{v \in \mathbf{V}_\Gamma} dx_v \wedge d\bar{x}_v, \tag{2.9}$$

where $\|x_{s(e)} - x_{t(e)}\| = \|p(z)_{s(e)} - p(z)_{t(e)}\|$ *and where the differential forms* dx_v *and* $d\bar{x}_v$ *denote, respectively, the holomorphic volume form* $dx_{v,1} \wedge \cdots \wedge dx_{v,D}$ *and its conjugate. The chain of integration is given, in this case, by the range of the projection*

$$\sigma^{(Z,y)} = X^{\mathbf{V}_\Gamma} \times \{y = (y_v)\} \subset Z^{\mathbf{V}_\Gamma} = X^{\mathbf{V}_\Gamma} \times X^{\mathbf{V}_\Gamma}, \tag{2.10}$$

for a fixed choice of a point $y = (y_v \,|\, v \in \mathbf{V}_\Gamma)$.

The form (2.9) is a closed $\mathcal{C}^\infty$ differential form on $F(X,\Gamma)$ of degree $2 \dim_{\mathbb{C}} X^{\mathbf{V}_\Gamma} = \dim_{\mathbb{C}} Z^{\mathbf{V}_\Gamma} = 2D|\mathbf{V}_\Gamma|$, hence it defines a cohomology class in $H^{2D|\mathbf{V}_\Gamma|}(F(X,\Gamma))$, and it gives a top form on the locus $\sigma^{(Z,y)}$.

In this case, the divergences of $\int_{\sigma^{(Z,y)}} \omega_\Gamma^{(Z)}$ are coming from the union of the diagonals $\cup_{e \in \mathbf{E}_\Gamma} \Delta_e^{(Z)}$ and from the divisor at infinity

$$\Delta_{\infty,\Gamma}^{(Z)} := \sigma^{(Z,y)} \smallsetminus \mathbb{A}^{D|\mathbf{V}_\Gamma|}(\mathbb{C}) \times \{y\}. \tag{2.11}$$

We will address the behavior of the integrand near the loci $\cup_{e \in \mathbf{E}_\Gamma} \Delta_e^{(Z)}$ and $\Delta_{\infty,\Gamma}^{(Z)}$ and the appropriate regularization of the Feynman amplitude and the chain of integration in §5 below. When convenient, we will choose coordinates so as to identify the affine space $\mathbb{A}^{D|\mathbf{V}_\Gamma|}(\mathbb{C}) \times \{y\} \subset \sigma^{(Z,y)}$ with a real affine space $\mathbb{A}^{2D|\mathbf{V}_\Gamma|}(\mathbb{R})$.

2.3.3. *Feynman amplitudes and complex Green forms.* A variant of the amplitude $\omega_\Gamma^{(Z)}$ of (2.9), which we will discuss briefly in §3, is related to the complex Green forms. In this setting, instead of (2.9) one considers the closely related form

$$\hat{\omega}_\Gamma = \prod_{e \in \mathbf{E}_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2D-2}} \bigwedge_{v \in \mathbf{V}_\Gamma} \sum_{k=1}^{D} (-1)^{k-1} dx_{v,[k]} \wedge d\bar{x}_{v,[k]}, \tag{2.12}$$

where the forms $dx_{v,[k]}$ and $d\bar{x}_{v,[k]}$ denote

$$dx_{v,[k]} = dx_{v,1} \wedge \cdots \wedge \widehat{dx_{v,k}} \wedge \cdots \wedge dx_{v,D},$$

$$d\bar{x}_{v,[k]} = d\bar{x}_{v,1} \wedge \cdots \wedge \widehat{d\bar{x}_{v,k}} \wedge \cdots \wedge d\bar{x}_{v,D},$$

respectively, with the factor $dx_{v,k}$ and $d\bar{x}_{v,k}$ removed.

Notice how, unlike the form considered in (2.9), which is defined on the affine $\mathbb{A}^{2D|\mathbf{V}_\Gamma|} \subset Z^{\mathbf{V}_\Gamma}$, the form (2.12) has the same degree of homogeneity $2D-2$ in the numerator and denominator, when the graph Γ has no multiple edges, hence it is invariant under rescaling of the coordinates by a common non-zero scalar factor.

2.3.4. *Distributional interpretation.* In the cases discussed above, the amplitudes defined by (2.2) and (2.9) can be given a distributional interpretation, as a pairing of a distribution

$$\prod_{e\in\mathbf{E}_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^\alpha}, \tag{2.13}$$

where α is either $2\lambda = D-2$ or $2D-2$, or $2D$, and test forms given in the various cases, respectively, by

(1) forms $\phi(x_v) \bigwedge_{v\in\mathbf{V}_\Gamma} dx_v$, with ϕ a test function in $\mathcal{C}^\infty(\mathbb{A}^{D|\mathbf{V}_\Gamma|}(\mathbb{R}))$;
(2) forms $\phi(z_v) \bigwedge_{v\in\mathbf{V}_\Gamma} dx_v \wedge d\bar{x}_v$, with $\phi(z_v) = \phi(p(z_v)) = \phi(x_v)$ a test function in $\mathcal{C}^\infty(X^{\mathbf{V}_\Gamma}(\mathbb{C}))$;

3. Feynman amplitudes and Bochner–Martinelli kernels

We describe the relation between the Feynman amplitude ω_Γ of (2.2) and the Green functions of the Laplacian and we compute $d\omega_\Gamma$ in terms of an integral kernel associated to the graph Γ and the affine space $\mathbb{A}^{D|\mathbf{V}_\Gamma|} \subset X^{\mathbf{V}_\Gamma}$. We also discuss the relation between the Feynman amplitude $\omega_\Gamma^{(Z)}$ of (2.9), in the complexified case, and the Bochner–Martinelli kernel.

3.1. **Real Green functions.** The Green function for the real Laplacian on $\mathbb{A}^D(\mathbb{R})$, with $D = 2\lambda+2$, is given by

$$G_\mathbb{R}(x,y) = \frac{1}{\|x-y\|^{2\lambda}} \tag{3.1}$$

Consider then the differential form $\omega = G_\mathbb{R}(x,y)\, dx \wedge dy$. This corresponds to the Feynman amplitude (2.2) in the case of the graph consisting of a single edge, with configuration space $(X \times X) \smallsetminus \Delta$, with $\Delta = \{(x,y)\,|\, x=y\}$ the diagonal.

Lemma 3.1. *The form* $\omega = G_{\mathbb{R}}(x,y)\, dx \wedge dy$ *is not closed. Its differential is given by*

$$d\omega = -\lambda \sum_{k=1}^{D} \frac{(x_k - y_k)}{\|x-y\|^D} \, (dx \wedge dy \wedge d\bar{x}_k - dx \wedge dy \wedge d\bar{y}_k)\,. \tag{3.2}$$

Proof. We have

$$d\omega = \bar{\partial}\omega = \sum_{k=1}^{D} \frac{\partial G_{\mathbb{R}}}{\partial \bar{x}_k} dx \wedge dy \wedge d\bar{x}_k + \sum_{k=1}^{D} \frac{\partial G_{\mathbb{R}}}{\partial \bar{y}_k} dx \wedge dy \wedge d\bar{y}_k.$$

We then see that

$$\frac{\partial \|x-y\|^{-2\lambda}}{\partial \bar{x}_k} = -\lambda \frac{(x_k - y_k)}{\|x-y\|^D}, \quad \text{and} \quad \frac{\partial \|x-y\|^{-2\lambda}}{\partial \bar{y}_k} = \lambda \frac{(x_k - y_k)}{\|x-y\|^D},$$

so that we obtain (3.2). □

3.2. **Feynman amplitudes and integral kernels on graphs.** We first consider the form defining the Feynman amplitude ω_Γ of (2.2). It is not a closed form. In fact, we compute here explicitly its differential $d\omega_\Gamma$ in terms of some integral kernels associated to graphs.

Recall that, given a graph Γ and a vertex $v \in \mathbf{V}_\Gamma$, the graph $\Gamma \smallsetminus \{v\}$ has

$$\mathbf{V}_{\Gamma \smallsetminus \{v\}} = \mathbf{V}_\Gamma \smallsetminus \{v\}, \quad \text{and} \quad \mathbf{E}_{\Gamma \smallsetminus \{v\}} = \mathbf{E}_\Gamma \smallsetminus \{e \in \mathbf{E}_\Gamma \,|\, v \in \partial(e)\}, \tag{3.3}$$

that is, one removes a vertex along with its star of edges.

Definition 3.2. *Suppose given a graph* Γ *and a vertex* $v \in \mathbf{V}_\Gamma$*. We define the differential form*

$$\kappa^{\mathbb{R}}_{\Gamma,v}(x) = (-1)^{|\mathbf{V}_\Gamma|} \epsilon_v \sum_{e:\, v \in \partial(e)} \epsilon_e \sum_{k=1}^{D} \frac{(x_{s(e),k} - x_{t(e),k})}{\|x_{s(e)} - x_{t(e)}\|^D} \, dx_v \wedge d\bar{x}_{v,k} \tag{3.4}$$

in the coordinates $x = (x_v \,|\, v \in \mathbf{V}_\Gamma)$*, where the sign* $\epsilon_e = \pm 1$ *is positive or negative according to whether* $v = s(e)$ *or* $v = t(e)$ *and the sign* $\epsilon_v = \pm 1$ *is defined by*

$$\epsilon_v (\bigwedge_{w \neq v} dx_w) \wedge dx_v = \bigwedge_{v' \in \mathbf{V}_\Gamma} dx_{v'}.$$

Given an oriented graph Γ*, we then consider the integral kernel*

$$\mathcal{K}_{\mathbb{R},\Gamma}(x) = \lambda \sum_{v \in \mathbf{V}_\Gamma} \omega_{\Gamma \smallsetminus \{v\}} \wedge \kappa^{\mathbb{R}}_{\Gamma,v}(x), \tag{3.5}$$

where $\omega_{\Gamma \smallsetminus \{v\}}$ *is the form* (2.2) *for the graph* (3.3).

Proposition 3.3. *The differential* $d\omega_\Gamma$ *is the integral kernel* $\mathcal{K}_{\mathbb{R},\Gamma}$ *of* (3.5).

Proof. First observe that, for ω_Γ the Feynman amplitude of (2.2), we have $d\omega_\Gamma = \bar\partial\omega_\Gamma$, that is,

$$d\omega_\Gamma = \sum_{v\in \mathbf{V}_\Gamma}\sum_{k=1}^{D}(-1)^{\mathbf{V}_\Gamma}\frac{\partial}{\partial \bar x_{v,k}}\omega_\Gamma \wedge d\bar x_{v,k}.$$

We have

$$\partial_{\bar x_{v,k}}\left(\prod_e \frac{1}{\|x_{s(e)}-x_{t(e)}\|^{2\lambda}}\right) =$$

$$\left(\prod_{e:v\notin\partial e}\frac{1}{\|x_{s(e)}-x_{t(e)}\|^{2\lambda}}\right)\cdot\left(\sum_{e:v=t(e)}\bar\partial_{x_{v,k}}\frac{1}{\|x_{s(e)}-x_v\|^{2\lambda}}\right)$$

$$-\left(\prod_{e:v\notin\partial e}\frac{1}{\|x_{s(e)}-x_{t(e)}\|^{2\lambda}}\right)\cdot\left(\sum_{e:v=s(e)}\bar\partial_{x_{v,k}}\frac{1}{\|x_{t(e)}-x_v\|^{2\lambda}}\right)$$

$$=\lambda\cdot\left(\prod_{e:v\notin\partial e}\frac{1}{\|x_{s(e)}-x_{t(e)}\|^{2\lambda}}\right)\cdot$$

$$\left(\sum_{e:v=s(e)}\frac{(x_{v,k}-x_{t(e),k})}{\|x_v-x_{t(e)}\|^D}-\sum_{e:v=t(e)}\frac{(x_{s(e),k}-x_{v,k})}{\|x_{s(e)}-x_v\|^D}\right),$$

where we used the fact that, for $z,w\in\mathbb{A}^D$, one has

$$\frac{\partial}{\partial\bar w_k}\frac{1}{\|z-w\|^{2\lambda}} = -\frac{\lambda\,(w_k-z_k)}{\|z-w\|^D}.$$

We also introduce the notation

$$\upsilon_v(x) = \prod_{e:v\notin\partial(e)}\frac{1}{\|x_{s(e)}-x_{t(e)}\|^{2\lambda}}, \tag{3.6}$$

so that we find

$$d\omega_\Gamma = \lambda(-1)^{\mathbf{V}_\Gamma}\sum_{v\in\mathbf{V}_\Gamma}\upsilon_v(x)\bigwedge_{w\neq v}dx_w\wedge\kappa^{\mathbb{R}}_{\Gamma,v}(x).$$

We then identify the term $\upsilon_v(x)\wedge_{w\neq v}dx_w$ with the form $\omega_{\Gamma\smallsetminus\{v\}}$. □

It is easy to see that this recovers (3.2) in the case of the graph consisting of two vertices and a single edge between them.

3.3. Complex Green functions and the Bochner–Martinelli kernel. On $\mathbb{A}^D(\mathbb{C}) \subset X$ the complex Laplacian

$$\Delta = \sum_{k=1}^{D} \frac{\partial^2}{\partial x_k \partial \bar{x}_k}.$$

has a fundamental solution of the form $(D > 1)$

$$G_{\mathbb{C}}(x,y) = \frac{-(D-2)!}{(2\pi i)^D \|x-y\|^{2D-2}}. \tag{3.7}$$

The Bochner–Martinelli kernel is given by

$$\mathcal{K}_{\mathbb{C}}(x,y) = \frac{(D-1)!}{(2\pi i)^D} \sum_{k=1}^{D} (-1)^{k-1} \frac{\bar{x}_k - \bar{y}_k}{\|x-y\|^{2D}} d\bar{x}_{[k]} \wedge dx, \tag{3.8}$$

where we write

$$dx = dx_1 \wedge \cdots \wedge dx_D \quad \text{ and } \quad d\bar{x}_{[k]} = d\bar{x}_1 \wedge \cdots \wedge \widehat{d\bar{x}_k} \wedge \cdots \wedge d\bar{x}_D,$$

with the k-th factor removed. The following facts are well known (see [24], §3.2 and [29]):

$$\begin{aligned} \mathcal{K}_{\mathbb{C}}(x,y) &= \sum_{k=1}^{D} (-1)^{k-1} \frac{\partial G_{\mathbb{C}}}{\partial x_k} d\bar{x}_{[k]} \wedge dx \\ &= (-1)^{D-1} \partial_x G_{\mathbb{C}} \wedge \sum_{k=1}^{D} d\bar{x}_{[k]} \wedge dx_{[k]}. \end{aligned} \tag{3.9}$$

where ∂_x and $\bar{\partial}_x$ denote the operators ∂ and $\bar{\partial}$ in the variables $x = (x_k)$. For fixed y, the coefficients of $\mathcal{K}_{\mathbb{C}}(x,y)$ are harmonic functions on $\mathbb{A}^D \smallsetminus \{y\}$ and $\mathcal{K}_{\mathbb{C}}(x,y)$ is closed, $d_x\mathcal{K}(x,y) = 0$. Moreover, the Bochner–Martinelli integral formula holds: for a bounded domain Σ with piecewise smooth boundary $\partial\Sigma$, a function $f \in \mathcal{C}^2(\overline{\Sigma})$ and for $y \in \Sigma$,

$$f(y) = \int_{\partial\Sigma} f(x)\mathcal{K}_{\mathbb{C}}(x,y) + \int_{\Sigma} \Delta(f)(x)\, G_{\mathbb{C}}(x,y) d\bar{x} \wedge dx - \int_{\partial\Sigma} G_{\mathbb{C}}(x,y)\, \mu_f(x), \tag{3.10}$$

where $\mu_f(x)$ is the form

$$\mu_f(x) = \sum_{k=1}^{D} (-1)^{D+k-1} \frac{\partial f}{\partial \bar{x}_k} dx_{[k]} \wedge d\bar{x}. \tag{3.11}$$

The integral (3.10) vanishes when $y \notin \Sigma$. A related Bochner–Martinelli integral, which can be derived from (3.10) (see Thm 1.3 of [29]) is of the form

$$f(y) = \int_{\partial\Sigma} f(x)\, \mathcal{K}_{\mathbb{C}}(x,y) - \int_{\Sigma} \bar{\partial} f \wedge \mathcal{K}_{\mathbb{C}}(x,y), \tag{3.12}$$

for $y \in \Sigma$ and $f \in \mathcal{C}^1(\overline{\Sigma})$, with $\bar{\partial} f = \sum_k \partial_{\bar{x}_k} f\, d\bar{x}_k$.

Similarly, one can consider Green forms associated to the Laplacians on $\Omega^{p,q}$ forms and related Bochner–Martinelli kernels, see [41].

3.4. **Feynman amplitude and the Bochner–Martinelli kernel.** We now consider the Feynman amplitude $\hat{\omega}_\Gamma$ of (2.12) in the complexified case discussed in §2.3.3. We first introduce a Bochner–Martinelli kernel for graphs.

3.4.1. *Bochner–Martinelli kernel for graphs.* We define Bochner–Martinelli kernels for graphs in the following way.

Definition 3.4. *Suppose given an oriented graph* Γ *and a vertex* $v \in \mathbf{V}_\Gamma$*. We set*

$$\kappa^{\mathbb{C}}_{\Gamma,v} = \sum_{e: v\in\partial(e)} \epsilon_e \sum_{k=1}^{D} (-1)^{k-1} \frac{(\bar{x}_{s(e),k} - \bar{x}_{t(e),k})}{\|x_{s(e)} - x_{t(e)}\|^{2D}} \, dx_v \wedge d\bar{x}_{v,[k]} \tag{3.13}$$

and

$$\kappa^{\mathbb{C},*}_{\Gamma,v} = \sum_{e: v\in\partial(e)} \epsilon_e \sum_{k=1}^{D} (-1)^{k-1} \frac{(x_{s(e),k} - x_{t(e),k})}{\|x_{s(e)} - x_{t(e)}\|^{2D}} \, dx_{v,[k]} \wedge d\bar{x}_v, \tag{3.14}$$

where the sign ϵ_e *is* ± 1 *depending on whether* $v = s(e)$ *or* $v = t(e)$*.*

3.4.2. *Bochner–Martinelli integral on graphs.* There is an analog of the classical Bochner–Martinelli integral (3.12) for the kernel (3.13) of graphs.

We first recall some well known facts about the Laplacian on graphs, see e.g. [7]. Given a graph Γ, one defines the exterior differential δ from functions on $\mathbf{V}_\Gamma$ to functions on $\mathbf{E}_\Gamma$ by

$$(\delta h)(e) = h(s(e)) - h(t(e))$$

and the δ^* operator from functions on edges to functions on vertices by

$$(\delta^*\xi)(v) = \sum_{e: v=s(e)} \xi(e) - \sum_{e: v=t(e)} \xi(e).$$

Thus, the Laplacian $\Delta_\Gamma = \delta^*\delta$ on Γ is given by

$$\begin{aligned}(\Delta_\Gamma f)(v) &= \sum_{e: v=s(e)} (h(v) - h(t(e))) - \sum_{e: v=t(e)} (h(s(e)) - h(v)) \\ &= N_v \, h(v) - \sum_{e: v\in\partial(e)} h(v_e),\end{aligned}$$

where N_v is the number of vertices connected to v by an edge, and v_e is the other endpoint of e (we assume as usual that Γ has no looping edges). Thus, a harmonic function h on a graph is a function on $\mathbf{V}_\Gamma$ satisfying

$$h(v) = \frac{1}{N_v} \sum_{e: v\in\partial(e)} h(v_e). \tag{3.15}$$

Motivated by the usual notion of graph Laplacian Δ_Γ and the harmonic condition (3.15) for graphs recalled here above, we introduce an operator

$$(\Delta_{\Gamma,v} f)(x) = \sum_{e: v\in\partial(e)} f(x_{v_e}), \tag{3.16}$$

which assigns to a complex valued function f defined on $\mathbb{A}^D \subset X$ a complex valued function $\Delta_{\Gamma,v} f$ defined on $X^{\mathbf{V}_\Gamma}$.

We then have the following result.

