

Structurable equivalence relations

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Abstract

For a class \mathcal{K} of countable relational structures, a countable Borel equivalence relation E is said to be \mathcal{K} -structurable if there is a Borel way to put a structure in \mathcal{K} on each E -equivalence class. We study in this paper the global structure of the classes of \mathcal{K} -structurable equivalence relations for various \mathcal{K} . We show that \mathcal{K} -structurability interacts well with several kinds of Borel homomorphisms and reductions commonly used in the classification of countable Borel equivalence relations. We consider the poset of classes of \mathcal{K} -structurable equivalence relations for various \mathcal{K} , under inclusion, and show that it is a distributive lattice; this implies that the Borel reducibility preordering among countable Borel equivalence relations contains a large sublattice. Finally, we consider the effect on \mathcal{K} -structurability of various model-theoretic properties of \mathcal{K} . In particular, we characterize the \mathcal{K} such that every \mathcal{K} -structurable equivalence relation is smooth, answering a question of Marks.

1 Introduction

(A) A countable Borel equivalence relation on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with the property that every equivalence class $[x]_E$, $x \in X$, is countable. We denote by \mathcal{E} the class of countable Borel equivalence relations. Over the last 25 years there has been an extensive study of countable Borel equivalence relations and their connection with group actions and ergodic theory. An important aspect of this work is an understanding of the kind of countable (first-order) structures that can be assigned in a uniform Borel way to each class of a given equivalence relation. This is made precise in the following definitions; see [JKL], Section 2.5.

Let $L = \{R_i \mid i \in I\}$ be a countable relational language, where R_i has arity n_i , and \mathcal{K} a class of countable structures in L closed under isomorphism. Let E be a countable Borel equivalence relation on a standard Borel space X . An L -**structure** on E is a Borel structure $\mathbb{A} = (X, R_i^{\mathbb{A}})_{i \in I}$ of L with universe X (i.e., each $R_i^{\mathbb{A}} \subseteq X^{n_i}$ is Borel) such that for $i \in I$ and $x_1, \dots, x_{n_i} \in X$, $R_i^{\mathbb{A}}(x_1, \dots, x_{n_i}) \implies x_1 E x_2 E \cdots E x_{n_i}$. Then each E -class C is the universe of the countable L -structure $\mathbb{A}|C$. If for all such C , $\mathbb{A}|C \in \mathcal{K}$, we say that \mathbb{A} is a \mathcal{K} -**structure** on E . Finally if E admits a \mathcal{K} -structure, we say that E is \mathcal{K} -**structurable**.

Many important classes of countable Borel equivalence relations can be described as the \mathcal{K} -structurable relations for appropriate \mathcal{K} . For example, the hyperfinite equivalence relations are exactly the \mathcal{K} -structurable relations, where \mathcal{K} is the class of linear orderings embeddable in \mathbb{Z} . The treeable equivalence relations are the \mathcal{K} -structurable relations, where \mathcal{K} is the class of countable trees (connected acyclic graphs). The equivalence relations generated by a free Borel action of a countable group Γ are the \mathcal{K} -structurable relations, where \mathcal{K} is the class of structures corresponding

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to free transitive Γ -actions. The equivalence relations admitting no invariant probability Borel measure are the \mathcal{K} -structurable relations, where $L = \{R, S\}$, R unary and S binary, and \mathcal{K} consists of all countably infinite structures $\mathbb{A} = (A, R^{\mathbb{A}}, S^{\mathbb{A}})$, with $R^{\mathbb{A}}$ an infinite, co-infinite subset of A and $S^{\mathbb{A}}$ the graph of a bijection between A and $R^{\mathbb{A}}$.

For $L = \{R_i \mid i \in I\}$ as before and countable set X , we denote by $\text{Mod}_X(L)$ the standard Borel space of countable L -structures with universe X . Clearly every countable L -structure is isomorphic to some $\mathbb{A} \in \text{Mod}_X(L)$, for $X \in \{1, 2, \dots, \mathbb{N}\}$. Given a class \mathcal{K} of countable L -structures, closed under isomorphism, we say that \mathcal{K} is Borel if $\mathcal{K} \cap \text{Mod}_X(L)$ is Borel in $\text{Mod}_X(L)$, for each countable set X . We are interested in Borel classes \mathcal{K} in this paper. For any $L_{\omega_1\omega}$ -sentence σ , the class of countable models of σ is Borel. By a classical theorem of Lopez-Escobar [LE], every Borel class \mathcal{K} of L -structures is of this form, for some $L_{\omega_1\omega}$ -sentence σ . We will also refer to such σ as a theory.

Adopting this model-theoretic point of view, given a theory σ and a countable Borel equivalence relation E , we put

$$E \models \sigma$$

if E is \mathcal{K} -structurable, where \mathcal{K} is the class of countable models of σ , and we say that E is **σ -structurable** if $E \models \sigma$. We denote by $\mathcal{E}_\sigma \subseteq \mathcal{E}$ the class of σ -structurable countable Borel equivalence relations. Finally we say that a class \mathcal{C} of countable Borel equivalence relations is **elementary** if it is of the form \mathcal{E}_σ , for some σ (which **axiomatizes** \mathcal{C}). In some sense the main goal of this paper is to study the global structure of elementary classes.

First we characterize which classes of countable Borel equivalence relations are elementary. We need to review some standard concepts from the theory of Borel equivalence relations. Given equivalence relations E, F on standard Borel spaces X, Y , resp., a **Borel homomorphism** of E to F is a Borel map $f: X \rightarrow Y$ with $x E y \implies f(x) F f(y)$. We denote this by $f: E \rightarrow_B F$. If moreover f is such that all restrictions $f|_{[x]_E}: [x]_E \rightarrow [f(x)]_F$ are bijective, we say that f is a **class-bijective homomorphism**, in symbols $f: E \rightarrow_B^{cb} F$. If such f exists we also write $E \rightarrow_B^{cb} F$. We similarly define the notion of **class-injective homomorphism**, in symbols \rightarrow_B^{ci} . A **Borel reduction** of E to F is a Borel map $f: X \rightarrow Y$ with $x E y \iff f(x) F f(y)$. We denote this by $f: E \leq_B F$. If f is also injective, it is called a **Borel embedding**, in symbols $f: E \sqsubseteq_B F$. If there is a Borel reduction of E to F we write $E \leq_B F$ and if there is a Borel embedding we write $E \sqsubseteq_B F$. An **invariant Borel embedding** is a Borel embedding f as above with $f(X) F$ -invariant. We use the notation $f: E \sqsubseteq_B^i F$ and $E \sqsubseteq_B^i F$ for these notions. By the usual Schroeder-Bernstein argument, $E \sqsubseteq_B^i F$ and $F \sqsubseteq_B^i E \iff E \cong_B F$, where \cong_B is Borel isomorphism.

Kechris-Solecki-Todorcevic [KST, 7.1] proved a universality result for theories of graphs, which was then extended to arbitrary theories by Miller; see Corollary 4.4.

Theorem 1.1 (Kechris-Solecki-Todorcevic, Miller). *For every theory σ , there is an invariantly universal σ -structurable countable Borel equivalence relation $E_{\infty\sigma}$, i.e., $E_{\infty\sigma} \models \sigma$, and $F \sqsubseteq_B^i E_{\infty\sigma}$ for any other $F \models \sigma$.*

Clearly $E_{\infty\sigma}$ is uniquely determined up to Borel isomorphism. In fact in Theorem 4.1 we formulate a “relative” version of this result and its proof that allows us to capture more information.

Next we note that clearly every elementary class is closed downwards under class-bijective Borel homomorphisms. We now have the following characterization of elementary classes (see Corollary 4.11).

Theorem 1.2. *A class $\mathcal{C} \subseteq \mathcal{E}$ of countable Borel equivalence relations is elementary iff it is (downwards-)closed under class-bijective homomorphisms and contains an invariantly universal element $E \in \mathcal{C}$.*

Examples of non-elementary classes include the class of non-smooth countable Borel equivalence relations (a countable Borel equivalence relation is **smooth** if it admits a Borel transversal), the class of equivalence relations admitting an invariant Borel probability measure, and the class of equivalence relations generated by a free action of *some* countable group. More generally, nontrivial unions of elementary classes are never elementary (see Corollary 4.5).

Next we show that every $E \in \mathcal{E}$ is contained in a (unique) smallest (under inclusion) elementary class (see Corollary 4.10).

Theorem 1.3. *For every $E \in \mathcal{E}$, there is a smallest elementary class containing E , namely $\mathcal{E}_E := \{F \in \mathcal{E} \mid F \rightarrow_B^{cb} E\}$.*

Many classes of countable Borel equivalence relations that have been extensively studied, like hyperfinite or treeable ones, are closed (downwards) under Borel reduction. It turns out that every elementary class is contained in a (unique) smallest (under inclusion) elementary class closed under Borel reduction (see Theorem 5.2).

Theorem 1.4. *For every elementary class \mathcal{C} , there is a smallest elementary class containing \mathcal{C} and closed under Borel reducibility, namely $\mathcal{C}^r := \{F \in \mathcal{E} \mid \exists E \in \mathcal{C} (F \leq_B E)\}$.*

We call an elementary class closed under reduction an **elementary reducibility class**. In analogy with Theorem 1.2, we have the following characterization of elementary reducibility classes (see Corollary 5.18). Below by a **smooth Borel homomorphism** of $E \in \mathcal{E}$ into $F \in \mathcal{E}$ we mean a Borel homomorphism for which the preimage of any point is smooth for E .

Theorem 1.5. *A class $\mathcal{C} \subseteq \mathcal{E}$ is an elementary reducibility class iff it is closed (downward) under smooth homomorphisms and contains an invariantly universal element $E \in \mathcal{C}$.*

We note that as a corollary of the proof of Theorem 1.4 every elementary reducibility class is also closed downward under class-injective Borel homomorphisms. Hjorth-Kechris [HK, D.3] proved (in our terminology and notation) that every \mathcal{C}^r (\mathcal{C} elementary) is closed under \subseteq , i.e., containment of equivalence relations on the same space. Since containment is a class-injective homomorphism (namely the identity), Theorem 1.4 generalizes this.

We also prove analogous results for Borel embeddability instead of Borel reducibility (see Theorem 5.1).

For each countably infinite group Γ denote by \mathcal{E}_Γ the elementary class of equivalence relations induced by free Borel actions of Γ . Its invariantly universal element is the equivalence relation induced by the free part of the shift action of Γ on \mathbb{R}^Γ . For trivial reasons this is not closed under Borel reducibility, so let \mathcal{E}_Γ^* be the elementary class of all equivalence relations whose aperiodic part is in \mathcal{E}_Γ . Then we have the following characterization (see Theorem 7.1).

Theorem 1.6. *Let Γ be a countably infinite group. Then the following are equivalent:*

- (i) \mathcal{E}_Γ^* is an elementary reducibility class.
- (ii) Γ is amenable.

We call equivalence relations of the form $E_{\infty\sigma}$ **universally structurable**. Denote by $\mathcal{E}_\infty \subseteq \mathcal{E}$ the class of universally structurable equivalence relations. In view of Theorem 1.1, an elementary class is uniquely determined by its invariantly universal such equivalence relation, and the poset of elementary classes under inclusion is isomorphic to the poset $(\mathcal{E}_\infty/\cong_B, \sqsubseteq_B^i)$ of Borel isomorphism classes of universally structurable equivalence relations under invariant Borel embeddability. It turns out that this poset has desirable algebraic properties (see Theorem 6.2).

Theorem 1.7. *The poset $(\mathcal{E}_\infty/\cong_B, \sqsubseteq_B^i)$ is an ω_1 -complete, distributive lattice. Moreover, the inclusion $(\mathcal{E}_\infty/\cong_B, \sqsubseteq_B^i) \subseteq (\mathcal{E}/\cong_B, \sqsubseteq_B^i)$ preserves (countable) meets and joins.*

This has implications concerning the structure of the class of universally structurable equivalence relations under Borel reducibility. The order-theoretic structure of the poset $(\mathcal{E}/\sim_B, \leq_B)$ of all bireducibility classes under \leq_B is not well-understood, apart from that it is very complicated (by [AK]). The first general study of this structure was made only recently by Kechris-Macdonald in [KMd]. In particular, they pointed out that it was even unknown whether there exists *any* pair of \leq_B -incomparable $E, F \in \mathcal{E}$ for which a \leq_B -meet exists. However it turns out that the subposet $(\mathcal{E}_\infty/\sim_B, \leq_B)$ behaves quite well (see Corollary 6.10).

Theorem 1.8. *The poset of universally structurable bireducibility classes, under \leq_B , $(\mathcal{E}_\infty/\sim_B, \leq_B)$ is an ω_1 -complete, distributive lattice. Moreover, the inclusion into the poset $(\mathcal{E}/\sim_B, \leq_B)$ of all bireducibility classes, under \leq_B , preserves (countable) meets and joins.*

Adapting the method of Adams-Kechris [AK], we also show that this poset is quite rich (see Theorem 6.21).

Theorem 1.9. *There is an order-embedding from the poset of Borel subsets of \mathbb{R} under inclusion into the poset $(\mathcal{E}_\infty/\sim_B, \leq_B)$.*

The combination of Theorem 1.8 and Theorem 1.9 answers the question mentioned in the paragraph following Theorem 1.7 by providing a large class of \leq_B -incomparable countable Borel equivalence relations for which \leq_B -meets exist.

An important question concerning structurability is which properties of a theory σ yield properties of the corresponding elementary class \mathcal{E}_σ . The next theorem provides the first instance of such a result. Marks [M, end of Section 4.3] asked (in our terminology) for a characterization of when the elementary class $\mathcal{E}_{\sigma_{\mathbb{A}}}$, where $\sigma_{\mathbb{A}}$ is a Scott sentence of a countable structure, consists of smooth equivalence relations, or equivalently, when $E_{\infty\sigma_{\mathbb{A}}}$ is smooth. We answer this problem in full generality, i.e., for an arbitrary theory σ . Although this result belongs purely in the category of Borel equivalence relations, our proof uses ideas and results from topological dynamics and ergodic theory (see Theorem 8.1).

Theorem 1.10. *Let σ be a theory. The following are equivalent:*

- (i) \mathcal{E}_σ contains only smooth equivalence relations, i.e., $E_{\infty\sigma}$ is smooth.
- (ii) There is an $L_{\omega_1\omega}$ -formula $\phi(x)$ which defines a finite nonempty subset in any countable model of σ .

Along these lines an interesting question is to find out what theories σ have the property that every aperiodic countable Borel equivalence relation is σ -structurable. A result that some particular

σ axiomatizes all aperiodic E shows that every such $E \in \mathcal{E}$ carries a certain type of structure, which can be useful in applications. A typical example is the very useful Marker Lemma (see [BK, 4.5.3]), which shows that every aperiodic E admits a decreasing sequence of Borel complete sections $A_0 \supseteq A_1 \supseteq \dots$ with empty intersection. This can be phrased as: every aperiodic countable Borel equivalence relation E is σ -structurable, where σ in the language $L = \{P_0, P_1, \dots\}$ asserts that each (unary) P_i defines a nonempty subset, $P_0 \supseteq P_1 \supseteq \dots$, and $\bigcap_i P_i = \emptyset$.

A particular case is when $\sigma = \sigma_{\mathbb{A}}$ is a Scott sentence of a countable structure. For convenience we say that E is **\mathbb{A} -structurable** if E is $\sigma_{\mathbb{A}}$ -structurable. Marks recently pointed out that the work of [AFP] implies a very general condition under which this happens (see Theorem 8.2).

Theorem 1.11 (Marks). *Let \mathbb{A} be a countable structure with trivial definable closure. Then every aperiodic countable Borel equivalence relation is \mathbb{A} -structurable.*

In particular (see Corollary 8.18) the following Fraïssé structures can structure every aperiodic countable Borel equivalence relation: $(\mathbb{Q}, <)$, the random graph, the random K_n -free graph (where K_n is the complete graph on n vertices), the random poset, and the rational Urysohn space.

Finally we mention an application to a model-theoretic question that has nothing to do with equivalence relations. The concept of amenability of a structure in the next result (see Corollary 8.19) can be either the one in [JKL, 2.6(iii)] or the one in [K91, 3.4]. This result was earlier proved by the authors by a different method (still using results of [AFP]) but it can also be seen as a corollary of Theorem 1.11.

Theorem 1.12. *Let \mathbb{A} be a countably infinite amenable structure. Then \mathbb{A} has non-trivial definable closure.*

(B) This paper is organized as follows: In Section 2 we review some basic background in the theory of Borel equivalence relations and model theory. In Section 3 we introduce the concept of structurability of equivalence relations and discuss various examples. In Section 4 we study the relationship between structurability and class-bijective homomorphisms, obtaining the tight correspondence given by Theorems 1.1 to 1.3; we also apply structurability to describe a product construction (class-bijective or “tensor” product) between countable Borel equivalence relations. In Section 5 we study the relationship between structurability and other kinds of homomorphisms, such as reductions; we also consider the relationship between reductions and compressible equivalence relations. In Section 6 we introduce some concepts from order theory convenient for describing the various posets of equivalence relations we are considering, and then study the poset $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$ of universally structurable equivalence relations (equivalently of elementary classes). In Section 7 we consider the elementary class \mathcal{E}_{Γ} of equivalence relations induced by free actions of a countable group Γ . In Section 8 we consider relationships between model-theoretic properties of a theory σ and the corresponding elementary class \mathcal{E}_{σ} . Finally, in Section 9 we list several open problems related to structurability.

The appendices contain some generalizations, alternative points of view, and miscellaneous results which are tangential to the main subject of this paper. In Appendix A we introduce fiber spaces (previously considered in [G] and [HK]), discuss their category-theoretic aspects, and discuss generalizations to that context of several concepts appearing in the body of this paper (including structurability and the various kinds of homomorphisms). In Appendix B we introduce a categorical structure on the class of all theories which interacts well with structurability. Finally, in Appendix C we prove a lattice-theoretic result generalizing the well-known Loomis-Sikorski

representation theorem for σ -Boolean algebras, which can be applied in particular to the lattice $(\mathcal{E}_\infty/\cong_B, \sqsubseteq_B^i)$ considered in Section 6.2.

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2 Preliminaries

For general model theory, see [Hod]. For general classical descriptive set theory, see [K95].

2.1 Theories and structures

By a **language**, we will always mean a countable first-order relational language, i.e., a countable set $L = \{R_i \mid i \in I\}$ of relation symbols, where each R_i has an associated arity $n_i \geq 1$. The only logic we will consider is the infinitary logic $L_{\omega_1\omega}$. We use letters like ϕ, ψ for formulas, and σ, τ for sentences. By a **theory**, we mean a pair (L, σ) where L is a language and σ is an $L_{\omega_1\omega}$ -sentence. When L is clear from context, we will often write σ instead of (L, σ) .

Let L be a language. By an **L -structure**, we mean in the usual sense of first-order logic, i.e., a tuple $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ where X is a set and for each n -ary relation symbol $R \in L$, $R^{\mathbb{A}} \subseteq X^n$ is an n -ary relation on X . Then as usual, for each formula $\phi(x_1, \dots, x_n) \in L_{\omega_1\omega}$ with n free variables, we have an interpretation $\phi^{\mathbb{A}} \subseteq X^n$ as an n -ary relation on X .

We write $\text{Mod}_X(L)$ for the set of L -structures with universe X . More generally, for a theory (L, σ) , we write $\text{Mod}_X(\sigma)$ for the set of models of σ with universe X . When X is countable, we equip $\text{Mod}_X(\sigma)$ with its usual standard Borel structure (see e.g., [K95, 16.C]).

If $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ is an L -structure and $f : X \rightarrow Y$ is a bijection, then we write $f(\mathbb{A})$ for the **pushforward structure**, with universe Y and

$$R^{f(\mathbb{A})}(\bar{y}) \iff R^{\mathbb{A}}(f^{-1}(\bar{y}))$$

for n -ary R and $\bar{y} \in Y^n$. When $X = Y$, this defines the **logic action** of S_X (the group of bijections of X) on $\text{Mod}_X(L, \sigma)$.

If $f : Y \rightarrow X$ is any function, then $f^{-1}(\mathbb{A})$ is the **pullback structure**, with universe Y and

$$R^{f^{-1}(\mathbb{A})}(\bar{y}) \iff R^{\mathbb{A}}(f(\bar{y})).$$

When f is the inclusion of a subset $Y \subseteq X$, we also write $\mathbb{A}|Y$ for $f^{-1}(\mathbb{A})$.

Every countable L -structure \mathbb{A} has a **Scott sentence** $\sigma_{\mathbb{A}}$, which is an $L_{\omega_1\omega}$ -sentence whose countable models are exactly the isomorphic copies of \mathbb{A} .

A **Borel class of L -structures** is a class \mathcal{K} of countable L -structures which is closed under isomorphism and such that $\mathcal{K} \cap \text{Mod}_X(L) \subseteq \text{Mod}_X(L)$ is Borel for every countable set X (equivalently, for $X \in \{1, 2, \dots, \mathbb{N}\}$). For example, for any $L_{\omega_1\omega}$ -sentence σ , the class of countable models of σ is Borel. By a classical theorem of Lopez-Escobar [LE], every Borel class \mathcal{K} of L -structures is of this form, for some σ . (While Lopez-Escobar's theorem is usually stated only for $\text{Mod}_{\mathbb{N}}(L)$, it is easily seen to hold also for $\text{Mod}_X(L)$ with X finite.)

2.2 Countable Borel equivalence relations

A **Borel equivalence relation** E on a standard Borel space X is an equivalence relation which is Borel as a subset of X^2 ; the equivalence relation E is **countable** if each of its classes is. We will also refer to the pair (X, E) as an equivalence relation.

Throughout this paper, we use \mathcal{E} to denote the class of countable Borel equivalence relations (X, E) .

If Γ is a group acting on a set X , then we let $E_\Gamma^X \subseteq X^2$ be the **orbit equivalence relation**:

$$x E_\Gamma^X y \iff \exists \gamma \in \Gamma (\gamma \cdot x = y).$$

If Γ is countable, X is standard Borel, and the action is Borel, then E_Γ^X is a countable Borel equivalence relation. Conversely, by the Feldman-Moore Theorem [FM], every countable Borel equivalence relation on a standard Borel space X is E_Γ^X for some countable group Γ with some Borel action on X .