Proposition 3.5. *Let f be a complex valued function defined on $\mathbb{A}^D \subset X$. Also suppose given a bounded domain Σ with piecewise smooth boundary $\partial\Sigma$ in $\mathbb{A}^D$ and assume that f is $\mathcal{C}^1$ on $\overline{\Sigma}$. For a given $v \in \mathbf{V}_\Gamma$ consider the set of $x = (x_w) \in \mathbb{A}^{D|\mathbf{V}_\Gamma|} \subset X^{\mathbf{V}_\Gamma}$, such that $x_{v_e} \in \Sigma$ for all $v_e \neq v$ endpoints of edges e with $v \in \partial(e)$. For such $x = (x_w)$ we have*

$$(\Delta_{\Gamma,v} f)(x) = \frac{(D-1)!}{(2\pi i)^D\, N_v}\left(\int_{\partial\Sigma} f(x_v)\kappa^{\mathbb{C}}_{\Gamma,v}(x) - \int_\Sigma \bar\partial_{x_v} f(x_v)\wedge\kappa^{\mathbb{C}}_{\Gamma,v}(x)\right), \tag{3.17}$$

where the integration on Σ and $\partial\Sigma$ is in the variable x_v and $\Delta_{\Gamma,v} f$ is defined as in (3.16)*.*

Proof. We have

$$\int_{\partial\Sigma} f(x_v)\kappa^{\mathbb{C}}_{\Gamma,v}(x) = \sum_{e:v\in\partial(e)} \epsilon_e \int_{\partial\Sigma} f(x_v)\sum_{k=1}^D (-1)^{k-1}\frac{(\bar x_{s(e),k} - \bar x_{t(e),k})}{\|x_{s(e)} - x_{t(e)}\|^{2D}}\,\eta_v$$

where

$$\eta_v = dx_v \wedge d\bar x_{v,[k]}.$$

We write the integral as

$$\sum_{e:v=s(e)} \int_{\partial\Sigma} f(x_v)\sum_{k=1}^D (-1)^{k-1}\frac{(\bar x_{v,k} - \bar x_{v_e,k})}{\|x_v - x_{v_e}\|^{2D}}\,\eta_v$$

$$-\sum_{e:v=t(e)} \int_{\partial\Sigma} f(x_v)\sum_{k=1}^D (-1)^{k-1}\frac{(\bar x_{v_e,k} - \bar x_{v,k})}{\|x_v - x_{v_e}\|^{2D}}\,\eta_v$$

$$= \sum_{e:v\in\partial(e)} \int_{\partial\Sigma} f(x_v)\sum_{k=1}^D (-1)^{k-1}\frac{(\bar x_{v,k} - \bar x_{v_e,k})}{\|x_v - x_{v_e}\|^{2D}}\,\eta_v.$$

The case of the integral on Σ is analogous. We then apply the classical result (3.12) about the Bochner–Martinelli integral and we obtain

$$\int_{\partial\Sigma} f(x)\kappa^{\mathbb{C}}_{\Gamma,v}(x) - \int_\Sigma \bar\partial_{x_v} f(x)\wedge\kappa^{\mathbb{C}}_{\Gamma,v}(x) = \frac{(2\pi i)^D}{(D-1)!}\sum_{e:v\in\partial(e)} f(x_{v_e}).$$

□

3.4.3. *Feynman amplitude and Bochner–Martinelli kernel.* The Bochner–Martinelli kernel of graphs defined above is related to the Feynman amplitude (2.12) by the following.

Proposition 3.6. *Let* $\hat{\omega}_\Gamma$ *be the Feynman amplitude* (2.12)*. Then*

$$\partial\hat{\omega}_\Gamma = \sum_{v\in\mathbf{V}_\Gamma} \epsilon_v \ \hat{\omega}_{\Gamma\smallsetminus\{v\}} \wedge \kappa^{\mathbb{C}}_{\Gamma,v} \tag{3.18}$$

$$\bar{\partial}\hat{\omega}_\Gamma = \sum_{v\in\mathbf{V}_\Gamma} \epsilon_v \ \hat{\omega}_{\Gamma\smallsetminus\{v\}} \wedge (-1)^{D-1}\kappa^{\mathbb{C},*}_{\Gamma,v}, \tag{3.19}$$

where the sign ϵ_v *is defined by*

$$\epsilon_v \left(\bigwedge_{w\neq v} \sum_k (-1)^{k-1} dx_{w,[k]} \wedge d\bar{x}_{w,[k]} \right) \wedge \left(\sum_k (-1)^{k-1} dx_{v,[k]} \wedge d\bar{x}_{v,[k]} \right) =$$

$$\bigwedge_{v'\in\mathbf{V}_\Gamma} \sum_k (-1)^{k-1} dx_{v',[k]} \wedge d\bar{x}_{v',[k]}.$$

Proof. The argument is analogous to Proposition 3.3. □

4. Integration over the real locus

In this section we consider the Feynman amplitudes ω_Γ defined in (2.2) and the domain σ_Γ defined in (2.5). We give an explicit formulation of the integral in terms of an expansion of the real Green functions $\|x_{s(e)}-x_{t(e)}\|^{-2\lambda}$ in Gegenbauer polynomials, based on a technique well known to physicists (the x-space method, see [13]). We consider the integrand restricted to the real locus $X(\mathbb{R})^{\mathbf{V}_\Gamma}$, and express it in polar coordinates, separating out an angular integral and a radial integral. We identify a natural subdivision of the domain of integration into chains that are indexed by acyclic orientations of the graph. In the special case of dimension $D = 4$, we express the integrand in terms closely related to multiple polylogarithm functions.

Spencer Bloch recently introduced a construction of cycles in the relative homology $H_*(X^{\mathbf{V}_\Gamma}, \cup_e \Delta_e)$ of the graph configuration spaces, that explicitly yield multiple zeta values as periods [10]. We use here a variant of his construction, which will have a natural interpretation in terms of the x-space method for the computation of the Feynman amplitudes in configuration spaces.

4.0.4. *Directed acyclic graph structures.*

Definition 4.1. *Let Γ be a finite graph without looping edges. Let $\Omega(\Gamma)$ denote the set of edge orientations on Γ such that the resulting directed graph is a directed acyclic graph.*

It is well known that all finite graphs without looping edges admit such orientations. In fact, the number of possible orientations that give it the structure of a directed acyclic graph are given by $(-1)^{\mathbf{V}_\Gamma} P_\Gamma(-1)$, where $P_\Gamma(t)$ is the chromatic polynomial of the graph Γ, see [39].

The following facts about directed acyclic graphs are also well known, and we recall them here for later use.

- Each orientation $\mathbf{o} \in \Omega(\Gamma)$ determines a partial ordering on the vertices of the graph Γ, by setting $w \geq_{\mathbf{o}} v$ whenever there is an oriented path of edges from v to w in the directed graph $(\Gamma, \mathbf{o})$.
- In every directed acyclic graph there is at least a vertex with no incoming edges and at least a vertex with no outgoing edges.

4.0.5. *Relative cycles from directed acyclic structures.* Given a graph Γ we consider the space $X^{\mathbf{V}_\Gamma}$. On the dense subset $\mathbb{A}^D(\mathbb{R}) = X(\mathbb{R}) \smallsetminus \Delta_\infty(\mathbb{R})$ of the chain of integration σ_Γ, we use polar coordinates with $x_v = r_v \omega_v$, with $r_v \in \mathbb{R}_+$ and $\omega_v \in S^{D-1}$.

Definition 4.2. *Let $\mathbf{o} \in \Omega(\Gamma)$ be an acyclic orientation. Consider the chain*

$$\Sigma_{\mathbf{o}} := \{(x_v) \in X^{\mathbf{V}_\Gamma}(\mathbb{R}) \,:\, r_w \geq r_v \text{ whenever } w \geq_{\mathbf{o}} v\}, \tag{4.1}$$

with boundary $\partial \Sigma_{\mathbf{o}}$ contained in $\cup_{e \in \mathbf{E}_\Gamma} \Delta_e$. It defines a middle dimensional relative homology class

$$[\Sigma_{\mathbf{o}}] \in H_{|\mathbf{V}_\Gamma|}(X^{\mathbf{V}_\Gamma}, \cup_{e \in \mathbf{E}_\Gamma} \Delta_e).$$

The following simple observation will be useful in the Feynman integral calculation we describe later in this section.

Lemma 4.3. *Let $\mathbf{o} \in \Omega(\Gamma)$ be an acyclic orientation and $\Sigma_{\mathbf{o}}$ the chain defined in* (4.1)*. Then $\Sigma_{\mathbf{o}} \smallsetminus \cup_v \{r_v = 0\}$ is a bundle with fiber $(S^{D-1})^{\mathbf{V}_\Gamma}$ over a base*

$$\overline{\Sigma}_{\mathbf{o}} = \{(r_v) \in (\mathbb{R}_+^*)^{\mathbf{V}_\Gamma} \,:\, r_w \geq r_v \text{ whenever } w \geq_{\mathbf{o}} v\}. \tag{4.2}$$

Proof. This is immediate from the polar coordinate form $x_v = r_v \omega_v$, with $r_v \in \mathbb{R}_+^*$ and $\omega_v \in S^{D-1}$. ∎

4.1. **Gegenbauer polynomials and angular integrals.** One of the techniques developed by physicists to compute Feynman amplitudes, by passing from momentum to configuration space, relies on the expansion in Gegenbauer polynomials, see for instance [13] and the recent [34].

The Gegenbauer polynomials (or ultraspherical polynomials) are defined through the generating function

$$\frac{1}{(1-2tx+t^2)^\lambda} = \sum_{n=0}^\infty C_n^{(\lambda)}(x) t^n, \tag{4.3}$$

for $|t|<1$. For $\lambda > -1/2$, they satisfy

$$\int_{-1}^1 C_n^{(\lambda)}(x) C_m^{(\lambda)}(x)\,(1-x^2)^{\lambda-1/2} dx = \delta_{n,m} \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}. \tag{4.4}$$

We use what is known in the physics literature as the x-space method (see [13]) to reformulate the integration involved in the Feynman amplitude calculation in a way that involves the relative chains of Definition 4.2.

Theorem 4.4. *In even dimension $D = 2\lambda+2$, the integral $\int_{\sigma_\Gamma}\omega_\Gamma$ of the form* (2.2) *on the chain σ_Γ can be rewritten in the form*

$$\sum_{\mathbf{o}\in\Omega(\Gamma)} m_{\mathbf{o}} \int_{\Sigma_{\mathbf{o}}} \prod_{e\in \mathbf{E}_\Gamma} r_{t_{\mathbf{o}}(e)}^{-2\lambda} \left(\sum_n (\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}})^n C_n^{(\lambda)}(\omega_{s_{\mathbf{o}}(e)}\cdot \omega_{t_{\mathbf{o}}(e)}) \right) dV, \tag{4.5}$$

for some positive integers $m_{\mathbf{o}}$, and with volume element $dV = \prod_v d^D x_v = \prod_v r_v^{D-1} dr_v\, d\omega_v$.

Proof. We write the integral in polar coordinates, with

$$d\omega = \sin^{D-2}(\phi_1)\sin^{D-3}(\phi_2)\cdots \sin(\phi_{D-2}) d\phi_1\cdots d\phi_{D-1}$$

the volume element on the sphere S^{D-1} and $d^D x_v = r_v^{D-1} dr_v\, d\omega_v$.

In dimension $D=2\lambda+2$, by (4.3), the Newton potential has an expansion in Gegenbauer polynomials, so that

$$\begin{aligned} \frac{1}{\|x_{s(e)}-x_{t(e)}\|^{2\lambda}} &= \frac{1}{\rho_e^{2\lambda}(1+(\frac{r_e}{\rho_e})^2 - 2\frac{r_e}{\rho_e}\omega_{s(e)}\cdot\omega_{t(e)})^\lambda} \\ &= \rho_e^{-2\lambda}\sum_{n=0}^\infty (\frac{r_e}{\rho_e})^n C_n^{(\lambda)}(\omega_{s(e)}\cdot\omega_{t(e)}), \end{aligned} \tag{4.6}$$

where $\rho_e = \max\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and $r_e = \min\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and with $\omega_v \in S^{D-1}$.

We can subdivide the integration into open sectors where, for each edge, either $r_{s(e)} < r_{t(e)}$ or the converse holds, so that each term $\rho_e^{-2\lambda}$ is $r_{t(e)}^{-2\lambda}$ (or $r_{s(e)}^{-2\lambda}$) and each term $(r_e/\rho_e)^n$ is $(r_{s(e)}/r_{t(e)})^n$ (or its reciprocal). In other words, let $\mathbf{b}$ denote an assignment of either $r_{s(e)} < r_{t(e)}$ or $r_{s(e)} > r_{t(e)}$ at each edge, which we write simply as $\mathbf{b}(r_{s(e)}, r_{t(e)})$, and let

$$\bar{\mathcal{R}}_{\mathbf{b}} = \{(r_v) \in (\mathbb{R}_+^*)^{\mathbf{V}_\Gamma} \,|\, \mathbf{b}(r_{s(e)}, r_{t(e)}) \text{ for } e \in \mathbf{E}_\Gamma\}$$

and $\mathcal{R}_{\mathbf{b}} = \bar{\mathcal{R}}_{\mathbf{b}} \times (S^{D-1})^{\mathbf{V}_\Gamma}$. Then we identify the domain of integration with $\cup_{\mathbf{b}} \mathcal{R}_{\mathbf{b}}$, up to a set of measure zero. The set $\mathcal{R}_{\mathbf{b}}$ is empty unless the assignment $\mathbf{b}$ defines a strict partial ordering of the vertices of Γ, in which case $\mathbf{b}$ determines an acyclic orientation $\mathbf{o} = \mathbf{o}(\mathbf{b})$ of Γ, as described in Definition 4.1. In this case, then, the chain of integration corresponding to the sector $\mathcal{R}_{\mathbf{b}}$ is the chain $\Sigma_{\mathbf{o}}$ of Definition 4.2, with $\rho_e = r_{t_{\mathbf{o}}(e)}$ and $r_e = r_{s_{\mathbf{o}}(e)}$. Thus, the domain of integration can be identified with $\cup_{\mathbf{o}\in\Omega(\Gamma)} m_{\mathbf{o}}\, \Sigma_{\mathbf{o}}$, where $m_{\mathbf{o}}$ is a multiplicity, taking into account the fact that different strict partial orderings may define the same acyclic orientation. ☐

The integral (4.5) can be approached by first considering a family of angular integrals

$$\mathcal{A}_{(n_e)_{e\in\mathbf{E}_\Gamma}} = \int_{(S^{D-1})^{\mathbf{V}_\Gamma}} \prod_e C_{n_e}^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}) \prod_v d\omega_v, \tag{4.7}$$

labelled by all choices of integers n_e for $e \in \mathbf{E}_\Gamma$. The evaluation of these angular integrals will lead to an expression $\mathcal{A}_{n_e}$ in the n_e, so that one obtains a radial integral

$$\sum_{\mathbf{o}\in\Omega(\Gamma)} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \prod_{e\in\mathbf{E}_\Gamma} \mathcal{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) \prod_v r_v^{D-1} dr_v \tag{4.8}$$

where

$$\mathcal{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) = r_{t_{\mathbf{o}}(e)}^{-2\lambda} \sum_{n_e} \mathcal{A}_{n_e} (\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}})^{n_e}. \tag{4.9}$$

4.2. **Polygons and polylogarithms.** We first discuss the very simple example of a polygon graph, where one sees polylogarithms and zeta values arising in the expression (4.9) and its integration on the domains $\bar{\Sigma}_{\mathbf{o}}$. In the following subsections we will analyze the more general structure of these integrals for more complicated graphs.

4.2.1. *The angular integral for polygons in arbitrary dimension.* The angular integral for polygon graphs has the following explicit expression.

Proposition 4.5. *Let Γ be a polygon with k edges. Then the angular integral (4.7) depends on a single variable $n \in \mathbb{N}$ and is given by*

$$\mathcal{A}_n = \left(\frac{\lambda 2\pi^{\lambda+1}}{\Gamma(\lambda+1)(n+\lambda)}\right)^k \cdot \dim \mathcal{H}_n(S^{2\lambda+1}), \tag{4.10}$$

and $\mathcal{H}_n(S^{2\lambda+1})$ is the space of harmonic functions of degree n on the sphere $S^{2\lambda+1}$.

Proof. The angular integral, in this case, is simply given by

$$\int_{(S^{D-1})^{\mathbf{V}_\Gamma}} C_{n_1}^{(\lambda)}(\omega_{v_k}\cdot\omega_{v_1}) C_{n_2}^{(\lambda)}(\omega_{v_1}\cdot\omega_{v_2})\cdots C_{n_k}^{(\lambda)}(\omega_{v_{k-1}}\cdot\omega_{v_k}) \prod_{v\in\mathbf{V}_\Gamma} d\omega_v,$$

which is independent of the orientation. We then use the fact that the Gegenbauer polynomials satisfy ([6] Vol.2, Lemma 4, §11.4)

$$\int_{S^{D-1}} C_m^{(\lambda)}(\omega_1\cdot\omega) C_n^{(\lambda)}(\omega\cdot\omega_2) d\omega = \delta_{n,m} \frac{\lambda\, Vol(S^{D-1})}{n+\lambda} C_n^{(\lambda)}(\omega_1\cdot\omega_2), \tag{4.11}$$

with $Vol(S^{D-1}) = 2\pi^{\lambda+1}/\Gamma(\lambda+1)$. Thus, we obtain

$$\mathcal{A}_n = \left(\frac{\lambda 2\pi^{\lambda+1}}{\Gamma(\lambda+1)}\right)^{k-1} \frac{1}{(n+\lambda)^{k-1}} \int_{S^{D-1}} C_n^{(\lambda)}(\omega\cdot\omega)\, d\omega, \tag{4.12}$$

where $n = n_1 = \cdots = n_k$ and where the remaining integral is just

$$\int_{S^{D-1}} C_n^{(\lambda)}(\omega\cdot\omega) d\omega = C_n^{(\lambda)}(1)\, Vol(S^{D-1}).$$

The value of $C_n^{(\lambda)}(1)$ can be seen using the fact that the Gegenbauer polynomials are related to the *zonal spherical harmonics* (see [40], §4, and [21], [33], [44]) $Z_{\omega_1}^{(n)}(\omega_2)$ by

$$C_n^{(\lambda)}(\omega_1\cdot\omega_2) = c_{D,n}\, Z_{\omega_1}^{(n)}(\omega_2), \tag{4.13}$$

for $D = 2\lambda+2$, with $\omega_1, \omega_2 \in S^{D-1}$, where the coefficient $c_{D,n}$ is given by

$$c_{D,n} = \frac{Vol(S^{D-1})\,(D-2)}{2n+D-2}. \tag{4.14}$$

In turn, the zonal spherical harmonics are expressed in terms of an orthonormal basis $\{Y_j\}$ of the Hilbert space $\mathcal{H}_n(S^{D-1})$ of spherical harmonics on S^{D-1} of degree n, as

$$Z_{\omega_1}^{(n)}(\omega_2) = \sum_{j=1}^{\dim \mathcal{H}_n(S^{D-1})} Y_j(\omega_1)\overline{Y_j(\omega_2)}. \tag{4.15}$$

The dimension of the space $\mathcal{H}_n(S^{D-1})$ of spherical harmonics is given by

$$\dim \mathcal{H}_n(S^{D-1}) = \binom{D-1+n}{n} - \binom{D-3+n}{n-2}.$$

Using (4.15), we then have

$$\int_{S^{D-1}} C_n^{(\lambda)}(\omega\cdot\omega) d\omega = c_{D,n} \int_{S^{D-1}} Z_\omega^{(n)}(\omega) d\omega$$

$$= c_{D,n} \sum_{j=1}^{\dim \mathcal{H}_n(S^{D-1})} \int_{S^{D-1}} |Y_j(\omega)|^2 d\omega = c_{D,n} \dim \mathcal{H}_n(S^{D-1}),$$

which gives $C_n^{(\lambda)}(1) = \frac{2\lambda \dim \mathcal{H}_n(S^{D-1})}{2(n+\lambda)}$. □

4.2.2. *Polygon amplitudes in dimension four.* We now specialize to the case where $D = 4$ and $\lambda = 1$ and we show how one obtains integrals of polylogarithm functions

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

Proposition 4.6. *Let Γ be a polygon with k edges and let $D = 4$. Then the Feynman amplitude is given by the integral*

$$(2\pi^2)^k \sum_{\mathbf{o}} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \mathrm{Li}_{k-2}(\prod_i \frac{r^2_{w_i}}{r^2_{v_i}}) \prod_v r_v \, dr_v, \tag{4.16}$$

where the vertices v_i and w_i are the sources and tails of the oriented paths determined by $\mathbf{o}$.