If Γ is a group and X is a set, the **(right) shift action** of Γ on X^Γ is given by

$$(\gamma \cdot \bar{x})(\delta) := \bar{x}(\delta\gamma)$$

for $\gamma \in \Gamma$, $\bar{x} \in X^\Gamma$, and $\delta \in \Gamma$. We let $E(\Gamma, X) := E_\Gamma^{X^\Gamma} \subseteq (X^\Gamma)^2$ denote the orbit equivalence of the shift action. If Γ is countable and X is standard Borel, then $E(\Gamma, X)$ is a countable Borel equivalence relation. If Γ already acts on X , then that action embeds into the shift action, via

$$\begin{aligned} X &\longrightarrow X^\Gamma \\ x &\longmapsto (\gamma \mapsto \gamma \cdot x). \end{aligned}$$

In particular, any action of Γ on a standard Borel space embeds into the shift action of Γ on \mathbb{R}^Γ .

The **free part** of a group action of Γ on X is

$$\{x \in X \mid \forall 1 \neq \gamma \in \Gamma (\gamma \cdot x \neq x)\};$$

the action is **free** if the free part is all of X . We let $F(\Gamma, X)$ denote the orbit equivalence of the free part of the shift action of Γ on X^Γ .

An **invariant measure** for a Borel group action of Γ on X is a nonzero σ -finite Borel measure μ on X such that $\gamma_*\mu = \mu$ for all $\gamma \in \Gamma$ (where $\gamma_*\mu$ is the pushforward). An **invariant measure** on a countable Borel equivalence relation (X, E) is an invariant measure for some Borel action of a countable group Γ on X which generates E , or equivalently for any such action (see [KM, 2.1]). An invariant measure μ on (X, E) is **ergodic** if for any E -invariant Borel set $A \subseteq X$, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

2.3 Homomorphisms

Let $(X, E), (Y, F) \in \mathcal{E}$ be countable Borel equivalence relations, and let $f : X \rightarrow Y$ be a Borel map (we write $f : X \rightarrow_B Y$ to denote that f is Borel). We say that f is:

- a **homomorphism**, written $f : (X, E) \rightarrow_B (Y, F)$, if

$$\forall x, y \in X (x E y \implies f(x) F f(y)),$$

i.e., f induces a map on the quotient spaces $X/E \rightarrow Y/F$;

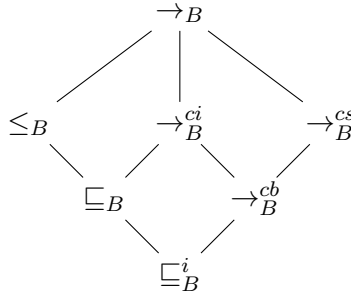
- a **reduction**, written $f : (X, E) \leq_B (Y, F)$, if f is a homomorphism and

$$\forall x, y \in X (f(x) F f(y) \implies x E y),$$

i.e., f induces an injection on the quotient spaces;

- a **class-injective homomorphism** (respectively, **class-surjective**, **class-bijective**), written $f : (X, E) \rightarrow_B^{ci} (Y, F)$ (respectively $f : (X, E) \rightarrow_B^{cs} (Y, F)$, $f : (X, E) \rightarrow_B^{cb} (Y, F)$), if f is a homomorphism such that for each $x \in X$, the restriction $f|_{[x]_E} : [x]_E \rightarrow [f(x)]_F$ to the equivalence class of x is injective (respectively, surjective, bijective);
- an **embedding**, written $f : (X, E) \sqsubseteq_B (Y, F)$, if f is an injective (or equivalently, class-injective) reduction;
- an **invariant embedding**, written $f : (X, E) \sqsubseteq_B^i (Y, F)$, if f is a class-bijective reduction, or equivalently an embedding such that the image $f(X) \subseteq Y$ is F -invariant.

Among these various kinds of homomorphisms, the reductions have received the most attention in the literature, while the class-bijective ones are most closely related to the notion of structurality. Here is a picture of the containments between these classes of homomorphisms, with the more restrictive classes at the bottom:



We say that (X, E) **(Borel) reduces** to (Y, F) , written $(X, E) \leq_B (Y, F)$ (or simply $E \leq_B F$), if there is a Borel reduction $f : (X, E) \leq_B (Y, F)$. Similarly for the other kinds of homomorphisms, e.g., E **embeds** into F , written $E \sqsubseteq_B F$, if there is some $f : E \sqsubseteq_B F$, etc. We also write:

- $E \sim_B F$ (E is **bireducible** to F) if $E \leq_B F$ and $F \leq_B E$;
- $E <_B F$ if $E \leq_B F$ and $F \not\leq_B E$, and similarly for \sqsubset_B and \sqsubset_B^i ;
- $E \leftrightarrow_B^{cb} F$ (E is **class-bijectively equivalent** to F) if $E \rightarrow_B^{cb} F$ and $F \rightarrow_B^{cb} E$;
- $E \cong_B F$ if E is Borel isomorphic to F , or equivalently (by the Borel Schröder-Bernstein theorem) $E \sqsubseteq_B^i F$ and $F \sqsubseteq_B^i E$.

Clearly \leq_B , \sqsubseteq_B , \rightarrow_B^{cb} , etc., are preorders on the class \mathcal{E} , and \sim_B , \leftrightarrow_B^{cb} , \cong_B are equivalence relations on \mathcal{E} . The \sim_B -equivalence classes are called **bireducibility classes**, etc.

2.4 Basic operations

We have the following basic operations on Borel equivalence relations. Let $(X, E), (Y, F)$ be Borel equivalence relations.

Their **disjoint sum** is $(X, E) \oplus (Y, F) = (X \oplus Y, E \oplus F)$ where $X \oplus Y$ is the disjoint union of X, Y , and $E \oplus F$ relates elements of X according to E and elements of Y according to F and does not relate elements of X with elements of Y . The canonical injections $\iota_1 : X \rightarrow_B X \oplus Y$ and $\iota_2 : Y \rightarrow_B X \oplus Y$ are then invariant embeddings $E, F \sqsubseteq_B^i E \oplus F$. Clearly the disjoint sum of countable equivalence relations is countable. We have obvious generalizations to disjoint sums of any countable family of equivalence relations.

Their **cross product** is $(X, E) \times (Y, F) = (X \times Y, E \times F)$, where

$$(x, y) (E \times F) (x', y') \iff x E x' \wedge y F y'.$$

(The “cross” adjective is to disambiguate from the *tensor products* to be introduced in Section 4.4.) The projections $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are class-surjective homomorphisms $E \times F \rightarrow_B^{cs} E, F$. Cross products also generalize to countably many factors; but note that only *finite* cross products of countable equivalence relations are countable.

2.5 Special equivalence relations

Recall that an equivalence relation (X, E) is **countable** if each E -class is countable; similarly, it is **finite** if each E -class is finite, and **aperiodic countable** if each E -class is countably infinite. A countable Borel equivalence relation is always the disjoint sum of a finite Borel equivalence relation and an aperiodic countable Borel equivalence relation. Since many of our results become trivial when all classes are finite, we will often assume that our equivalence relations are aperiodic.

For any set X , the **indiscrete** equivalence relation on X is $I_X := X \times X$.

A Borel equivalence relation (X, E) is **smooth** if $E \leq_B \Delta_Y$ where Δ_Y is the equality relation on some standard Borel space Y . When E is countable, this is equivalent to E having a Borel **transversal**, i.e., a Borel set $A \subseteq X$ meeting each E -class exactly once, or a Borel **selector**, i.e., a Borel map $f : X \rightarrow_B X$ such that $x E f(x)$ and $x E y \implies f(x) = f(y)$ for all $x, y \in X$. Any finite Borel equivalence relation is smooth. Up to bireducibility, the smooth Borel equivalence relations consist exactly of

$$\Delta_0 <_B \Delta_1 <_B \Delta_2 <_B \cdots <_B \Delta_{\mathbb{N}} <_B \Delta_{\mathbb{R}};$$

and these form an initial segment of the preorder (\mathcal{E}, \leq_B) (Silver’s dichotomy; see [MK, 9.1.1]).

We will sometimes use the standard fact that a countable Borel equivalence relation (X, E) is smooth iff every ergodic invariant (σ -finite Borel) measure on E is atomic. (For the converse direction, use e.g., that if E is not smooth, then $E_t \sqsubseteq_B^i E$ (see Theorem 2.1 below), and E_t is isomorphic to the orbit equivalence of the translation action of \mathbb{Q} on \mathbb{R} , which admits Lebesgue measure as an ergodic invariant nonatomic σ -finite measure.)

If $f : X \rightarrow Y$ is any function between sets, the **kernel** of f is the equivalence relation $\ker f$ on X given by $x (\ker f) y \iff f(x) = f(y)$. So a Borel equivalence relation is smooth iff it is the kernel of some Borel map.

A countable Borel equivalence relation E is **universal** if E is \leq_B -greatest in \mathcal{E} , i.e., for any other countable Borel equivalence relation F , we have $F \leq_B E$. An example is $E(\mathbb{F}_2, 2)$ (where \mathbb{F}_2 is the free group on 2 generators) [DJK, 1.8].

A countable Borel equivalence relation E is **invariantly universal** if E is \sqsubseteq_B^i -greatest in \mathcal{E} , i.e., for any other countable Borel equivalence relation F , we have $F \sqsubseteq_B^i E$. We denote by E_∞ any such E ; in light of the Borel Schröder-Bernstein theorem, E_∞ is unique up to isomorphism. Clearly E_∞ is also \leq_B -universal. (Note: in the literature, E_∞ is commonly used to denote any \leq_B -universal countable Borel equivalence relation (which is determined only up to bireducibility).) One realization of E_∞ is $E(\mathbb{F}_\omega, \mathbb{R})$. (This follows from the Feldman-Moore Theorem.)

A (countable) Borel equivalence relation (X, E) is **hyperfinite** if E is the increasing union of a sequence of finite Borel equivalence relations on X . We will use the following facts (see [DJK, 5.1, 7.2, 9.3]):

Theorem 2.1. *Let $(X, E), (Y, F) \in \mathcal{E}$ be countable Borel equivalence relations.*

- (a) E is hyperfinite iff $E = E_{\mathbb{Z}}^X$ for some action of \mathbb{Z} on X .
- (b) E is hyperfinite iff there is a Borel binary relation $<$ on X such that on each E -class, $<$ is a linear order embeddable in $(\mathbb{Z}, <)$.
- (c) If E, F are both hyperfinite and non-smooth, then $E \sqsubseteq_B F$. Thus there is a unique bireducibility (in fact biembeddability) class of non-smooth hyperfinite Borel equivalence relations.
- (d) Let E_0, E_t be the equivalence relations on $2^{\mathbb{N}}$ given by

$$\begin{aligned} x E_0 y &\iff \exists i \in \mathbb{N} \forall j \in \mathbb{N} (x(i+j) = y(i+j)), \\ x E_t y &\iff \exists i, j \in \mathbb{N} \forall k \in \mathbb{N} (x(i+k) = y(j+k)). \end{aligned}$$

Up to isomorphism, the non-smooth, aperiodic, hyperfinite Borel equivalence relations are

$$E_t \sqsubseteq_B^i E_0 \sqsubseteq_B^i 2 \cdot E_0 \sqsubseteq_B^i 3 \cdot E_0 \sqsubseteq_B^i \cdots \sqsubseteq_B^i \aleph_0 \cdot E_0 \sqsubseteq_B^i 2^{\aleph_0} \cdot E_0,$$

where $n \cdot E_0 := \Delta_n \times E_0$. Each $n \cdot E_0$ has exactly n ergodic invariant probability measures.

- (e) (Glimm-Effros dichotomy) E is not smooth iff $E_t \sqsubseteq_B^i E$.

A countable Borel equivalence relation (X, E) is **compressible** if there is a $f : E \sqsubseteq_B E$ such that $f(C) \subsetneq C$ for every E -class $C \in X/E$. The basic example is $I_{\mathbb{N}}$. A fundamental theorem of Nadkarni [N] asserts that E is compressible iff it does not admit an invariant probability measure. Thus, E_t is compressible. For more on compressibility, see [DJK, Section 2]; we will use the results therein extensively in Section 5.4.

A countable Borel equivalence relation (X, E) is **treeable** if E is generated by an acyclic Borel graph on X . For properties of treeability which we use later on, see [JKL, Section 3].

2.6 Fiber products

Let $(X, E), (Y, F), (Z, G)$ be Borel equivalence relations, and let $f : (Y, F) \rightarrow_B (X, E)$ and $g : (Z, G) \rightarrow_B (X, E)$ be homomorphisms. The **fiber product** of F and G (with respect to f and g) is $(Y, F) \times_{(X, E)} (Z, G) = (Y \times_X Z, F \times_E G)$, where

$$Y \times_X Z := \{(y, z) \in Y \times Z \mid f(y) = g(z)\}, \quad F \times_E G := (F \times G) \upharpoonright (Y \times_X Z).$$

The projections $\pi_1 : F \times_E G \rightarrow F$ and $\pi_2 : F \times_E G \rightarrow G$ fit into a commutative diagram:

$$\begin{array}{ccc} F \times_E G & \xrightarrow{\pi_2} & G \\ \pi_1 \downarrow & & \downarrow g \\ F & \xrightarrow{f} & E \end{array}$$

It is easily verified that if g is class-injective, class-surjective, or a reduction, then so is π_1 .

2.7 Some categorical remarks

For each of the several kinds of homomorphisms mentioned in Section 2.3, we have a corresponding category of countable Borel equivalence relations and homomorphisms of that kind. We use e.g., $(\mathcal{E}, \rightarrow_B^{cb})$ to denote the category of countable Borel equivalence relations and class-bijective homomorphisms, etc.

(Depending on context, we also use $(\mathcal{E}, \rightarrow_B^{cb})$ to denote the preorder \rightarrow_B^{cb} on \mathcal{E} , i.e., the preorder gotten by collapsing all morphisms in the category $(\mathcal{E}, \rightarrow_B^{cb})$ between the same two objects.)

From a categorical standpoint, among these categories, the two most well-behaved ones seem to be $(\mathcal{E}, \rightarrow_B)$ and $(\mathcal{E}, \rightarrow_B^{cb})$. The latter will be treated in Sections 4.4 and 4.5. As for $(\mathcal{E}, \rightarrow_B)$, we note that (countable) disjoint sums, (finite) cross products, and fiber products give respectively coproducts, products, and pullbacks in that category. It follows that $(\mathcal{E}, \rightarrow_B)$ is finitely complete, i.e., has all finite categorical limits (see e.g., [ML, V.2, Exercise III.4.10]). (However, it does not have coequalizers: let E_0 on $2^{\mathbb{N}}$ be generated by a Borel automorphism $T : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$; then it is easy to see that $T : (2^{\mathbb{N}}, \Delta_{2^{\mathbb{N}}}) \rightarrow_B (2^{\mathbb{N}}, \Delta_{2^{\mathbb{N}}})$ and the identity map do not have a coequalizer.)

For later reference, let us note that the category of (not necessarily countable) Borel equivalence relations and Borel homomorphisms has inverse limits of countable chains. That is, for each $n \in \mathbb{N}$, let (X_n, E_n) be a Borel equivalence relation, and $f_n : E_{n+1} \rightarrow_B E_n$ be a Borel homomorphism. Then the **inverse limit** of the system is $\varprojlim_n (X_n, E_n) = (\varprojlim_n X_n, \varprojlim_n E_n)$, where

$$\begin{aligned} \varprojlim_n X_n &:= \{\bar{x} = (x_0, x_1, \dots) \in \prod_n X_n \mid \forall n (x_n = f_n(x_{n+1}))\}, \\ \varprojlim_n E_n &:= \prod_n E_n \mid \varprojlim_n X_n. \end{aligned}$$

It is easily seen that $\varprojlim_n E_n$ together with the projections $\pi_m : \varprojlim_n E_n \rightarrow_B E_m$ has the universal property of an inverse limit, i.e., for any other Borel equivalence relation (Y, F) and homomorphisms $g_m : F \rightarrow_B E_m$ such that $g_m = f_m \circ g_{m+1}$ for each m , there is a unique homomorphism $\tilde{g} : F \rightarrow_B \varprojlim_n E_n$ such that $\pi_m \circ \tilde{g} = g_m$ for each m . This is depicted in the following commutative diagram:

$$\begin{array}{ccccccc} F & \overset{\tilde{g}}{\dashrightarrow} & \varprojlim_n E_n & & & & \\ & \searrow & \downarrow \pi_0 & \searrow \pi_1 & \searrow \pi_0 & & \\ & & \downarrow \pi_2 & \searrow \pi_1 & \searrow \pi_0 & & \\ & \searrow g_2 & \downarrow \pi_1 & \searrow \pi_0 & \searrow \pi_0 & & \\ & & \downarrow \pi_0 & \searrow \pi_0 & \searrow \pi_0 & & \\ \dots & \xrightarrow{f_2} & E_2 & \xrightarrow{f_1} & E_1 & \xrightarrow{f_0} & E_0 \end{array}$$

It follows that the category of Borel equivalence relations and Borel homomorphisms is countably complete, i.e., has all limits of countable diagrams (again see [ML, V.2, Exercise III.4.10]).

3 Structures on equivalence relations

We now define the central notion of this paper.

Let L be a language and X be a standard Borel space. We say that an L -structure $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ with universe X is **Borel** if $R^{\mathbb{A}} \subseteq X^n$ is Borel for each n -ary $R \in L$.

Now let (X, E) be a countable Borel equivalence relation. We say that a Borel L -structure $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ is a **Borel L -structure on E** if for each n -ary $R \in L$, $R^{\mathbb{A}}$ only relates elements within the same E -class, i.e.,

$$R^{\mathbb{A}}(x_1, \dots, x_n) \implies x_1 E x_2 E \cdots E x_n.$$

For an $L_{\omega_1\omega}$ -sentence σ , we say that \mathbb{A} is a **Borel σ -structure on E** , written

$$\mathbb{A} : E \models \sigma,$$

if for each E -class $C \in X/E$, the structure $\mathbb{A}|C$ satisfies σ . We say that E is **σ -structurable**, written

$$E \models \sigma,$$

if there is some Borel σ -structure on E . Similarly, if \mathcal{K} is a Borel class of L -structures, we say that \mathbb{A} is a **Borel \mathcal{K} -structure on E** if $\mathbb{A}|C \in \mathcal{K}$ for each $C \in X/E$, and that E is **\mathcal{K} -structurable** if there is some Borel \mathcal{K} -structure on E . Note that E is \mathcal{K} -structurable iff it is σ -structurable, for any $L_{\omega_1\omega}$ -sentence σ axiomatizing \mathcal{K} .

We let

$$\mathcal{E}_\sigma \subseteq \mathcal{E}, \quad \mathcal{E}_\mathcal{K} \subseteq \mathcal{E}$$

denote respectively the classes of σ -structurable and \mathcal{K} -structurable countable Borel equivalence relations. For any class $\mathcal{C} \subseteq \mathcal{E}$ of countable Borel equivalence relations, we say that \mathcal{C} is **elementary** if $\mathcal{C} = \mathcal{E}_\sigma$ for some theory (L, σ) , in which case we say that (L, σ) **axiomatizes \mathcal{C}** .

3.1 Examples of elementary classes

Several notions of “sufficiently simple” countable Borel equivalence relations which have been considered in the literature are given by an elementary class.

For example, a countable Borel equivalence relation E is smooth iff E is structurable by pointed sets (i.e., sets with a distinguished element). By Theorem 2.1, E is hyperfinite iff E is structurable by linear orders which embed in \mathbb{Z} . Hyperfiniteness can also be axiomatized by the sentence in the language $L = \{R_0, R_1, R_2, \dots\}$ which asserts that each R_i is a finite equivalence relation and $R_0 \subseteq R_1 \subseteq \dots$ with union the indiscrete equivalence relation. Similarly, it is straightforward to verify that for each $\alpha < \omega_1$, α -Fréchet-amenability (see [JKL, 2.11–12]) is axiomatizable. Also, E is compressible iff it is structurable via structures in the language $L = \{R\}$ where R is the graph of a non-surjective injection.

For some trivial examples: every E is σ -structurable for logically valid σ , or for the (non-valid) sentence σ in the language $L = \{R_0, R_1, \dots\}$ asserting that the R_i 's form a separating family of unary predicates (i.e., $\forall x, y (\bigwedge_i (R_i(x) \leftrightarrow R_i(y)) \leftrightarrow x = y)$); thus \mathcal{E} is elementary. The class of aperiodic countable Borel equivalence is axiomatized by the theory of infinite sets, etc.

Let \mathcal{T}_1 denote the class of trees (i.e., acyclic connected graphs), and more generally, \mathcal{T}_n denote the class of contractible n -dimensional (abstract) simplicial complexes. Then E is \mathcal{T}_1 -structurable iff E is treeable. Gaboriau [G] has shown that $\mathcal{E}_{\mathcal{T}_1} \subsetneq \mathcal{E}_{\mathcal{T}_2} \subsetneq \dots$.

For any language L and countable L -structure \mathbb{A} , if $\sigma_{\mathbb{A}}$ denotes the Scott sentence of \mathbb{A} , then E is $\sigma_{\mathbb{A}}$ -structurable iff it is structurable via isomorphic copies of \mathbb{A} . For example, if $L = \{<\}$ and $(X, \mathbb{A}) = (\mathbb{Z}, <)$, then E is $\sigma_{\mathbb{A}}$ -structurable iff it is aperiodic hyperfinite. We write

$$\mathcal{E}_{\mathbb{A}} := \mathcal{E}_{\sigma_{\mathbb{A}}}$$

for the class of \mathbb{A} -**structurable** countable Borel equivalence relations.

Let Γ be a countable group, and regard Γ as a structure in the language $L_{\Gamma} = \{R_{\gamma} \mid \gamma \in \Gamma\}$, where R_{γ} is the graph of the map $\delta \mapsto \gamma \cdot \delta$. Then a model of σ_{Γ} is a Γ -action isomorphic to Γ , i.e., a free transitive Γ -action. Thus a countable Borel equivalence relation E is Γ -structurable (i.e., σ_{Γ} -structurable) iff it is generated by a free Borel action of Γ .

Finally, we note that several important classes of countable Borel equivalence relations are *not* elementary. This includes all classes of “sufficiently complex” equivalence relations, such as (invariantly) universal equivalence relations, non-smooth equivalence relations, and equivalence relations admitting an invariant probability measure; these classes are not elementary by Proposition 3.1. Another example of a different flavor is the class of equivalence relations generated by a free action of *some* countable group; more generally, nontrivial unions of elementary classes are never elementary (see Corollary 4.5).

3.2 Classwise pullback structures

Let $(X, E), (Y, F)$ be countable Borel equivalence relations and $f : E \rightarrow_B^{cb} F$ be a class-bijective homomorphism. For an L -structure \mathbb{A} on F , recall that the pullback structure of \mathbb{A} along f , denoted $f^{-1}(\mathbb{A})$, is the L -structure with universe X given by

$$R^{f^{-1}(\mathbb{A})}(\bar{x}) \iff R^{\mathbb{A}}(f(\bar{x}))$$

for each n -ary $R \in L$ and $\bar{x} \in X^n$. Let $f_E^{-1}(\mathbb{A})$ denote the **classwise pullback structure**, given by

$$R^{f_E^{-1}(\mathbb{A})}(\bar{x}) \iff R^{\mathbb{A}}(f(\bar{x})) \wedge x_1 E \cdots E x_n.$$

Then $f_E^{-1}(\mathbb{A})$ is a Borel L -structure on E , such that for each E -class $C \in X/E$, the restriction $f|_C : C \rightarrow f(C)$ is an isomorphism between $f_E^{-1}(\mathbb{A})|_C$ and $\mathbb{A}|_{f(C)}$. In particular, if \mathbb{A} is a σ -structure for some $L_{\omega_1\omega}$ -sentence σ , then so is $f_E^{-1}(\mathbb{A})$. We record the consequence of this simple observation for structurability:

Proposition 3.1. *Every elementary class $\mathcal{E}_{\sigma} \subseteq \mathcal{E}$ is (downwards-)closed under class-bijective homomorphisms, i.e., if $E \rightarrow_B^{cb} F$ and $F \in \mathcal{E}_{\sigma}$, then $E \in \mathcal{E}_{\sigma}$.*

This connection between structurability and class-bijective homomorphisms will be significantly strengthened in the next section.

4 Basic universal constructions

In this section we present the two main constructions relating structures on equivalence relations to class-bijective homomorphisms. Both are “universal” constructions: the first turns any theory (L, σ) into a universal equivalence relation with a σ -structure, while the second turns any equivalence relation into a universal theory.

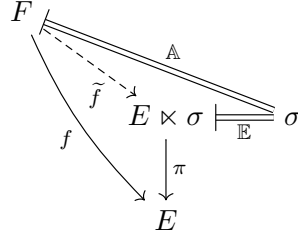
4.1 The universal σ -structured equivalence relation

Kechris-Solecki-Todorćević [KST, 7.1] proved a universality result for graphs, which was then extended to arbitrary Borel classes of structures by Miller. Here, we formulate a version of this result and its proof that allows us to capture more information.

Theorem 4.1. *Let $(X, E) \in \mathcal{E}$ be a countable Borel equivalence relation and (L, σ) be a theory. Then there is a “universal σ -structured equivalence relation lying over E ”, i.e., a triple $(E \times \sigma, \pi, \mathbb{E})$ where*

$$E \times \sigma \in \mathcal{E}, \quad \pi : E \times \sigma \rightarrow_{cb}^B E, \quad \mathbb{E} : E \times \sigma \models \sigma,$$

such that for any other $F \in \mathcal{E}$ with $f : F \rightarrow_{cb}^B E$ and $\mathbb{A} : F \models \sigma$, there is a unique class-bijective homomorphism $\tilde{f} : F \rightarrow_{cb}^B E \times \sigma$ such that $f = \pi \circ \tilde{f}$ and $\mathbb{A} = \tilde{f}_F^{-1}(\mathbb{E})$. This is illustrated by the following “commutative” diagram:



Proof. First we describe $E \times \sigma$ while ignoring all questions of Borelness, then we verify that the construction can be made Borel.

Ignoring Borelness, $E \times \sigma$ will live on a set Z and will have the following form: for each E -class $C \in X/E$, and each σ -structure \mathbb{B} on the universe C , there will be one $(E \times \sigma)$ -class lying over C (i.e., projecting to C via π), which will have the σ -structure given by pulling back \mathbb{B} . Thus we put

$$\begin{aligned} Z &:= \{(x, \mathbb{B}) \mid x \in X, \mathbb{B} \in \text{Mod}_{[x]_E}(\sigma)\}, \\ (x, \mathbb{B}) (E \times \sigma) (x', \mathbb{B}') &\iff x E x' \wedge \mathbb{B} = \mathbb{B}', \\ \pi(x, \mathbb{B}) &:= x, \end{aligned}$$

with the σ -structure \mathbb{E} on $E \times \sigma$ given by

$$R^{\mathbb{E}}((x_1, \mathbb{B}), \dots, (x_n, \mathbb{B})) \iff R^{\mathbb{B}}(x_1, \dots, x_n)$$

for n -ary $R \in L$, $x_1 E \dots E x_n$, and $\mathbb{B} \in \text{Mod}_{[x_1]_E}(\sigma)$. It is immediate that π is class-bijective and that \mathbb{E} satisfies σ . The universal property is also straightforward: given (Y, F) , f, \mathbb{A} as above, the map \tilde{f} is given by

$$\tilde{f}(y) := (f(y), f(\mathbb{A}[y]_F)) \in Z,$$

and this choice is easily seen to be unique by the requirements $f = \pi \circ \tilde{f}$ and $\mathbb{A} = \tilde{f}_F^{-1}(\mathbb{E})$.

Now we indicate how to make this construction Borel. The only obstruction is the use of $\text{Mod}_{[x]_E}(\sigma)$ which depends on x in the definition of Z above. We restrict to the case where E is aperiodic; in general, we may split E into its finite part and aperiodic part, and it will be clear that the finite case can be handled similarly. In the aperiodic case, the idea is to replace $\text{Mod}_{[x]_E}(\sigma)$ with $\text{Mod}_{\mathbb{N}}(\sigma)$, where $[x]_E$ is identified with \mathbb{N} but in a manner which varies depending on x .

Let $T : X \rightarrow X^{\mathbb{N}}$ be a Borel map such that each $T(x)$ is a bijection $\mathbb{N} \rightarrow [x]_E$ (the existence of such T is easily seen from Lusin-Novikov uniformization), and replace $\text{Mod}_{[x]_E}(\sigma)$ with $\text{Mod}_{\mathbb{N}}(\sigma)$ while inserting $T(x)$ into the appropriate places in the above definitions:

$$\begin{aligned} Z &:= \{(x, \mathbb{B}) \mid x \in X, \mathbb{B} \in \text{Mod}_{\mathbb{N}}(\sigma)\} = X \times \text{Mod}_{\mathbb{N}}(\sigma), \\ (x, \mathbb{B}) (E \times \sigma) (x', \mathbb{B}') &\iff x E x' \wedge T(x)(\mathbb{B}) = T(x')(\mathbb{B}'), \\ R^{\mathbb{E}}((x_1, \mathbb{B}_1), \dots, (x_n, \mathbb{B}_n)) &\iff R^{T(x_1)(\mathbb{B}_1)}(x_1, \dots, x_n), \\ \tilde{f}(y) &:= (f(y), T(f(y))^{-1}(f(\mathbb{A}[y]_F))). \end{aligned}$$

These are easily seen to be Borel and still satisfy the requirements of the theorem. \square

Remark 4.2. It is clear that $E \times \sigma$ satisfies a universal property in the formal sense of category theory. This in particular means that $(E \times \sigma, \pi, \mathbb{E})$ is unique up to *unique* (Borel) isomorphism.

Remark 4.3. The construction of $E \times \sigma$ for aperiodic E in the proof of Theorem 4.1 can be seen as an instance of the following general notion (see e.g., [K10, 10.E]):

Let (X, E) be a Borel equivalence relation, and let Γ be a (Borel) group. Recall that a **Borel cocycle** $\alpha : E \rightarrow \Gamma$ is a Borel map satisfying $\alpha(y, z)\alpha(x, y) = \alpha(x, z)$ for all $x, y, z \in X, x E y E z$. Given a cocycle α and a Borel action of Γ on a standard Borel space Y , the **skew product** $E \times_{\alpha} Y$ is the Borel equivalence relation on $X \times Y$ given by

$$(x, y) (E \times_{\alpha} Y) (x', y') \iff x E x' \wedge \alpha(x, x') \cdot y = y'.$$

Note that for such a skew product, the first projection $\pi_1 : X \times Y \rightarrow X$ is always a class-bijective homomorphism $E \times_{\alpha} Y \rightarrow_B^{cb} E$.

Now given a family $T : X \rightarrow X^{\mathbb{N}}$ of bijections $\mathbb{N} \cong_B [x]_E$, as in the proof of Theorem 4.1, we call $\alpha_T : E \rightarrow S_{\infty}$ given by $\alpha_T(x, x') := T(x')^{-1} \circ T(x)$ the **cocycle induced by T** . Then the construction of $E \times \sigma$ for aperiodic E can be seen as the skew product $E \times_{\alpha_T} \text{Mod}_{\mathbb{N}}(\sigma)$ (with the logic action of S_{∞} on $\text{Mod}_{\mathbb{N}}(\sigma)$). (However, the structure \mathbb{E} on E depends on T , not just on α .)

Theorem 4.1 has the following consequence:

Corollary 4.4 (Kechris-Solecki-Todorćević, Miller). *For every theory (L, σ) , there is an invariantly universal σ -structurable countable Borel equivalence relation $E_{\infty\sigma}$, i.e., $E_{\infty\sigma} \models \sigma$, and $F \sqsubseteq_B^i E_{\infty\sigma}$ for any other $F \models \sigma$.*

Proof. Put $E_{\infty\sigma} := E_{\infty} \times \sigma$. For any $F \models \sigma$, we have an invariant embedding $f : F \sqsubseteq_B^i E_{\infty}$, whence there is $\tilde{f} : F \rightarrow_B^{cb} E_{\infty} \times \sigma = E_{\infty\sigma}$ such that $f = \pi \circ \tilde{f}$; since f is injective, so is \tilde{f} . \square

In other words, every elementary class \mathcal{E}_{σ} of countable Borel equivalence relations has an invariantly universal element $E_{\infty\sigma}$ (which is unique up to isomorphism). For a Borel class of structures \mathcal{K} ,

we denote the invariantly universal \mathcal{K} -structurable equivalence relation by $E_{\infty\mathcal{K}}$. For an L -structure \mathbb{A} , we denote the invariantly universal \mathbb{A} -structurable equivalence relation by $E_{\infty\mathbb{A}}$.

As a basic application, we can now rule out the elementarity of a class of equivalence relations mentioned in Section 3.1:

Corollary 4.5. *If $(\mathcal{C}_i)_{i \in I}$ is a collection of elementary classes of countable Borel equivalence relations, then $\bigcup_i \mathcal{C}_i$ is not elementary, unless there is some j such that $\mathcal{C}_i \subseteq \mathcal{C}_j$ for every i .*

In particular, the class of equivalence relations generated by a free Borel action of some countable group is not elementary.

Proof. If $\bigcup_i \mathcal{C}_i$ is elementary, then it has an invariantly universal element E , which is in some \mathcal{C}_j ; then for every i and $F \in \mathcal{C}_i$, we have $F \sqsubseteq_B^i E \in \mathcal{C}_j$, whence $F \in \mathcal{C}_j$ since \mathcal{C}_j is elementary.

For the second statement, the class in question is $\bigcup_{\Gamma} \mathcal{E}_{\Gamma}$ where Γ ranges over countable groups (and \mathcal{E}_{Γ} is the class of equivalence relations generated by a free Borel action of Γ); and there cannot be a single \mathcal{E}_{Γ} which contains all others, since if Γ is amenable then \mathcal{E}_{Γ} does not contain $F(\mathbb{F}_2, 2)$ (see [HK, A4.1]), while if Γ is not amenable then \mathcal{E}_{Γ} does not contain E_0 (see [K91, 2.3]). \square

We also have

Proposition 4.6. *Let \mathcal{C} denote the class of countable increasing unions of equivalence relations generated by free Borel actions of (possibly different) countable groups. Then \mathcal{C} does not have a \leq_B -universal element, hence is not elementary.*

Proof. Let $E = \bigcup_n E_n \in \mathcal{C}$ be the countable increasing union of countable Borel equivalence relations $E_0 \subseteq E_1 \subseteq \dots$ on X , where each E_n is generated by a free Borel action of a countable group Γ_n . Since there are uncountably many finitely generated groups, there is a finitely generated group L such that L does not embed in any Γ_n . Put $\Delta := SL_3(\mathbb{Z}) \times (L * \mathbb{Z})$, and let $F(\Delta, 2)$ live on $Y \subseteq 2^{\Delta}$ (the free part of the shift action), with its usual product probability measure μ . By [T2, 3.6] (see also 3.7–9 of that paper), $F(\Delta, 2)|Z \not\leq_B E_n$ for each n and $Z \subseteq Y$ of μ -measure 1.

If E were \leq_B -universal in \mathcal{C} , then we would have some $f : F(\Delta, 2) \leq_B E$. Let $F_n := f^{-1}(E_n)$, so that $F(\Delta, 2) = \bigcup_n F_n$. By [GT, 1.1] (using that $SL_3(\mathbb{Z})$ acts strongly ergodically [HK, A4.1]), there is an n and a Borel $A \subseteq Y$ with $\mu(A) > 0$ such that $F(\Delta, 2)|A = F_n|A$. By ergodicity of μ , $Z := [A]_{F(\Delta, 2)}$ has μ -measure 1; but $F(\Delta, 2)|Z \sim_B F(\Delta, 2)|A = F_n|A \leq_B E_n$, a contradiction. \square

We conclude this section by explicitly describing the invariantly universal equivalence relation in several elementary classes:

- The \sqsubseteq_B^i -universal finite Borel equivalence relation is $\bigoplus_{1 \leq n \in \mathbb{N}} (\Delta_{\mathbb{R}} \times I_n)$.
- The \sqsubseteq_B^i -universal aperiodic smooth countable Borel equivalence relation is $\Delta_{\mathbb{R}} \times I_{\mathbb{N}}$.
- The \sqsubseteq_B^i -universal aperiodic hyperfinite Borel equivalence relation is $2^{\aleph_0} \cdot E_0 = \Delta_{\mathbb{R}} \times E_0$, and the \sqsubseteq_B^i -universal compressible hyperfinite Borel equivalence relation is E_t (see Theorem 2.1).
- The \sqsubseteq_B^i -universal countable Borel equivalence relation is E_{∞} , and the \sqsubseteq_B^i -universal compressible Borel equivalence relation is $E_{\infty} \times I_{\mathbb{N}}$ (see Section 5.4).
- For a countable group Γ , the \sqsubseteq_B^i -universal equivalence relation $E_{\infty\Gamma}$ generated by a free Borel action of Γ is $F(\Gamma, \mathbb{R})$.

4.2 The “Scott sentence” of an equivalence relation

We now associate to every $E \in \mathcal{E}$ a “Scott sentence” σ_E . Just as the Scott sentence $\sigma_{\mathbb{A}}$ of an ordinary first-order structure \mathbb{A} axiomatizes structures isomorphic to \mathbb{A} , the “Scott sentence” σ_E will axiomatize equivalence relations class-bijectively mapping to E .

Theorem 4.7. *Let $(X, E) \in \mathcal{E}$ be a countable Borel equivalence relation. Then there is a sentence σ_E (in some fixed language not depending on E) and a σ_E -structure $\mathbb{H} : E \models \sigma_E$, such that for any $F \in \mathcal{E}$ and $\mathbb{A} : F \models \sigma_E$, there is a unique class-bijective homomorphism $f : F \rightarrow_B^{cb} E$ such that $\mathbb{A} = f_F^{-1}(\mathbb{H})$. This is illustrated by the following diagram:*

$$\begin{array}{ccc} F & & \\ \downarrow f & \swarrow \mathbb{A} & \\ E & \xlongequal[\mathbb{H}]{} & \sigma_E \end{array}$$

Proof. We may assume that X is a Borel subspace of $2^{\mathbb{N}}$. Let $L = \{R_0, R_1, \dots\}$ where each R_i is unary. The idea is that a Borel L -structure will code a Borel map to $X \subseteq 2^{\mathbb{N}}$. Note that since L is unary, there is no distinction between Borel L -structures on X and Borel L -structures on E , or between pullback L -structures and classwise pullback L -structures.

Let \mathbb{H}' be the Borel L -structure on $2^{\mathbb{N}}$ given by

$$R_i^{\mathbb{H}'}(x) \iff x(i) = 1.$$

It is clear that for any standard Borel space Y , we have a bijection

$$\begin{aligned} \{\text{Borel maps } Y \rightarrow_B 2^{\mathbb{N}}\} &\longleftrightarrow \{\text{Borel } L\text{-structures on } Y\} \\ f &\longmapsto f^{-1}(\mathbb{H}') \\ (y \mapsto (i \mapsto R_i^{\mathbb{A}}(y))) &\longleftarrow \mathbb{A}. \end{aligned} \quad (*)$$

It will suffice to find an $L_{\omega_1\omega}$ -sentence σ_E such that for all $(Y, F) \in \mathcal{E}$ and $f : Y \rightarrow_B 2^{\mathbb{N}}$,

$$f^{-1}(\mathbb{H}') : F \models \sigma_E \iff f(Y) \subseteq X \wedge f : F \rightarrow_B^{cb} E. \quad (**)$$

Indeed, we may then put $\mathbb{H} := \mathbb{H}'|_X$, and $(*)$ will restrict to a bijection between class-bijective homomorphisms $F \rightarrow_B^{cb} E$ and σ_E -structures on F , as claimed in the theorem.

Now we find σ_E satisfying $(**)$. The conditions $f(Y) \subseteq X$ and $f : F \rightarrow_B^{cb} E$ can be rephrased as: for each F -class $D \in Y/F$, the restriction $f|_D : D \rightarrow 2^{\mathbb{N}}$ is a bijection between D and some E -class. Using $(*)$, this is equivalent to: for each F -class $D \in Y/F$, the structure $\mathbb{B} := f^{-1}(\mathbb{H}')|_D$ on D is such that

$$y \mapsto (i \mapsto R_i^{\mathbb{B}}(y)) \text{ is a bijection from the universe of } \mathbb{B} \text{ to some } E\text{-class.} \quad (***)$$

So it suffices to show that the class \mathcal{K} of L -structures \mathbb{B} satisfying $(***)$ is Borel (so we may let σ_E be any $L_{\omega_1\omega}$ -sentence axiomatizing \mathcal{K}), i.e., that for any $I = 1, 2, \dots, \mathbb{N}$, $\mathcal{K} \cap \text{Mod}_I(L) \subseteq \text{Mod}_I(L)$ is Borel. Using $(*)$ again, $\mathcal{K} \cap \text{Mod}_I(L)$ is the image of the Borel injection

$$\begin{aligned} \{\text{bijections } I \rightarrow (\text{some } E\text{-class})\} &\longrightarrow \text{Mod}_I(L) \\ f &\longmapsto f^{-1}(\mathbb{H}'). \end{aligned}$$

The domain of this injection is clearly a Borel subset of X^I , whence its image is Borel. \square

In the rest of this section, we give an alternative, more “explicit” construction of σ_E (rather than obtaining it from Lopez-Escobar’s definability theorem as in the above proof). Using the same notations as in the proof, we want to find σ_E satisfying (**).

By Lusin-Novikov uniformization, write $E = \bigcup_i G_i$ where $G_0, G_1, \dots \subseteq X^2$ are graphs of (total) Borel functions. For each i , let $\phi_i(x, y)$ be a quantifier-free $L_{\omega_1\omega}$ -formula whose interpretation in the structure \mathbb{H}' is $\phi_i^{\mathbb{H}'} = G_i \subseteq (2^{\mathbb{N}})^2$. (Such a formula can be obtained from a Borel definition of $G_i \subseteq (2^{\mathbb{N}})^2$ in terms of the basic rectangles $R_j^{\mathbb{H}'} \times R_k^{\mathbb{H}'}$, by replacing each $R_j^{\mathbb{H}'} \times R_k^{\mathbb{H}'}$ with $R_j(x) \wedge R_k(y)$.) Define the $L_{\omega_1\omega}$ -sentences:

$$\begin{aligned}\sigma_E^h &:= \forall x \forall y \bigvee_i \phi_i(x, y), \\ \sigma_E^{ci} &:= \forall x \forall y (\bigwedge_i (R_i(x) \leftrightarrow R_i(y)) \rightarrow x = y), \\ \sigma_E^{cs} &:= \forall x \bigwedge_i \exists y \phi_i(x, y).\end{aligned}$$

Lemma 4.8. *In the notation of the proof of Theorem 4.7,*

$$\begin{aligned}f^{-1}(\mathbb{H}') : F \models \sigma_E^h &\iff f(Y) \subseteq X \wedge f : F \rightarrow_B E, \\ f^{-1}(\mathbb{H}') : F \models \sigma_E^{ci} &\iff f|_D : D \rightarrow 2^{\mathbb{N}} \text{ is injective } \forall D \in Y/F, \\ f^{-1}(\mathbb{H}') : F \models \sigma_E^{cs} &\iff f(Y) \subseteq X \wedge f(D) \text{ is } E\text{-invariant } \forall D \in Y/F.\end{aligned}$$

Proof. $f^{-1}(\mathbb{H}') : F \models \sigma_E^h$ iff for all $(y, y') \in F$, there is some i such that $\phi_i^{f^{-1}(\mathbb{H}')} (y, y')$; $\phi_i^{f^{-1}(\mathbb{H}')} (y, y')$ is equivalent to $\phi_i^{\mathbb{H}'} (f(y), f(y'))$, i.e., $f(y) G_i f(y')$, so we get that $f^{-1}(\mathbb{H}') : F \models \sigma_E^h$ iff for all $(y, y') \in F$, we have $f(y) E f(y')$. (Taking $y = y'$ yields $f(y) \in X$.)

$f^{-1}(\mathbb{H}') : F \models \sigma_E^{ci}$ iff for all $(y, y') \in F$ with $y \neq y'$, there is some i such that $R_i^{f^{-1}(\mathbb{H}')} (y) \iff R_i^{f^{-1}(\mathbb{H}')} (y')$, i.e., $R_i^{\mathbb{H}'} (f(y)) \iff R_i^{\mathbb{H}'} (f(y'))$, i.e., $f(y) \neq f(y')$.

$f^{-1}(\mathbb{H}') : F \models \sigma_E^{cs}$ iff for all $y \in Y$ and all $i \in \mathbb{N}$, there is some $y' F y$ such that $\phi_i^{f^{-1}(\mathbb{H}')} (y, y')$, i.e., $\phi_i^{\mathbb{H}'} (f(y), f(y'))$, i.e., $f(y) G_i f(y')$; from the definition of the G_i , this is equivalent to: for all $y \in Y$, we have $f(y) \in X$, and for every $x' E f(y)$ there is some $y' F y$ such that $f(y') = x'$. \square

So defining $\sigma_E := \sigma_E^h \wedge \sigma_E^{ci} \wedge \sigma_E^{cs}$, we have that $f^{-1}(\mathbb{H}') : F \models \sigma_E$ iff $f : F \rightarrow_B^{cb} E$, as desired. Moreover, by modifying these sentences, we may obtain theories for which structures on F correspond to other kinds of homomorphisms $F \rightarrow E$. We will take advantage of this later, in Sections 5.1 and 5.2.