Proof. We write the terms in the integrand (4.5) as

$$\prod_{e\in\mathbf{E}_\Gamma} \rho_e^{-2\lambda} \left(\sum_n (\frac{r_e}{\rho_e})^n \; C_n^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}) \right) = \tag{4.17}$$

$$(\rho_1 \cdots \rho_k)^{-2\lambda} \sum_{n_1,\ldots,n_k} (\frac{r_1}{\rho_1})^{n_1} \cdots (\frac{r_k}{\rho_N})^{n_k} \; C_{n_1}^{(\lambda)}(\omega_{s(e_1)}\cdot\omega_{t(e_1)}) \cdots C_{n_k}^{(\lambda)}(\omega_{s(e_k)}\cdot\omega_{t(e_k)})$$

and we perform the angular integral as in (4.10). In the case $D = 4$ we have

$$\dim \mathcal{H}_n(S^3) = \binom{n+3}{n} - \binom{n+1}{n-2} = (n+1)^2.$$

Thus, the angular integral of Proposition 4.5 becomes

$$\mathcal{A}_n = \frac{(2\pi^2)^k}{(n+1)^{k-2}}. \tag{4.18}$$

We then write the radial integrand as in (4.8). An acyclic orientation $\mathbf{o} \in \Omega(\Gamma)$, subdivides the polygon Γ into oriented paths γ_i such that $s_{\gamma_i} = s_{\gamma_{i-1}}$ and $t_{\gamma_i} = t_{\gamma_{i+1}}$ or $s_{\gamma_i} = s_{\gamma_{i+1}}$ and $t_{\gamma_i} = t_{\gamma_{i-1}}$. Correspondingly, the set of vertices is subdivided into $\mathbf{V}_\Gamma = \{v_i\} \cup \{w_i\} \cup \{v \notin \{v_i, w_i\}\}$ with v_i the sources and w_i the tails of the oriented paths and the remaining vertices partitioned into internal vertices of each oriented path. We then have

$$\mathcal{F}(r_{s(e)}, r_{t(e)}) = (2\pi^2)^k (\prod_e r^{-2}_{t(e)}) \sum_{n\geq 0} \frac{1}{(n+1)^{k-2}} (\prod_i \frac{r^2_{v_i}}{r^2_{w_i}})^n$$

$$= (2\pi^2)^k (\prod_e r^{-2}_{t(e)}) (\prod_i \frac{r^2_{w_i}}{r^2_{v_i}}) \sum_{n\geq 1} \frac{1}{n^{k-2}} (\prod_i \frac{r^2_{v_i}}{r^2_{w_i}})^n.$$

Since each w_i is counted twice as target of an edge and the internal vertices of the oriented paths are counted only once, we obtain

$$\prod_e \mathcal{F}(r_{s(e)}, r_{t(e)}) \prod_v r_v^3 dv = (2\pi^2)^k \cdot \mathrm{Li}_{k-2}(\prod_i \frac{r^2_{w_i}}{r^2_{v_i}}) \prod_v r_v dr_v. \tag{4.19}$$

□

4.2.3. *Zeta values.* After a cutoff regularization, these integrals produce combinations of zeta values with coefficients that are rationals combinations of powers of $2\pi i$. To see how this happens, we look explicitly at the contribution of an acyclic orientations of the polygon consisting of just two oriented paths γ_1 and γ_2 with source v and target w, respectively with k_1 and k_2 internal vertices. The other summands can be handled similarly. By changing variables to $t = r_v^2/r_w^2$, $t_i = r_{v_i}^2/r_w^2$ for v_i the internal edges of γ_1 and $s_i = r_{v_i}^2/r_w^2$ for v_i the internal edges of γ_2, we obtain

$$\bigwedge_{v\in \mathbf{V}_\gamma} r_v\, dr_v = \pm 2^{1-k} r_w^{2k+1} dr_w\, dt \bigwedge_i dt_i \wedge ds_i.$$

After factoring out a divergence along $\Delta_\infty = X \smallsetminus \mathbb{A}^4$, coming from the integration of the r_w term, which gives a pole along the divisor Δ_∞, one obtains an integral of the form

$$2\pi^{2k} \int_{\bar\Sigma_1\cap\bar\Sigma_2} \mathrm{Li}_{k-2}(t)\; dt \prod_i dt_i ds_i,$$

where $\bar\Sigma_{\mathbf{o}} = \bar\Sigma_1 \cap \bar\Sigma_2$ with $\bar\Sigma_1 = \{(t, t_i, s_i)\,|\, t \leq t_1 \leq \cdots \leq t_{k_1-1} \leq 1\}$ and $\bar\Sigma_2 = \{(t, t_i, s_i)\,|\, t \leq s_1 \leq \cdots \leq s_{k_2-1} \leq 1\}$. One can use the relation [17]

$$\int x^m \mathrm{Li}_n(x)\, dx = \frac{1}{m+1} x^{m+1} \mathrm{Li}_n(x) - \frac{1}{m+1}\int x^m \mathrm{Li}_{n-1}(x) dx, \tag{4.20}$$

to reduce the integral to a combination of zeta values.

4.3. **Stars of vertices and isoscalars.** To see how more complicated expressions can arise in the integrands, which eventually lead to the presence of multiple zeta values, it is convenient to regard graphs as being built out of stars of vertices pasted together by suitably matching the half edges, where by the *star of a vertex* we mean a single vertex v of valence k with k half-edges e_j attached to it. One can then built the Feynman integral by first identifying the contribution of the star of a vertex, which is obtained by integrating in the variables r_v and ω_v of the central vertex v, and results in a function of variables r_{v_j} and ω_{v_j}, for each of the edges e_j. One then obtains the integral for the graph, which gives a number (possibly after a regularization), by matching the half edges and identifying the corresponding variables and integrating over them.

We first introduce the analog of the angular integral (4.7) for the case of a graph with half-edges. While in the case of a usual graph, where all half-edges are paired to form edges, the angular integral (4.7) is a number, in the case with open (unpaired) half-edges it is a function of the variables of the half edges, which we denote by $\mathcal{A}_{\underline{n}}(\underline{\omega})$, where $\underline{n} = (n_1, \ldots, n_\ell)$ and $\underline{\omega} = (\omega_1, \ldots, \omega_\ell)$ are vectors of integers $n_j \in \mathbb{N}$, for each half-edge e_j, and of variables $\omega_j \in S^{D-1}$. We will sometime denote these variables by ω_{v_j}, where v_j simply denotes the end of the half-edge e_j. We will equivalently use the notation $\mathcal{A}_{\underline{n}}(\underline{\omega})$ or $\mathcal{A}_{(n_j)}(\omega_{v_j})$. In the case of the star of a vertex, the angular integral is of the form

(4.21)
$$\mathcal{A}_{\underline{n}}(\underline{\omega}) = \int_{S^{D-1}} \prod_j C^{(\lambda)}_{n_j}(\omega_j \cdot \omega)\, d\omega, \text{ with } \underline{n} = (n_j)_{e_j \in \mathbf{E}_\Gamma},\ \underline{\omega} = (\omega_j)_{e_j \in \mathbf{E}_\Gamma}.$$

Lemma 4.7. *Let* Γ *be the star of a valence* v *vertex. Then the angular integral* (4.21) *is given by the function*

$$\mathcal{A}_{(n_j)}(\omega_{v_j}) = c_{D,n_1} \cdots c_{D,n_k}\ \tilde{\mathcal{A}}_{(n_j)}(\omega_{v_j}),$$

(4.22)
$$\tilde{\mathcal{A}}_{(n_j)}(\omega_{v_j}) = \sum_{\ell_1,\ldots,\ell_k} \overline{Y^{(n_1)}_{\ell_1}(\omega_1) \cdots Y^{(n_k)}_{\ell_k}(\omega_k)} \int_{S^{D-1}} Y^{(n_1)}_{\ell_1}(\omega) \cdots Y^{(n_k)}_{\ell_k}(\omega)\, d\omega,$$

where $\{Y^{(n)}_\ell\}_{\ell=1,\ldots,d_n}$ *is an orthonormal basis of the space* $\mathcal{H}_n(S^{D-1})$ *of spherical harmonics of degree* n*, and* $d_n = \dim \mathcal{H}_n(S^{D-1})$*, with the coefficients* $c_{D,n}$ *as in* (4.14)*.*

Proof. Using the relation (4.13), (4.15) between the Gegenbauer polynomials and the spherical harmonics, we rewrite the angular integral (4.21) in the form (4.22). □

Thus, the evaluation of the angular integrals (4.22) for stars of vertices relates to the well known problem of evaluating coupling coefficients for spherical harmonics,

$$\langle Y^{(n_1)}_{\ell_1}, \ldots, Y^{(n_k)}_{\ell_k} \rangle_D := \int_{S^{D-1}} Y^{(n_1)}_{\ell_1}(\omega) \cdots Y^{(n_k)}_{\ell_k}(\omega)\, d\omega. \tag{4.23}$$

In the following we will be using the standard labeling of the basis $\{Y^{(n)}_\ell\}$ where, for fixed n, the indices ℓ run over a set of $(D-2)$-tuples

$$(m_{D-2}, m_{D-1}, \ldots, m_2, m_1) \quad \text{with} \quad n \geq m_{D-2} \geq \cdots \geq m_2 \geq |m_1|.$$

The spherical harmonics $Y^{(n)}_\ell$ have the symmetry

$$\overline{Y^{(n)}_\ell} = (-1)^{m_1} Y^{(n)}_{\bar{\ell}}, \tag{4.24}$$

where, for $\ell = (m_{D-2}, m_{D-1}, \ldots, m_2, m_1)$, one has

$$\bar{\ell} := (m_{D-2}, m_{D-1}, \ldots, m_2, -m_1).$$

In the simplest case of a tri-valent vertex, these coefficients are also referred to as the Gaunt coefficients, and have been extensively studied, see for instance [4], [28], [36]. The Gaunt coefficients arising from the integration of three harmonic functions determine the coefficients of the expansion formula

$$Y^{(n_1)}_{\ell_1} Y^{(n_2)}_{\ell_2} = \sum_{n,\ell} \mathcal{K}_{D,n_i,n,\ell_i,\ell}\ Y^{(n)}_\ell \tag{4.25}$$

that expresses the product of two harmonic functions in terms of a linear combination of other harmonic functions, with the cases where some of the factors are conjugated taken care of by the symmetry (4.24). In the more general case (4.23) one can therefore repeatedly apply (4.25), hence we focus here on the example of the star of a tri-valent vertex.

The Gaunt coefficients $\langle Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2} Y^{(n_3)}_{\ell_3}\rangle_D$ can be computed via Racah's factorization lemma ([4], [28]) in terms of *isoscalar factors* and the Gaunt coefficients for $D-1$, according to

(4.26)
$$\langle Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2}, Y^{(n_3)}_{\ell_3}\rangle_D = \begin{pmatrix} n_1 & n_2 & n_3 \\ n_1' & n_2' & n_3' \end{pmatrix}_{D:D-1} \langle Y^{(n_1')}_{\ell_1'}, Y^{(n_2')}_{\ell_2'}, Y^{(n_3')}_{\ell_3'}\rangle_{D-1},$$

where $\ell_i = (n_i', \ell_i')$ with $n_i' = m_{D-2,i}$ and $\ell_i' = (m_{D-3,i}, \dots, m_{1,i})$. An explicit expression of the isoscalar factors

$$\begin{pmatrix} n_1 & n_2 & n_3 \\ n_1' & n_2' & n_3' \end{pmatrix}_{D:D-1} \tag{4.27}$$

is given in [4], [28]. We will discuss this more in detail in §4.4 and §4.5 below.

4.4. **Gluing two stars along an edge.** We now consider the effect of patching together two trivalent stars by gluing two half edges with matching orientations.

Lemma 4.8. *Let $\mathcal{A}_{n,n_1,n_2}(\omega,\omega_1,\omega_2)$ and $\mathcal{A}_{n',n_3,n_4}(\omega',\omega_3,\omega_4)$ be the angular integrals associated to two trivalent stars, as in Lemma 4.7. Then the angular integral of the graph obtained by joining the two stars at an edge is*

$$\mathcal{A}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4}) = \sum_{\ell_i} \prod_{i=1}^{4} c_{D,n_i}\, \overline{Y^{(n_i)}_{\ell_i}(\omega_i)}\, \mathcal{K}_{n_i,\ell_i}(n), \tag{4.28}$$

with coefficients $\mathcal{K}_{\underline{n},\underline{\ell}}(n)$ given by

$$\mathcal{K}_{n_i,\ell_i}(n) = c^2_{D,n} \sum_{\ell=1}^{d_n} \langle Y^{(n)}_{\ell}, Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2}\rangle_D \cdot \langle Y^{(n)}_{\ell}, Y^{(n_3)}_{\ell_3}, Y^{(n_4)}_{\ell_4}\rangle_D. \tag{4.29}$$

Proof. The angular integral $\mathcal{A}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4})$ is obtained by integrating along the variables of the matched half edges,

$$\mathcal{A}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4}) = c_{D,n} c_{D,n'} (\prod_{i=1}^{4} c_{D,n_i}) \cdot \tilde{\mathcal{A}}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4}),$$

where $\tilde{\mathcal{A}}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4})$ is given by

$$\int_{S^{D-1}} d\omega \sum_{\ell,\ell',\ell_i} \overline{Y^{(n)}_{\ell}(\omega) Y^{(n')}_{\ell'}(\omega)} \prod_i \overline{Y^{(n_i)}_{\ell_i}(\omega_i)} \begin{pmatrix} n & n_1 & n_2 \\ \ell & \ell_1 & \ell_2 \end{pmatrix}_D \begin{pmatrix} n' & n_3 & n_4 \\ \ell' & \ell_3 & \ell_4 \end{pmatrix}_D$$

where we used the shorthand notation

$$\left(\begin{array}{ccc} n & n_1 & n_2 \\ \ell & \ell_1 & \ell_2 \end{array} \right)_D := \langle Y^{(n)}_\ell, Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2} \rangle_D. \tag{4.30}$$

Using the orthogonality relations for the spherical harmonics, this gives

$$(\prod_{i=1}^4 c_{D,n_i}) \sum_{\ell_1,\ell_2,\ell_3,\ell_4} \overline{Y^{(n_1)}_{\ell_1}(\omega_1) Y^{(n_2)}_{\ell_2}(\omega_2) Y^{(n_3)}_{\ell_3}(\omega_3) Y^{(n_4)}_{\ell_4}(\omega_4)} \; \mathcal{K}_{\underline{n},\underline{\ell}}(n),$$

with the coefficients as in (4.29). ∎

The coefficients $\mathcal{K}_{\underline{n},\underline{\ell}}(n)$ are usually very involved to compute explicitly (see (3.3), (3.6) and (4.7) of [4]). However, some terms simplify greatly in the case $D = 4$, and that will allow us to show the occurrence of functions closely related to multiple polylogarithm functions in §4.5 below. For later use, we give here the explicit computation, in dimension $D = 4$, of the coefficients $\mathcal{K}_{\underline{n},\underline{\ell}}(n)$, in the particular case with $\underline{\ell} = 0$.

Proposition 4.9. *In the case where $D = 4$, the coefficient $\mathcal{K}_{\underline{n},\underline{\ell}}(n)$ with $\ell_i = 0$ has the form*

$$\mathcal{K}^{(D=4)}_{\underline{n},\underline{0}}(n) = (\prod_{i=1}^4 \frac{1}{(n_i+1)^{1/2}}) \frac{4\pi^4}{(n+1)^3}, \tag{4.31}$$

in the range where $n+n_1+n_2$ and $n+n_3+n_4$ are even and the inequalities $|n_j - n_k| \leq n_i \leq n_j + n_k$ hold for (n_i, n_j, n_k) equal to (n, n_1, n_2) or (n, n_3, n_4) and transpositions, and are equal to zero outside of this range.

Proof. We use the fact that ([4], (4.9) and [28], (22) and (23)) the coefficients $\langle Y^{(n_1)}_0, Y^{(n_2)}_0, Y^{(n_3)}_0 \rangle_D$ are zero outside the range where

$$\sum_i n_i \quad \text{is even and} \quad |n_j - n_k| \leq n_i \leq n_j + n_k, \tag{4.32}$$

while within this range they are given by the expression

$$\epsilon_D \frac{1}{\Gamma(D/2)} \left(\frac{(J+D-3)!}{(D-3)!\,\Gamma(J+D/2)} \prod_i \frac{(n_i + \frac{D}{2} - 1)\,\Gamma(J - n_i + \frac{D}{2} - 1)}{d^{(D)}_{n_i}\,(J-n_i)!} \right)^{1/2}, \tag{4.33}$$

where ϵ_D is a sign, $J = \frac{1}{2}\sum_i n_i$, and $d^{(D)}_{n_i} = \dim \mathcal{H}_{n_i}(S^{D-1})$. In the particular case where $D = 4$, the expression (4.33) reduces to

$$\left(\begin{array}{ccc} n_1 & n_2 & n_3 \\ 0 & 0 & 0 \end{array} \right)_4 = \epsilon_4 \prod_i \frac{(n_i+1)^{1/2}}{(d^{(4)}_{n_i})^{1/2}} = \epsilon_4 \prod_i (n_i+1)^{-1/2},$$

using again the fact that $\dim \mathcal{H}_n(S^3) = (n+1)^2$. Thus, we obtain

$$\mathcal{K}^{(D=4)}_{n_i,\ell_i=0}(n) = c^2_{4,n} \left(\begin{array}{ccc} n & n_1 & n_2 \\ 0 & 0 & 0 \end{array} \right)_4 \left(\begin{array}{ccc} n & n_3 & n_4 \\ 0 & 0 & 0 \end{array} \right)_4 =$$

$$= \left(\frac{2\,Vol(S^3)}{2(n+1)}\right)^2 \frac{1}{(n+1)} \prod_{i=1}^{4} \frac{1}{(n_i+1)^{1/2}} = \prod_{i=1}^{4} (n_i+1)^{-1/2} \frac{4\pi^4}{(n+1)^3}.$$

□

4.5. **Gluing stars of vertices.** We now consider the full integrand, including the radial variables and again look at the effect of gluing together two half edges of two trivalent stars. We will see that one can explicitly identify the leading term in the resulting expression in the integrand with a function closely related to multiple polylogarithms.

Lemma 4.10. *Consider the star of a trivalent vertex, and let $D = 4$. After a change of variables $t_i = r_{v_i}/r$, with $r = r_v$ for v the central vertex of the star, the integrand (4.8), for an orientation $\mathbf{o}$, can be written as an expression $\mathcal{I}_{\mathbf{o}}(r, t_1, t_2, t_3, \omega_1, \omega_2, \omega_3)\, dr dt_1 dt_2 dt_3$ of the form*

$$r^9 \prod_{i=1}^{3} t_i^{\alpha_i} \sum_{n_1,n_2,n_3} \mathcal{A}_{(n_1,n_2,n_3)}(\omega_1, \omega_2, \omega_3) t_1^{\epsilon_1 n_1} t_2^{\epsilon_2 n_2} t_3^{\epsilon_3 n_3}\, dr \prod_{i=1}^{3} dt_i, \tag{4.34}$$

where $\alpha_i = 1$ and $\epsilon_i = 1$ if the half-edge e_i is outgoing in the orientation $\mathbf{o}$ and $\alpha_i = 3$ and $\epsilon_i = -1$ if it is incoming, and where $\mathcal{A}_{\underline{n}}(\underline{\omega}) = \mathcal{A}_{(n_j)}(\omega_j)$ is the angular integral of Lemma 4.7 with $D = 4$.

Proof. The integrand of (4.8), for the case of a trivalent star, is of the form

$$\prod_{i=1}^{3} \mathcal{F}(r_{s(e_i)}, r_{t(e_i)}) r^3 dr \prod_{i=1}^{3} r_{v_i}^3 dr_{v_i}, \tag{4.35}$$

with

$$\prod_{i=1}^{3} \mathcal{F}(r_{s(e_i)}, r_{t(e_i)}) = (\prod_{i=1}^{3} r_{t(e_i)}^{-2}) \sum_{\underline{n}} \mathcal{A}_{\underline{n}}(\underline{\omega}) \left(\frac{r_{s(e_1)}}{r_{t(e_1)}}\right)^{n_1} \left(\frac{r_{s(e_2)}}{r_{t(e_2)}}\right)^{n_2} \left(\frac{r_{s(e_3)}}{r_{t(e_3)}}\right)^{n_3}.$$

When combined with the volume form as in (4.35), this can be rewritten as

$$r^{\alpha_0} r_1^{\alpha_1} r_2^{\alpha_2} r_3^{\alpha_3} \sum_{\underline{n}} \mathcal{A}_{\underline{n}}(\underline{\omega}) \left(\frac{r_1}{r}\right)^{\epsilon_1 n_1} \left(\frac{r_2}{r}\right)^{\epsilon_2 n_2} \left(\frac{r_3}{r}\right)^{\epsilon_3 n_3} dr\, dr_1 dr_2 dr_3, \tag{4.36}$$

where the exponents α_i are given by the table

	$\mathbf{o}_0$	$\mathbf{o}_1$	$\mathbf{o}_2$	$\mathbf{o}_3$
α_0	-3	3	-1	1
α_1	1	3	1	1
α_2	1	3	3	1
α_3	1	3	3	3

where the orientation $\mathbf{o}_0$ has all the half-edges of the star pointing outward, $\mathbf{o}_1$ all pointing inward, $\mathbf{o}_2$ has e_1 outward and e_2 e_3 inward and $\mathbf{o}_3$ has e_1 and e_2 outward and e_3 inward. All the other cases are obtained by relabeling of indices. After we change variables to $t_i = r_{v_i}/r$, we obtain $dr \wedge \wedge_{i=1}^3 dr_i = r^3 dr \wedge \wedge_{i=1}^3 dt_i$ and (4.36) becomes

(4.37)
$$\mathcal{I}_{\mathbf{o}}(r,(t_i),\underline{\omega}) = r^9 t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \sum_{\underline{n}} \mathcal{A}_{\underline{n}}(\underline{\omega})\ (t_1)^{\epsilon_1 n_1}\ (t_2)^{\epsilon_2 n_2}\ (t_3)^{\epsilon_3 n_3}\ dr\, dt_1 dt_2 dt_3.$$

□

We can now perform the gluing of two stars by matching an oriented half-edge of one trivalent star to an oriented half edge of the other, so that one obtains an oriented edge. This means integrating

(4.38)
$$\int_0^\infty \int_{\bar\Sigma} \int_{S^{D-1}} \mathcal{I}_{\mathbf{o}}(r,t,t_1,t_2,\omega,\omega_1,\omega_2)\mathcal{I}_{\mathbf{o}}(r,t,t_3,t_4,\omega,\omega_3,\omega_4)\, dr\, dt\, d\omega.$$

There is an overall divergent factor arising from the integration of (4.38) in the variable r, which can be taken care of by a cutoff regularization. Up to this divergence, one obtains an integrand $\mathcal{I}(t_1,t_2,t_3,t_4,\omega_1,\omega_2,\omega_3,\omega_4)$, as the result of gluing two trivalent stars by matching oriented half-edges to form an oriented edge e, which is given by

$$(4.39)\quad \mathcal{I}_{\mathbf{o}}(t_i,\omega_i) = \int_{\bar\Sigma} \int_{S^{D-1}} \mathcal{I}_{\mathbf{o}}(t,t_1,t_2,\omega,\omega_1,\omega_2)\mathcal{I}_{\mathbf{o}}(t,t_3,t_4,\omega,\omega_3,\omega_4)\, dt d\omega,$$

where the domain of integration $\bar\Sigma = \bar\Sigma(t_1,t_2,t_3,t_4)$ for the variable t is given by

$$\bar\Sigma = \cap_{i,j:t(e_i)=s(e),s(e_j)=t(e)} \{t \,|\, t_i \le t \le t_j\}.$$

In the following we write $\underline{t} = (t_1,t_2,t_3,t_4)$ and similarly for $\underline{\omega}$, $\underline{n}$ and $\underline{\ell}$.

By combining (4.28) with (4.22), we can rephrase (4.39) in terms of isoscalars. This gives a decomposition of $\mathcal{I}_{\mathbf{o}}(t_i,\omega_i)$ into a sum of terms of the form

$$\mathcal{I}_{\mathbf{o}}(t_i,\omega_i) = \sum_{\underline{n},\underline{\ell}} \mathcal{I}_{\mathbf{o},\underline{\mathbf{n}},\underline{\ell}}(t_i,\omega_i).$$

We denote by $\mathcal{I}_{\mathbf{o},0}(\underline{t},\underline{\omega})$ the leading term

$$(4.40)\qquad \mathcal{I}_{\mathbf{o},\mathbf{0}}(t_i,\omega_i) = \sum_{\underline{n}} \mathcal{I}_{\mathbf{o},\underline{\mathbf{n}},\underline{\mathbf{0}}}(t_i,\omega_i),$$

involving only the isoscalars with all $\ell_i = 0$. We have the following result computing the terms $\mathcal{I}_{\mathbf{o},0}(t_i,\omega_i)$.

Lemma 4.11. *In the case $D = 4$, the integrands $\mathcal{I}_{\mathbf{o},0}(\underline{t},\underline{\omega})$ are explicitly given by*

(4.41)
$$\mathcal{I}_{\mathbf{o},0}(\underline{t},\underline{\omega}) = \sum_{\underline{n}} (\prod_{i=1}^4 c_{D,n_i} \overline{Y_0^{(n_i)}(\omega_i)} \frac{t_i^{\alpha_i+\epsilon_i n_i}\, dt_i}{(n_i+1)^{1/2}}) \int_{\bar\Sigma} t^4\, dt\ \sum_n \frac{4\pi^2}{(n+1)^3} t^{\epsilon n},$$

where the sum over the indices n and $\underline{n}$ is restricted by the constraints $n + n_1 + n_2$ and $n + n_3 + n_4$ are even and the inequalities $|n_j - n_k| \leq n_i \leq n_j + n_k$ hold for (n_i, n_j, n_k) equal to (n, n_1, n_2) or (n, n_3, n_4) and transpositions.

Proof. Using (4.39) and (4.28), (4.29), (4.34), we obtain for $\mathcal{I}_{\mathbf{o}}(\underline{t}, \underline{\omega})$ the expression

$$\sum_{\underline{n}} \sum_{\underline{\ell}} (\prod_{i=1}^{4} c_{D,n_i} \overline{Y_{\ell_i}^{(n_i)}(\omega_i)} t_i^{\alpha_i + \epsilon_i n_i} \, dt_i) \int_{\bar{\Sigma}} t^4 \, dt \sum_n \mathcal{K}_{\underline{n},\underline{\ell}}(n) \; t^{\epsilon n}.$$

The expression (4.41) then follows directly from the form (4.31) of the coefficients $\mathcal{K}_{\underline{n},\underline{0}}^{(D=4)}(n)$. The factor t^4 in the integral comes from the exponents $\alpha = 1$ and $\alpha = 3$ of the two half edges, which have matching orientations. □

Notice that, without the constraints on the summation range of the indices n, n_i, we would obtain again an integral of the general form (4.20), involving polylogarithm functions $\mathrm{Li}_s(t^\epsilon)$, with $s = 3 = k-2$ as in the case of polygons analyzed above. However, because not all values of n, n_i are allowed and one needs to impose the constraints of the form (4.32), one obtains more interesting expressions. We first introduce some notation.

4.5.1. *Summation domains and even condition.* In the following we let $\mathcal{R}$ denote a domain of summation for integers $(n_1, \ldots, n_k)$. We consider in particular the cases

$$\begin{aligned} \mathcal{R} = \mathcal{R}_P^{(k)} \quad &:= \{(n_1, \ldots, n_k) \,|\, n_i > 0, \;\; i = 1, \ldots, k\} \\ \mathcal{R} = \mathcal{R}_{MP}^{(k)} \quad &:= \{(n_1, \ldots, n_k) \,|\, n_k > \cdots > n_2 > n_1 > 0\} \\ \mathcal{R} = \mathcal{R}_T^{(3)} \quad &:= \{(n_1, n_2, n_3) \,|\, n_2 > n_1, \quad n_2 - n_1 < n_3 < n_2 + n_1\}. \end{aligned} \tag{4.42}$$

We denote by $\mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(z_1, \ldots, z_k)$ the associated series

$$\mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(z_1, \ldots, z_k) = \sum_{(n_1,\ldots,n_k) \in \mathcal{R}} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}. \tag{4.43}$$

In the first two cases of (4.42), this is, respectively, a product of polylogarithms $\mathrm{Li}^{\mathcal{R}_P}_{s_1,\ldots,s_k}(z_1, \ldots, z_k) = \prod_j \mathrm{Li}_{s_j}(z_j)$ and and a multiple polylogarithm $\mathrm{Li}^{\mathcal{R}_{MP}}_{s_1,\ldots,s_k}(z_1, \ldots, z_k) = \mathrm{Li}_{s_1,\ldots,s_k}(z_1, \ldots, z_k)$. We will discuss the third case more in detail below. We then define

$$\begin{aligned} \mathrm{Li}^{\mathcal{R},\mathrm{even}}_{s_1,\ldots,s_k}(z_1, \ldots, z_k) \quad &:= \tfrac{1}{2} \left(\mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(z_1, \ldots, z_k) + \mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(-z_1, \ldots, -z_k) \right) \\ \mathrm{Li}^{\mathcal{R},\mathrm{odd}}_{s_1,\ldots,s_k}(z_1, \ldots, z_k) \quad &:= \tfrac{1}{2} \left(\mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(z_1, \ldots, z_k) - \mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(-z_1, \ldots, -z_k) \right). \end{aligned} \tag{4.44}$$

The odd $\mathrm{Li}^{\mathcal{R},\mathrm{odd}}_{s_1,\ldots,s_k}(z_1, \ldots, z_k)$ is a direct generalization of the Legendre χ function, while the even $\mathrm{Li}^{\mathcal{R},\mathrm{even}}_{s_1,\ldots,s_k}(z_1, \ldots, z_k)$ corresponds to summing only over those indices in $\mathcal{R}$ whose sum is even,

$$\mathrm{Li}^{\mathcal{R},\text{even}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \sum_{(n_1,\ldots,n_k)\in\mathcal{R},\,\sum_i n_i\in 2\mathbb{N}} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}. \tag{4.45}$$

More generally, one can also consider summations of the form

$$\mathrm{Li}^{\mathcal{R},\mathcal{E}_1,\ldots,\mathcal{E}_k}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \sum_{(n_1,\ldots,n_k)\in\mathcal{R},\,n_i\in\mathcal{E}_i} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}, \tag{4.46}$$

where, for each $i=1,\ldots,k$, $\mathcal{E}_i = 2\mathbb{N}$ or $\mathcal{E}_i = \mathbb{N}\smallsetminus 2\mathbb{N}$, that is, some of the summation indices are even and some odd.

4.5.2. *Matching half-edges.* We now illustrate in one sufficiently simple and explicit case, what the leading $\ell=0$ term looks like when all the half-edges of stars are joined together. We look at the case of two stars of trivalent vertices with the half edges pairwise joined, that is, the 3-banana graph (two vertices and three parallel edges between them).

Proposition 4.12. *In the case of $D=4$, consider the graph with two vertices and three parallel edges between them. The $\underline{\ell}=0$ amplitude $\mathcal{I}_{\mathbf{o},0}$ is given by*

$$\mathcal{I}_{\mathbf{o},0} = \int_0^1 t^9\,(2^6\mathrm{Li}^{\mathcal{R}_{MP},odd,even}_{6,3}(t,t) + 2\mathrm{Li}^{\mathcal{R}_T,even}_{3,3,3}(t,t,t))\,dt. \tag{4.47}$$

Proof. There is a unique acyclic orientation of this graph, with the three edges oriented in the same direction. Thus, there is a single variable $t\in[0,1]=\bar{\Sigma}$ in the integrand of $\mathcal{I}_{\mathbf{o},0}$, and the latter has the form

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}} \mathcal{K}_{n_1,n_2,n_3} t^{n_1+n_2+n_3},$$

where the coefficients $\mathcal{K}_{n_1,n_2,n_3}$ are given by

$$\mathcal{K}_{n_1,n_2,n_3} = \frac{c^2_{4,n_1}c^2_{4,n_2}c^4_{4,n_3}}{(n_1+1)(n_2+1)(n_3+1)} = \frac{(4\pi^2)^3}{(n_1+1)^3(n_2+1)^3(n_3+1)^3},$$

according to Proposition 4.9, and the fact that all the half-edges of the two trivalent stars are matched. The summation domain $\mathcal{D}$ is given by

$$\mathcal{D} = \{(n_1,n_2,n_3)\,|\,n_i\geq 0\ |n_j-n_k|\leq n_i\leq n_j+n_k,\ \sum_i n_i \text{ even}\}.$$

We subdivide this into separate domains $\mathcal{D}=\mathcal{D}_1\cup\mathcal{D}_2\cup\mathcal{D}_3\cup\mathcal{D}_4\cup\mathcal{D}_5$, where

$$\begin{aligned}
\mathcal{D}_1 &= \{n_1=0,\ n_2\geq 0,\ n_3=n_2\}\\
\mathcal{D}_2 &= \{n_2=0,\ n_1>0,\ n_3=n_1\}\\
\mathcal{D}_3 &= \{n_1>0,\ n_2=n_1,\ 0\leq n_3\leq 2n_1,\ n_3 \text{ even }\}\\
\mathcal{D}_4 &= \{0<n_2<n_1,\ n_1-n_2\leq n_3\leq n_1+n_2,\ \textstyle\sum_i n_i \text{ even }\}\\
\mathcal{D}_5 &= \{0<n_1<n_2,\ n_2-n_1\leq n_3\leq n_1+n_2,\ \textstyle\sum_i n_i \text{ even }\}.
\end{aligned}$$

We have

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}_1} \frac{t^{n_1+n_2+n_3}}{(n_1+1)^3(n_2+1)^3(n_3+1)^3} = t^{-2}\sum_{n\geq 1}\frac{t^{2n}}{n^6} = t^{-2}\mathrm{Li}_6(t^2),$$

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}_2} \frac{t^{n_1+n_2+n_3}}{(n_1+1)^3(n_2+1)^3(n_3+1)^3} = t^{-2}\sum_{n\geq 2}\frac{t^{2n}}{n^6} = t^{-2}\mathrm{Li}_6(t^2)-1$$

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}_3} \frac{t^{n_1+n_2+n_3}}{(n_1+1)^3(n_2+1)^3(n_3+1)^3} = \sum_{n>0, 0\leq\ell\leq n}\frac{t^{2(n+\ell)}}{(2\ell+1)^3(n+1)^6}$$

$$= -1+2^6t^{-3}\sum_{n\geq 0,\, 0\leq \ell\leq n}\frac{t^{2n+2+2\ell+1}}{(2\ell+1)^3(2n+2)^6}$$

$$= -1+2^6t^{-3}\sum_{\substack{0<m_1<m_2\\ m_1 \text{ odd }, m_2 \text{ even}}}\frac{t^{m_1+m_2}}{m_1^6m_2^3} = -1+2^6t^{-3}\mathrm{Li}^{\mathcal{R}_{MP},\text{odd,even}}_{6,3}(t,t)$$

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}_4} \frac{t^{n_1+n_2+n_3}}{(n_1+1)^3(n_2+1)^3(n_3+1)^3} = t^{-3}\mathrm{Li}^{\mathcal{R}_T,\text{even}}_{3,3,3}(t,t,t)+1-t^{-2}\mathrm{Li}_6(t^2)$$

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}_5} \frac{t^{n_1+n_2+n_3}}{(n_1+1)^3(n_2+1)^3(n_3+1)^3} = t^{-3}\mathrm{Li}^{\mathcal{R}_T,\text{even}}_{3,3,3}(t,t,t)+1-t^{-2}\mathrm{Li}_6(t^2),$$

where in the last two cases the term $t^{-2}\mathrm{Li}_6(t^2)-1$ corresponds to the summation over $m_2=1$, $m_1>1$ and $m_3=m_1$ (respectively, $m_1=1$, $m_2>1$, $m_3=m_2$), with $m_i=n_1+1$. The integrand has a factor of t^4 for each edge, as in Lemma 4.11, which gives a power of t^{12} that combines with the t^{-3} factor in the result of the sum of the terms above to give the t^9 factor in (4.47). □

For more general graphs, where more vertices and more stars are involved, one gets summations involving several "triangular conditions" $|n_j-n_k|\leq n_i\leq n_j+n_k$ around each vertex, and the integrand can correspondingly be expressed in terms of series with a higher depth. Moreover, notice that we have focused here on the leading terms $\mathcal{K}^{D=4}_{\underline{n},\underline{\ell}=0}(n)$ only. When one includes all the other terms $\mathcal{K}_{\underline{n},\underline{\ell}}(n)$ with $\ell_i\neq 0$, the expressions become much more involved, as these coefficients are expressed in terms of the isoscalars (4.27) and of the standard $3j$-symbols for $SO(3)$, through the factorization (4.26). The isofactors are known explicitly [4], [28] so the computation can in principle be carried out in full, but it becomes much more cumbersome. We will discuss this elsewhere.

Next we show that the functions $\mathrm{Li}^{\mathcal{R}_T}_{s_1,s_2,s_3}(z_1,z_2,z_3)$ that appear in these Feynman amplitude computations can be related, via the Euler–Maclaurin summation formula, to some well known generalizations of multiple zeta values and multiple polylogarithms.

4.5.3. *Mordell–Tornheim and Apostol–Vu series.* We consider two generalizations of the multiple polylogarithm series, which arise in connection to the Mordell–Tornheim and the Apostol–Vu multiple series. The Mordell–Tornheim multiple series is given by [32], [42]

(4.48)
$$\zeta_{MT,k}(s_1,\dots,s_k;s_{k+1}) = \sum_{(n_1,\dots,n_k)\in\mathcal{R}_P^{(k)}} n_1^{-s_1}\cdots n_k^{-s_k}(n_1+\cdots+n_k)^{-s_{k+1}},$$

with an associated multiple polylogarithm-type function

(4.49)
$$\mathrm{Li}^{MT}_{s_1,\dots,s_k;s_{k+1}}(z_1,\dots,z_k;z_{k+1}) = \sum_{(n_1,\dots,n_k)\in\mathcal{R}_P^{(k)}} \frac{z_1^{n_1}\cdots z_k^{n_k} z_{k+1}^{(n_1+\cdots+n_k)}}{n_1^{s_1}\cdots n_k^{s_k}(n_1+\cdots+n_k)^{s_{k+1}}}.$$

Similarly, the Apostol–Vu multiple series [5] is defined as

(4.50)
$$\zeta_{AV,k}(s_1,\dots,s_k;s_{k+1}) = \sum_{(n_1,\dots,n_k)\in\mathcal{R}_{MP}^{(k)}} n_1^{-s_1}\cdots n_k^{-s_k}(n_1+\cdots+n_k)^{-s_{k+1}},$$

and we consider the associated multiple polylogarithm-type series

(4.51)
$$\mathrm{Li}^{AV}_{s_1,\dots,s_k;s_{k+1}}(z_1,\dots,z_k;z_{k+1}) = \sum_{(n_1,\dots,n_k)\in\mathcal{R}_{MP}^{(k)}} \frac{z_1^{n_1}\cdots z_k^{n_k} z_{k+1}^{(n_1+\cdots+n_k)}}{n_1^{s_1}\cdots n_k^{s_k}(n_1+\cdots+n_k)^{s_{k+1}}}.$$

4.5.4. *Euler–Maclaurin formula.* A way to understand better the behavior of the functions (4.45) with $\mathcal{R} = \mathcal{R}_T^{(3)}$ that appear in this result, is in terms of the Euler–Maclaurin summation formula.

Lemma 4.13. *Let $f(t) = x^t t^{-s}$. Then*

(4.52)
$$f^{(k)}(t) = \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}\binom{s+k-j-1}{k-j}(k-j)!\,t^{-(s+k-j)}x^t \log(x)^j.$$

Proof. Inductively, we have

$$f^{(k)}(t) = \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} s(s+1)\cdots(s+k-j-1)\, t^{-(s+k-j)}\, x^t \log(x)^j,$$

where $s(s+1)\cdots(s+k-j-1) = \binom{s+k-j-1}{k-j}(k-j)!$. □

The Euler–Maclaurin summation formula gives

$$\sum_{n=a}^{b} f(n) = \int_a^b f(t)dt + \frac{1}{2}(f(b)+f(a)) + \sum_{k=2}^{N} \frac{b_k}{k!}(f^{(k-1)}(b) - f^{(k-1)}(a)) - \int_a^b \frac{B_N(t-[t])}{N!} f^{(N)}(t)\, dt, \tag{4.53}$$

where b_k are the Bernoulli numbers and B_k the Bernoulli polynomials. We then have the following result.

Proposition 4.14. *Consider the series* $\mathrm{Li}^{\mathcal{R}}_{s_1,s_2,s_3}(z_1,z_2,z_3)$ *defined as in* (4.43)*, with* $\mathcal{R} = \mathcal{R}^{(3)}_T$*. When applying the Euler–Maclaurin formula to the innermost sum, the summation terms in* (4.53) *give rise to terms of the form*

$$\pm F_{j,k}(s_3,z_3)\ \mathrm{Li}^{AV}_{s_1,s_2;s_{3+k-j}}(z_1,z_2;z_3) \tag{4.54}$$

or

$$\pm F_{j,k}(s_3,z_3)\ \mathrm{Li}^{MT}_{s_1,s_{3+k-j};s_2}(z_1,z_2;z_3), \tag{4.55}$$

where

$$F_{j,k}(s,z) = \frac{b_k}{k!}\binom{k}{j}\binom{s+k-j-1}{k-j}(k-j)!\ \log(z)^j \tag{4.56}$$

Proof. For $\mathrm{Li}^{\mathcal{R}}_{s_1,s_2,s_3}(z_1,z_2,z_3)$, with $\mathcal{R} = \mathcal{R}^{(3)}_T$, the summation

$$\sum_{n_2-n_1<n_3<n_2+n_1} \frac{z_3^{n_3}}{n_3^{s_3}} \tag{4.57}$$

can be expressed, using Lemma 4.13, through the Euler–Maclaurin summation formula (4.53). Up to a sign, each summation term in the right-hand-side of (4.53) is the product of a function of z_3 of the form $F_{j,k}(s_3,z_3)$, as in (4.56), and a term of the form

$$\frac{z_3^{n_2+n_1}}{(n_2+n_1)^{s_3+k-j}} \quad \text{or} \quad \frac{z_3^{n_2-n_1}}{(n_2-n_1)^{s_3+k-j}}.$$

When inserted back into the summation on the remaining indices $n_2 > n_1$, this gives summations of the form

$$\sum_{n_2>n_1>0} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_1+n_2}}{n_1^{s_1} n_2^{s_2} (n_1+n_2)^{s_3+k-j}}, \tag{4.58}$$

in the first case, or in the second case, after a change of variables $m = n_2 - n_1$, $n = n_1$ in the indices

$$\sum_{n>0,m>0} \frac{z_1^n z_3^m z_2^{n+m}}{n^{s_1} m^{s_3+k-j} (n+m)^{s_2}}, \tag{4.59}$$

which are respectively of the form (4.49) and (4.51). □

5. Wonderful compactifications and the Feynman amplitudes

In this section, we consider the case of the Feynman amplitude (2.9) introduced in §2.3.2. As in Definition 2.6, the locus of integration is, in this case, the complex variety $X^{\mathbf{V}_\Gamma} \times \{y = (y_v)\}$, for a fixed choice of a point $y = (y_v)$, inside the configuration space $Z^{\mathbf{V}_\Gamma}$ with $Z = X \times X$.

To discuss an appropriate regularization procedure for the Feynman integral and interpret the result in terms of periods, we need first some basic facts about the wonderful compactifications of the configuration spaces $F(X, \Gamma)$.

We described in detail in our previous work [12] the geometry of the wonderful compactifications of the configuration spaces $\mathrm{Conf}_\Gamma(X)$. We recall here the main definitions and statements, adapted from $\mathrm{Conf}_\Gamma(X)$ to $F(X, \Gamma)$. The arguments are essentially the same as in [12].

5.1. **Arrangements of diagonals.** A *simple arrangement* of subvarieties of a smooth quasi-projective ambient variety Y is a finite collection of nonsingular closed subvarieties $\mathcal{S} = \{S_i \subset Y, i \in I\}$ such that

- all nonempty intersections $\bigcap_{i\in J} S_i$ for $J \subset I$ are in the collection $\mathcal{S}$.
- for any pair $S_i, S_j \in \mathcal{S}$, the intersection $S_i \cap S_j$ is *clean*, that is, the tangent bundle of the intersection is the intersection of the restrictions of the tangent bundles.