4.3 Structurability and class-bijective homomorphisms

The combination of Theorems 4.1 and 4.7 gives the following (closely related) corollaries, which imply a tight connection between structurability and class-bijective homomorphisms.

Corollary 4.9. *For $E, F \in \mathcal{E}$, we have $F \models \sigma_E$ iff $F \rightarrow_B^{cb} E$.*

Proof. By Theorem 4.7 and Proposition 3.1. \square

Corollary 4.10. *For every $E \in \mathcal{E}$, there is a smallest elementary class containing E , namely $\mathcal{E}_{\sigma_E} = \{F \in \mathcal{E} \mid F \rightarrow_B^{cb} E\}$.*

Proof. By Proposition 3.1, \mathcal{E}_{σ_E} is contained in every elementary class containing E . \square

We define $\mathcal{E}_E := \mathcal{E}_{\sigma_E} = \{F \in \mathcal{E} \mid F \rightarrow_B^{cb} E\}$, and call it the **elementary class of E** .

Corollary 4.11. *A class $\mathcal{C} \subseteq \mathcal{E}$ of countable Borel equivalence relations is elementary iff it is (downwards-)closed under class-bijective homomorphisms and contains an invariantly universal element $E \in \mathcal{C}$, in which case $\mathcal{C} = \mathcal{E}_E$.*

Proof. One implication is Proposition 3.1 and Corollary 4.4. Conversely, if \mathcal{C} is closed under \rightarrow_B^{cb} and $E \in \mathcal{C}$ is invariantly universal, then clearly $\mathcal{C} = \{F \mid F \rightarrow_B^{cb} E\} = \mathcal{E}_E$. \square

So every elementary class \mathcal{C} is determined by a canonical isomorphism class contained in \mathcal{C} , namely the invariantly universal elements of \mathcal{C} . We now characterize the class of equivalence relations which are invariantly universal in some elementary class.

Corollary 4.12. *Let $E \in \mathcal{E}$. The following are equivalent:*

- (i) $E \cong_B E_{\infty\sigma_E}$, i.e., E is invariantly universal in \mathcal{E}_E .
- (ii) $E \cong_B E_{\infty\sigma}$ for some σ , i.e., E is invariantly universal in some elementary class.
- (iii) For every $F \in \mathcal{E}$, $F \rightarrow_B^{cb} E$ iff $F \sqsubseteq_B^i E$.

Proof. Clearly (i) \implies (ii) \implies (iii), and if (iii) holds, then $\mathcal{E}_E = \{F \mid F \rightarrow_B^{cb} E\} = \{F \mid F \sqsubseteq_B^i E\}$ so E is invariantly universal in \mathcal{E}_E . \square

Remark 4.13. The awkward notation $E_{\infty\sigma_E}$ will be replaced in the next section (with $E_\infty \otimes E$).

We say that $E \in \mathcal{E}$ is **universally structurable** if the equivalent conditions in Corollary 4.12 hold. We let $\mathcal{E}_\infty \subseteq \mathcal{E}$ denote the class of universally structurable countable Borel equivalence relations. The following summarizes the relationship between \mathcal{E}_∞ and elementary classes:

Corollary 4.14. *We have an order-isomorphism of posets*

$$\begin{aligned} (\{\text{elementary classes}\}, \subseteq) &\longleftrightarrow (\mathcal{E}_\infty / \cong_B, \sqsubseteq_B^i) = (\mathcal{E}_\infty / \leftrightarrow_B^{cb}, \rightarrow_B^{cb}) \\ \mathcal{C} &\longmapsto \{\sqsubseteq_B^i\text{-universal elements of } \mathcal{C}\} \\ \mathcal{E}_E &\longleftarrow E. \end{aligned}$$

We will study the purely order-theoretical aspects of the poset $(\mathcal{E}_\infty / \cong_B, \sqsubseteq_B^i)$ (equivalently, the poset of elementary classes) in Section 6.

We conclude this section by pointing out the following consequence of universal structurability:

Corollary 4.15. *If $E \in \mathcal{E}$ is universally structurable, then $E \cong_B \Delta_{\mathbb{R}} \times E$. In particular, E has either none or continuum many ergodic invariant probability measures.*

Proof. Clearly $E \sqsubseteq_B^i \Delta_{\mathbb{R}} \times E$, and $\Delta_{\mathbb{R}} \times E \rightarrow_B^{cb} E$, so $\Delta_{\mathbb{R}} \times E \sqsubseteq_B^i E$. \square

4.4 Class-bijective products

In this section and the next, we use the theory of the preceding sections to obtain some structural results about the category $(\mathcal{E}, \rightarrow_B^{cb})$ of countable Borel equivalence relations and class-bijective homomorphisms. For categorical background, see [ML].

This section concerns a certain product construction between countable Borel equivalence relations, which, unlike the cross product $E \times F$, is well-behaved with respect to class-bijective homomorphisms.

Proposition 4.16. *Let $E, F \in \mathcal{E}$ be countable Borel equivalence relations. There is a countable Borel equivalence relation, which we denote by $E \otimes F$ and call the **class-bijective product** (or **tensor product**) of E and F , which is the categorical product of E and F in the category $(\mathcal{E}, \rightarrow_B^{cb})$. In other words, there are canonical class-bijective projections*

$$\pi_1 : E \otimes F \rightarrow_B^{cb} E, \quad \pi_2 : E \otimes F \rightarrow_B^{cb} F,$$

such that the triple $(E \otimes F, \pi_1, \pi_2)$ is universal with these properties: for any other $G \in \mathcal{E}$ with $f : G \rightarrow_B^{cb} E$ and $g : G \rightarrow_B^{cb} F$, there is a unique class-bijective homomorphism $\langle f, g \rangle : G \rightarrow_B^{cb} E \otimes F$ such that $f = \pi_1 \circ \langle f, g \rangle$ and $g = \pi_2 \circ \langle f, g \rangle$. This is illustrated by the following commutative diagram:

$$\begin{array}{ccc} & G & \\ f \swarrow & \vdots & \searrow g \\ E & \xrightarrow{\langle f, g \rangle} & E \otimes F & \xrightarrow{\pi_2} & F \\ \pi_1 \swarrow & & & & \searrow \pi_2 \end{array}$$

Proof. Put $E \otimes F := E \times \sigma_F$. The rest follows from chasing through the universal properties in Theorem 4.1 and Theorem 4.7 (or equivalently, the Yoneda lemma). For the sake of completeness, we give the details.

From Theorem 4.1, we have a canonical projection $\pi_1 : E \otimes F \rightarrow_B^{cb} E$. We also have a canonical σ_F -structure on $E \otimes F$, namely $\mathbb{E} : E \otimes F = E \times \sigma_F \models \sigma_F$. This structure corresponds to a unique class-bijective map $\pi_2 : E \otimes F \rightarrow_B^{cb} F$ such that $\mathbb{E} = (\pi_2)_{E \otimes F}^{-1}(\mathbb{H})$, where $\mathbb{H} : F \models \sigma_F$ is the canonical structure from Theorem 4.7.

Now given G, f, g as above, the map $\langle f, g \rangle$ is produced as follows. We have the classwise pullback structure $g_G^{-1}(\mathbb{H}) : G \models \sigma_F$, which, together with $f : G \rightarrow_B^{cb} E$, yields (by Theorem 4.1) a unique map $\langle f, g \rangle : G \rightarrow_B^{cb} E \times \sigma_F = E \otimes F$ such that $f = \pi_1 \circ \langle f, g \rangle$ and $g_G^{-1}(\mathbb{H}) = \langle f, g \rangle_G^{-1}(\mathbb{E})$. Since $\mathbb{E} = (\pi_2)_{E \otimes F}^{-1}(\mathbb{H})$, we get $g_G^{-1}(\mathbb{H}) = \langle f, g \rangle_G^{-1}((\pi_2)_{E \otimes F}^{-1}(\mathbb{H})) = (\pi_2 \circ \langle f, g \rangle)_G^{-1}(\mathbb{H})$; since (by Theorem 4.7) g is the unique map $h : G \rightarrow_B^{cb} F$ such that $g_G^{-1}(\mathbb{H}) = h_G^{-1}(\mathbb{H})$, we get $g = \pi_2 \circ \langle f, g \rangle$, as desired. It remains to check uniqueness of $\langle f, g \rangle$. If $h : G \rightarrow_B^{cb} E \otimes F$ is such that $f = \pi_1 \circ h$ and $g = \pi_2 \circ h$, then (reversing the above steps) we have $g_G^{-1}(\mathbb{H}) = h_G^{-1}(\mathbb{E})$; since $\langle f, g \rangle$ was unique with these properties, we get $h = \langle f, g \rangle$, as desired. \square

Remark 4.17. As with all categorical products, \otimes is unique up to unique (Borel) isomorphism, as well as associative and commutative up to (Borel) isomorphism. Note that the two latter properties are not immediately obvious from the definition $E \otimes F := E \times \sigma_F$.

Remark 4.18. However, by unravelling the proofs of Theorems 4.1 and 4.7, we may explicitly describe $E \otimes F$ in a way that makes associativity and commutativity more obvious. Since this explicit description also sheds some light on the structure of $E \otimes F$, we briefly give it here.

Let E live on X , F live on Y , and $E \otimes F$ live on Z . We have one $(E \otimes F)$ -class for each E -class C , F -class D , and bijection $b : C \cong D$; the elements of the $(E \otimes F)$ -class corresponding to (C, D, b) are the elements of C , or equivalently via the bijection b , the elements of D . Thus, ignoring Borelness, we put

$$\begin{aligned} Z &:= \{(x, y, b) \mid x \in X, y \in Y, b : [x]_E \cong [y]_F, b(x) = y\}, \\ (x, y, b) (E \otimes F) (x', y', b') &\iff x E x' \wedge y F y' \wedge b = b', \\ \pi_1(x, y, b) &:= x, \quad \pi_2(x, y, b) := y. \end{aligned}$$

Given G, f, g as in Proposition 4.16 (G living on W , say), the map $\langle f, g \rangle : G \rightarrow_B^{cb} E \otimes F$ is given by $\langle f, g \rangle(w) = (f(w), g(w), (g|[w]_G) \circ (f|[w]_G)^{-1})$, where $(g|[w]_G) \circ (f|[w]_G)^{-1} : [f(w)]_E \cong [g(w)]_F$ since f, g are class-bijective.

To make this construction Borel, we assume that E, F are aperiodic, and replace Z in the above with a subspace of $X \times Y \times S_\infty$, where bijections $b : [x]_E \cong [y]_F$ are transported to bijections $\mathbb{N} \cong \mathbb{N}$ via Borel enumerations of the E -classes and F -classes, as with the map $T : X \rightarrow X^{\mathbb{N}}$ in the proof of Theorem 4.1. (If E, F are not aperiodic, then we split them into the parts consisting of classes with each cardinality $n \in \{1, 2, \dots, \aleph_0\}$; then there will be no $(E \otimes F)$ -classes lying over an E -class and an F -class with different cardinalities.)

The tensor product \otimes and the cross product \times are related as follows: we have a canonical homomorphism $(\pi_1, \pi_2) : E \otimes F \rightarrow_B^{ci} E \times F$, where $(\pi_1, \pi_2)(z) = (\pi_1(z), \pi_2(z))$ (where $\pi_1 : E \otimes F \rightarrow_B^{cb} E$ and $\pi_2 : E \otimes F \rightarrow_B^{cb} F$ are the projections from the tensor product), which is class-injective because $\pi'_1 \circ (\pi_1, \pi_2) = \pi_1$ is class-injective (where $\pi'_1 : E \times F \rightarrow_B E$ is the projection from the cross product).

When we regard $E \otimes F$ as in Remark 4.18, (π_1, π_2) is the obvious projection from Z to $X \times Y$. This in particular shows that

Proposition 4.19. (a) $(\pi_1, \pi_2) : E \otimes F \rightarrow_B^{ci} E \times F$ is surjective iff E and F have all classes of the same cardinality (in particular, if both are aperiodic);

(b) $(\pi_1, \pi_2) : E \otimes F \rightarrow_B^{ci} E \times F$ is an isomorphism if $E = \Delta_X$ and $F = \Delta_Y$.

We now list some formal properties of \otimes :

Proposition 4.20. Let $E, E_i, F, G \in \mathcal{E}$ for $i < n \leq \mathbb{N}$ and let (L, σ) be a theory.

(a) If $E \models \sigma$ then $E \otimes F \models \sigma$.

(b) If $f : E \sqsubseteq_B^i F$ and $g : E \rightarrow_B^{cb} G$, then $\langle f, g \rangle : E \sqsubseteq_B^i F \otimes G$.

(c) If E is universally structurable, then so is $E \otimes F$.

(d) $(E \otimes F) \times \sigma \cong_B E \otimes (F \times \sigma)$ (and the isomorphism is natural in E, F).

(e) $\bigoplus_i (E_i \otimes F) \cong_B (\bigoplus_i E_i) \otimes F$ (and the isomorphism is natural in E_i, F).

Proof. (a): follows from $\pi_1 : E \otimes F \rightarrow_B^{cb} E$.

(b): since $f = \pi_1 \circ \langle f, g \rangle$ is class-injective, so is $\langle f, g \rangle$.

(c): if $f : G \rightarrow_B^{cb} E \otimes F$, then $\pi_1 \circ f : G \rightarrow_B^{cb} E$, whence there is some $g : G \sqsubseteq_B^i E$ since E is universally structurable, whence $\langle g, \pi_2 \circ f \rangle : G \sqsubseteq_B^i E \otimes F$ (by (b)).

(d): follows from a chase through the universal properties of \otimes and \times (or the Yoneda lemma). (A class-bijective homomorphism $G \rightarrow_B^{cb} (E \otimes F) \times \sigma$ is the same thing as pair of class-bijective homomorphisms $G \rightarrow_B^{cb} E$ and $G \rightarrow_B^{cb} F$ together with a σ -structure on G , which is the same thing as a class-bijective homomorphism $G \rightarrow_B^{cb} E \otimes (F \times \sigma)$.)

(e): this is an instance of the following more general fact, which follows easily from the construction of $E \times \sigma$ in Theorem 4.1 (and which could have been noted earlier, in Section 4.1):

Proposition 4.21. $\bigoplus_i (E_i \times \sigma) \cong_B (\bigoplus_i E_i) \times \sigma$.

Moreover, the isomorphism can be taken to be the map $d : \bigoplus_i (E_i \times \sigma) \rightarrow_B^{cb} (\bigoplus_i E_i) \times \sigma$ such that for each i , the restriction $d|(E_i \times \sigma) : E_i \times \sigma \rightarrow_B^{cb} (\bigoplus_i E_i) \times \sigma$ is the canonical such map induced by the inclusion $E_i \sqsubseteq_B^i \bigoplus_i E_i$.

(In other words, the functor $E \mapsto E \times \sigma$ preserves countable coproducts.)

To get from this to (e), simply put $\sigma := \sigma_F$. Naturality is straightforward. \square

Remark 4.22. The analog of Proposition 4.20(a) for cross products is false: the class of treeable countable Borel equivalence relations is *not* closed under cross products (see e.g., [JKL, 3.28]).

We note that for any $E \in \mathcal{E}$, the equivalence relation $E_{\infty\sigma_E}$ (i.e., the invariantly universal element of \mathcal{E}_E) can also be written as the less awkward $E_{\infty} \otimes E$, which is therefore how we will write it from now on.

Here are some sample computations of class-bijective products:

- $\Delta_m \otimes \Delta_n \cong_B \Delta_m \times \Delta_n = \Delta_{m \times n}$ for $m, n \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ (by Proposition 4.19(b)).
- $I_{\mathbb{N}} \otimes I_{\mathbb{N}} \cong_B \Delta_{\mathbb{R}} \times I_{\mathbb{N}}$, since $I_{\mathbb{N}} \otimes I_{\mathbb{N}}$ is aperiodic smooth (Proposition 4.20(a)) and there are continuum many bijections $\mathbb{N} \cong \mathbb{N}$.
- $E_{\infty} \otimes E_{\infty} \cong_B E_{\infty}$, since $E_{\infty} \sqsubseteq_B^i E_{\infty} \otimes E_{\infty}$ (Proposition 4.20(b)).
- If E is universally structurable and $E \rightarrow_B^{cb} F$, then $E \otimes F \cong_B E$, since $E \sqsubseteq_B^i E \otimes F$ (Proposition 4.20(b)), and $\pi_1 : E \otimes F \rightarrow_B^{cb} E$ so $E \otimes F \sqsubseteq_B^i E$.
- $E_0 \otimes E_0 \cong_B \Delta_{\mathbb{R}} \times E_0$, since $E_0 \otimes E_0$ is aperiodic hyperfinite (Proposition 4.20(a)), and there are 2^{\aleph_0} pairwise disjoint copies of E_0 in $E_0 \otimes E_0$. This last fact can be seen by taking a family $(f_r)_{r \in \mathbb{R}}$ of Borel automorphisms $f_r : E_0 \cong_B E_0$ such that f_r, f_s disagree on every E_0 -class whenever $r \neq s$, and then considering the embeddings $\langle 1_{E_0}, f_r \rangle : E_0 \sqsubseteq_B^i E_0 \otimes E_0$, which will have pairwise disjoint images (by Remark 4.18).

(The existence of the family $(f_r)_r$ is standard; one construction is by regarding E_0 as the orbit equivalence of the translation action of \mathbb{Z} on \mathbb{Z}_2 , the 2-adic integers, then taking $f_r : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ for $r \in \mathbb{Z}_2$ to be translation by r .)

Note that the last example can be used to compute \otimes of all hyperfinite equivalence relations.

We conclude by noting that we do not currently have a clear picture of tensor products of general countable Borel equivalence relations. For instance, the examples above suggest that perhaps $E \otimes E$ is universally structurable (equivalently, $E \otimes E \cong_B E_{\infty} \otimes E$, since $E_{\infty} \otimes E \otimes E \cong_B E_{\infty} \otimes E$) for all aperiodic E ; but we do not know if this is true.

4.5 Categorical limits in $(\mathcal{E}, \rightarrow_B^{cb})$

This short section concerns general categorical limits in the category $(\mathcal{E}, \rightarrow_B^{cb})$ of countable Borel equivalence relations and class-bijective homomorphisms. Throughout this section, we use categorical terminology, e.g., “product” means categorical product (i.e., class-bijective product), “pullback” means categorical pullback (i.e., fiber product), etc. For definitions, see [ML, III.3–4, V].

We have shown that $(\mathcal{E}, \rightarrow_B^{cb})$ contains binary products. By iterating (or by generalizing the construction outlined in Remark 4.18), we may obtain all finite (nontrivial) products. The category $(\mathcal{E}, \rightarrow_B^{cb})$ also contains pullbacks (see Section 2.6). It follows that it contains all finite nonempty limits, i.e., limits of all diagrams $F : J \rightarrow (\mathcal{E}, \rightarrow_B^{cb})$ where the indexing category J is finite and nonempty (see [ML, V.2, Exercise III.4.9]).

Remark 4.23. $(\mathcal{E}, \rightarrow_B^{cb})$ does *not* contain a terminal object, i.e., a limit of the empty diagram. This would be a countable Borel equivalence relation E such that any other countable Borel equivalence relation F has a *unique* class-bijective map $F \rightarrow_B^{cb} E$; clearly such E does not exist.

We now verify that $(\mathcal{E}, \rightarrow_B^{cb})$ has inverse limits of countable chains, and that these coincide with the same limits in the category of all Borel equivalence relations and Borel homomorphisms (Section 2.7):

Proposition 4.24. *Let $(X_n, E_n)_{n \in \mathbb{N}}$ be countable Borel equivalence relations, and $(f_n : E_{n+1} \rightarrow_B^{cb} E_n)_n$ be class-bijective homomorphisms. Then the inverse limit $\varprojlim_n (X_n, E_n)$ in the category of Borel equivalence relations and Borel homomorphisms, as defined in Section 2.7, is also the inverse limit in $(\mathcal{E}, \rightarrow_B^{cb})$. More explicitly,*

- (a) *the projections $\pi_m : \varprojlim_n E_n \rightarrow_B E_m$ are class-bijective (so in particular $\varprojlim_n E_n$ is countable);*
- (b) *if $(Y, F) \in \mathcal{E}$ is a countable Borel equivalence relation with class-bijective homomorphisms $g_n : F \rightarrow_B^{cb} E_n$ such that $g_n = f_n \circ g_{n+1}$, then the unique homomorphism $\tilde{g} : F \rightarrow_B \varprojlim_n E_n$ such that $\pi_n \circ \tilde{g} = g_n$ for each n (namely $\tilde{g}(y) = (g_n(y))_n$) is class-bijective.*

Proof. For (a), note that since $\pi_m = f_m \circ \pi_{m+1}$ and the f_m are class-bijective, it suffices to check that π_0 is class-bijective, since we may then inductively get that π_1, π_2, \dots are class-bijective. Let $\bar{x} = (x_n)_n \in \varprojlim_n X_n$ and $x_0 = \pi_0(\bar{x}) \in E_0$; we must find a unique $\bar{x}' \in \varprojlim_n E_n$ such that $x'_0 = \pi_0(\bar{x}')$. For the coordinate $x'_1 = \pi_1(\bar{x}')$, we must have $x'_0 = f_0(x'_1)$ (in order to have $\bar{x}' \in \varprojlim_n X_n$) and $x'_1 \in E_1$ (in order to have $\bar{x}' \in \varprojlim_n E_n$); since $x'_0 \in E_0$, by class-bijectivity of f_0 , there is a unique such x'_1 . Continuing inductively, we see that there is a unique choice of $x'_n = \pi_n(\bar{x}')$ for each $n > 0$. Then $\bar{x}' := (x'_n)_n$ is the desired element.

For (b), simply note that since $\pi_n \circ \tilde{g} = g_n$ and π_n, g_n are class-bijective, so must be \tilde{g} . \square

Corollary 4.25. $(\mathcal{E}, \rightarrow_B^{cb})$ has all countable (nontrivial) products.

Proof. To compute the product $\bigotimes_i E_i$ of $E_0, E_1, E_2, \dots \in \mathcal{E}$, take the inverse limit of the chain $\dots \rightarrow_B^{cb} E_0 \otimes E_1 \otimes E_2 \rightarrow_B^{cb} E_0 \otimes E_1 \rightarrow_B^{cb} E_0$ (where the maps are the projections). \square

Remark 4.26. Countable products can also be obtained by generalizing Remark 4.18.

Corollary 4.27. $(\mathcal{E}, \rightarrow_B^{cb})$ has all countable nonempty limits, i.e., limits of all diagrams $F : J \rightarrow (\mathcal{E}, \rightarrow_B^{cb})$, where the indexing category J is countable and nonempty.

Proof. Follows from countable products and pullbacks; again see [ML, V.2, Exercise III.4.9]. \square

5 Structurability and reducibility

This section has two parts: the first part (Sections 5.1 to 5.3) relates structurability to various classes of homomorphisms, in the spirit of Sections 4.1 to 4.3; while the second part (Section 5.4) relates reductions to compressibility, using results from the first part and from [DJK, Section 2].

We describe here the various classes of homomorphisms that we will be considering. These fit into the following table:

Global	Local
\sqsubseteq_B^i	\rightarrow_B^{cb}
\sqsubseteq_B	\rightarrow_B^{ci}
\leq_B	\rightarrow_B^{sm}

Table 5.1: global and local classes of homomorphisms

The last entry in the table denotes the following notion: we say that a Borel homomorphism $f : (X, E) \rightarrow_B (Y, F)$ is **smooth**, written $f : E \rightarrow_B^{sm} F$, if the f -preimage of every smooth set is smooth (where by a smooth subset of Y (resp., X) we mean a subset to which the restriction of F (resp., E) is smooth). This notion was previously considered by Clemens-Conley-Miller [CCM], under the name **smooth-to-one homomorphism** (because of Proposition 5.8(ii)). See Proposition 5.8 for basic properties of smooth homomorphisms.

Let $(\mathcal{G}, \mathcal{L})$ be a row in Table 5.1. We say that \mathcal{G} is a “global” class of homomorphisms, while \mathcal{L} is the corresponding “local” class. Note that $\mathcal{G} \subseteq \mathcal{L}$. The idea is that \mathcal{G} is a condition on homomorphisms requiring injectivity between classes (i.e., \mathcal{G} consists only of reductions), while \mathcal{L} is an analogous “classwise” condition which can be captured by structurability.

Our main results in this section state the following: for any elementary class $\mathcal{C} \subseteq \mathcal{E}$, the downward closure of \mathcal{C} under \mathcal{G} is equal to the downward closure under \mathcal{L} , and is elementary. In particular, when $\mathcal{C} = \mathcal{E}_E$, this implies that the downward closure of $\{E\}$ under \mathcal{L} is elementary. In the case $(\mathcal{G}, \mathcal{L}) = (\sqsubseteq_B^i, \rightarrow_B^{cb})$, these follow from Section 4.3; thus, our results here generalize our results therein to the other classes of homomorphisms appearing in Table 5.1.

Theorem 5.1. *Let $\mathcal{C} \subseteq \mathcal{E}$ be an elementary class. Then the downward closures of \mathcal{C} under \sqsubseteq_B and \rightarrow_B^{ci} , namely*

$$\begin{aligned} \mathcal{C}^e &:= \{F \in \mathcal{E} \mid \exists E \in \mathcal{C} (F \sqsubseteq_B E)\}, \\ \mathcal{C}^{cih} &:= \{F \in \mathcal{E} \mid \exists E \in \mathcal{C} (F \rightarrow_B^{ci} E)\}, \end{aligned}$$

are equal and elementary.

In particular, when $\mathcal{C} = \mathcal{E}_E$, we get that

$$\begin{aligned} \mathcal{E}_E^e = \mathcal{E}_E^{cih} &= \{F \in \mathcal{E} \mid F \rightarrow_B^{ci} E\} \\ & (= \{F \in \mathcal{E} \mid F \sqsubseteq_B E\}, \text{ if } E \text{ is universally structurable}) \end{aligned}$$

is the smallest elementary class containing E and closed under \sqsubseteq_B .

Theorem 5.2. *Let $\mathcal{C} \subseteq \mathcal{E}$ be an elementary class. Then the downward closures of \mathcal{C} under \leq_B and \rightarrow_B^{sm} , namely*

$$\begin{aligned}\mathcal{C}^r &:= \{F \in \mathcal{E} \mid \exists E \in \mathcal{C} (F \leq_B E)\}, \\ \mathcal{C}^{smh} &:= \{F \in \mathcal{E} \mid \exists E \in \mathcal{C} (F \rightarrow_B^{sm} E)\},\end{aligned}$$

are equal and elementary.

In particular, when $\mathcal{C} = \mathcal{E}_E$, we get that

$$\begin{aligned}\mathcal{E}_E^r = \mathcal{E}_E^{smh} &= \{F \in \mathcal{E} \mid F \rightarrow_B^{sm} E\} \\ &= \{F \in \mathcal{E} \mid F \leq_B E\}, \text{ if } E \text{ is universally structurable}\end{aligned}$$

is the smallest elementary class containing E and closed under \leq_B .

Our proof strategy is as follows. For each $(\mathcal{G}, \mathcal{L}) = (\sqsubseteq_B, \rightarrow_B^{ci})$ or $(\leq_B, \rightarrow_B^{sm})$, we prove a ‘‘factorization lemma’’ which states that \mathcal{L} consists precisely of composites of homomorphisms in \mathcal{G} followed by class-bijective homomorphisms (in that order). This yields that the closures of \mathcal{C} under \mathcal{G} and \mathcal{L} are equal, since \mathcal{C} is already closed under class-bijective homomorphisms. We then prove that for any $E \in \mathcal{E}$, a variation of the ‘‘Scott sentence’’ from Theorem 4.7 can be used to code \mathcal{L} -homomorphisms to E . This yields that for any $E \in \mathcal{E}$, the \mathcal{L} -downward closure of $\{E\}$ is elementary, which completes the proof.

5.1 Embeddings and class-injective homomorphisms

We begin with embeddings and class-injective homomorphisms, for which we have the following factorization lemma:

Proposition 5.3. *Let $(X, E), (Y, F) \in \mathcal{E}$ be countable Borel equivalence relations and $f : E \rightarrow_B^{ci} F$ be a class-injective homomorphism. Then there is a countable Borel equivalence relation $(Z, G) \in \mathcal{E}$, an embedding $g : E \sqsubseteq_B G$, and a class-bijective homomorphism $h : G \rightarrow_B^{cb} F$, such that $f = h \circ g$:*

$$\begin{array}{ccc} (X, E) & \xrightarrow{f} & (Y, F) \\ & \searrow \text{dashed } g & \nearrow \text{dashed } h \\ & (Z, G) & \end{array}$$

Furthermore, g can be taken to be a complete section embedding, i.e., $[g(X)]_G = Z$.

Proof. Consider the equivalence relation (W, D) where

$$\begin{aligned}W &:= \{(x, y) \in X \times Y \mid f(x) F y\}, \\ (x, y) D (x', y') &\iff x E x' \wedge y = y'.\end{aligned}$$

Then (W, D) is a countable Borel equivalence relation. We claim that it is smooth. Indeed, by Lusin-Novikov uniformization, write $F = \bigcup_i G_i$ where $G_i \subseteq Y^2$ for $i \in \mathbb{N}$ are graphs of Borel functions $g_i : Y \rightarrow Y$. Then a Borel selector for D is found by sending $(x, y) \in W$ to $((f|[x]_E)^{-1}(g_i(y)), y)$ for the least i such that $g_i(y)$ is in the image of $f|[x]_E$. (Here we are using that f is class-injective.)

Now put $Z := W/D$, and let G be the equivalence relation on Z given by

$$[(x, y)]_D G [(x', y')]_D \iff x E x' \wedge y F y'.$$

Then (Z, G) is a countable Borel equivalence relation. Let $g : X \rightarrow Z$ and $h : Z \rightarrow Y$ be given by

$$g(x) := [(x, f(x))]_D, \quad h([(x, y)]_D) := y.$$

It is easily seen that $g : E \sqsubseteq_B G$ is a complete section embedding, $h : G \rightarrow_B^{cb} F$, and $f = h \circ g$, as desired. \square

Remark 5.4. It is easy to see that the factorization produced by Proposition 5.3 (with the requirement that g be a complete section embedding) is unique up to unique Borel isomorphism. In other words, if $(Z', G') \in \mathcal{E}$, $g' : E \sqsubseteq_B G'$, and $h' : G' \rightarrow_B^{cb} F$ with $f = h' \circ g'$ are another factorization, with g' a complete section embedding, then there is a unique Borel isomorphism $i : G \cong_B G'$ such that $i \circ g = g'$ and $h = h' \circ i$.

Corollary 5.5. *If $E \in \mathcal{E}$ is universally structurable, then $F \sqsubseteq_B E \iff F \rightarrow_B^{ci} E$, for all $F \in \mathcal{E}$. Similarly, if \mathcal{C} is an elementary class, then $\mathcal{C}^e = \mathcal{C}^{cih}$.*

Proof. If E is universally structurable and $F \rightarrow_B^{ci} E$, then by Proposition 5.3, $F \sqsubseteq_B G \rightarrow_B^{cb} E$ for some G ; then $G \sqsubseteq_B^i E$, whence $F \sqsubseteq_B E$. The second statement is similar. \square

We now have the following analog of Theorem 4.7 for class-injective homomorphisms, which we state in the simpler but slightly weaker form of Corollary 4.9 since that is all we will need:

Proposition 5.6. *Let $E \in \mathcal{E}$ be a countable Borel equivalence relation. Then there is a sentence σ_E^{cih} (in some fixed language) such that for all $F \in \mathcal{E}$, we have $F \models \sigma_E^{cih}$ iff $F \rightarrow_B^{ci} E$.*

Proof. We may either modify the proof of Theorem 4.7 (by considering “injections $I \rightarrow$ (some E -class)” instead of bijections in the last few lines of the proof), or take $\sigma_E^{cih} := \sigma_E^h \wedge \sigma_E^{ci}$ where σ_E^h and σ_E^{ci} are as in Lemma 4.8. \square

Corollary 5.7. *If $\mathcal{C} = \mathcal{E}_E$ is an elementary class, then so is $\mathcal{C}^e = \mathcal{C}^{cih} = \{F \in \mathcal{E} \mid F \rightarrow_B^{ci} E\}$.*

This completes the proof of Theorem 5.1.

5.2 Reductions, smooth homomorphisms, and class-surjectivity

Recall that a Borel homomorphism $f : E \rightarrow_B F$ between countable Borel equivalence relations E, F is **smooth** if the preimage of every smooth set is smooth. We have the following equivalent characterizations of smooth homomorphisms, parts of which are implicit in [CCM, 2.1–2.3]:

Proposition 5.8. *Let $(X, E), (Y, F) \in \mathcal{E}$ and $f : E \rightarrow_B F$. The following are equivalent:*

- (i) f is smooth.
- (ii) For every $y \in Y$, $f^{-1}(y)$ is smooth (i.e., $E|f^{-1}(y)$ is smooth).
- (iii) $E \cap \ker f$ is smooth (as a countable Borel equivalence relation on X).

(iv) f can be factored into a surjective reduction $g : E \leq_B G$, followed by a complete section embedding $h : G \sqsubseteq_B H$, followed by a class-bijective homomorphism $k : H \rightarrow_B^{cb} F$, for some $G, H \in \mathcal{E}$:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \searrow g & & \nearrow k \\ & G \dashrightarrow H & \end{array}$$

(In particular, f can be factored into a reduction $h \circ g$ (with image a complete section) followed by a class-bijective homomorphism k , or a surjective reduction g followed by a class-injective homomorphism $k \circ h$.)

(v) f belongs to the smallest class of Borel homomorphisms between countable Borel equivalence relations which is closed under composition and contains all reductions and class-injective homomorphisms.

Proof. (i) \implies (ii) is obvious.

(ii) \implies (iii): [CCM, 2.2] If $E \cap \ker f$ is not smooth, then it has an ergodic invariant σ -finite non-atomic measure μ . The pushforward $f_*\mu$ is then a Δ_Y -ergodic (because μ is $(\ker f)$ -ergodic) measure on Y , hence concentrates at some $y \in Y$, i.e., $\mu(f^{-1}(y)) > 0$, whence $f^{-1}(y)$ is not smooth.

(iii) \implies (iv): Letting $g : X \rightarrow X/(E \cap \ker f)$ be the projection and G be the equivalence relation on $X/(E \cap \ker f)$ induced by E , we have that $g : E \leq_B G$ is a surjective reduction, and f descends along g to a class-injective homomorphism $f' : G \rightarrow_B^{ci} F$. By Proposition 5.3, f' factors as a complete section embedding $h : G \sqsubseteq_B H$ followed by a class-bijective homomorphism $k : H \rightarrow_B^{cb} F$, for some $H \in \mathcal{E}$.

(iv) \implies (v) is obvious.

(v) \implies (i): Clearly reductions are smooth, as are class-bijective homomorphisms; it follows that so are class-injective homomorphisms, by Proposition 5.3 (see also [CCM, 2.3]). \square

Similarly to before we now have

Corollary 5.9. *If $E \in \mathcal{E}$ is universally structurable, then $F \leq_B E \iff F \rightarrow_B^{sm} E$, for all $F \in \mathcal{E}$. Similarly, if \mathcal{C} is an elementary class, then $\mathcal{C}^r = \mathcal{C}^{smh}$.*

Proposition 5.10. *Let $E \in \mathcal{E}$ be a countable Borel equivalence relation. Then there is a sentence σ_E^{smh} (in some fixed language) such that for all $F \in \mathcal{E}$, we have $F \models \sigma_E^{smh}$ iff $F \rightarrow_B^{sm} E$.*

Proof. The language is $L = \{R_0, R_1, \dots\} \cup \{P\}$ where R_i, P are unary predicates, and the sentence is $\sigma_E^{smh} := \sigma_E^h \wedge \sigma_E^{sm}$, where σ_E^h is as in Lemma 4.8, and

$$\sigma_E^{sm} := \forall x \exists! y (P(y) \wedge \bigwedge_i (R_i(x) \leftrightarrow R_i(y))).$$

It is easily seen that for any L -structure \mathbb{A} on F , we will have $\mathbb{A} : F \models \sigma_E^{sm}$ iff the interpretation $P^{\mathbb{A}}$ is a Borel transversal of $F \cap \ker f$, where f is the Borel map to E coded by \mathbb{A} . \square

Corollary 5.11. *If $\mathcal{C} = \mathcal{E}_E$ is an elementary class, then so is $\mathcal{C}^r = \mathcal{C}^{smh} = \{F \in \mathcal{E} \mid F \rightarrow_B^{sm} E\}$.*

This completes the proof of Theorem 5.2.

Remark 5.12. Theorem 5.2 generalizes [CCM, 2.3], which shows that some particular classes of the form \mathcal{C}^r , \mathcal{C} elementary, are closed under \rightarrow_B^{sm} .

Hjorth-Kechris [HK, D.3] proved that every \mathcal{C}^r (\mathcal{C} elementary) is closed under \subseteq , i.e., containment of equivalence relations on the same space. Since containment is a class-injective homomorphism (namely the identity), Theorem 5.2 also generalizes this.

See Appendix A.5 for more on the relation between [HK, Appendix D] and the above.

We end this section by pointing out that exactly analogous proofs work for yet another pair of (“global” resp. “local”) classes of homomorphisms (which we did not include in Table 5.1), which forms a natural counterpart to $(\sqsubseteq_B, \rightarrow_B^{ci})$. We write \leq_B^{cs} to denote a **(Borel) class-surjective reduction**, and \rightarrow_B^{cssm} to denote a **class-surjective smooth homomorphism**. Then we have

Theorem 5.13. *Let $\mathcal{C} = \mathcal{E}_E$ be an elementary class. Then*

$$\begin{aligned}\mathcal{C}^{csr} &:= \{F \in \mathcal{E} \mid \exists E \in \mathcal{C} (F \leq_B^{cs} E)\}, \\ \mathcal{C}^{cssmh} &:= \{F \in \mathcal{E} \mid \exists E \in \mathcal{C} (F \rightarrow_B^{cssm} E)\}\end{aligned}$$

are equal and elementary, and $\mathcal{C}^{csr} = \{F \in \mathcal{E} \mid F \rightarrow_B^{cssm} E\}$.

Proof. Exactly as before, we have the following chain of results:

Proposition 5.14. *Let $(X, E), (Y, F) \in \mathcal{E}$ and $f : E \rightarrow_B^{cssm} F$. Then there is a $(Z, G) \in \mathcal{E}$, a surjective reduction $g : E \leq_B G$, and a class-bijective homomorphism $h : G \rightarrow_B^{cb} F$, such that $f = h \circ g$.*

Proof. By Proposition 5.8, f can be factored into a surjective reduction g followed by a class-injective homomorphism h ; since $h \circ g = f$ is class-surjective and g is surjective, h must be class-surjective, i.e., class-bijective. \square

Corollary 5.15. *If $E \in \mathcal{E}$ is universally structurable, then $F \leq_B^{cs} E \iff F \rightarrow_B^{cssm} E$, for all $F \in \mathcal{E}$. Similarly, if \mathcal{C} is an elementary class, then $\mathcal{C}^{csr} = \mathcal{C}^{cssmh}$.*

Proposition 5.16. *Let $E \in \mathcal{E}$. Then there is a sentence σ_E^{cssmh} (in some fixed language) such that for all $F \in \mathcal{E}$, we have $F \models \sigma_E^{cssmh}$ iff $F \rightarrow_B^{cssm} E$.*

Proof. Like Proposition 5.10, but put $\sigma_E^{cssmh} := \sigma_E^h \wedge \sigma_E^{cs} \wedge \sigma_E^{sm}$. \square

It follows that $\mathcal{C}^{csr} = \mathcal{C}^{cssmh} = \{F \in \mathcal{E} \mid F \rightarrow_B^{cssm} E\}$ is elementary. \square

Remark 5.17. Since any reduction $f : E \leq_B F$ can be factored into a surjective reduction onto its image followed by an embedding, we could have alternatively proved that \mathcal{C}^r is elementary (for elementary \mathcal{C}) by combining Theorem 5.1 with Theorem 5.13.

5.3 Elementary reducibility classes

We say that an elementary class $\mathcal{C} \subseteq \mathcal{E}$ is an **elementary reducibility class** if it is closed under reductions. The following elementary classes mentioned in Section 3.1 are elementary reducibility classes: smooth equivalence relations, hyperfinite equivalence relations, treeable equivalence relations [JKL, 3.3], \mathcal{E} . The following classes are *not* elementary reducibility classes: finite equivalence relations, aperiodic equivalence relations, compressible equivalence relations, compressible hyperfinite equivalence relations. In Section 7, we will prove that for a countably infinite group Γ , \mathcal{E}_Γ^* is an elementary reducibility class iff Γ is amenable, where \mathcal{E}_Γ^* consists of equivalence relations whose aperiodic part is generated by a free action of Γ .

By Theorem 5.2, for every $E \in \mathcal{E}$, \mathcal{E}_E^r is the smallest elementary reducibility class containing E ; this is analogous to Corollary 4.10. We also have the following analog of Corollary 4.11:

Corollary 5.18. *A class $\mathcal{C} \subseteq \mathcal{E}$ is an elementary reducibility class iff it is closed under smooth homomorphisms and contains an invariantly universal element $E \in \mathcal{C}$, in which case $\mathcal{C} = \mathcal{E}_E^r$.*

As well, there is the analog of Corollary 4.12:

Corollary 5.19. *Let $E \in \mathcal{E}$. The following are equivalent:*

- (i) *E is invariantly universal in \mathcal{E}_E^r .*
- (ii) *E is invariantly universal in some elementary reducibility class.*
- (iii) *For every $F \in \mathcal{E}$, $F \rightarrow_B^{sm} E$ iff $F \sqsubseteq_B^i E$.*

We call $E \in \mathcal{E}$ **stably universally structurable** if these equivalent conditions hold. We write $\mathcal{E}_\infty^r \subseteq \mathcal{E}$ for the class of stably universally structurable countable Borel equivalence relations. For any $E \in \mathcal{E}$, we write $E_{\infty E}^r := E_{\infty \sigma_E^{smh}}$ for the \sqsubseteq_B^i -universal element of \mathcal{E}_E^r .

Remark 5.20. We emphasize that being stably universally structurable is a *stronger* notion than being universally structurable (\mathcal{E}_∞^r is a transversal of \leftrightarrow_B^{sm} , which is a coarser equivalence relation than \leftrightarrow_B^{cb}). In particular, note that “stably universally structurable” does *not* mean “universal with respect to \leq_B ”.

Remark 5.21. By Proposition 5.8, the preorder \rightarrow_B^{sm} on \mathcal{E} is the composite $(\rightarrow_B^{cb}) \circ (\leq_B)$ of the two preorders \leq_B and \rightarrow_B^{cb} on \mathcal{E} , hence also the join of \leq_B and \rightarrow_B^{cb} in the complete lattice of all preorders on \mathcal{E} (that are \cong_B -invariant, say), i.e., \rightarrow_B^{sm} is the finest preorder on \mathcal{E} coarser than both \leq_B and \rightarrow_B^{cb} . Similarly, \leftrightarrow_B^{sm} is the join of \sim_B and \leftrightarrow_B^{cb} in the lattice of equivalence relations on \mathcal{E} ; this follows from noting that $E \leftrightarrow_B^{cb} E_\infty \otimes E \sim_B E_{\infty E}^r$.

One may ask what is the meet of the preorders \leq_B and \rightarrow_B^{cb} . We do not know of a simple answer. Note that the meet is strictly coarser than \sqsubseteq_B^i ; indeed, $2 \cdot E_0 \leq_B E_0$ and $2 \cdot E_0 \rightarrow_B^{cb} E_0$, but $2 \cdot E_0 \not\sqsubseteq_B^i E_0$. (Similarly, the meet of \sim_B and \leftrightarrow_B^{cb} is strictly coarser than \cong_B .)

Remark 5.22. Clearly one can define similar notions of “elementary embeddability class” and “elementary class-surjective reducibility class”.

5.4 Reductions and compressibility

Dougherty-Jackson-Kechris proved several results relating Borel reducibility to compressibility [DJK, 2.3, 2.5, 2.6], which we state here in a form suited for our purposes.

Proposition 5.23 (Dougherty-Jackson-Kechris). *Let E, F be countable Borel equivalence relations.*

- (a) E is compressible iff $E \cong_B E \times I_{\mathbb{N}}$ (and the latter is always compressible).
- (b) If E is compressible and $E \sqsubseteq_B F$, then $E \sqsubseteq_B^i F$.
- (c) If F is compressible and $E \leq_B^{cs} F$, then $E \sqsubseteq_B^i F$.
- (d) If E, F are compressible and $E \leq_B F$, then $E \sqsubseteq_B^i F$.
- (e) $E \leq_B F$ iff $E \times I_{\mathbb{N}} \sqsubseteq_B^i F \times I_{\mathbb{N}}$.

Proof. While these were all proved at some point in [DJK], not all of them were stated in this form. For (a), see [DJK, 2.5]. For (b), see [DJK, 2.3]. Clearly (e) follows from (a) and (d) (and that $E \sim_B E \times I_{\mathbb{N}}$). We now sketch (c) and (d), which are implicit in the proof of [DJK, 2.6].