5.1.1. *Diagonals of induced subgraphs and their arrangement.* For each induced and not necessarily connected subgraph $\gamma \subseteq \Gamma$, the corresponding (poly)diagonal is

(5.1)
$$\Delta_\gamma^{(Z)} = \{z = (z_v) \in Z^{\mathbf{V}_\Gamma} \mid p(z_v) = p(z_{v'}) \text{ for } \{v, v'\} = \partial_\Gamma(e),\ e \in \mathbf{E}_\gamma\}.$$

We have the following simple description.

Lemma 5.1. *The diagonal* $\Delta_\gamma^{(Z)}$ *is isomorphic to* $X^{\mathbf{V}_{\Gamma//\gamma}} \times X^{\mathbf{V}_\Gamma}$*, and has dimension*

$$\dim \Delta_\gamma^{(Z)} = \dim X^{\mathbf{V}_{\Gamma//\gamma}} \times X^{\mathbf{V}_\Gamma} = \dim(X)(2|\mathbf{V}_\Gamma| - |\mathbf{V}_\gamma| + b_0(\gamma)) \tag{5.2}$$

where $b_0(\gamma)$ *is the number of connected components of* γ*.*

We then observe that the diagonals form an arrangement of subvarieties. This is the analog of Lemma 5 of [12].

Lemma 5.2. *For a given graph* Γ*, the collection*

$$\mathcal{S}_\Gamma = \{\Delta_\gamma^{(Z)} \mid \gamma \in \mathbf{SG}(\Gamma)\}, \tag{5.3}$$

with $\mathbf{SG}(\Gamma)$ *the set of all induced subgraphs of* Γ*, is a simple arrangement of diagonal subvarieties in* $Z^{\mathbf{V}_\Gamma}$*.*

Proof. Let γ_1 and γ_2 be a pair of induced subgraphs. If $\gamma_1 \cap \gamma_2 = \emptyset$, then $\gamma = \gamma_1 \cup \gamma_2$ is in $\mathbf{SG}(\Gamma)$, and the corresponding diagonal $\Delta_\gamma^{(Z)}$ is given by the intersection $\Delta_{\gamma_1}^{(Z)} \cap \Delta_{\gamma_2}^{(Z)}$. On the other hand, if $\gamma_1 \cap \gamma_2 \neq \emptyset$, we consider the connected components γ_j of the union γ. Then, the intersection $\Delta_{\gamma_1}^{(Z)} \cap \Delta_{\gamma_2}^{(Z)}$ can be written as $\cap_j \Delta_{i(\gamma_j)}^{(Z)}$ where $i(\gamma_j)$ is the smallest induced subgraph of Γ containing γ_j. All diagonals are smooth and the ideal sheaf of intersection $\Delta_\gamma^{(Z)}$ is the direct sum of the ideal sheaves of the intersecting diagonals $\Delta_{\gamma_j}^{(Z)}$. By Lemma 5.1 of [31], their intersections are clean. □

5.1.2. *Building set of the arrangements of diagonals.* A subset $\mathcal{G} \subset \mathcal{S}$ is called a *building set* of the simple arrangement $\mathcal{S}$ if for any $S \in \mathcal{S}$, the minimal elements in $\{G \in \mathcal{G} : G \supseteq S\}$ intersect transversely and the intersection is S.

A $\mathcal{G}$-building set for the arrangement $\mathcal{S}_\Gamma$ can be identified by considering further combinatorial properties of graphs. A graph Γ is *2-vertex-connected* (or *biconnected*) if it cannot be disconnected by the removal of a single vertex along with the open star of edges around it. The graph consisting of a single edge is assumed to be biconnected.

We then have the analog of Proposition 1 of [12].

Proposition 5.3. *For a given graph* Γ*, the set*

$$\mathcal{G}_\Gamma = \{\Delta_\gamma^{(Z)} \,|\, \gamma \subseteq \Gamma \textit{ induced, biconnected}\} \tag{5.4}$$

is a $\mathcal{G}$*-building set for the arrangement* $\mathcal{S}_\Gamma$ *of* (5.3).

Proof. The intersection of a bi-connected subgraph of Γ with an induced subgraph is either empty or a union of bi-connected induced subgraphs attached at cut-vertices. We decompose induced subgraphs into bi-connected components. The diagonals corresponding to these bi-connected components are the minimal elements in the collection $\mathcal{S}_\Gamma$. For a pair of bi-connected induced subgraphs γ_1, γ_2 with $\gamma = \gamma_1 \cup \gamma_2$, we have the following equalities due to Lemma 5.1; $\dim \Delta_{\gamma_1}^{(Z)} + \dim \Delta_{\gamma_2}^{(Z)} - \dim \Delta_\gamma^{(Z)} = \dim(X^{\mathbf{V}_{\Gamma//\gamma_1}} \times X^{\mathbf{V}_\Gamma}) + \dim(X^{\mathbf{V}_{\Gamma//\gamma_2}} \times X^{\mathbf{V}_\Gamma}) - \dim(X^{\mathbf{V}_{\Gamma//\gamma}} \times X^{\mathbf{V}_\Gamma}) = 2\dim(X)|\mathbf{V}_\Gamma| = \dim Z^{\mathbf{V}_\Gamma}$; and these guarantee the transversality of the intersection $\Delta_{\gamma_1}^{(Z)} \cap \Delta_{\gamma_2}^{(Z)}$. □

5.2. **The wonderful compactifications of arrangements.** To be able to analyze the residues of Feynman integrals, we need a compactification $Z[\Gamma]$ of the configuration space $F(X,\Gamma)$ satisfying certain properties. In particular, $Z[\Gamma]$ must contain $F(X,\Gamma)$ as the top dimensional stratum, and the complement $Z[\Gamma] \setminus F(X,\Gamma)$ of this principal stratum must be a union of transversally intersecting divisors in $Z[\Gamma]$. The transversality is essential for the use of iterated Poincaré residues, which we will discuss in §7 below.

There is a smooth wonderful compactification $Z[\Gamma]$ of the configuration space $F(X,\Gamma)$ which is a generalization of the Fulton–MacPherson compactification [20]. The construction is completely analogous to the construction of the wonderful compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$ considered in our previous paper [12]. Again, we illustrate here briefly what changes in passing from the case of $\mathrm{Conf}_\Gamma(X)$ to the case of $F(X,\Gamma)$.

5.2.1. *The iterated blowup description.* As in the case of $\mathrm{Conf}_\Gamma(X)$ (see §2.3 of [12]), the wonderful compactification $Z[\Gamma]$ is obtained by an iterated sequence of blowups.

The following is the direct analog of Proposition 2 of [12].

Let $|\mathbf{V}_\Gamma| = n$ and let $\mathcal{G}_{k,\Gamma} \subseteq \mathcal{G}_\Gamma$ be the subcollection

$$\mathcal{G}_{k,\Gamma} = \{\Delta_\gamma^{(Z)} \,|\, \gamma \in \mathbf{SG}_k(\Gamma) \text{ and biconnected}\} \quad \text{for} \quad k = 1, \dots, n-1. \tag{5.5}$$

Let $Y_0 = Z^{\mathbf{V}_\Gamma}$ and let Y_k be the blowup of Y_{k-1} along the (iterated) dominant transform of $\Delta_\gamma^{(Z)} \in \mathcal{G}_{n-k+1,\Gamma}$. If Γ is itself biconnected, then Y_1 is the blowup of Y_0 along the deepest diagonal $\Delta_\Gamma^{(Z)}$, and otherwise $Y_1 = Y_0$. Similarly, we have $Y_k = Y_{k-1}$ if there are no biconnected induced subgraphs with exactly $n-k+1$ vertices. The resulting sequence of blowups

$$Y_{n-1} \to \cdots \to Y_2 \to Y_1 \to Z^{\mathbf{V}_\Gamma} \tag{5.6}$$

does not depend on the order in which the blowups are performed along the (iterated) dominant transforms of the diagonals $\Delta_\gamma^{(Z)}$, for $\gamma \in \mathcal{G}_{n-k+1,\Gamma}$, for a fixed k. Thus, the intermediate varieties Y_k in the sequence (5.6) are all well defined. The variety Y_{n-1} obtained through this sequence of iterated blowups is called the wonderful compactification;

$$Z[\Gamma] := Y_{n-1}. \tag{5.7}$$

Note that $Z[\Gamma]$ is a smooth quasi-projective variety as can be seen through its iterated blow-up construction, see [31].

5.2.2. *Divisors and their intersections.* Recall from [31] that a $\mathcal{G}_\Gamma$-*nest* is a collection $\{\gamma_1, \dots, \gamma_\ell\}$ of biconnected induced subgraphs with the property that any two subgraphs γ and γ' in the set satisfy one of the following: (1) $\gamma \cap \gamma' = \emptyset$; (2) $\gamma \cap \gamma' = \{v\}$, a single vertex; (3) $\gamma \subseteq \gamma'$ or $\gamma' \subseteq \gamma$.

We then have the following analog of Proposition 4 of [12].

Proposition 5.4. *For a given biconnected induced subgraph $\gamma \subseteq \Gamma$, let $D_\gamma^{(Z)}$ be the divisor obtained as the iterated dominant transform of $\Delta_\gamma^{(Z)}$ in the iterated blowup construction* (5.6) *of $Z[\Gamma]$. Then*

$$Z[\Gamma] \setminus F(X,\Gamma) = \bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma^{(Z)}. \tag{5.8}$$

The divisors $D^{(Z)}_\gamma$ *have the property that*

$$D^{(Z)}_{\gamma_1} \cap \cdots \cap D^{(Z)}_{\gamma_\ell} \neq \emptyset \Leftrightarrow \{\gamma_1, \ldots, \gamma_\ell\} \text{ is a } \mathcal{G}_\Gamma\text{-nest.} \tag{5.9}$$

5.3. **Motives of wonderful compactifications.** As in the case of the wonderful compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$ analyzed in [12], one can obtain the explicit formula for the motive of the compactifications $Z[\Gamma]$ directly from the formula for the motive of blow-ups and the iterated construction of §5.2.1.

We first introduce the following notation as in [12], [30]. Given a $\mathcal{G}_\Gamma$-nest $\mathcal{N}$, and a biconnected induced subgraph γ such that $\mathcal{N}' = \mathcal{N} \cup \{\gamma\}$ is still a $\mathcal{G}_\Gamma$-nest, we set

$$r_\gamma = r_{\gamma,\mathcal{N}} := \dim(\cap_{\gamma' \in \mathcal{N}: \gamma' \subset \gamma} \Delta_{\gamma'}) - \dim \Delta_\gamma, \tag{5.10}$$

$$M_\mathcal{N} := \{(\mu_\gamma)_{\Delta_\gamma \in \mathcal{G}_\Gamma} \,:\, 1 \leq \mu_\gamma \leq r_\gamma - 1,\ \mu_\gamma \in \mathbb{Z}\}, \tag{5.11}$$

$$\|\mu\| := \sum_{\Delta_\gamma \in \mathcal{G}_\Gamma} \mu_\gamma. \tag{5.12}$$

We write here $\mathfrak{m}(X)$ for the motive in the Voevodsky category. This corresponds to the notation M_{gm} of [45].

The following result is the analog of Proposition 8 of [12], see also [30] for the formulation in the case of Chow motives.

Proposition 5.5. *Let X be a smooth projective variety. The Voevodsky motive $\mathfrak{m}(Z[\Gamma])$ of the wonderful compactification is given by*

$$\mathfrak{m}(Z[\Gamma]) = \mathfrak{m}(Z^{\mathbf{V}_\Gamma}) \oplus \bigoplus_{\mathcal{N} \in \mathcal{G}_\Gamma\text{-nests}} \bigoplus_{\mu \in M_\mathcal{N}} \mathfrak{m}(X^{\mathbf{V}_{\Gamma/\delta_\mathcal{N}(\Gamma)}} \times X^{\mathbf{V}_\Gamma})(\|\mu\|)[2\|\mu\|] \tag{5.13}$$

where $\Gamma/\delta_\mathcal{N}(\Gamma)$ is the quotient $\Gamma/\delta_\mathcal{N}(\Gamma) := \Gamma//(\gamma_1 \cup \cdots \cup \gamma_r)$ for the $\mathcal{G}_\Gamma$-nest $\mathcal{N} = \{\gamma_1, \ldots, \gamma_r\}$.

Proof. Let $\widetilde{Y} \to Y$ be the blow-up of a smooth scheme Y along a smooth closed subscheme $V \subset Y$. Then $\mathfrak{m}(\widetilde{Y})$ is canonically isomorphic to

$$\mathfrak{m}(Y) \oplus \bigoplus_{k=1}^{\mathrm{codim}_Y(V)-1} \mathfrak{m}(V)(k)[2k],$$

see Proposition 3.5.3 in [45]. The result then follows by applying this blow-up formula for Voevodsky's motives to the iterated blow-up construction given in Section 5.2.1. □

We obtain then from Proposition 5.5 the following simple corollary (see §3.2 of [12]).

Corollary 5.6. *If the motive of the smooth projective variety X is mixed Tate, then the motive of $Z[\Gamma]$ is mixed Tate, for all graphs Γ. Moreover, the exceptional divisors $D^{(Z)}_\gamma$ associated to the biconnected induced subgraphs $\gamma \subseteq \Gamma$ and the intersections $D^{(Z)}_{\gamma_1} \cap \cdots \cap D^{(Z)}_{\gamma_\ell}$ associated to the $\mathcal{G}_\Gamma$-nests $\{\gamma_1, \dots, \gamma_\ell\}$ are also mixed Tate.*

Proof. This is an immediate consequence of the construction of $Z[\Gamma]$ since the motive of $Z[\Gamma]$ depends upon the motive of X only through products, Tate twists, sums, and shifts. All these operations preserve the subcategory of mixed Tate motives. The reason why the intersections $D^{(Z)}_{\gamma_1} \cap \cdots \cap D^{(Z)}_{\gamma_\ell}$ are also mixed Tate is because one has an explicit stratification, as described in [12] and [31], where one has a description of the intersections of diagonals in terms of configuration spaces of quotient graphs and by repeated use of the blowup formula for motives. □

Remark 5.7. One can also see easily that, if the variety X is defined over $\mathbb{Z}$, then so is $Z[\Gamma]$ and so are the $D^{(Z)}_\gamma$ and their unions and intersections. Moreover, all these varieties then satisfy the unramified criterion of §3.5 and Proposition 3.10 of [22].

5.4. **Feynman amplitude and wonderful compactifications.** We now consider the form $\omega^{(Z)}_\Gamma$ defined as in (2.9) and discuss its behavior when puled back from $Z^{\mathbf{V}_\Gamma}$ to the wonderful compactification $Z[\Gamma]$.

5.4.1. *Loci of divergence.* For massless scalar Euclidean field theories, the pole locus $\{\omega^{(Z)}_\Gamma = \infty\}$ in $Z^{\mathbf{V}_\Gamma}$ is

$$\mathcal{Z}_\Gamma := \left\{ \prod_{e \in \mathbf{E}_\Gamma} \| p(z_{v_{s(e)}}) - p(z_{v_{t(e)}}) \|^2 = 0 \right\}. \tag{5.14}$$

This definition can be rephrased as follows.

Lemma 5.8. *The divergent locus of the density $\omega^{(Z)}_\Gamma$ of (2.9) in $Z^{\mathbf{V}_\Gamma}$ is given by the union $\bigcup_{e \in \mathbf{E}_\Gamma} \Delta^{(Z)}_e$.*

5.4.2. *Order of singularities in the blowups.* Let $\pi^*_\gamma(\omega^{(Z)}_\Gamma)$ denote the pullbacks of the form $\omega^{(Z)}_\Gamma$ of (2.9) to the iterated blowups of $Z^{\mathbf{V}_\Gamma}$ along the (dominant transforms of) the diagonals $\Delta^{(Z)}_\gamma$, for $\gamma \subset \Gamma$ a biconnected induced subgraph.

Proposition 5.9. *Let Γ be a connected graph. Then for every biconnected induced subgraph $\gamma \subset \Gamma$, the pullback $\pi^*_\gamma(\omega^{(Z)}_\Gamma)$ of $\omega^{(Z)}_\Gamma$ to the blowup along* *the (dominant transform of)* $\Delta_\gamma^{(Z)}$ *has singularities of order*

$$\begin{aligned} \text{ord}_\infty(\pi_\gamma^*(\omega_\Gamma), D_\gamma^{(Z)}) &= (D-2)|\mathbf{E}_\gamma| - 2D(|\mathbf{V}_\gamma|-1)+2 \\ &= 2Db_1(\gamma)-(D+2)|\mathbf{E}_\gamma|+2 \end{aligned} \tag{5.15}$$

along the exceptional divisors $D_\gamma^{(Z)}$ *in the blowup. Here* $b_1(\gamma)$ *denotes the first Betti number of graph* γ*.*

Proof. Let $m = D|\mathbf{V}_\Gamma|$ and $L \subset \mathbb{A}^{2m}$ be the coordinate subspace given by the equations $\{x_1 = \cdots = x_k = 0\}$, and $\pi : \widetilde{\mathbb{A}}^{2m} \to \mathbb{A}^{2m}$ be the blowup along $L \subset \mathbb{A}^{2m}$. If one chooses the coordinates w_i in the blow up, such that,$w_i = x_i$ for $i = k, \dots, 2m$, and $w_i w_p = x_i$ for $i < k$. The exceptional divisor given by $w_p = 0$ in these coordinates. Then, one obtains

$$\pi^*(dx_1 \wedge dx_1^* \wedge \cdots \wedge dx_m \wedge dx_m^*) = |w|^{2(k-1)} dw_1 \wedge dw_1^* \wedge \cdots \wedge dw_d \wedge dw_D^*.$$

This form has a zero of order $2\cdot(\text{codim}(L)-1)$ along the exceptional divisor of the blowup.

The codimension of the diagonal $\Delta_\gamma \subset X^{\mathbf{V}_\Gamma}$ associated to a connected subgraph $\gamma \subset \Gamma$ is $D\ (|\mathbf{V}_\gamma|-1)$. On the other hand, the form $\omega_\Gamma^{(Z)}$ has singularity along $\Delta_\gamma^{(Z)}$ of order $(D-2)|\mathbf{E}_\gamma|$. Hence,

$$\begin{aligned} \text{ord}_\infty(\pi_\gamma^*(\omega_\Gamma^{(Z)}), D_\gamma^{(Z)}) &= (\text{order of } \infty) - (\text{order of zeros}) \\ &= (D-2)|\mathbf{E}_\gamma| - 2D(|\mathbf{V}_\gamma|-1)+2. \end{aligned}$$

□

Note that the orders of pole are different from the case of the form ω_Γ on $\overline{\text{Conf}}_\Gamma(X)$, see §4.3 of [12]. Lemma 5.8 and Proposition 5.9 then imply the following.

Corollary 5.10. *Let* $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ *denote the pullback of* $\omega_\Gamma^{(Z)}$ *to the wonderful compactification* $Z[\Gamma]$*. The divergence locus of* $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ *in* $Z[\Gamma]$ *is given by the union of divisors*

$$\bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma^{(Z)}. \tag{5.16}$$

5.5. **Chain of integration and divergence locus.** When pulling back the form $\omega_\Gamma^{(Z)}$ along the projection $\pi_\Gamma : Z[\Gamma] \to Z^{\mathbf{V}_\Gamma}$, one also replaces the chain of integration $\sigma_\Gamma^{(Z,y)} = X^{\mathbf{V}_\Gamma} \times \{y\}$ of (2.10) with $\tilde\sigma_\Gamma^{(Z,y)} \subset Z[\Gamma]$ with $\pi_\Gamma(\tilde\sigma_\Gamma^{(Z,y)}) = \sigma_\Gamma^{(Z,y)}$, which gives

$$\tilde\sigma_\Gamma^{(Z,y)} = \overline{\text{Conf}}_\Gamma(X) \times \{y\} \subset Z[\Gamma] = \overline{\text{Conf}}_\Gamma(X) \times X^{\mathbf{V}_\Gamma}. \tag{5.17}$$

Lemma 5.11. *The chain of integration* $\tilde{\sigma}_\Gamma^{(Z,y)}$ *of* (5.17) *intersects the locus of divergence* (5.16) *in*

$$\bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma \times \{y\} \subset \overline{\mathrm{Conf}}_\Gamma(X) \times \{y\}. \tag{5.18}$$

Proof. This follows directly from Corollary 5.10 and (5.17). □

Notice that, since $\mathcal{G}_\Gamma$-factors intersect transversely (see Proposition 2.8 of [31] and Proposition 4 of [12]), the intersection (5.18) of the chain of integration $\tilde{\sigma}_\Gamma^{(Z,y)}$ with the locus of divergence consists of a union of divisors D_γ inside $X^{\mathbf{V}_\Gamma}$ intersecting transversely, with $D_{\gamma_1} \cap \cdots \cap D_{\gamma_\ell} \neq \emptyset$ whenever $\{\gamma_1, \ldots, \gamma_\ell\}$ form a $\mathcal{G}_\Gamma$-nest (see [12] and [31]).

5.6. **Smooth and algebraic differential forms.** Consider the restriction of the amplitude $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ to the chain $\tilde{\sigma}_\Gamma^{(Z,y)}$. It is defined on the complement of the divergence locus, namely on

$$\tilde{\sigma}_\Gamma^{(Z,y)} \smallsetminus \left(\bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma \times \{y\} \right) \simeq \overline{\mathrm{Conf}}_\Gamma(X) \smallsetminus \left(\bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma \right), \tag{5.19}$$

which is a copy of $\mathrm{Conf}_\Gamma(X)$ inside $Z[\Gamma]$. The form $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ is a closed form of top dimension on this domain.

We recall the following general fact. Let $\mathcal{X}$ be a smooth projective variety of dimension m and let $\mathcal{D}$ be a union of smooth hypersurfaces intersecting transversely (strict normal crossings divisor). Let $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{D}$.

Lemma 5.12. *Let* ω *be a* $\mathcal{C}^\infty$ *closed differential form of degree* m *on* $\mathcal{U}$*, and let* $[\omega]$ *be the corresponding de Rham cohomology class in* $H^m(\mathcal{U})$*. Then there exists an algebraic differential form* η *with logarithmic poles along* $\mathcal{D}$*, that is cohomologous,* $[\eta] = [\omega] \in H^m(\mathcal{U})$*, to the given form* ω*.*

Proof. First we use the fact that de Rham cohomology of a smooth quasi-projective variety $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{D}$ can always be computed using algebraic differential forms, [25], [26]. Thus, the cohomology class $[\omega]$ in $H^m(\mathcal{U})$ can be realized by a form $\alpha \in H^0(\mathcal{U}, \Omega^m)$, with Ω^m the sheaf of algebraic differential forms. Moreover, by §3.2 of [16], the algebraic de Rham cohomology $H^*(\mathcal{U})$ satisfies

$$H^*(\mathcal{U}) \simeq \mathbb{H}^*(\mathcal{X}, \Omega^*_{\mathcal{X}}(\log(\mathcal{D}))), \tag{5.20}$$

where $\Omega^*_{\mathcal{X}}(\log(\mathcal{D}))$ denotes the sheaf of meromorphic differential forms on $\mathcal{X}$ with logarithmic poles along $\mathcal{D}$. Thus, we can find a form $\eta \in \Omega^*_{\mathcal{X}}(\log(\mathcal{D}))$ so that $[\eta] = [\omega] \in H^m(\mathcal{U})$. □

We then have the following consequence.

Lemma 5.13. *Let $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ be the pullback of the Feynman amplitude* (2.9) *to the wonderful compactification $Z[\Gamma]$. Then there exists a meromorphic differential form $\eta_\Gamma^{(Z)}$, the* algebraic Feynman amplitude, *on $\tilde{\sigma}_\Gamma^{(Z,y)} = \overline{\mathrm{Conf}}_\Gamma(X) \times \{y\}$, with logarithmic poles along*

$$\mathcal{D}_\Gamma = \bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma \times \{y\},$$

such that

$$[\eta_\Gamma^{(Z)}] = [\pi_\Gamma^*(\omega_\Gamma^{(Z)})] \in H^{2D|\mathbf{V}_\Gamma|}(\tilde{\sigma}_\Gamma^{(Z)} \smallsetminus \mathcal{D}_\Gamma).$$

Proof. This follows directly from Lemma 5.12 above. □

5.7. **Iterated Poincaré residues.** One can associate to the holomorphic differential form η_Γ with logarithmic poles along $\mathcal{D}_\Gamma$ a Poincaré residue on each non-empty intersection of a collection of divisors $D_\gamma^{(Z)}$ that corresponds to a $\mathcal{G}_\Gamma$-nest $\mathcal{N} = \{\gamma_1, \ldots, \gamma_r\}$.

Proposition 5.14. *For every $\mathcal{G}_\Gamma$-nest $\mathcal{N} = \{\gamma_1, \ldots, \gamma_r\}$, there is a Poincaré residue $\mathcal{R}_\mathcal{N}(\eta_\Gamma)$, which defines a cohomology class in $H^{2D|\mathbf{V}_\Gamma|-r}(V_\mathcal{N})$, on the complete intersection $V_\mathcal{N}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$. The pairing of $\mathcal{R}_\mathcal{N}(\eta_\Gamma)$ with an $(2D|\mathbf{V}_\Gamma| - r)$-cycle $\Sigma_\mathcal{N}$ in $V_\mathcal{N}^{(Z)}$ is equal to*

$$\int_{\Sigma_\mathcal{N}} \mathcal{R}_\mathcal{N}(\eta_\Gamma) = \frac{1}{(2\pi i)^r} \int_{\mathcal{L}_\mathcal{N}(\Sigma_\mathcal{N})} \eta_\Gamma, \tag{5.21}$$

where $\mathcal{L}_\mathcal{N}(\Sigma_\mathcal{N})$ is the $2D|\mathbf{V}_\Gamma|$-cycle in $Z[\Gamma]$ given by an iterated Leray coboundary of $\Sigma_\mathcal{N}$, which is a T^r-torus bundle over $\Sigma_\mathcal{N}$. Under the assumption that the variety X is a mixed Tate motive, the integrals (5.21) *are periods of mixed Tate motives.*

Proof. As shown in Proposition 5.4, the divisors $D_\gamma^{(Z)}$ in $Z[\Gamma]$ have the property that

$$V_\mathcal{N}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)} \neq \emptyset \Leftrightarrow \{\gamma_1, \ldots, \gamma_r\} \quad \text{is a } \mathcal{G}_\Gamma - \text{n}est, \tag{5.22}$$

with transverse intersections.

Consider the first divisor $D_{\gamma_1}^{(Z)}$ in the $\mathcal{G}_\Gamma$-nest $\mathcal{N}$, and a tubular neighborhood $N_{\Gamma,\gamma_1} = N_{Z[\Gamma]}(D_{\gamma_1}^{(Z)})$ of $D_{\gamma_1}^{(Z)}$ in N_{Γ,γ_1}. This is a unit disk bundle over $D_{\gamma_1}^{(Z)}$ with projection $\pi : N_{\Gamma,\gamma_1} \to D_{\gamma_1}^{(Z)}$ and with $\sigma : D_{\gamma_1}^{(Z)} \hookrightarrow N_{\Gamma,\gamma_1}$ the zero section. The Gysin long exact sequence in homology gives

$$\cdots \to H_k(N_{\Gamma,\gamma_1} \smallsetminus D_{\gamma_1}^{(Z)}) \stackrel{\iota_*}{\to} H_k(N_{\Gamma,\gamma_1})$$

$$\stackrel{\sigma^!}{\to} H_{k-2}(D_{\gamma_1}^{(Z)}) \stackrel{\pi^!}{\to} H_{k-1}(N_{\Gamma,\gamma_1} \smallsetminus D_{\gamma_1}^{(Z)}) \to \cdots$$

where $\pi^!$ is the Leray coboundary map, which assigns to a chain Σ in $D^{(Z)}_{\gamma_1}$ the homology class in $N_{\Gamma,\gamma_1} \smallsetminus D^{(Z)}_{\gamma_1}$ of the boundary $\partial\pi^{-1}(\Sigma)$ of the disk bundle $\pi^{-1}(\Sigma)$ over Σ, which is an S^1-bundle over Σ. Its dual is a morphism

$$\mathcal{R}_{\gamma_1} : H^{k+1}(N_{\Gamma,\gamma_1} \smallsetminus D^{(Z)}_{\gamma_1}) \to H^k(D^{(Z)}_{\gamma_1}),$$

which is the residue map. The iterated residue map is obtained by considering the complements $\mathcal{U}_0 = N_{\Gamma,\gamma_1} \smallsetminus D^{(Z)}_{\gamma_1}$ and

$$\mathcal{U}_1 = D^{(Z)}_{\gamma_1} \smallsetminus \bigcup_{1<k\leq r} D^{(Z)}_{\gamma_k},$$

$$\mathcal{U}_2 = (D^{(Z)}_{\gamma_1} \cap D^{(Z)}_{\gamma_2}) \smallsetminus \bigcup_{2<k\leq r} D^{(Z)}_{\gamma_k},$$

and so on. One obtains a sequence of maps

$$H^k(\mathcal{U}_0) \stackrel{\mathcal{R}_{\gamma_1}}{\to} H^{k-1}(\mathcal{U}_1) \stackrel{\mathcal{R}_{\gamma_2}}{\to} H^{k-2}(\mathcal{U}_2) \to \cdots \stackrel{\mathcal{R}_{\gamma_r}}{\to} H^{k-r}(V^{(Z)}_{\mathcal{N}}).$$

The composition $\mathcal{R}_{\mathcal{N}} = \mathcal{R}_{\gamma_r} \circ \cdots \circ \mathcal{R}_{\gamma_1}$ is the iterated residue map. Because the residue map is dual to Leray coboundary, under the pairing of homology and cohomology one obtains

$$\langle \mathcal{R}_{\mathcal{N}}(\eta), \Sigma \rangle = \langle \eta, \mathcal{L}_{\mathcal{N}}(\Sigma) \rangle,$$

where $\mathcal{L}_{\mathcal{N}} = \mathcal{L}_{\gamma_1} \circ \cdots \circ \mathcal{L}_{\gamma_r}$ is the compositions of the Leray coboundary maps of the divisors $D^{(Z)}_{\gamma_k}$. The resulting $\mathcal{L}_{\mathcal{N}}(\Sigma)$ is therefore, by construction, a T^r-torus bundle over Σ. At the level of differential forms, the residue map $\mathcal{R}_{\gamma_1}$ is given by integration along the circle fibers of the S^1-bundle $\partial\pi^{-1}(\Sigma) \to \Sigma$. Thus, the pairing $\langle \mathcal{R}_{\mathcal{N}}(\eta), \Sigma \rangle$ contains a $2\pi i$ factor, coming from the integration of a form df/f, with f the local defining equation of the hypersurface, along the circle fibers. This means that, when writing the pairings in terms of differential forms, one obtains (5.21). As shown in [12], if $\mathfrak{m}(X)$ is mixed Tate, the divisors D_γ and their intersections in $\overline{\mathrm{Conf}}_\Gamma(X)$ are mixed Tate motives, and so are the $D^{(Z)}_\gamma$ and their intersections $V^{(Z)}_{\mathcal{N}}$ in $Z[\Gamma]$. ☐

Corollary 5.15. *Given a $\mathcal{G}_\Gamma$-nest $\{\gamma_1, \ldots, \gamma_r\}$ let $V_{\mathcal{N}} = D_{\gamma_1} \cap \cdots \cap D_{\gamma_r}$ be the intersection of the corresponding divisors in $\overline{\mathrm{Conf}}_\Gamma(X)$. The residues $\mathcal{R}_{\mathcal{N}}(\eta_\Gamma)$ of Proposition 5.14 pair with the $2D|\mathbf{V}_\Gamma| - r$-dimensional cycles in $V^{(Z)}_{\mathcal{N}}$ given by $V_{\mathcal{N}} \times \{y\}$,*

$$\langle \mathcal{R}_{\mathcal{N}}(\eta_\Gamma), V_{\mathcal{N}} \rangle = \int_{V_{\mathcal{N}} \times \{y\}} \mathcal{R}_{\mathcal{N}}(\eta_\Gamma). \tag{5.23}$$

Proof. The chain of integration $\tilde{\sigma}^{(Z,y)}_\Gamma = \overline{\mathrm{Conf}}_\Gamma(X) \times \{y\}$ intersects the loci $V^{(Z)}_{\mathcal{N}} = V_{\mathcal{N}} \times X^{\mathbf{V}_\Gamma}$ along $V_{\mathcal{N}} \times \times \{y\}$, where the $V_{\mathcal{N}}$ are the intersections $V_{\mathcal{N}} = D_{\gamma_1} \cap \cdots \cap D_{\gamma_r}$ of the divisors D_{γ_k} in $\overline{\mathrm{Conf}}_\Gamma(X)$. Thus, $\Sigma = V_{\mathcal{N}} \times \{y\}$

defines a $2D|\mathbf{V}_\Gamma| - r$-dimensional cycle in $V_{\mathcal{N}}^{(Z)}$, which can be paired with the form $\mathcal{R}_{\mathcal{N}}(\eta_\Gamma)$ of degree $2D|\mathbf{V}_\Gamma| - r$ on $V_{\mathcal{N}}^{(Z)}$. □

Remark 5.16. The reason for passing to the wonderful compactification $Z[\Gamma]$ and pulling back the form $\omega_\Gamma^{(Z)}$ along the projection $\pi_\Gamma : Z[\Gamma] \to Z^{\mathbf{V}_\Gamma}$ is in order to pass to a setting where the locus of divergence is described by divisors D_γ intersecting transversely in $\overline{\mathrm{Conf}}_\Gamma(X)$, while the intersections of the diagonals Δ_γ in $X^{\mathbf{V}_\Gamma}$ can be non-transverse. These transversality issues are discussed in more detail in [12]. There is a generalization of the theory of forms with logarithmic poles and Poincaré residues [35], that extends the case of [15] of normal crossings divisors, but in this more general setting the Poincaré residue gives meromorphic instead of holomorphic forms.

6. Regularization and integration

In this section, we describe a regularization of the Feynman integral

$$\int_{\tilde{\sigma}_\Gamma^{(Z,y)} \smallsetminus \mathcal{D}_\Gamma} \eta_\Gamma^{(Z)} \tag{6.1}$$

in distributional terms, using the theory of principal value and residue currents. This will show that one can express ambiguities in the regularization in terms of the iterated residues along the intersections of the divisors $D_\gamma^{(Z)}$, described in §5.7 above.

6.1. **Current-regularized Feynman amplitudes.**

We review briefly some well known facts about residue and principal value currents and we apply them to the Feynman amplitude regularization.

6.1.1. *Residue currents and Mellin transforms.* Recall that, for a single smooth hypersurface defined by an equation $\{f = 0\}$, the residue current $[Z_f]$, supported on the hypersurface, is defined as

$$[Z_f] = \frac{1}{2\pi i}\bar{\partial}[\frac{1}{f}] \wedge df := \frac{1}{2\pi i}\bar{\partial}\partial \log |f|^2.$$

This is known as the Poincaré–Lelong formula. It can also be seen as a limit

$$\int_{Z_f} \varphi = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{|f|=\epsilon} \frac{df}{f} \wedge \varphi.$$

A generalization is given by the Coleff–Herrera residue current [14], associated to a collection of functions $\{f_1, \ldots, f_r\}$. Under the assumption that these define a complete intersection $V = \{f_1 = \cdots = f_r = 0\}$, the residue current

$$\mathcal{R}_f = \bar{\partial}[\frac{1}{f_1}] \wedge \cdots \wedge \bar{\partial}[\frac{1}{f_r}] \tag{6.2}$$

is obtained as a limit

$$\mathcal{R}_f(\varphi) = \lim_{\delta\to 0} \int_{T_{\epsilon(\delta)}(f)} \frac{\varphi}{f_1\cdots f_r},$$

with $T_{\epsilon(\delta)}(f) = \{|f_k| = \epsilon_k(\delta)\}$, with the limit taken over "admissible paths" $\epsilon(\delta)$, which satisfy the properties

$$\lim_{\delta\to 0} \epsilon_r(\delta) = 0, \quad \lim_{\delta\to 0} \frac{\epsilon_k(\delta)}{(\epsilon_{k+1}(\delta))^\ell} = 0,$$

for $k = 1, \dots, r$ and any positive integer ℓ. The test form φ is a $(2n-r)$-form of type $(n, n-r)$, where $2n$ is the real dimension of the ambient space, and the residual current obtained in this way is a $(0,r)$-current. For more details, see §3 of [8] and [43]. Notice that, while in general one cannot take products of distributions, the Coleff–Herrera product (6.2) is well defined for residue currents, as well as between residue and principal value currents.

Moreover, the Mellin transform

$$\Gamma_f^\varphi(\lambda) = \int_{\mathbb{R}_+^r} \mathcal{I}_f^\varphi(\epsilon)\, \epsilon^{\lambda - I}\, d\epsilon, \tag{6.3}$$

with

$$\mathcal{I}_f^\varphi(\epsilon) = \int_{T_\epsilon(f)} \frac{\varphi}{f_1\cdots f_r}$$

and with

$$\epsilon^{\lambda-I}\, d\epsilon = \epsilon_1^{\lambda_1-1}\cdots\epsilon_r^{\lambda_r-1}\, d\epsilon_1\wedge\cdots\wedge d\epsilon_r,$$

can also be written, as in [8], [43], as

$$\Gamma_f^\varphi(\lambda) = \frac{1}{(2\pi i)^r}\int_{\mathcal{X}} |f|^{2(\lambda-I)}\overline{df}\wedge\varphi, \tag{6.4}$$

where the integration is on the ambient variety $\mathcal{X}$ and where

$$|f|^{2(\lambda-I)} = |f_1|^{2(\lambda_1-1)}\cdots|f_r|^{2(\lambda_r-1)}, \quad \text{and} \quad \overline{df} = \overline{df_1}\wedge\cdots\wedge\overline{df_r}.$$

When $\{f_1,\dots,f_r\}$ define a complete intersection, the function $\lambda_1\cdots\lambda_r\Gamma_f^\varphi(\lambda)$ is holomorphic in a neighborhood of $\lambda = 0$ and the value at zero is given by the residue current ([8], [43])

$$\mathcal{R}_f(\varphi) = \lambda_1\cdots\lambda_r\Gamma_f^\varphi(\lambda)|_{\lambda=0}. \tag{6.5}$$

Equivalently, (6.4) and (6.5) can also be written as

$$\lim_{\lambda\to 0}\lambda\Gamma_f^\varphi(\lambda) = \lim_{\lambda\to 0}\frac{1}{(2\pi i)^r}\int_{\mathcal{X}} \frac{\bar\partial|f_r|^{2\lambda_r}\wedge\cdots\wedge\bar\partial|f_1|^{2\lambda_1}}{f_r\cdots f_1}\wedge\varphi, \tag{6.6}$$

where the factor λ on the left-hand-side stands for $\lambda_1\cdots\lambda_r$ as in (6.5).

The Poincaré–Lelong formula, in this more general case of a complete intersection defined by a collection $\{f_1,\dots,f_r\}$, expresses the integration current Z_f as

$$[Z_f] = \frac{1}{(2\pi i)^r}\bar\partial[\frac{1}{f_r}]\wedge\cdots\wedge\bar\partial[\frac{1}{f_1}]\wedge df_1\wedge\cdots\wedge df_r. \tag{6.7}$$

The correspondence between residue currents and the Poincaré residues on complete intersections, discussed above in §5.7, is given for instance in Theorem 4.1 of [3].

6.1.2. *Principal value current.* The principal value current $[1/f]$ of a single holomorphic function f can be computed as [27], [38]

$$\langle [\frac{1}{f}], \phi \rangle = \lim_{\epsilon \to 0} \int_{|f|>\epsilon} \frac{\phi \, d\zeta \wedge d\bar{\zeta}}{f}, \tag{6.8}$$

where ϕ is a test function. More generally, for $\{f_1, \ldots, f_r\}$ as above, the principal value current is given by

$$\langle [\frac{1}{f}], \phi \rangle = \lim_{\epsilon \to 0} \int_{N_\epsilon(f)} \frac{\phi}{f_r \cdots f_1}, \tag{6.9}$$

with ϕ a test form and with

$$N_\epsilon(f) = \{|f_k| > \epsilon_k\}. \tag{6.10}$$

More generally we will use the following notation.

Definition 6.1. *Given a meromorphic (p,q)-form η on an m-dimensional smooth projective variety $\mathcal{X}$, with poles along an effective divisor $\mathcal{D} = D_1 \cup \cdots D_r$, where the components D_k are smooth hypersurfaces defined by equations $f_k = 0$, the principal value current $PV(\eta)$ is defined by*

$$\langle PV(\eta), \phi \rangle = \lim_{\epsilon \to 0} \int_{N_\epsilon(f)} \eta \wedge \phi, \tag{6.11}$$

for an $(m-p, m-q)$ test form ϕ, with $N_\epsilon(f)$ defined as in (6.10).

The following simple Lemma describes the source of ambiguities and its relation to residues.

Lemma 6.2. *When the test form ϕ is modified to $\phi + \bar{\partial}\psi$, the principal value current satisfies*

$$\langle [\frac{1}{f}], \phi + \bar{\partial}\psi \rangle = \langle [\frac{1}{f}], \phi \rangle - \langle \bar{\partial}[\frac{1}{f}], \psi \rangle, \tag{6.12}$$

where $\bar{\partial}[1/f]$ is the residue current $\mathcal{R}_f$ of (6.2).

Proof. By Stokes theorem, we have

$$\langle [\frac{1}{f}], \bar{\partial}\psi \rangle = \lim_{\epsilon \to 0} \int_{|f|>\epsilon} \frac{\bar{\partial}\psi}{f} = \lim_{\epsilon \to 0} - \int_{|f|=\epsilon} \frac{\psi}{f} = -\langle \bar{\partial}[\frac{1}{f}], \psi \rangle.$$

□

We now return to the case of the Feynman amplitudes and describe the corresponding regularization and ambiguities.

6.1.3. *Principal value and Feynman amplitude.* We can regularize the Feynman amplitude given by the integral (6.1), interpreted in the distributional sense, as in §2.3.4, using the principal value current.

Definition 6.3. *The principal value regularization of the Feynman amplitude* (6.1) *is given by the current* $PV(\eta_\Gamma^{(Z)})$ *defined as in* (6.11),

$$\langle PV(\eta_\Gamma^{(Z)}), \varphi\rangle = \lim_{\epsilon\to 0} \int_{N_\epsilon(f)} \varphi\ \eta,$$

for a test function φ.

We can also write the regularized integral in the following form.