For (c), take $f : E \leq_B^{cs} F$, and let $G \sqsubseteq_B^i F$ be the image of f . Then f is a surjective reduction $E \leq_B G$, hence we can find a $g : G \sqsubseteq_B E$ such that $f \circ g = 1_G$; in particular, g is a complete section embedding. Now G is compressible, so applying [DJK, 2.2], we get $G \cong_B E$, whence $E \cong_B G \sqsubseteq_B^i F$.

For (d), take $f : E \leq_B F$, and let $G \sqsubseteq_B F$ be the image of f . Then $f : E \leq_B^{cs} G$ and $G \sqsubseteq_B F$, whence $E \times I_{\mathbb{N}} \leq_B^{cs} G \times I_{\mathbb{N}} \sqsubseteq_B F \times I_{\mathbb{N}}$. By (a–c), $E \cong_B E \times I_{\mathbb{N}} \sqsubseteq_B^i G \times I_{\mathbb{N}} \sqsubseteq_B^i F \times I_{\mathbb{N}} \cong_B F$. \square

Remark 5.24. In passing, we note that Proposition 5.23(b) and Proposition 5.3 together give the following: if E is compressible and $E \rightarrow_B^{ci} F$, then $E \rightarrow_B^{cb} F$.

It follows from Proposition 5.23 that the compressible equivalence relations (up to isomorphism) form a transversal of bireducibility, with corresponding selector $E \mapsto E \times I_{\mathbb{N}}$, which is moreover compatible with the reducibility ordering. We summarize this as follows. Let $\mathcal{E}_c \subseteq \mathcal{E}$ denote the compressible countable Borel equivalence relations.

Corollary 5.25. *We have an order-isomorphism of posets*

$$\begin{aligned} (\mathcal{E}/\sim_B, \leq_B) &\longleftrightarrow (\mathcal{E}_c/\cong_B, \sqsubseteq_B^i) \\ E &\longmapsto E \times I_{\mathbb{N}}. \end{aligned}$$

Remark 5.26. Unlike the selector $E \mapsto E_{\infty} \otimes E$ for \leftrightarrow_B^{cb} , the selector $E \mapsto E \times I_{\mathbb{N}}$ for \sim_B does not take E to the \sqsubseteq_B^i -greatest element of its \sim_B -class (e.g., $E_0 \times I_{\mathbb{N}} \cong_B E_t \sqsubseteq_B^i E_0$). Nor does it always take E to the \sqsubseteq_B^i -least element of its \sim_B -class, or even to an element \sqsubseteq_B^i -less than E : for finite E clearly $E \times I_{\mathbb{N}} \not\sqsubseteq_B^i E$, while for aperiodic E , a result of Thomas [T] (see also [HK, 3.9]) states that there are aperiodic E such that $E \times I_2 \not\sqsubseteq_B^i E$.

We now relate compressibility to structurability. Let $E_{\infty c}$ denote the invariantly universal compressible countable Borel equivalence relation, i.e., the \sqsubseteq_B^i -universal element of \mathcal{E}_c . Aside from $E \mapsto E \times I_{\mathbb{N}}$, we have another canonical way of turning any E into a compressible equivalence relation, namely $E \mapsto E_{\infty c} \otimes E$. These two maps are related as follows:

Proposition 5.27. *Let $E \in \mathcal{E}$ be a countable Borel equivalence relation.*

(a) $E_{\infty c} \otimes E \rightarrow_B^{cb} E \times I_{\mathbb{N}}$.

(b) Suppose E is universally structurable. Then:

(i) $E \times I_{\mathbb{N}}$ is universally structurable, so $E_{\infty c} \otimes E \sqsubseteq_B^i E \times I_{\mathbb{N}}$;

(ii) $E \times I_{\mathbb{N}} \sqsubseteq_B E$ iff $E \times I_{\mathbb{N}} \sqsubseteq_B^i E_{\infty c} \otimes E$ (iff $E \times I_{\mathbb{N}} \cong_B E_{\infty c} \otimes E$).

Proof. For (a), we have $E_{\infty c} \otimes E \rightarrow_B^{cb} E$, whence $E_{\infty c} \otimes E \cong_B (E_{\infty c} \otimes E) \times I_{\mathbb{N}} \rightarrow_B^{cb} E \times I_{\mathbb{N}}$.

Now suppose E is universally structurable. Then similarly to (a), from $E_{\infty c} \otimes E \sqsubseteq_B^i E$ we get $E_{\infty c} \otimes E \sqsubseteq_B^i E \times I_{\mathbb{N}}$, proving the latter part of (i).

Next we prove (ii). If $E \times I_{\mathbb{N}} \sqsubseteq_B^i E_{\infty c} \otimes E$, then $E_{\infty c} \otimes E \sqsubseteq_B^i E$ gives $E \times I_{\mathbb{N}} \sqsubseteq_B^i E$. Conversely, if $E \times I_{\mathbb{N}} \sqsubseteq_B E$, then since $E \times I_{\mathbb{N}}$ is compressible, $E \times I_{\mathbb{N}} \sqsubseteq_B^i E$, and also $E \times I_{\mathbb{N}} \sqsubseteq_B^i E_{\infty c}$, whence $E \times I_{\mathbb{N}} \sqsubseteq_B^i E_{\infty c} \otimes E$. This proves (ii) (the parenthetical follows from the latter part of (i)).

Finally, to finish proving (i), find by Theorem 5.13 a universally structurable $F \sim_B E$ such that \mathcal{E}_F is closed under class-surjective reductions. Then $F \times I_{\mathbb{N}} \leq_B^{cs} F$ gives $F \times I_{\mathbb{N}} \sqsubseteq_B^i F$, whence by (ii), $E \times I_{\mathbb{N}} \cong_B F \times I_{\mathbb{N}} \cong_B E_{\infty c} \otimes F$ is universally structurable. \square

Remark 5.28. We do not know if there is an aperiodic universally structurable E with $E \times I_{\mathbb{N}} \not\sqsubseteq_B E$. The example of Thomas [T] mentioned above is far from universally structurable, since it has a unique ergodic invariant probability measure.

We call a bireducibility class $\mathcal{C} \subseteq \mathcal{E}$ **universally structurable** if it contains a universally structurable element. In this case, by Theorem 5.2, \mathcal{C} contains an invariantly universal (stably universally structurable) element, namely $E_{\infty E}^r$ for any $E \in \mathcal{C}$; and by Proposition 5.27, it also contains a compressible universally structurable element, namely $E \times I_{\mathbb{N}}$ for any $E \in \mathcal{C}$. Between these two (in the ordering \sqsubseteq_B^i) lie all those universally structurable $E \in \mathcal{C}$ such that $E \times I_{\mathbb{N}} \sqsubseteq_B E$.

Let $\mathcal{E}_{\infty c} := \mathcal{E}_{\infty} \cap \mathcal{E}_c$ denote the class of compressible universally structurable equivalence relations. Since \mathcal{E}_{∞} forms a transversal (up to isomorphism) of the equivalence relation \leftrightarrow_B^{cb} , while \mathcal{E}_c forms a transversal of \sim_B , we would expect $\mathcal{E}_{\infty c}$ to form a transversal of \leftrightarrow_B^{sm} , the join of \leftrightarrow_B^{cb} and \sim_B . That this is the case follows from the fact that the two corresponding selectors $E \mapsto E_{\infty} \otimes E$ (for \leftrightarrow_B) and $E \mapsto E \times I_{\mathbb{N}}$ (for \sim_B) commute:

Proposition 5.29. For any $E \in \mathcal{E}$, $(E_{\infty} \otimes E) \times I_{\mathbb{N}} \cong_B E_{\infty} \otimes (E \times I_{\mathbb{N}})$.

Proof. We have $E_{\infty} \otimes E \rightarrow_B^{cb} E$, whence $(E_{\infty} \otimes E) \times I_{\mathbb{N}} \rightarrow_B^{cb} E \times I_{\mathbb{N}}$, and so $(E_{\infty} \otimes E) \times I_{\mathbb{N}} \sqsubseteq_B^i E_{\infty} \otimes (E \times I_{\mathbb{N}})$. Conversely, we have $E \sqsubseteq_B^i E_{\infty} \otimes E$, whence $E \times I_{\mathbb{N}} \sqsubseteq_B^i (E_{\infty} \otimes E) \times I_{\mathbb{N}}$, and so $E_{\infty} \otimes (E \times I_{\mathbb{N}}) \sqsubseteq_B^i (E_{\infty} \otimes E) \times I_{\mathbb{N}}$, since the latter is universally structurable by Proposition 5.27. \square

6 The poset of elementary classes

In this section, we consider the order-theoretic structure of the poset of elementary classes under inclusion (equivalently the poset $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$), as well as the poset of elementary reducibility classes under inclusion (equivalently $(\mathcal{E}_{\infty}^r/\cong_B, \sqsubseteq_B^i)$, or $(\mathcal{E}_{\infty c}/\cong_B, \sqsubseteq_B^i)$, or $(\mathcal{E}_{\infty}/\sim_B, \leq_B)$).

In Section 6.1, we introduce some concepts from order theory which give us a convenient way of concisely stating several results from previous sections. In Section 6.2, we discuss meets and joins in the poset $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$. In Section 6.4, we extend a well-known result of Adams-Kechris [AK] to show that $(\mathcal{E}_{\infty}/\sim_B, \leq_B)$ is quite complicated, by embedding the poset of Borel subsets of reals.

We remark that we always consider the empty equivalence relation \emptyset on the empty set to be a countable Borel equivalence relation; this is particularly important in this section. Note that \emptyset is (vacuously) σ -structurable for any σ , hence is the \sqsubseteq_B^i -universal \perp -structurable equivalence relation, where \perp denotes an inconsistent theory.

6.1 Projections and closures

Among the various posets (or preordered sets) of equivalence relations we have considered so far (e.g., $(\mathcal{E}, \rightarrow_B^{cb})$, $(\mathcal{E}_\infty, \sqsubseteq_B^i)$, $(\mathcal{E}_c, \sqsubseteq_B^i)$), there is one which is both the finest and the most inclusive, namely $(\mathcal{E}/\cong_B, \sqsubseteq_B^i)$. Several of the other posets and preorders may be viewed as derived from $(\mathcal{E}/\cong_B, \sqsubseteq_B^i)$ via the following general order-theoretic notion.

Let (P, \leq) be a poset. A **projection operator** on P is an idempotent order-preserving map $e : P \rightarrow P$, i.e.,

$$\forall x, y \in P (x \leq y \implies e(x) \leq e(y)), \quad e \circ e = e.$$

The image $e(P)$ of a projection operator e is a **retract** of P , i.e., the inclusion $i : e(P) \rightarrow P$ has a one-sided (order-preserving) inverse $e : P \rightarrow e(P)$, such that $e \circ i = 1_{e(P)}$. A projection operator e also gives rise to an **induced preorder** \lesssim on P , namely the pullback of \leq along e , i.e.,

$$x \lesssim y \iff e(x) \leq e(y).$$

Letting $\sim := \ker e$, which is also the equivalence relation associated with \lesssim , we thus have two posets derived from (P, \leq) associated with each projection operator e , namely the quotient poset $(P/\sim, \lesssim)$ and the subposet $(e(P), \leq)$. These are related by an order-isomorphism:

$$\begin{aligned} (P/\sim, \lesssim) &\longleftrightarrow (e(P), \leq) = (e(P), \lesssim) \\ [x]_\sim &\longmapsto e(x) \\ e^{-1}(y) = [y]_\sim &\longleftarrow y. \end{aligned}$$

(There is the following analogy with equivalence relations: set \leftrightarrow poset, equivalence relation \leftrightarrow preorder, selector \leftrightarrow projection, and transversal \leftrightarrow retract.)

Summarizing previous results, we list here several projection operators on $(\mathcal{E}/\cong_B, \sqsubseteq_B^i)$ that we have encountered, together with their images and induced preorders.

- $E \mapsto E_\infty \otimes E$, which has image $\mathcal{E}_\infty/\cong_B$ (the universally structurable equivalence relations) and induces the preorder \rightarrow_B^{cb} (Section 4.3);
- $E \mapsto E \times I_{\mathbb{N}}$, which has image \mathcal{E}_c/\cong_B (the compressible equivalence relations) and induces the preorder \leq_B (Proposition 5.23);
- $E \mapsto (E_\infty \otimes E) \times I_{\mathbb{N}} \cong_B E_\infty \otimes (E \times I_{\mathbb{N}})$ (Proposition 5.29), which has image $\mathcal{E}_{\infty c}/\cong_B$ (the compressible universally structurable equivalence relations) and induces the preorder \rightarrow_B^{sm} ;
- $E \mapsto E_{\infty E}^r$ (the \sqsubseteq_B^i -universal element of \mathcal{E}_E^r), which has image $\mathcal{E}_\infty^r/\cong_B$ (the stably universally structurable equivalence relations) and also induces the preorder \rightarrow_B^{sm} ;
- similarly, $E \mapsto$ the \sqsubseteq_B^i -universal element of \mathcal{E}_E^e , which induces \rightarrow_B^{ci} .

Also note that some of these projection operators can be restricted to the images of others; e.g., the restriction of $E \mapsto E \times I_{\mathbb{N}}$ to \mathcal{E}_{∞} is a projection operator on $\mathcal{E}_{\infty}/\cong_B$ (by Proposition 5.27), with image $\mathcal{E}_{\infty c}/\cong_B$.

Again let (P, \leq) be a poset, and let $e : P \rightarrow P$ be a projection operator. We say that e is a **closure operator** if

$$\forall x \in P (x \leq e(x)).$$

In other words, each $e(x)$ is the (\leq -)greatest element of its \sim -class. In that case, the induced preorder \lesssim satisfies

$$x \lesssim y \iff x \leq e(y) \quad (\iff e(x) \leq e(y)).$$

Among the projection operators on $(\mathcal{E}/\cong_B, \sqsubseteq_B^i)$ listed above, three are closure operators, namely $E \mapsto$ the \sqsubseteq_B^i -universal element of \mathcal{E}_E , \mathcal{E}_E^r , or \mathcal{E}_E^e (the first of these being $E \mapsto E_{\infty} \otimes E$).

For another example, let us say that a countable Borel equivalence relation $E \in \mathcal{E}$ is **idempotent** if $E \cong_B E \oplus E$. This is easily seen to be equivalent to $E \cong_B \aleph_0 \cdot E$; hence, the idempotent elements of \mathcal{E} form the image of the closure operator $E \mapsto \aleph_0 \cdot E$ on $(\mathcal{E}/\cong_B, \sqsubseteq_B^i)$. Note that all universally structurable equivalence relations are idempotent (Corollary 4.15).

6.2 The lattice structure

We now discuss the lattice structure of the poset of elementary classes under inclusion, equivalently the poset $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$ of universally structurable isomorphism classes under \sqsubseteq_B^i .

Let us first introduce the following notation. For theories (L, σ) and (L', τ) , we write

$$(L, \sigma) \Rightarrow^* (L', \tau) \quad (\text{or } \sigma \Rightarrow^* \tau)$$

to mean that $\mathcal{E}_{\sigma} \subseteq \mathcal{E}_{\tau}$, i.e., for every $E \in \mathcal{E}$, if $E \models \sigma$, then $E \models \tau$. Thus \Rightarrow^* is a preorder on the class of theories which is equivalent to the poset of elementary classes (via $\sigma \mapsto \mathcal{E}_{\sigma}$), and hence also to the poset $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$ (via $\sigma \mapsto E_{\infty\sigma}$). We denote the associated equivalence relation by \Leftrightarrow^* .

Remark 6.1. We stress that in the notation $\sigma \Rightarrow^* \tau$, σ and τ may belong to *different* languages. Of course, if they happen to belong to the same language and σ logically implies τ , then also $\sigma \Rightarrow^* \tau$; but the latter is in general a weaker condition.

Let (P, \leq) be a poset. We say that P is an ω_1 -**complete lattice** if every countable subset $A \subseteq P$ has a meet (i.e., greatest lower bound) $\bigwedge A$, as well as a join (i.e., least upper bound) $\bigvee A$. We say that P is an ω_1 -**distributive lattice** if it is an ω_1 -complete lattice which satisfies the ω_1 -**distributive laws**

$$x \wedge \bigvee_i y_i = \bigvee_i (x \wedge y_i), \quad x \vee \bigwedge_i y_i = \bigwedge_i (x \vee y_i),$$

where i runs over a countable index set.

Theorem 6.2. *The poset $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$ is an ω_1 -distributive lattice, in which joins are given by \bigoplus , nonempty meets are given by \bigotimes , the greatest element is E_{∞} , and the least element is \emptyset .*

Moreover, the inclusion $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i) \subseteq (\mathcal{E}/\cong_B, \sqsubseteq_B^i)$ preserves (countable) meets and joins. In other words, if $E_0, E_1, \dots \in \mathcal{E}_{\infty}$ are universally structurable equivalence relations, then $\bigotimes_i E_i$ (respectively $\bigoplus_i E_i$) is their meet (respectively join) in $(\mathcal{E}/\cong_B, \sqsubseteq_B^i)$ as well as in $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$.

Before giving the proof of Theorem 6.2, we discuss the operations on theories which correspond to the operations \otimes and \oplus . That is, let $((L_i, \sigma_i))_i$ be a countable family of theories; we want to find theories (L', σ') and (L'', σ'') such that $\otimes_i E_{\infty\sigma_i} \cong_B E_{\infty\sigma'}$ and $\oplus_i E_{\infty\sigma_i} \cong_B E_{\infty\sigma''}$.

Proposition 6.3. *Let $\otimes_i(L_i, \sigma_i) = (\bigsqcup_i L_i, \otimes_i \sigma_i)$ be the theory where $\bigsqcup_i L_i$ is the disjoint union of the L_i , and $\otimes_i \sigma_i$ is the conjunction of the σ_i 's regarded as being in the language $\bigsqcup_i L_i$ (so that the different σ_i 's have disjoint languages). Then $\otimes_i E_{\infty\sigma_i} \cong_B E_{\infty\otimes_i \sigma_i}$.*

Proof. For each i , the $(\otimes_j \sigma_j)$ -structure on $E_{\infty\otimes_j \sigma_j}$ has a reduct which is a σ_i -structure, so $E_{\infty\otimes_j \sigma_j} \models \sigma_i$, i.e., $E_{\infty\otimes_j \sigma_j} \sqsubseteq_B^i E_{\infty\sigma_i}$; hence $E_{\infty\otimes_j \sigma_j} \sqsubseteq_B^i \otimes_i E_{\infty\sigma_i}$. Conversely, for each j we have $\otimes_i E_{\infty\sigma_i} \xrightarrow{cb} E_{\infty\sigma_j} \models \sigma_j$ so $\otimes_i E_{\infty\sigma_i} \models \sigma_j$; combining these σ_j -structures yields a $\otimes_j \sigma_j$ -structure, so $\otimes_i E_{\infty\sigma_i} \sqsubseteq_B^i E_{\infty\otimes_j \sigma_j}$. \square

While we can similarly prove that $\oplus_i E_{\infty\sigma_i}$ corresponds to the theory given by the disjunction of the σ_i 's, we prefer to work with the following variant, which is slightly better behaved with respect to structurability. Let $\oplus_i(L_i, \sigma_i) = (\oplus_i L_i, \oplus_i \sigma_i)$ be the theory where $\oplus_i L_i := \bigsqcup_i (L_i \sqcup \{P_i\})$ where each P_i is a unary relation symbol, and

$$\oplus_i \sigma_i := \bigvee_i ((\forall x P_i(x)) \wedge \sigma_i \wedge \bigwedge_{j \neq i} \bigwedge_{R \in L_i \sqcup \{P_i\}} \forall \bar{x} \neg R(\bar{x}))$$

(where on the right-hand side, σ_i is regarded as having language $\oplus_i L_i$). In other words, $\oplus_i \sigma_i$ asserts that for some (unique) i , P_i holds for all elements, and we have a σ_i -structure; and for all $j \neq i$, P_j and all relations in L_j hold for no elements. Then for a countable Borel equivalence relation $(X, E) \in \mathcal{E}$, a $\oplus_i \sigma_i$ -structure $\mathbb{A} : E \models \oplus_i \sigma_i$ is the same thing as a Borel E -invariant partition $(P_i^{\mathbb{A}})_i$ of X , together with a σ_i -structure $\mathbb{A}|P_i^{\mathbb{A}} : E|P_i^{\mathbb{A}} \models \sigma_i$ for each i .

Proposition 6.4. $\oplus_i E_{\infty\sigma_i} \cong_B E_{\infty\oplus_i \sigma_i}$.

Proof. The σ_i -structure on each $E_{\infty\sigma_i}$ yields a $\oplus_j \sigma_j$ -structure (with $P_i^{\mathbb{A}} = \text{everything}$, and $P_j^{\mathbb{A}} = \emptyset$ for $j \neq i$); so $\oplus_i E_{\infty\sigma_i} \sqsubseteq_B^i E_{\infty\oplus_j \sigma_j}$. Conversely, letting $\mathbb{A} : E_{\infty\oplus_j \sigma_j} \models \oplus_j \sigma_j$, we have $E_{\infty\oplus_j \sigma_j} = \oplus_i E_{\infty\oplus_j \sigma_j}|P_i^{\mathbb{A}}$ and $\mathbb{A}|P_i^{\mathbb{A}} : E_{\infty\oplus_j \sigma_j}|P_i^{\mathbb{A}} \models \sigma_i$ for each i , whence $E_{\infty\oplus_i \sigma_i} \sqsubseteq_B^i \oplus_i E_{\infty\sigma_i}$. \square

Remark 6.5. For a more formal viewpoint on the operations \otimes and \oplus on theories, see Appendix B.2.

As noted in [KMd, 2.C], the next lemma follows from abstract properties of the poset $(\mathcal{E}/\cong_B, \sqsubseteq_B^i)$; for the convenience of the reader, we include a direct proof.

Lemma 6.6. *Let $E_0, E_1, \dots \in \mathcal{E}$ be countably many idempotent countable Borel equivalence relations. Then $\oplus_i E_i$ is their join in the preorder $(\mathcal{E}, \sqsubseteq_B^i)$.*

Proof. Clearly each $E_j \sqsubseteq_B^i \oplus_i E_i$. Let $F \in \mathcal{E}$ and $E_i \sqsubseteq_B^i F$ for each i ; we must show that $\oplus_i E_i \sqsubseteq_B^i F$. Since each $E_i \sqsubseteq_B^i F$, we have $F \cong_B E_i \oplus F_i$ for some F_i ; since $E_i \cong_B E_i \oplus E_i$, we have $F \cong_B E_i \oplus E_i \oplus F_i \cong_B E_i \oplus F$. So an invariant embedding $\oplus_i E_i \sqsubseteq_B^i F$ is built by invariantly embedding E_0 into $E_0 \oplus F \cong_B F$ so that the remainder (complement of the image) is isomorphic to F , then similarly embedding E_1 into the remainder, etc. \square

Proof of Theorem 6.2. By Propositions 6.3 and 6.4, we may freely switch between the operations \otimes and \oplus on universally structurable equivalence relations, and the same operations on theories.

First we check that \otimes is meet and \oplus is join. Let $(L_0, \sigma_0), (L_1, \sigma_1), \dots$ be theories; it suffices to show that $\otimes_i \sigma_i$, resp., $\oplus_i \sigma_i$, is their meet, resp., join, in the preorder \Rightarrow^* . For \otimes this is clear, since a $(\otimes_i \sigma_i)$ -structure on $E \in \mathcal{E}$ is the same thing as a σ_i -structure for each i . For \oplus , a σ_i -structure on E for any i yields a $(\oplus_j \sigma_j)$ -structure (corresponding to the partition of E where the i th piece is everything); thus $\sigma_i \Rightarrow^* \oplus_j \sigma_j$ for each i . And if (L', τ) is another theory with $\sigma_i \Rightarrow^* \tau$ for each i , then given a $(\oplus_i \sigma_i)$ -structure on E , we have a partition of E into pieces which are σ_i -structured for each i , so by $\sigma_i \Rightarrow^* \tau$ we can τ -structure each piece of the partition; thus $\oplus_i \sigma_i \Rightarrow^* \tau$.

That the inclusion $(\mathcal{E}_\infty / \cong_B, \sqsubseteq_B^i) \rightarrow (\mathcal{E} / \cong_B, \sqsubseteq_B^i)$ preserves (*all existing*) meets follows from the fact that $\mathcal{E}_\infty / \cong_B \subseteq \mathcal{E} / \cong_B$ is the image of the closure operator $E \mapsto E_\infty \otimes E$. That it preserves countable joins follows from Lemma 6.6.

Now we check the ω_1 -distributive laws. Distributivity of \otimes over \oplus follows from Proposition 4.20(e). To check distributivity of \oplus over \otimes , we again work with theories. Let $\sigma, \tau_0, \tau_1, \dots$ be theories; we need to show that $\sigma \oplus \otimes_i \tau_i \Leftrightarrow^* \otimes_i (\sigma \oplus \tau_i)$. The \Rightarrow^* inequality, as in any lattice, is trivial. For the converse inequality, let $(X, E) \in \mathcal{E}$ and $\mathbb{A} : E \models \otimes_i (\sigma \oplus \tau_i)$, which amounts to a $\mathbb{A}_i : E \models \sigma \oplus \tau_i$ for each i . Then for each i , we have a Borel E -invariant partition $X = A_i \cup B_i$ such that $\mathbb{A}_i|_{A_i} : E|_{A_i} \models \sigma$ and $\mathbb{A}_i|_{B_i} : E|_{B_i} \models \tau_i$. By combining the various \mathbb{A}_i , we get $E|_{\bigcup_i A_i} \models \sigma$ and $E|_{\bigcap_i B_i} \models \otimes_i \tau_i$; and so the partition $X = (\bigcup_i A_i) \cup (\bigcap_i B_i)$ witnesses that $E \models \sigma \oplus \otimes_i \tau_i$. \square

Remark 6.7. It is not true that \oplus is join in $(\mathcal{E} / \cong_B, \sqsubseteq_B^i)$, since there exist $E \in \mathcal{E}$ such that $E \not\cong_B E \oplus E$ (e.g., $E = E_0$). Similarly, it is not true that \otimes is meet in $(\mathcal{E} / \cong_B, \sqsubseteq_B^i)$, since there are E with $E \not\cong_B E \otimes E$ (see examples near end of Section 4.4).

Remark 6.8. That the inclusion $(\mathcal{E}_\infty / \cong_B, \sqsubseteq_B^i) \rightarrow (\mathcal{E} / \cong_B, \sqsubseteq_B^i)$ preserves countable joins suggests that perhaps $\mathcal{E}_\infty / \cong_B \subseteq \mathcal{E} / \cong_B$ is also the image of an “interior operator”. This would mean that every countable Borel equivalence relation $E \in \mathcal{E}$ contains (in the sense of \sqsubseteq_B^i) a greatest universally structurable equivalence relation. We do not know if this is true.

By restricting Theorem 6.2 to the class \mathcal{E}_c of compressible equivalence relations, which is downward-closed under \sqsubseteq_B^i , closed under \oplus , and has greatest element $E_{\infty c}$, we immediately obtain

Corollary 6.9. *The poset $(\mathcal{E}_{\infty c} / \cong_B, \sqsubseteq_B^i)$ is an ω_1 -distributive lattice, in which joins are given by \oplus , nonempty meets are given by \otimes , the greatest element is $E_{\infty c}$, and the least element is \emptyset . Moreover, the inclusion $(\mathcal{E}_{\infty c} / \cong_B, \sqsubseteq_B^i) \subseteq (\mathcal{E}_c / \cong_B, \sqsubseteq_B^i)$ preserves (countable) meets and joins.*

Now using that $(\mathcal{E}_c / \cong_B, \sqsubseteq_B^i)$ is isomorphic to $(\mathcal{E} / \sim_B, \leq_B)$, we may rephrase this as

Corollary 6.10. *The poset of universally structurable bireducibility classes under \leq_B is an ω_1 -distributive lattice. Moreover, the inclusion into the poset $(\mathcal{E} / \sim_B, \leq_B)$ of all bireducibility classes under \leq_B preserves (countable) meets and joins.*

Remark 6.11. We stress that the \leq_B -meets in Corollary 6.10 must be computed using the *compressible* elements of bireducibility classes. That is, if E, F are universally structurable, then their \leq_B -meet is $(E \times I_{\mathbb{N}}) \otimes (F \times I_{\mathbb{N}})$, but *not necessarily* $E \otimes F$. For example, if E is invariantly universal finite and F is invariantly universal aperiodic, then $E \otimes F = \emptyset$ is clearly not the \leq_B -meet of E, F . Also, if there is an aperiodic universally structurable E with $E \times I_{\mathbb{N}} \not\leq_B E$, then (by Proposition 5.27) $E \otimes E_{\infty c}$ is not the \leq_B -meet of E and $E_{\infty c} \sim_B E_\infty$.

The order-theoretic structure of the poset $(\mathcal{E}/\sim_B, \leq_B)$ of *all* bireducibility classes under \leq_B is not well-understood, apart from that it is very complicated (by [AK]). The first study of this structure was made by Kechris-Macdonald in [KMd]. In particular, they raised the question of whether there exists any pair of \leq_B -incomparable $E, F \in \mathcal{E}$ for which a \leq_B -meet exists. Corollary 6.10, together with the existence of many \leq_B -incomparable universally structurable bireducibility classes (Theorem 6.21), answers this question by providing a large class of bireducibility classes for which \leq_B -meets always exist.

There are some natural order-theoretic questions one could ask about the posets $(\mathcal{E}_\infty/\cong_B, \sqsubseteq_B^i)$ and $(\mathcal{E}_\infty/\sim, \leq_B)$, which we do not know how to answer. For example, is either a complete lattice? If so, is it completely distributive? Is it a “zero-dimensional” ω_1 -complete lattice, in that it embeds (preserving all countable meets and joins) into 2^X for some set X ? (See Corollary 6.16 below for some partial results concerning this last question.)

Remark 6.12. It can be shown that every ω_1 -distributive lattice is a *quotient* of a sublattice of 2^X for some set X (see Appendix C). In particular, this implies that the set of algebraic identities involving \otimes and \oplus which hold in $(\mathcal{E}_\infty/\cong_B, \sqsubseteq_B^i)$ is “completely understood”, in that it consists of exactly those identities which hold in $2 = \{0 < 1\}$.

6.3 Closure under independent joins

We mention here some connections with recent work of Marks [M].

Let E_0, E_1, \dots be countably many countable Borel equivalence relations on the same standard Borel space X . We say that the E_i are **independent** if there is no sequence x_0, x_1, \dots, x_n of distinct elements of X , where $n \geq 1$, such that $x_0 E_{i_0} x_1 E_{i_1} \dots E_{i_{n-1}} x_n E_{i_n} x_0$ for some i_0, \dots, i_n with $i_j \neq i_{j+1}$ for each j . In that case, their **independent join** is the smallest equivalence relation on X containing each E_i . For example, the independent join of treeable equivalence relations is still treeable. Marks proves the following for elementary classes closed under independent joins [M, 4.15, 4.16]:

Theorem 6.13 (Marks). *If \mathcal{E}_σ is an elementary class of aperiodic equivalence relations closed under binary independent joins, then for any Borel homomorphism $p : E_{\infty\sigma} \rightarrow_B \Delta_X$ (where X is any standard Borel space), there is some $x \in X$ such that $E_{\infty\sigma} \sim_B E_{\infty\sigma}|p^{-1}(x)$.*

Theorem 6.14 (Marks). *If \mathcal{E}_σ is an elementary class of aperiodic equivalence relations closed under countable independent joins, then for any $E \in \mathcal{E}$, if $E_{\infty\sigma} \leq_B E$, then $E_{\infty\sigma} \sqsubseteq_B E$.*

Remark 6.15. Clearly the aperiodicity condition in Theorems 6.13 and 6.14 can be loosened to the condition that $I_{\mathbb{N}} \in \mathcal{E}_\sigma$ (so that restricting \mathcal{E}_σ to the aperiodic elements does not change $E_{\infty\sigma}$ up to biembeddability).

Above we asked whether the ω_1 -distributive lattice $(\mathcal{E}_\infty/\sim_B, \leq_B)$ is zero-dimensional, i.e., embeds into 2^X for some set X . This is equivalent to asking whether there are enough ω_1 -**prime filters** (i.e., filters closed under countable meets whose complements are closed under countable joins) in $(\mathcal{E}_\infty/\sim_B, \leq_B)$ to separate points. Theorem 6.13 gives some examples of ω_1 -prime filters:

Corollary 6.16. *If \mathcal{E}_σ contains $I_{\mathbb{N}}$ and is closed under binary independent joins, then*

$$\{E \in \mathcal{E}_\infty \mid E_{\infty\sigma} \leq_B E\}$$

is an ω_1 -prime filter in $(\mathcal{E}_\infty/\sim_B, \leq_B)$.

Proof. If $g : E_{\infty\sigma} \leq_B \bigoplus_i E_i$ then we have a homomorphism $E_{\infty\sigma} \rightarrow_B \Delta_{\mathbb{N}}$ sending the g -preimage of E_i to i ; by Theorem 6.13, it follows that $E_{\infty\sigma} \leq_B E_i$ for some i . \square

We also have the following simple consequence of Theorem 6.14:

Corollary 6.17. *If \mathcal{E}_σ is an elementary class such that \mathcal{E}_σ^r is closed under countable independent joins, then $\mathcal{E}_\sigma^e = \mathcal{E}_\sigma^r$.*

Proof. The \sqsubseteq_B^i -universal element of \mathcal{E}_σ^r reduces to $E_{\infty\sigma}$, whence by Theorem 6.14 it embeds into $E_{\infty\sigma}$, i.e., belongs to \mathcal{E}_σ^e . \square

Remark 6.18. Although the conclusions of Theorems 6.13 and 6.14 are invariant with respect to bireducibility (respectively biembeddability), Marks has pointed out that the notion of being closed under independent joins is not similarly invariant: there are $E_{\infty\sigma} \sim_B E_{\infty\tau}$ such that \mathcal{E}_σ is closed under independent joins but \mathcal{E}_τ is not. In particular, if σ axiomatizes trees while τ axiomatizes trees of degree ≤ 3 , then $E_{\infty\sigma} \sim_B E_{\infty\tau}$ by [JKL, 3.10]; but it is easy to see (using an argument like that in Proposition 6.19 below) that independent joins of τ -structurable equivalence relations can have arbitrarily high cost, so are not all τ -structurable.

Clearly if $\mathcal{E}_\sigma, \mathcal{E}_\tau$ are closed under independent joins, then so is $\mathcal{E}_{\sigma \otimes \tau} = \mathcal{E}_\sigma \cap \mathcal{E}_\tau$. In particular, the class \mathcal{E}_c of compressible equivalence relations is closed under independent joins, since the join of compressible equivalence relations contains a compressible equivalence relation; thus the class of compressible treeable equivalence relations is closed under independent joins. We note that this is the smallest nontrivial elementary class to which Theorems 6.13 and 6.14 apply:

Proposition 6.19. *If \mathcal{E}_σ is an elementary class containing $I_{\mathbb{N}}$ and closed under binary independent joins, then \mathcal{E}_σ contains all compressible treeable equivalence relations.*

Proof. Since \mathcal{E}_σ is elementary and contains $I_{\mathbb{N}}$, it contains all aperiodic smooth countable Borel equivalence relations. Now let $(X, E) \in \mathcal{E}$ be compressible treeable. By [JKL, 3.11], there is a Borel treeing $T \subseteq E$ with degree ≤ 3 . By [KST, 4.6] (see also remarks following [KST, 4.10]), there is a Borel edge coloring $c : T \rightarrow 5$. Then E is the independent join of the equivalence relations $E_i := c^{-1}(i) \cup \Delta_X$ for $i = 0, 1, 2, 3, 4$. Since the E_i are not aperiodic, consider the following modification. Let

$$X' := X \sqcup (X \times 5 \times \mathbb{N}),$$

let T' be the tree on X' consisting of T on X and the edges $(x, (x, i, 0))$ and $((x, i, n), (x, i, n+1))$ for $x \in X$, $i \in 5$, and $n \in \mathbb{N}$, and let $c' : T' \rightarrow 5$ extend c with $c'(x, (x, i, 0)) = c'((x, i, n), (x, i, n+1)) = i$ (note that c' is not an edge coloring). Then the inclusion $X \rightarrow X'$ is a complete section embedding of each E_i into the equivalence relation E'_i generated by $c'^{-1}(i)$, and of E into the equivalence relation E' generated by T' . It follows that each E'_i is (aperiodic) smooth (because E_i is), hence in \mathcal{E}_σ , while $E \cong_B E'$ (because E is compressible). But it is easily seen that E' is the independent join of the E'_i , whence $E \in \mathcal{E}_\sigma$. \square

6.4 Embedding the poset of Borel sets

Adams-Kechris [AK] proved the following result showing that the poset $(\mathcal{E}/\sim_B, \leq_B)$ is extremely complicated:

Theorem 6.20 (Adams-Kechris). *There is an order-embedding from the poset of Borel subsets of \mathbb{R} under inclusion into the poset $(\mathcal{E}/\sim_B, \leq_B)$.*

In this short section, we show that their proof may be strengthened to yield

Theorem 6.21. *There is an order-embedding from the poset of Borel subsets of \mathbb{R} under inclusion into the poset $(\mathcal{E}_\infty/\sim_B, \leq_B)$.*

Proof. By [AK, 4.2], there is a countable Borel equivalence relation (X, E) , a Borel homomorphism $p : (X, E) \rightarrow_B (\mathbb{R}, \Delta_{\mathbb{R}})$, and a Borel map $x \mapsto \mu_x$ taking each $x \in \mathbb{R}$ to a Borel probability measure μ_x on X , such that, putting $E_x := E|_{p^{-1}(x)}$, we have

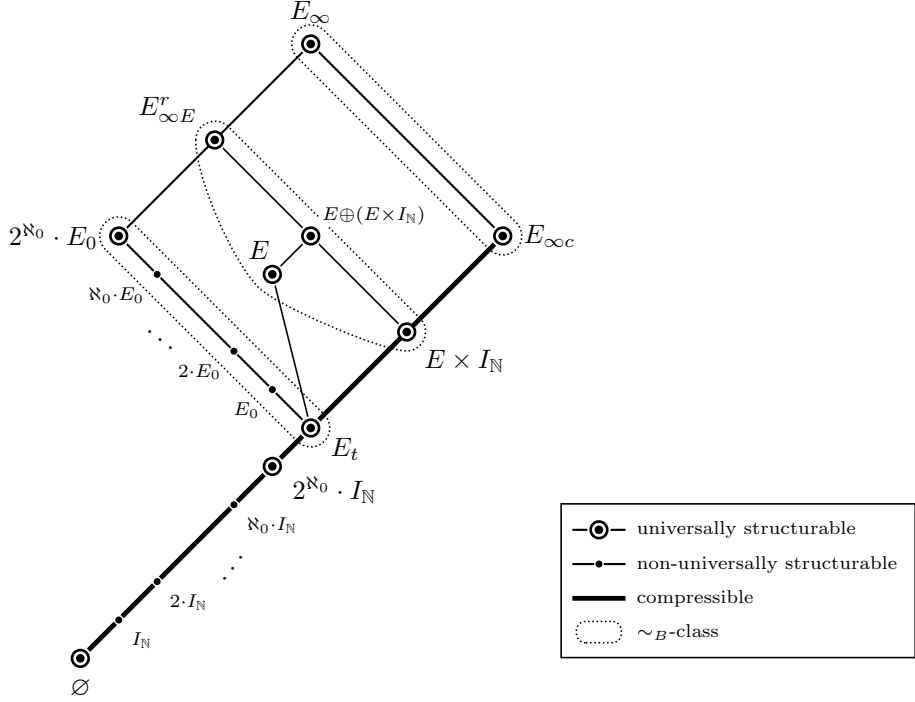
- (i) for each $x \in \mathbb{R}$, μ_x is nonatomic, concentrated on $p^{-1}(x)$, E_x -invariant, and E_x -ergodic;
- (ii) if $x, y \in \mathbb{R}$ with $x \neq y$, then every Borel homomorphism $f : E_x \rightarrow_B E_y$ maps a Borel E_x -invariant set $M \subseteq p^{-1}(x)$ of μ_x -measure 1 to a single E_y -class.

For Borel $A \subseteq \mathbb{R}$, put $E_A := E|_{p^{-1}(A)}$ and $F_A := E_\infty \otimes E_A$. We claim that $A \mapsto F_A$ gives the desired order-embedding. It is clearly order-preserving. Now suppose $A, B \subseteq \mathbb{R}$ with $A \not\subseteq B$ but $F_A \leq_B F_B$. By taking $x \in A \setminus B$, we get $x \notin B$ but $E_x \sqsubseteq_B^i E_\infty \otimes E_x = F_{\{x\}} \leq_B F_A \leq_B F_B$. Let $f : E_x \leq_B F_B = E_\infty \otimes E_B$, and let $\pi_2 : E_\infty \otimes E_B \rightarrow_B^{cb} E_B$ be the second projection. Then $p \circ \pi_2 \circ f : E_x \rightarrow_B \Delta_B$, whence by E_x -ergodicity of μ_x , there is a $y \in B$ and an E_x -invariant $M \subseteq p^{-1}(x)$ of μ_x -measure 1 such that $(p \circ \pi_2 \circ f)(M) = \{y\}$, i.e., $(\pi_2 \circ f)(M) \subseteq p^{-1}(y)$. By (ii) above, there is a further E_x -invariant $N \subseteq M$ of μ_x -measure 1 such that $(\pi_2 \circ f)(N)$ is contained in a single E_y -class. But since π_2 is class-bijective and f is a reduction, this implies that $E|_N$ is smooth, a contradiction. \square

Remark 6.22. If in Theorem 6.21 we replace $(\mathcal{E}_\infty/\sim_B, \leq_B)$ with $(\mathcal{E}_\infty/\cong_B, \sqsubseteq_B^i)$ (thus weakening the result), then a simpler proof may be given, using groups of different costs (see [KM, 36.4]) instead of [AK].

6.5 A global picture

The picture below is a simple visualization of the poset $(\mathcal{E}/\cong_B, \sqsubseteq_B^i)$. For the sake of clarity, among the hyperfinite equivalence relations, only the aperiodic ones are shown.



Six landmark universally structurable equivalence relations are shown (circled dots): \emptyset , $2^{N_0} \cdot I_{\mathbb{N}}$ (\sqsubseteq_B^i -universal aperiodic smooth), E_t (\sqsubseteq_B^i -universal compressible hyperfinite), $2^{N_0} \cdot E_0$ (\sqsubseteq_B^i -universal aperiodic hyperfinite), $E_{\infty c}$ (\sqsubseteq_B^i -universal compressible), and E_{∞} (\sqsubseteq_B^i -universal).

Also shown is the “backbone” of compressible equivalence relations (bold line), which contains one element from each bireducibility class (dotted loops).

The middle of the picture shows a “generic” universally structurable E and its relations to some canonical elements of its bireducibility class: the \sqsubseteq_B^i -universal element $E_{\infty E}^r$ and the compressible element $E \times I_{\mathbb{N}}$. Note that $E \times I_{\mathbb{N}}$ is not depicted as being below E , in accordance with Remark 5.28. Note also that for non-smooth E , the \sqsubseteq_B^i -universal element $E_{\infty E}^r$ of its bireducibility class would indeed be above $2^{N_0} \cdot E_0$, as shown: $E_0 \leq_B E$ implies $2^{N_0} \cdot E_0 \sqsubseteq_B^i E_{\infty E}^r$ since $E_{\infty E}^r$ is stably universally structurable.

Finally, note that the picture is somewhat misleading in a few ways. It is not intended to suggest that the compressibles form a linear order. Nor is it intended that any of the pairs $E \sqsubseteq_B^i F$ do not have anything strictly in between them (except of course for the things below $2^{N_0} \cdot E_0$, which are exactly as shown).

7 Free actions of a group

Let Γ be a countably infinite group. Recall (from Section 3.1) that we regard Γ as a structure in the language $L_{\Gamma} = \{R_{\gamma} \mid \gamma \in \Gamma\}$, where $R_{\gamma}^{\Gamma} \subseteq \Gamma^2$ is the graph of the left multiplication of γ on Γ . Thus, $\mathcal{E}_{\Gamma} = \mathcal{E}_{\sigma_{\Gamma}}$ (where σ_{Γ} is the Scott sentence of Γ in L_{Γ}) is the class of Borel equivalence relations generated by a free Borel action of Γ . Our main goal in this section is to characterize when \mathcal{E}_{Γ} is an elementary reducibility class.

Actually, to deal with a technicality, we need to consider the following variant of \mathcal{E}_{Γ} . Let $\mathcal{E}_{\Gamma}^* := \mathcal{E}_{\sigma_{\Gamma} \oplus \sigma_f}$, where σ_f is a sentence axiomatizing the finite equivalence relations. Thus \mathcal{E}_{Γ}^* consists

of countable Borel equivalence relations whose aperiodic part is generated by a free Borel action of Γ . This is needed because every equivalence relation in \mathcal{E}_Γ must have all classes of the same cardinality as Γ .

Theorem 7.1. *Let Γ be a countably infinite group. The following are equivalent:*

- (i) Γ is amenable.
- (ii) \mathcal{E}_Γ^* is closed under \sqsubseteq_B .
- (iii) \mathcal{E}_Γ^* is closed under \leq_B , i.e., \mathcal{E}_Γ^* is an elementary reducibility class.