Lemma 6.4. *The regularized integral satisfies*

$$\langle PV(\eta_\Gamma^{(Z)}), \varphi\rangle = \lim_{\lambda\to 0} \int_{\tilde\sigma_\Gamma^{(Z,y)}} |f_n|^{2\lambda_n}\cdots|f_1|^{2\lambda_1}\eta_\Gamma^{(Z,y)}\ \varphi$$

where $n = n_\Gamma$ *is the cardinality* $n_\Gamma = \#\mathcal{G}_\Gamma$ *of the building set* $\mathcal{G}_\Gamma$ *and the* f_k *are the defining equations of the* $D_\gamma^{(Z)}$ *in* $\mathcal{G}_\Gamma$

Proof. The form $\eta_\Gamma^{(Z)}$ has poles along the divisor $\mathcal{D}_\Gamma = \cup_{\Delta_\gamma^{(Z)}\in\mathcal{G}_\Gamma} D_\gamma^{(Z)}$. Thus, if we denote by f_k, with $k = 1,\dots,n$ with $n = \#\mathcal{G}_\Gamma$ the defining equations of the $D_\gamma^{(Z)}$, we can write the principal value current in the form

$$\lim_{\lambda\to 0} \int_{\tilde\sigma_\Gamma^{(Z,y)}} \frac{|f_n|^{2\lambda_n}\cdots|f_1|^{2\lambda_1}}{f_n\cdots f_1} h\,\varphi,$$

with h an algebraic from without poles and φ is a test function. □

6.1.4. *Pseudomeromorphic currents.* If $\{f_1,\dots,f_n\}$ define a complete intersection $V = \{f_1 = \cdots = f_r = 0\}$ in a smooth projective variety $\mathcal{X}$, an *elementary pseudomeromorphic current* is a current of the form

$$C_{r,n} := [\frac{1}{f_n}]\wedge\cdots\wedge[\frac{1}{f_{r+1}}]\wedge\bar\partial[\frac{1}{f_r}]\wedge\cdots\wedge\bar\partial[\frac{1}{f_1}], \tag{6.13}$$

for some $1 \le r \le n$, where the products of principal value and residue currents are well defined Coleff–Herrera products and the resulting current is commuting in the principal value factors and anticommuting in the residue factors. These distributions also have a Mellin transform formulation (see [8]) as

$$\langle C_{r,n}, \phi\rangle = \lim_{\lambda\to 0} \frac{1}{(2\pi i)^r} \int_{\mathcal{X}} \prod_{k=r+1}^{n} \frac{|f_k|^{2\lambda_k}}{f_k} \bigwedge_{j=1}^{r} \bar\partial\left(\frac{|f_j|^{2\lambda_j}}{f_j}\right)\wedge\phi. \tag{6.14}$$

6.2. **Ambiguities of regularized Feynman integrals.** We can use the formalism of residue currents recalled above to describe the ambiguities in the principal value regularization of Feynman amplitudes of Definition 6.3.

6.2.1. *Feynman amplitude and residue currents.* As in §5.7, consider a $\mathcal{G}_\Gamma$-nest $\{\gamma_1, \dots, \gamma_r\}$ and the associated intersection $V_{\mathcal{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$. Also let $n_\Gamma = \#\mathcal{G}_\Gamma$ and let f_k, for $k = 1, \dots, n_\Gamma$ be the defining equations for the $D_\gamma^{(Z)}$, for γ ranging over the subgraphs defining the building set $\mathcal{G}_\Gamma$. For $\epsilon = (\epsilon_k)$, we define

$$\tilde{\sigma}_{\Gamma,\epsilon}^{(Z,y)} := \tilde{\sigma}_\Gamma^{(Z,y)} \cap N_\epsilon(f), \tag{6.15}$$

with $N_\epsilon(f)$ defined as in (6.10). The principal value regularization of Definition 6.3 can then be written as

$$\langle PV(\eta_\Gamma^{(Z,y)}), \varphi \rangle = \lim_{\epsilon \to 0} \int_{\tilde{\sigma}_{\Gamma,\epsilon}^{(Z,y)}} \varphi \, \eta_\Gamma^{(Z,y)},$$

where the limit is taken over admissible paths.

Similarly, given a $\mathcal{G}_\Gamma$-nest $\mathcal{N} = \{\gamma_1, \dots, \gamma_r\}$, we introduce the notation

$$\tilde{\sigma}_{\Gamma,\mathcal{N},\epsilon}^{(Z,y)} := \tilde{\sigma}_\Gamma^{(Z,y)} \cap T_{\mathcal{N},\epsilon}(f) \cap N_{\mathcal{N},\epsilon}(f), \tag{6.16}$$

where $T_{\mathcal{N},\epsilon}(f) = \{|f_k| = \epsilon_k, \, k = 1, \dots, r\}$ and $N_{\mathcal{N},\epsilon}(f) = \{|f_k| > \epsilon, \, k = r+1, \dots, n\}$, where we have ordered the n subgraphs γ in $\mathcal{G}_\Gamma$ so that the first r belong to the nest $\mathcal{N}$.

Proposition 6.5. *For a $\mathcal{G}_\Gamma$-nest $\mathcal{N} = \{\gamma_1, \dots, \gamma_r\}$, as above, the limit*

$$\lim_{\epsilon \to 0} \int_{\tilde{\sigma}_{\Gamma,\mathcal{N},\epsilon}^{(Z,y)}} \varphi \, \eta_\Gamma^{(Z,y)} \tag{6.17}$$

determines a pseudomeromorphic current, whose residue part is an iterated residue supported on $V_{\mathcal{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$.

Proof. Let f_k, with $k = 1, \dots, n$ with $n = \#\mathcal{G}_\Gamma$, be the defining equations of the $D_\gamma^{(Z)}$. We assume the subgraphs in $\mathcal{G}_\Gamma$ are ordered so that the first r belong to the given $\mathcal{G}_\Gamma$-nest $\mathcal{N}$. We can then write the current (6.17) in the form $\langle C_{r,n}, h\varphi \rangle$, where $C_{r,n}$ is the elementary pseudomeromorphic current of (6.13) and h is algebraic without poles. ☐

6.2.2. *Residue currents as ambiguities.* With the same setting as in Proposition 6.5, we then have the following characterization of the ambiguities of the principal value regularization.

Proposition 6.6. *The ambiguities in the current-regularization $PV(\eta_\Gamma^{(Z,y)})$ are given by iterated residues supported on the intersections $V_{\mathcal{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$, of divisors corresponding to $\mathcal{G}_\Gamma$-nests $\mathcal{N} = \{\gamma_1, \dots, \gamma_r\}$.*

Proof. As above, we have

$$\langle PV(\eta_\Gamma^{(Z,y)}), \varphi \rangle = \lim_{\epsilon \to 0} \int_{\tilde{\sigma}_{\Gamma,\epsilon}^{(Z,y)}} \varphi \, \eta_\Gamma^{(Z,y)} = \lim_{\lambda \to 0} \int_{\tilde{\sigma}_\Gamma^{(Z,y)}} \frac{|f_n|^{2\lambda_n} \cdots |f_1|^{2\lambda_1}}{f_n \cdots f_1} \, h \, \varphi$$

If we replace the form $h\,\varphi$ with a form $h\varphi + \bar\partial_{\mathcal{N}}\psi$, where $\mathcal{N}$ is a $\mathcal{G}_\Gamma$-nest and the notation $\bar\partial_{\mathcal{N}}\psi$ means a form

$$\bar\partial_{\mathcal{N}}\psi := \psi_n \cdots \psi_{r+1}\, \bar\partial\psi_r \wedge \cdots \wedge \bar\partial\psi_1,$$

for test functions ψ_k, $k = 1, \ldots, n$, we obtain a pseudomeromorphic current

$$\langle PV(\eta_\Gamma^{(Z,y)}), \bar\partial_{\mathcal{N}}\psi\rangle = \langle [\frac{1}{f_n}] \wedge \cdots \wedge [\frac{1}{f_{r+1}}] \wedge \bar\partial[\frac{1}{f_r}] \wedge \cdots \wedge \bar\partial[\frac{1}{f_1}], \psi\rangle,$$

with $\psi = \psi_n \cdots \psi_1$. Notice then that the residue part

$$\langle \bar\partial[\frac{1}{f_r}] \wedge \cdots \wedge \bar\partial[\frac{1}{f_1}], \psi\rangle = \mathcal{R}_{\mathcal{N}}(\psi)$$

is an iterated residue supported on $V_{\mathcal{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$. □

By the results of §5.7, and the relation between residue currents and iterated Poincaré residues (see [3]), when evaluated on algebraic test forms on the varieties $V_{\mathcal{N}}^{(Z)}$, these ambiguities can be expressed in terms of periods of mixed Tate motives, that is, by the general result of [11], in terms of multiple zeta values.

7. Other regularization methods

We now discuss a regularization method for the evaluation of the Feynman integral

$$\int_{\tilde\sigma_\Gamma^{(Z,y)}} \pi_\Gamma^*(\omega_\Gamma^{(Z)}),$$

with the pullback $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ to $Z[\Gamma]$ as in Corollary 5.10 and the chain of integration $\tilde\sigma_\Gamma^{(Z,y)}$ as in (5.17), obtained from the form $\omega_\Gamma^{(Z)}$ and the chain $\sigma_\Gamma^{(Z)}$ of Definition 2.6. The geometric method of regularization we adopt is based on the *deformation to the normal cone.*

A general method of regularization consists of deforming the chain of integration so that it no longer intersects the locus of divergences. We first describe briefly why this cannot be done directly within the space $Z[\Gamma]$ considered above, and then we introduce a simultaneous deformation of the form and of the space where integration happens, so that the integral can be regularized according to the general method mentioned above.

To illustrate where the problem arises, if one tries to deform the chain of integration away from the locus of divergence in $Z[\Gamma]$, consider the local problem near a point $z \in D_\gamma^{(Z)}$ in the intersection of $\tilde\sigma_\Gamma^{(Z,y)}$ with one of the divisors in the divergence locus of the form $\pi_\Gamma^*(\omega_\Gamma)$. Near this point, the locus of divergence is a product $D_\gamma \times X^{\mathbf{V}_\Gamma}$. We look at the intersection of the integration chain $\tilde\sigma_\Gamma^{(Z,y)}$ with a small tubular neighborhood T_ϵ of $D_\gamma \times X^{\mathbf{V}_\Gamma}$. We have

$$\tilde\sigma_\Gamma^{(Z,y)} \cap \partial T_\epsilon = \partial\pi_\epsilon^{-1}(D_\gamma) \times \{y\},$$

with $\pi_\epsilon : T_\epsilon(D_\gamma) \to D_\gamma$ the projection of the 2-disc bundle and $\partial\pi_\epsilon^{-1}(D_\gamma)$ a circle bundle, locally isomorphic to $D_\gamma \times S^1$. Thus, locally, $\tilde{\sigma}_\Gamma^{(Z,y)} \cap T_\epsilon$ looks like a ball $B^{2D|\mathbf{V}_\Gamma|} \times \{0\}$ inside a ball $B^{4D|\mathbf{V}_\Gamma|}$. Locally, we can think of the problem of deforming the chain of integration in a neighborhood of the divergence locus as the question of deforming a ball $B^{2D|\mathbf{V}_\Gamma|} \times \{0\}$ leaving fixed the boundary $S^{2D|\mathbf{V}_\Gamma|-1} \times \{0\}$ inside a ball $B^{4D|V_\Gamma|}$ so as to avoid the locus $\{0\} \times B^{2D|\mathbf{V}_\gamma|}$ that lies in the divergence locus. However, one can check that the spheres $S^{2D|\mathbf{V}_\Gamma|-1} \times \{0\}$ and $\{0\} \times S^{2D|\mathbf{V}_\gamma|-1}$ are linked inside the sphere $S^{4D|\mathbf{V}_\Gamma|-1}$. This can be seen, for instance, by computing their Gauss linking integral (see [37])

$$\mathrm{Lk}(M,N) = \frac{1}{\mathrm{Vol}(S)} \int_{M\times N} \frac{\Omega_{k,\ell}(\alpha)}{\sin^n(\alpha)} [x, dx, y, dy] \tag{7.1}$$

with $M = S^k \times \{0\}$, $N = \{0\} \times S^\ell$, $S = S^n$, and with $k = \ell = 2D|\mathbf{V}_\Gamma| - 1$ and $n = 4D|\mathbf{V}_\Gamma| - 1$, where

$$\Omega_{k,\ell}(\alpha) := \int_{\theta=\alpha}^{\pi} \sin^k(\theta-\alpha)\sin^\ell(\theta)d\theta,$$

$$\alpha(x,y) := \mathrm{dist}_{S^n}(x,y), \quad x \in M,\ y \in N,$$

$$[x, dx, y, dy] := \det(x, \frac{\partial x}{\partial s_1}, \dots, \frac{\partial x}{\partial s_k}, y, \frac{\partial y}{\partial t_1}, \dots, \frac{\partial y}{\partial t_\ell})\, ds\, dt,$$

with x, y the embeddings of S^k and S^ℓ in S^n and s, t the local coordinates on S^k and S^ℓ. Then one can see (§4 of [37]) that in S^n with $n = k + \ell + 1$ the linking number is $\mathrm{Lk}(S^k \times \{0\}, \{0\} \times S^\ell) = 1$.

This type of problem can be easily avoided by introducing a simultaneous deformation of the form $\pi_\Gamma^*(\omega_\Gamma)$ and of the space $Z[\Gamma]$ as we show in the following.

7.1. **Form regularization.** We first regularize the form $\omega_\Gamma^{(Z)}$ by embedding the configuration space $Z^{\mathbf{V}_\Gamma}$ as the fiber over zero in a one parameter family $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$ and using the additional coordinate $\zeta \in \mathbb{P}^1$ to alter the differential form in a suitable way.

Definition 7.1. *The regularization of the Feynman amplitude $\omega_\Gamma^{(Z)}$ on the space $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$ is given by*

$$\tilde{\omega}_\Gamma^{(Z)} = \prod_{e\in\mathbf{E}_\Gamma} \frac{1}{(\|x_{s(e)} - x_{t(e)}\|^2 + |\zeta|^2)^{D-1}} \bigwedge_{v\in\mathbf{V}_\Gamma} dx_v \wedge d\bar{x}_v \wedge d\zeta \wedge d\bar{\zeta}, \tag{7.2}$$

where ζ is the local coordinate on $\mathbb{P}^1$.

Lemma 7.2. *The divergent locus $\{\tilde{\omega}_\Gamma^{(Z)} = \infty\}$ of the form (7.2) on $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$ is given by the locus $\cup_{e\in\mathbf{E}_\Gamma} \Delta_e^{(Z)} \subset Z^{\mathbf{V}_\Gamma} \times \{0\}$.*

Proof. The locus of divergence is the intersection of $\{\zeta = 0\}$ and the union of the products $\Delta_e^{(Z)} \times \mathbb{P}^1 = \{x_{s(e)} - x_{t(e)} = 0\}$. □

Notice that we have introduced in the form (7.2) an additional variable of integration, $d\zeta \wedge d\bar{\zeta}$. The reason for shifting the degree of the form will become clear later in this section (see §7.5 below), where we see that, when using the deformation to the normal cone, the chain of integration $\sigma_\Gamma^{(Z,y)}$ is also extended by an additional complex dimension to $\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1$, of which one then takes a proper transforms and deforms it inside the deformation to the normal cone. In terms of the distributional interpretation of the Feynman amplitudes of §2.3.4, the relation between the form (7.2) and the original amplitude (2.9) can be written as

$$\omega_\Gamma^{(Z)} = \int \prod_{e \in \mathbf{E}_\Gamma} \frac{\delta(\zeta = 0)}{(\|x_{s(e)} - x_{t(e)}\|^2 + |\zeta|^2)^{D-1}} \bigwedge_{v \in \mathbf{V}_\Gamma} dx_v \wedge d\bar{x}_v \wedge d\zeta \wedge d\bar{\zeta}, \tag{7.3}$$

where the distributional delta constraint can be realized as a limit of normalized integrations on small tubular neighborhoods of the central fiber $\zeta = 0$ in the trivial fibration $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$.

7.2. **Deformation to the normal cone.** The deformation to the normal cone is the natural algebro-geometric replacement for tubular neighborhoods in smooth geometry, see [18]. We use it here to extend the configuration space $Z^{\mathbf{V}_\Gamma}$ to a trivial fibration $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$ and then replacing the fiber over $\{0\} \in \mathbb{P}^1$ with the wonderful compactification $Z[\Gamma]$. This will allow us to simultaneously regularize the form and the chain of integration. For simplicity we illustrate the construction for the case where the graph Γ is itself biconnected.

Proposition 7.3. *Let Γ be a biconnected graph. Starting with the product $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$, a sequence of blowups along loci parameterized by the $\Delta_\gamma^{(Z)} \times \{0\}$, with γ induced biconnected subgraphs yields a variety $\mathcal{D}(Z[\Gamma])$ fibered over $\mathbb{P}^1$ such that the fiber over all points $\zeta \in \mathbb{P}^1$ with $\zeta \neq 0$ is still equal to $Z^{\mathbf{V}_\Gamma}$, while the fiber over $\zeta = 0$ has a component equal to the wonderful compactification $Z[\Gamma]$ and other components given by projectivizations $\mathbb{P}(C \oplus 1)$ with C the normal cone of the blowup locus.*

Proof. We start with the product $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$. We then perform the first blowup of the iterated sequence of §5.2.1 on the fiber over $\zeta = 0$ namely we blowup the locus $\Delta_\Gamma^{(Z)} \times \{0\}$, with $\Delta_\Gamma^{(Z)}$ the deepest diagonal, inside $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$. (Note that this is where we are using the biconnected hypothesis on Γ, otherwise the first blowup may be along induced biconnected subgraphs with a smaller number of vertices.) The blowup $\mathrm{Bl}_{\Delta_\Gamma^{(Z)} \times \{0\}}(Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1)$ is equal to $Z^{\mathbf{V}_\Gamma} \times (\mathbb{P}^1 \smallsetminus \{0\})$ away from $\zeta = 0$, while over the point $\zeta = 0$ it has a fiber with two components. One of the components is isomorphic to the blowup of

$Z^{\mathbf{V}_\Gamma}$ along $\Delta_\Gamma^{(Z)}$, that is, $\mathrm{Bl}_{\Delta_\Gamma^{(Z)}}(Z^{\mathbf{V}_\Gamma}) = Y_1$, with the notation of §5.2.1. The other component is equal to $\mathbb{P}(C_{Z^{\mathbf{V}_\Gamma}}(\Delta_\Gamma^{(Z)}) \oplus 1)$ where $C_{Z^{\mathbf{V}_\Gamma}}(\Delta_\Gamma^{(Z)})$ is the normal cone of $\Delta_\Gamma^{(Z)}$ in $Z^{\mathbf{V}_\Gamma}$. Since $\Delta_\Gamma^{(Z)} \simeq X \times X^{\mathbf{V}_\Gamma}$ is smooth, the normal cone is the normal bundle of $\Delta_\Gamma^{(Z)}$ in $Z^{\mathbf{V}_\Gamma}$. The two Cartier divisors Y_1 and $\mathbb{P}(C_{Z^{\mathbf{V}_\Gamma}}(\Delta_\Gamma^{(Z)}) \oplus 1)$ meet along $\mathbb{P}(C_{Z^{\mathbf{V}_\Gamma}}(\Delta_\Gamma^{(Z)}))$. We can then proceed to blow up the further loci $\Delta_\gamma^{(Z)}$ with $\gamma \in \mathcal{G}_{n-1,\Gamma}$ inside the special fiber $\tilde{\pi}^{-1}(0)$ in $\mathrm{Bl}_{\Delta_\Gamma^{(Z)} \times \{0\}}(Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1)$, where

$$\tilde{\pi} : \mathrm{Bl}_{\Delta_\Gamma^{(Z)} \times \{0\}}(Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1) \to Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$$

is the projection. These loci lie in the intersection of the two components of the special fiber $\tilde{\pi}^{-1}(0)$. Thus, at the next stage we obtain a variety that again agrees with $Z^{\mathbf{V}_\Gamma} \times (\mathbb{P}^1 \smallsetminus \{0\})$ away from the central fiber, while over $\zeta = 0$ it now has a component equal to Y_2 and further components coming from the normal cone after this additional blowup. After iterating this process as in §5.2.1 we obtain a variety that has fiber $Z^{\mathbf{V}_\Gamma}$ over all points $\zeta \neq 0$ and over $\zeta = 0$ it has a component equal to the wonderful compactification $Z[\Gamma]$ and other components coming from normal cones. □

Notice that one can also realize the iterated blowup of §5.2.1 as a single blowup over a more complicated locus and perform the deformation to the normal cone for that single blowup. We proceed as in Proposition 7.3, as it will be easier in this way to follow the effect that this deformation has on the motive.

The main reason for introducing the deformation to the normal cone, as we discuss more in detail in §7.5 below, is the fact that it will provide us with a natural mechanism for deforming the chain of integration away from the locus of divergences. The key idea is depicted in Figure 1, where one considers a variety $\mathcal{X}$ and the deformation $\mathrm{Bl}_{\mathcal{Y}\times\{0\}}(\mathcal{X} \times \mathbb{P}^1)$. If $\pi : \mathrm{Bl}_{\mathcal{Y}\times\{0\}}(\mathcal{X} \times \mathbb{P}^1) \to \mathbb{P}^1$ denotes the projection, the special fiber $\pi^{-1}(0)$ has two components, one given by the blowup $\mathrm{Bl}_{\mathcal{Y}}(\mathcal{X})$ of $\mathcal{X}$ along $\mathcal{Y}$ and the other is the normal cone $\mathbb{P}(C_{\mathcal{X}}(\mathcal{Y}) \oplus 1)$ of $\mathcal{Y}$ inside $\mathcal{X}$. The two components meet along $\mathbb{P}(C_{\mathcal{X}}(\mathcal{Y}))$. As shown in §2.6 of [19], one can use the deformation to the normal cone to deform $\mathcal{Y}$ to the zero section of the normal cone. Thus, given a subvariety $\mathcal{Z} \subset \mathcal{Y}$ the proper transform $\overline{\mathcal{Z} \times \mathbb{P}^1}$ in $\mathrm{Bl}_{\mathcal{Y}\times\{0\}}(\mathcal{X} \times \mathbb{P}^1)$ gives a copy of $\mathcal{Z}$ inside the special fiber $\pi^{-1}(0)$ lying in the normal cone component, see Figure 1.

7.3. **Deformation and the motive.** We check that passing from the space $Z^{\mathbf{V}_\Gamma}$ to the deformation $\mathcal{D}(Z[\Gamma])$ described in Proposition 7.3 does not alter the nature of the motive.

It is easy to see that this is the case at the level of virtual motives, that is, classes in the Grothendieck ring of varieties.

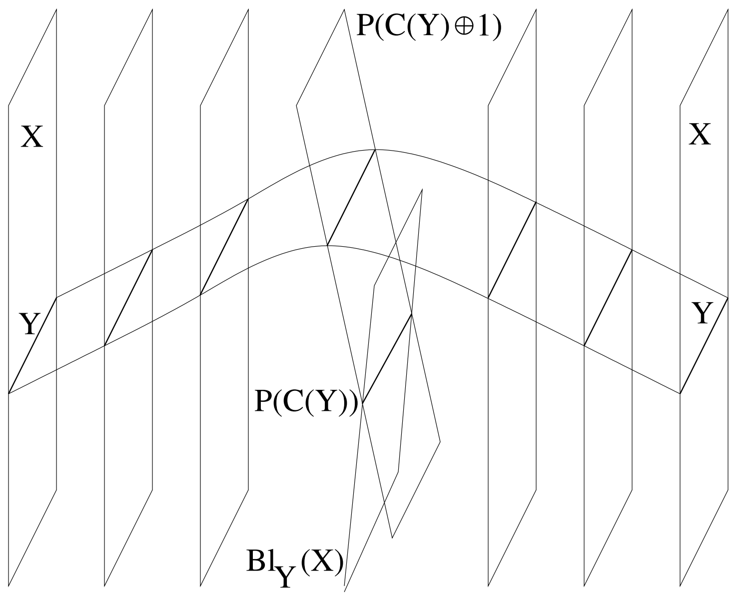

FIGURE 1. Deformation to the normal cone.

Proposition 7.4. *If the class $[X]$ in the Grothendieck ring of varieties $K_0(\mathcal{V})$ is a virtual mixed Tate motive, that is, it lies in the subring $\mathbb{Z}[\mathbb{L}]$ generated by the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$, then the class of $\mathcal{D}(Z[\Gamma])$ is also in $\mathbb{Z}[\mathbb{L}]$.*

Proof. As shown in Proposition 7.3, the space $\mathcal{D}(Z[\Gamma])$ is a fibration over $\mathbb{P}^1$, which is a trivial fibration over $\mathbb{P}^1 \smallsetminus \{0\}$ with fiber $Z^{\mathbf{V}_\Gamma}$. By inclusion-exclusion, we can write the class $[\mathcal{D}(Z[\Gamma])]$ in $K_0(\mathcal{V})$ as a sum of the class of the fibration over $\mathbb{P}^1 \smallsetminus \{0\}$, which is

$$[Z^{\mathbf{V}_\Gamma} \times (\mathbb{P}^1 \smallsetminus \{0\})] = [X]^{2\mathbf{V}_\Gamma} \mathbb{L}$$

and the class $[\pi^{-1}(0)]$ of the fiber over $\zeta = 0$, with $\pi : \mathcal{D}(Z[\Gamma]) \to \mathbb{P}^1$ the fibration. The component $[X]^{2\mathbf{V}_\Gamma}\mathbb{L}$ is in $\mathbb{Z}[\mathbb{L}]$ if the class $[X] \in \mathbb{Z}[\mathbb{L}]$ as we are assuming, so we need to check that the class $[\pi^{-1}(0)]$ is also in $\mathbb{Z}[\mathbb{L}]$. The locus $\pi^{-1}(0)$ is constructed in a sequence of steps as shown in Proposition 7.3. At the first step, we are dealing with the deformation to the normal cone $\mathrm{Bl}_{\Delta_\Gamma^{(Z)}}(Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1)$ and the fiber over zero is the union of Y_1 and $\mathbb{P}(C_{Z^{\mathbf{V}_\Gamma}}(\Delta_\Gamma^{(Z)}) \oplus 1)$, intersecting along $\mathbb{P}(C_{Z^{\mathbf{V}_\Gamma}}(\Delta_\Gamma^{(Z)}))$. Since $\Delta_\Gamma^{(Z)} \simeq X \times X^{\mathbf{V}_\Gamma}$ is smooth and a Tate motive, $\mathbb{P}(C_{Z^{\mathbf{V}_\Gamma}}(\Delta_\Gamma^{(Z)}) \oplus 1)$ is a projective bundle over a Tate motive so it is itself a Tate motive. So is $\mathbb{P}(C_{Z^{\mathbf{V}_\Gamma}}(\Delta_\Gamma^{(Z)}))$, for the same reason. So is also Y_1 because of the blowup formula for Grothendieck classes [9],

$$[Y_1] = [Z^{\mathbf{V}_\Gamma}] + \sum_{k=1}^{\operatorname{codim}(\Delta_\Gamma^{(Z)}\times\{0\})-1} [\Delta_\Gamma^{(Z)}]\,\mathbb{L}^k.$$

By the inclusion-exclusion relations in the Grothendieck ring, it then follows that if the two components of the fiber over zero are in $\mathbb{Z}[\mathbb{L}]$ and the class of their intersection also is, then so is also the class of the union, which is the class of the fiber itself. At the next step the fiber over zero is blown up again, this time along the (dominant transforms of) $\Delta_\gamma^{(Z)}$ with $\gamma \in \mathcal{G}_{n-1,\Gamma}$. Each of these is a blowup of a variety whose class is a virtual mixed Tate motive along a locus whose class is also a virtual mixed Tate motive, hence repeated application of the blowup formula in the Grothendieck ring and an argument analogous to the one used in the first step shows that the Grothendieck class of the fiber over zero is also in $\mathbb{Z}[\mathbb{L}]$. □

We can then, with a similar technique, improve the result from the level of Grothendieck classes to the level of motives.

Proposition 7.5. *If the motive $\mathfrak{m}(X)$ of the variety X is mixed Tate, then the motive $\mathfrak{m}(\mathcal{D}(Z[\Gamma])$ of the deformation $\mathcal{D}(Z[\Gamma])$ is also mixed Tate.*

Proof. As in the case of the Grothendieck classes, it suffices to check that, at each step in the construction of $\mathcal{D}(Z[\Gamma])$, the result remains inside the category of mixed Tate motives. It is clear that, if $\mathfrak{m}(X)$ is mixed Tate, then $\mathfrak{m}(Z)$, $\mathfrak{m}(Z^{\mathbf{V}_\Gamma})$ and $\mathfrak{m}(Z^{\mathbf{V}_\Gamma}\times\mathbb{P}^1)$ also are. At the next step, we use the blowup formula for Voevodsky motives (Proposition 3.5.3 of [45]) and we obtain

$$\mathfrak{m}(\mathrm{Bl}_{\Delta_\Gamma^{(Z)}\times\{0\}}(Z^{\mathbf{V}_\Gamma}\times\mathbb{P}^1)) =$$

$$\mathfrak{m}(Z^{\mathbf{V}_\Gamma}\times\mathbb{P}^1))\oplus \bigoplus_{k=1}^{\operatorname{codim}(\Delta_\Gamma^{(Z)}\times\{0\})-1} \mathfrak{m}(\Delta_\Gamma^{(Z)})(k)[2k].$$

This implies that $\mathfrak{m}(\mathrm{Bl}_{\Delta_\Gamma^{(Z)}\times\{0\}}(Z^{\mathbf{V}_\Gamma}\times\mathbb{P}^1))$ is mixed Tate if $\mathfrak{m}(X)$ is. The successive steps are again obtained by blowing up loci $\Delta_\gamma^{(Z)}$ whose motive $\mathfrak{m}(\Delta_\gamma^{(Z)})$ is mixed Tate, inside a variety whose motive is mixed Tate by the previous step, hence repeated application of the blowup formula for motives yields the result. □

The analog of Remark 5.7 also holds for the motive $\mathfrak{m}(\mathcal{D}(Z[\Gamma])$.

7.4. **Form regularization on the deformation.** Let $\tilde{\omega}_\Gamma^{(Z)}$ be the regularized form defined in (7.2). In order to allow room for a regularization of the chain of integration, we pull it back to the deformation to the normal cone described above.

Definition 7.6. *The regularization of the form* $\omega_\Gamma^{(Z)}$ *on the deformation space* $\mathcal{D}(Z[\Gamma])$ *is the pullback*

$$\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)}), \tag{7.4}$$

where $\tilde{\pi}_\Gamma : \mathcal{D}(Z[\Gamma]) \to Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$ *is the projection and* $\tilde{\omega}_\Gamma^{(Z)}$ *is the regularization of* (7.2)*.*

The locus of divergence $\{\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)}) = \infty\}$ inside the deformation space $\mathcal{D}(Z[\Gamma])$ is then given by the following.

Lemma 7.7. *The locus of divergence of the regularized Feynman amplitude* $\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})$ *on the space* $\mathcal{D}(Z[\Gamma])$ *is a union of divisors inside the central fiber,*

$$\bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma^{(Z)} \subset \pi^{-1}(0), \tag{7.5}$$

where $\pi : \mathcal{D}(Z[\Gamma]) \to \mathbb{P}^1$ *is the projection of the fibration.*

Proof. When pulling back the regularized form $\tilde{\omega}_\Gamma^{(Z)}$ from $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$ to $\mathcal{D}(Z[\Gamma])$, the poles of $\tilde{\omega}_\Gamma^{(Z)}$ along the diagonals $\Delta_\gamma^{(Z)} \times \{0\}$ yield (as in Proposition 5.9 and Corollary 5.10) poles along the divisors $D_\gamma^{(Z)}$, contained in the central fiber $\pi^{-1}(0)$ at $\zeta = 0$ of $\mathcal{D}(Z[\Gamma])$. □

7.5. **Deformation of the chain of integration.**

We now describe a regularization of the chain of integration, based on the deformation to the normal cone.

Proposition 7.8. *The proper transform of the chain* $\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1$ *inside* $\mathcal{D}(Z[\Gamma])$ *gives a deformation of the chain of integration, which does not intersect the locus of divergences of the form* $\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})$*.*

Proof. Consider the chain $\sigma_\Gamma^{(Z,y)} = X^{\mathbf{V}_\Gamma} \times \{y\}$ of (2.10), inside $Z^{\mathbf{V}_\Gamma}$. Extend it to a chain $\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1$ inside $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$. Let $\overline{\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1}$ denote the proper transform in the blowup $\mathcal{D}(Z[\Gamma])$. Then, as illustrated in Figure 1, we obtain a deformation of $\sigma_\Gamma^{(Z,y)}$ inside the normal cone component of the special fiber $\pi^{-1}(0)$ in $\mathcal{D}(Z[\Gamma])$ that is separated from the intersection with the component given by the blowup $Z[\Gamma]$. □

7.6. **Regularized integral.**

Using the deformation of the chain of integration and of the form, one can regularize the Feynman integral by

$$\int_{\Sigma_\Gamma^{(Z,y)}} \tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)}), \tag{7.6}$$

where $\Sigma_\Gamma^{(Z,y)}$ denotes the $(2\mathbf{V}_\Gamma+2)$-chain on $\mathcal{D}(Z[\Gamma])$ obtained as in Proposition 7.8. As in (7.3), one also has a corresponding integral on the intersection of the deformed chain $\Sigma^{(Z,y)}$ with the central fiber, which we can write as

$$\int_{\Sigma_\Gamma^{(Z,y)}} \delta(\pi^{-1}(0))\ \tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)}).$$

7.6.1. *Behavior at infinity.* The regularization (7.6) described above avoids divergences along the divisors $D_\gamma^{(Z)}$ in $Z[\Gamma]$. It remains to check the behavior at infinity, both in the $\mathbb{P}^1$-direction added in the deformation construction, and along the locus $\mathcal{D}_\infty$ in $\mathcal{D}(Z[\Gamma])$ defined, in the intersection of each fiber $\pi^{-1}(\zeta)$ with the chain of integration $\Sigma_\Gamma^{(Z,y)}$, by $\Delta_{\Gamma,\infty} := X^{\mathbf{V}_\Gamma} \smallsetminus \mathbb{A}^{D\mathbf{V}_\Gamma}$.

Proposition 7.9. *The integral* (7.6) *is convergent at infinity when* $D > 2$.

Proof. For the behavior of (7.6) when $\zeta \to \infty$ in $\mathbb{P}^1$, we see that the form behaves like $r^{-2D+2}\, rdr$, where $r = |\zeta|$. This gives a convergent integral for $2D - 3 > 1$. For the behavior at $\Delta_{\Gamma,\infty}$, consider first the case where a single radial coordinate $r_v = |x_v| \to \infty$. In polar coordinates, we then have a radial integral $r_v^{-(2D-2)\mathbf{E}_{\Gamma,v}}\, r^{D-1} dr$, where $\mathbf{E}_{\Gamma,v} = \{e \in \mathbf{E}_\Gamma \,|\, v \in \partial(e)\}$ is the valence $\upsilon(v)$ of the vertex v. This gives a convergent integral when $(2D-2)\upsilon(v) - D + 1 > 1$. Since $\upsilon(v) \geq 1$ and $2D - 2 \geq 0$, we have $(2D-2)\mathbf{E}_{\Gamma,v} - D + 1 \geq D - 1$, so the condition is satisfied whenever $D > 2$. More generally, one can have several $r_v \to \infty$. The strongest constraint comes from the case that behaves like $r^{-(2D-2)\sum_v \upsilon(v)} r^{D|\mathbf{V}_\Gamma|-1}$. In this case the convergence condition is given by $(2D-2)\upsilon_\Gamma - D|\mathbf{V}_\Gamma| > 0$, where $\upsilon_\Gamma = \sum_{v \in \mathbf{V}_\Gamma} \upsilon(v)$. Again we have $\upsilon_\Gamma \geq |\mathbf{V}_\Gamma|$, and we obtain

$$(2D-2)\upsilon_\Gamma - D|\mathbf{V}_\Gamma| \geq (D-2)|\mathbf{V}_\Gamma| > 0,$$

whenever $D > 2$. In this case the condition for convergence at $|\zeta| \to \infty$ is also satisfied. □

Acknowledgments. Parts of this work have been carried out during visits of the first author to the California Institute of Technology, the Institut des Hautes Études Scientifiques and the Max Planck Institut für Mathematik. We thank these institutions for their support. The second author acknowledges support from NSF grants DMS-0901221, DMS-1007207, DMS-1201512, and PHY-1205440. The authors thank Paolo Aluffi and Spencer Bloch for many useful conversations.

The first author's son, Uzay, was diagnosed with neuroblastoma, at the time when we were in the early stages of this project. His doctor, Tanju Başarır Özkan, not only saved Uzay with her exceptional professional skills, but also gave constant personal support, so that Ö.C. could return to work and continue the project. This paper is dedicated to her, to express the author's deepest gratitude: *Ellerin dert görmesin Tanju abla.*

References

[1] L.A. Aizenberg, A.P. Yuzhakov, *Integral representations in multidimensional complex analysis*, Transl. Amer. Math. Soc. Vol.58, 1980.

[2] L.A. Aizenberg, A.K. Tsikh, A.P. Yuzhakov, *Multidimensional residues and applications*, in "Several Complex Variables, II", Encyclopedia of Mathematical Sciences, Vol.8, Springer Verlag, 1994.

[3] A.G. Aleksandrov, A.K. Tsikh, *Multi-logarithmic differential forms on complete intersections*, J. Siberian Fed. Univ. Math and Phys. 2 (2008) 105–124.

[4] S. Ališauskas, *Coupling coefficients of $SO(n)$ and integrals involving Jacobi and Gegenbauer polynomials*, J. Phys. A, 35 (2002) 7323–7345.

[5] T.M. Apostol, T.H. Vu, *Dirichlet series related to the Riemann zeta function*, J. Number Theory, 19 (1984) 85–102.

[6] H. Bateman, A. Erdelyi, *Higher transcendental functions*, Vol.2, McGraw-Hill, 1953.

[7] I. Benjamini, L. Lovász, *Harmonic and analytic functions on graphs*, J. Geom. 76 (2003) 3–15.

[8] C.A. Berenstein, R. Gay, A. Vidras, A. Yger, *Residue currents and Bezout identities*, Progress in Math. Vol.114, Birkäuser, 1993.

[9] F. Bittner, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. 140 (2004), no. 4, 1011–1032.

[10] S. Bloch, lecture at Caltech, May 2012.

[11] F. Brown, *Mixed Tate motives over $\mathbb{Z}$*, Ann. of Math. 175 (2012) N.2, 949–976.

[12] Ö. Ceyhan, M. Marcolli *Feynman integrals and motives of configuration spaces*, Communications in Mathematical Physics, Vol.313 (2012) N.1, 35–70

[13] K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, *New approach to evaluation of multiloop Feynman integrals: the Gegenbauer polynomial x-space technique*, Nuclear Phys. B 174 (1980) 345–377.

[14] N. Coleff, M. Herrera, *Les courants résidus associés à une forme méromorphe*, Lect. Notes Math. 633, Springer, 1978.

[15] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Math., 163, Springer, Berlin, 1970.

[16] P. Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 5–57.

[17] P. Freitas, *Integrals of polylogarithmic functions, recurrence relations, and associated Euler sums*, Mathematics of Computation, Vol. 74 (2005) N. 251, 1425–1440.

[18] W. Fulton, *Intersection theory*, Second Edition, Springer, 1998.

[19] W. Fulton, *Introduction to Intersection Theory in Algebraic Geometry*, American Mathematical Soc., 1984.

[20] W. Fulton, R. MacPherson, *A compactification of configuration spaces.* Ann. of Math. (2) 139 (1994), no. 1, 183–225.

[21] J. Gallier, *Notes on spherical harmonics and linear representations of Lie groups*, preprint, 2009.

[22] A.B. Goncharov, *Periods and mixed motives*, arXiv.math/0202154v2.

[23] Ph. Griffiths, *On the periods of certain rational integrals. I, II*, Ann. of Math. (2) 90 (1969), 460–495; 496–541.

[24] Ph. Griffiths, J. Harris, *Principles of algebraic geometry*, Wiley, 1994.

[25] A. Grothendieck, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. No. 29 (1966) 95–103.

[26] R. Hartshorne, *On the De Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. No. 45 (1975), 5–99.

[27] M. Herrera, D. Lieberman, *Residues and principal values on complex space*, Math. Ann. 194 (1971) 259–294.

[28] G. Junker, *Explicit evaluation of coupling coefficients for the most degenerate representations of $SO(n)$*, J. Phys. A, 26 (1993) 1649–1661.

[29] A.M. Kytmanov, *The Bochner–Martinelli integral and its applications*, Birkhäuser, 1995.

[30] L. Li, *Chow Motive of FultonMacPherson Configuration Spaces and Wonderful Compactifications.* Michigan Math. J. 58 (2009).

[31] L. Li, *Wonderful compactification of an arrangement of subvarieties.* Michigan Math. J. 58 (2009), no. 2, 535–563.

[32] L.J. Mordell, *On the evaluation of some multiple series*, J. London Math. Soc. 33 (1958) 368–371.

[33] M. Morimoto, *Analytic functionals on the sphere*, Translations of Mathematical Monographs, Vol.178, AMS, 1998.

[34] N.M. Nikolov, R. Stora, I. Todorov, *Configuration space renormalization of massless QFT as an extension problem for associate homogeneous distributions*, IHES preprints 2011: IHES/P/11/07.

[35] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), no. 2, 265–291.

[36] D. Sébilleau, *On the computation of the integrated products of three spherical harmonics*, J. Phys. A, 31 (1998) 7157–7168.

[37] C. Shonkwiler, D.S. Vela-Vick, *Higher-dimensional linking integrals*, Proc. Amer. Math. Soc. 139 (2010) N.4, 1511–1519.

[38] L. Schwartz, *Division par une fonction holomorphe sur une variété analytique complexe*, Summa Brasil. Math. 3 (1955) 181–209.

[39] R. Stanley, *Acyclic orientations of graphs*, Discrete Math. 5 (1973) 171–178.

[40] E. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton, 1971.

[41] N.N. Tarkhanov, A.A. Shlapunov, *Green's formulas in complex analysis*, Journal of Mathematical Sciences, Vol.120 (2004) N.6, 1868–1900.

[42] L. Tornheim, *Harmonic double series*, Amer. J. Math. 72 (1950) 303–314.

[43] A. Tsikh, A. Yger, *Residue currents. Complex analysis*, J. Math. Sci. (N.Y.) 120 (2004) N. 6, 1916–1971.

[44] N.J. Vilenkin, *Special functions and the theory of group representations*, Translations of Mathematical Monographs, Vol.22, AMS, 1968.

[45] V. Voevodsky, *Triangulated categories of motives over a field* in Cycles, transfer and motivic homology theories, pp. 188238, Annals of Mathematical Studies, Vol. 143, Princeton, 2000.

[46] M. Waldschmidt, *Multiple polylogarithms: an introduction*, in "Number theory and discrete mathematics (Chandigarh, 2000)", pp.1–12, Trends Math., Birkhäuser, 2002.

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