To motivate Theorem 7.1, consider the following examples. By Theorem 2.1, $\mathcal{E}_{\mathbb{Z}}^*$ is the class of all hyperfinite equivalence relations, which is closed under \leq_B . On the other hand, for every $2 \leq n \leq \aleph_0$, the free group \mathbb{F}_n on n generators is such that $(\mathcal{E}_{\mathbb{F}_n}^*)^r$ is the class of treeable equivalence relations, by [JKL, 3.17]; but $\mathcal{E}_{\mathbb{F}_n}^*$ is not itself the class of all treeables, since every $E \in \mathcal{E}_{\mathbb{F}_n}^*$ with a nonatomic invariant probability measure has cost n (see [KM, 36.2]).

Recall that the \sqsubseteq_B^i -universal element of \mathcal{E}_Γ is $F(\Gamma, \mathbb{R})$, the orbit equivalence of the free part of the shift action of Γ on \mathbb{R}^Γ . Thus the \sqsubseteq_B^i -universal element of \mathcal{E}_Γ^* is $F(\Gamma, \mathbb{R}) \oplus E_{\infty f}$, where $E_{\infty f}$ is the \sqsubseteq_B^i -universal finite equivalence relation (given by $E_{\infty f} = \bigoplus_{1 \leq n \in \mathbb{N}} 2^{\aleph_0} \cdot I_n$).

Remark 7.2. Seward and Tucker-Drob [ST] have shown that for countably infinite Γ , every free Borel action of Γ admits an equivariant class-bijective map into $F(\Gamma, 2)$ (clearly the same holds for finite Γ). It follows that $F(\Gamma, 2)$ is \rightarrow_B^{cb} -universal in \mathcal{E}_Γ .

A well-known open problem asks whether every orbit equivalence of a Borel action of a countable amenable group Γ is hyperfinite. In the purely Borel context, the best known general result is the following [SS]:

Theorem 7.3 (Schneider-Seward). *If Γ is a countable locally nilpotent group, i.e., every finitely generated subgroup of Γ is nilpotent, then every orbit equivalence E_Γ^X of a Borel action of Γ is hyperfinite.*

Remark 7.4. Recently Conley, Jackson, Marks, Seward, and Tucker-Drob have found examples of solvable but not locally nilpotent countable groups for which the conclusion of Theorem 7.3 still holds.

If Theorem 7.3 generalizes to arbitrary countable amenable Γ , then it would follow that \mathcal{E}_Γ^* is the class of all hyperfinite equivalence relations (since it contains $F(\Gamma, \mathbb{R})$ which admits an invariant probability measure); then the main implication (i) \implies (iii) in Theorem 7.1 would trivialize.

In the measure-theoretic context, a classical result of Ornstein-Weiss [OW] states that the orbit equivalence of a Borel action of an amenable group Γ is hyperfinite almost everywhere with respect to every probability measure. We will need a version of this result which is uniform in the measure, which we now state. For a standard Borel space X , we let $P(X)$ denote the **space of probability Borel measures on X** (see [K95, 17.E]).

Lemma 7.5. *Let X, Y be standard Borel spaces, $E = E_\Gamma^X$ be the orbit equivalence of a Borel action of a countable amenable group Γ on X , and $m : Y \rightarrow_B P(X)$. Then there is a Borel set $A \subseteq Y \times X$, with $\pi_1(A) = Y$ (where $\pi_1 : Y \times X \rightarrow_B Y$ is the first projection), such that*

(i) for each $y \in Y$, $A_y := \{x \in X \mid (y, x) \in A\}$ has $m(y)$ -measure 1 and is E -invariant;

(ii) $(\Delta_Y \times E)|_A$ is hyperfinite.

Proof. This follows from verifying that the proofs of [KM, 9.2, 10.1] can be made uniform. We omit the details, which are tedious but straightforward. \square

We now have the following, which forms the core of Theorem 7.1:

Proposition 7.6. *Let Γ be a countable amenable group, and let $(X, E), (Y, F) \in \mathcal{E}$ be countable Borel equivalence relations. If $E \leq_B F$ and $F = E_\Gamma^Y$ for some Borel action of Γ on Y , then E is the disjoint sum of a hyperfinite equivalence relation and a compressible equivalence relation.*

Proof. If E is compressible, then we are done. Otherwise, E has an invariant probability measure. Consider the ergodic decomposition of E ; see e.g., [KM, 3.3]. This gives a Borel homomorphism $p : E \rightarrow_B \Delta_{P(X)}$ such that

(i) p is a surjection onto the Borel set $P_e(E) \subseteq P(X)$ of ergodic invariant probability measures on E ;

(ii) for each $e \in P_e(E)$, we have $e(p^{-1}(e)) = 1$.

Let $f : E \leq_B F$, and apply Lemma 7.5 to F and $f_* : P_e(E) \rightarrow_B P(Y)$, where f_* is the pushforward of measures. This gives Borel $A \subseteq P_e(E) \times Y$ such that

(iii) for each $e \in P_e(E)$, $e(f^{-1}(A_e)) = (f_*e)(A_e) = 1$, and $A_e \subseteq Y$ is F -invariant (so A is $(\Delta_{P_e(E)} \times F)$ -invariant);

(iv) $(\Delta_{P_e(E)} \times F)|_A$ is hyperfinite.

Now consider the homomorphism $g := (p, f) : E \rightarrow_B \Delta_{P_e(E)} \times F$, i.e., $g(x) = (p(x), f(x))$. Then g is a reduction because f is. It follows that $B := g^{-1}(A)$ is E -invariant and $E|_B$ is hyperfinite. It now suffices to note that $E|(X \setminus B)$ is compressible. Indeed, otherwise it would have an ergodic invariant probability measure, i.e., there would be some $e \in P_e(E)$ such that $e(X \setminus B) = 1$. But then $e(p^{-1}(e) \cap f^{-1}(A_e)) = 1$, while $p^{-1}(e) \cap f^{-1}(A_e) \subseteq B$, a contradiction. \square

Proof of Theorem 7.1. Clearly (iii) \implies (ii). If (ii) holds, then $E_0 \sqsubseteq_B F(\Gamma, \mathbb{R})$ implies that E_0 is generated by a free action of Γ , so since E_0 is hyperfinite and has an invariant probability measure, Γ is amenable (see [JKL, 2.5(ii)]). So it remains to prove (i) \implies (iii).

Let $E \leq_B F \in \mathcal{E}_\Gamma^*$. Then E splits into a smooth part, which is clearly in \mathcal{E}_Γ^* , and a part which reduces to some $F' \in \mathcal{E}_\Gamma$; so we may assume $F \in \mathcal{E}_\Gamma$. Factor the reduction $E \leq_B F$ into a surjective reduction $f : E \leq_B G$ (onto the image) followed by an embedding $G \sqsubseteq_B F$. By Proposition 7.6, $G = G' \oplus G''$, where G' is hyperfinite and G'' is compressible. Then $E = f^{-1}(G') \oplus f^{-1}(G'')$. Since $f^{-1}(G'') \leq_B^{cs} G'' \sqsubseteq_B F$ and G'' is compressible, we have $f^{-1}(G'') \sqsubseteq_B^i F$ (Proposition 5.23) and so $f^{-1}(G'') \in \mathcal{E}_\Gamma$. Finally, we have $f^{-1}(G') \in \mathcal{E}_\Gamma^*$, since \mathcal{E}_Γ^* contains all hyperfinite equivalence relations (because $E_0 \sqsubseteq_B^i F(\Gamma, \mathbb{R})$, by Ornstein-Weiss's theorem and Theorem 2.1). \square

8 Structurability and model theory

In the previous sections, we have studied the relationship between structurability and common notions from the theory of countable Borel equivalence relations. This section, by contrast, concerns the other side of the \models relation, i.e., logic. In particular, we are interested in model-theoretic properties of theories (L, σ) which are reflected in the elementary class \mathcal{E}_σ that they axiomatize.

A general question one could ask is when two theories $(L, \sigma), (L', \tau)$ axiomatize the same elementary class, i.e., in the notation of Section 6.2, when does $\sigma \Leftrightarrow^* \tau$. Our main result here answers one instance of this question. Let σ_{sm} denote any sentence axiomatizing the smooth countable Borel equivalence relations.

Theorem 8.1. *Let (L, σ) be a theory. The following are equivalent:*

- (i) *There is an $L_{\omega_1\omega}$ -formula $\phi(x)$ which defines a finite nonempty subset in any countable model of σ .*
- (ii) *$\sigma \Rightarrow^* \sigma_{sm}$, i.e., any σ -structurable equivalence relation is smooth, or equivalently $E_{\infty\sigma}$ is smooth.*
- (iii) *For any countably infinite group Γ , we have $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_{sm}$, i.e., any σ -structurable equivalence relation generated by a free Borel action of Γ is smooth.*
- (iv) *There is a countably infinite group Γ such that $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_{sm}$.*

In particular, this answers a question of Marks [M, end of Section 4.3], who asked for a characterization of when $E_{\infty\sigma_\mathbb{A}}$ ($\sigma_\mathbb{A}$ a Scott sentence) is smooth. The proof uses ideas from topological dynamics and ergodic theory.

Marks observed that recent work of Ackerman-Freer-Patel [AFP] implies the following sufficient condition for a structure \mathbb{A} to structure every aperiodic countable Borel equivalence relation. In Section 8.2, we present his proof of this result, as well as several corollaries and related results. The result refers to the model-theoretic notion of **trivial definable closure**; see Section 8.2 for details. Let σ_a denote any sentence axiomatizing the aperiodic countable Borel equivalence relations.

Theorem 8.2 (Marks). *Let L be a language and \mathbb{A} be a countable L -structure with trivial definable closure. Then $\sigma_a \Rightarrow^* \sigma_\mathbb{A}$, i.e., every aperiodic countable Borel equivalence relation is \mathbb{A} -structurable.*

In Section 8.3 we discuss the problem of when an elementary class can be axiomatized by a Scott sentence.

8.1 Smoothness of $E_{\infty\sigma}$

We now begin the proof of Theorem 8.1. The implication (i) \implies (ii) is easy: given a formula ϕ as in (i), ϕ may be used to uniformly pick out a finite nonempty subset of each $E_{\infty\sigma}$ -class, thus $E_{\infty\sigma}$ is smooth. The implications (ii) \implies (iii) \implies (iv) are obvious. So let Γ be as in (iv).

Consider the logic action of S_Γ on $\text{Mod}_\Gamma(L)$, the space of L -structures with universe Γ . Recall that this is given as follows: for $f \in S_\Gamma$, $\bar{\delta} \in \Gamma^n$, n -ary $R \in L$, and $\mathbb{A} \in \text{Mod}_\Gamma(L)$, we have

$$R^{f(\mathbb{A})}(\bar{\delta}) \iff R^\mathbb{A}(f^{-1}(\bar{\delta})).$$

We regard Γ as a subgroup of S_Γ via the left multiplication action, so that Γ acts on $\text{Mod}_\Gamma(L)$.

Lemma 8.3. *Let Γ be a countably infinite group. Then $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_{sm}$ iff $E_\Gamma^{\text{Mod}_\Gamma(\sigma)}$ is smooth.*

Proof. The proof is essentially that of [KM, 29.5], with an additional wrinkle in the use of [SW] below (to deal with σ -finite measures).

\Leftarrow : suppose (X, E) is generated by a free Borel action of Γ , $\mathbb{A} : E \models \sigma$, and E is not smooth (hence witnessing that $\sigma \otimes \sigma_\Gamma \not\Rightarrow^* \sigma_{sm}$). Then it admits an ergodic non-atomic invariant (σ -finite) measure μ . Define $f : X \rightarrow \text{Mod}_\Gamma(\sigma)$ by

$$R^{f(x)}(\gamma_1, \dots, \gamma_n) \iff R^\mathbb{A}(\gamma_1^{-1} \cdot x, \dots, \gamma_n^{-1} \cdot x).$$

Then f is Γ -equivariant, whence $f_*\mu$ is an ergodic non-atomic invariant (σ -finite) measure on $E_\Gamma^{\text{Mod}_\Gamma(\sigma)}$, which is therefore not smooth.

\Rightarrow : let ν be an ergodic non-atomic invariant measure on $E_\Gamma^{\text{Mod}_\Gamma(\sigma)}$. Consider the free part $Y \subseteq 2^\Gamma$ of the shift action of Γ on 2^Γ , with orbit equivalence $F = F(\Gamma, 2)$. The usual product measure ρ on 2^Γ concentrates on Y , and is invariant and mixing with respect to the action of Γ on Y (see [KM, 3.1]). Then consider the product action of Γ on $Y \times \text{Mod}_\Gamma(\sigma)$, which is free since Γ acts freely on Y . By [SW, 2.3, 2.5], this product action admits $\rho \times \nu$ as an ergodic non-atomic invariant (σ -finite) measure. Thus $E_\Gamma^{Y \times \text{Mod}_\Gamma(\sigma)}$ is not smooth.

Now observe that $E_\Gamma^{Y \times \text{Mod}_\Gamma(\sigma)}$ is the skew product $F \ltimes \text{Mod}_\Gamma(\sigma)$ with respect to the cocycle $\alpha : F \rightarrow \Gamma$ associated to the free action of Γ on Y ; and that α , when regarded as a cocycle $F \rightarrow S_\Gamma$, is induced, in the sense of Remark 4.3, by $T : Y \rightarrow Y^\Gamma$ where $T(y)(\gamma) := \gamma^{-1} \cdot y$. So (as in the proof of Theorem 4.1) $E_\Gamma^{Y \times \text{Mod}_\Gamma(\sigma)}$ is σ -structurable, hence witnesses that $\sigma \otimes \sigma_\Gamma \not\Rightarrow^* \sigma_{sm}$. \square

So we have converted (iv) in Theorem 8.1 into a property of the action of Γ on $\text{Mod}_\Gamma(\sigma)$. Our next step requires some preparation.

Let L be a language and $\mathbb{A} = (X, R^\mathbb{A})_{R \in L}$ be a countable L -structure. We say that \mathbb{A} has the **weak duplication property (WDP)** if for any finite sublanguage $L' \subseteq L$ and finite subset $F \subseteq X$, there is a finite subset $G \subseteq X$ disjoint from F such that $(\mathbb{A}|L')|F \cong (\mathbb{A}|L')|G$ (here $\mathbb{A}|L'$ denotes the reduct in the sublanguage L').

Remark 8.4. If we define the **duplication property (DP)** for \mathbb{A} by replacing L' in the above by L , then clearly the DP is equivalent to the **strong joint embedding property (SJEP)** for the age of \mathbb{A} : for any $\mathbb{F}, \mathbb{G} \in \text{Age}(\mathbb{A})$, there is $\mathbb{H} \in \text{Age}(\mathbb{A})$ and embeddings $\mathbb{F} \rightarrow \mathbb{H}$ and $\mathbb{G} \rightarrow \mathbb{H}$ with disjoint images. (Recall that $\text{Age}(\mathbb{A})$ is the class of finite L -structures embeddable in \mathbb{A} .)

For a countable group Γ acting continuously on a topological space X , we say that a point $x \in X$ is **recurrent** if x is not isolated in the orbit $\Gamma \cdot x$ (with the subspace topology). When X is a Polish space, a basic fact is that E_Γ^X is smooth iff it does not have a recurrent point; see e.g., [K10, 22.3].

Thus far, we have only regarded the space $\text{Mod}_X(L)$ of L -structures on a countable set X as a standard Borel space. Below we will also need to consider the topological structure on $\text{Mod}_X(L)$. See e.g., [K95, 16.C]. In particular, we will use the system of basic clopen sets consisting of

$$N_{\mathbb{F}} := \{\mathbb{A} \in \text{Mod}_X(L) \mid (\mathbb{A}|L')|F = \mathbb{F}\}$$

where $L' \subseteq L$ is a finite sublanguage and $\mathbb{F} = (F, R^\mathbb{F})_{R \in L'}$ is an L' -structure on a finite nonempty subset $F \subseteq X$.

The next lemma, which translates between the dynamics of $\text{Mod}_\Gamma(\sigma)$ and a model-theoretic property of σ , is the heart of the proof of (iv) \implies (i) in Theorem 8.1:

Lemma 8.5. *Let Γ be a countably infinite group. Let L be a language, and let σ be an $L_{\omega_1\omega}$ -sentence such that $\text{Mod}_\Gamma(\sigma) \subseteq \text{Mod}_\Gamma(L)$ is a G_δ subspace. Suppose there is a countable model $\mathbb{A} \models \sigma$ with the WDP, such that the interpretation $R_0^\mathbb{A}$ of some $R_0 \in L$ is not definable (without parameters) from equality. Then the action of Γ on $\text{Mod}_\Gamma(\sigma)$ has a recurrent point, thus $E_\Gamma^{\text{Mod}_\Gamma(\sigma)}$ is not smooth.*

Proof. We claim that it suffices to show that

- (*) every basic clopen set $N_\mathbb{F} \subseteq \text{Mod}_\Gamma(L)$ containing some isomorphic copy of \mathbb{A} also contains two distinct isomorphic copies, $\mathbb{B} \cong \mathbb{A}$ and $\gamma \cdot \mathbb{B} \neq \mathbb{B}$, from the same Γ -orbit.

Suppose this has been shown; we complete the proof. Note that since \mathbb{A} has WDP, \mathbb{A} must be infinite. Let $\overline{\text{Mod}_\Gamma(\sigma_\mathbb{A})}$ denote the closure in $\text{Mod}_\Gamma(\sigma)$ of $\text{Mod}_\Gamma(\sigma_\mathbb{A})$ (where $\sigma_\mathbb{A}$ is the Scott sentence of \mathbb{A}). Since $\text{Mod}_\Gamma(\sigma) \subseteq \text{Mod}_\Gamma(L)$ is G_δ , $\overline{\text{Mod}_\Gamma(\sigma_\mathbb{A})}$ is a Polish space, which is nonempty because it contains an isomorphic copy of \mathbb{A} . For each basic clopen set $N_\mathbb{F} \subseteq \text{Mod}_\Gamma(L)$, the set of $\mathbb{B} \in \overline{\text{Mod}_\Gamma(\sigma_\mathbb{A})}$ such that

$$\mathbb{B} \in N_\mathbb{F} \implies \exists \gamma \in \Gamma (\mathbb{B} \neq \gamma \cdot \mathbb{B} \in N_\mathbb{F}) \quad (**)$$

is clearly open; and by (*), it is also dense. Thus the set of recurrent points in $\overline{\text{Mod}_\Gamma(\sigma_\mathbb{A})}$, i.e., the set of $\mathbb{B} \in \overline{\text{Mod}_\Gamma(\sigma_\mathbb{A})}$ for which (**) holds for every $N_\mathbb{F}$, is comeager.

So it remains to prove (*). Let \mathbb{F} be such that $N_\mathbb{F}$ contains an isomorphic copy of \mathbb{A} . Let $\mathbb{A} = (X, R^\mathbb{A})_{R \in L}$, and let $\mathbb{F} = (F, R^\mathbb{F})_{R \in L'}$ where $F \subseteq \Gamma$ is finite nonempty and $L' \subseteq L$ is finite. We may assume $R_0 \in L'$.

Since $N_\mathbb{F}$ contains an isomorphic copy of \mathbb{A} , there is a map $f : F \rightarrow X$ which is an embedding $\mathbb{F} \rightarrow \mathbb{A}$. We will extend f to a bijection $\Gamma \rightarrow X$, and then define $\mathbb{B} := f^{-1}(\mathbb{A})$, thus ensuring that $\mathbb{B} \in N_\mathbb{F}$; we need to choose f appropriately so that there is $\gamma \in \Gamma$ with $\mathbb{B} \neq \gamma \cdot \mathbb{B} \in N_\mathbb{F}$.

Put $G := f(F) \subseteq X$. By WDP, there is a $G' \subseteq X$ disjoint from G such that $(\mathbb{A}|L')|G \cong (\mathbb{A}|L')|G'$, say via $g : G \rightarrow G'$. By the hypothesis that $R_0^\mathbb{A}$ is not definable from equality, there are $\bar{x} = (x_1, \dots, x_n) \in X^n$ and $\bar{x}' = (x'_1, \dots, x'_n) \in X^n$, where n is the arity of R_0 , such that $\bar{x} \in R_0^\mathbb{A}$, $\bar{x}' \notin R_0^\mathbb{A}$, and \bar{x}, \bar{x}' have the same equality type, i.e., we have a bijection $\{x_1, \dots, x_n\} \rightarrow \{x'_1, \dots, x'_n\}$ sending x_i to x'_i . Again by WDP, we may find \bar{x}, \bar{x}' disjoint from G, G' , and each other.

Now pick $\bar{\delta} = (\delta_1, \dots, \delta_n) \in \Gamma^n$ disjoint from F and with the same equality type as \bar{x} , and pick $\gamma \in \Gamma$ such that $\gamma^{-1}F$ and $\gamma^{-1}\bar{\delta}$ are disjoint from F and $\bar{\delta}$. Extend $f : F \rightarrow X$ to a bijection $f : \Gamma \rightarrow X$ such that

$$f|\gamma^{-1}F = g \circ (f|F) \circ \gamma : \gamma^{-1}F \rightarrow G', \quad f(\bar{\delta}) = \bar{x}, \quad f(\gamma^{-1}\bar{\delta}) = \bar{x}'.$$

Then putting $\mathbb{B} := f^{-1}(\mathbb{A})$, it is easily verified that $(\gamma \cdot \mathbb{B}|L')|F = \mathbb{F}$, i.e., $\gamma \cdot \mathbb{B} \in N_\mathbb{F}$; but $R_0^{\gamma \cdot \mathbb{B}}(\bar{\delta}) \iff R_0^\mathbb{B}(\gamma^{-1}\bar{\delta}) \iff R_0^\mathbb{A}(\bar{x}') \iff \neg R_0^\mathbb{A}(\bar{x}) \iff \neg R_0^\mathbb{B}(\bar{\delta})$, so $\gamma \cdot \mathbb{B} \neq \mathbb{B}$. \square

Corollary 8.6. *Let Γ be a countably infinite group. Let L be a language, and let σ be an $L_{\omega_1\omega}$ -sentence such that $\text{Mod}_\Gamma(\sigma) \subseteq \text{Mod}_\Gamma(L)$ is a G_δ subspace. If $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_{sm}$, then no countable model of σ has the WDP.*

Proof. Suppose a countable (infinite) $\mathbb{A} \models \sigma$ has the WDP. If for some $R_0 \in L$, the interpretation $R_0^\mathbb{A}$ is not definable (without parameters) from equality, then $\sigma \otimes \sigma_\Gamma \not\Rightarrow^* \sigma_{sm}$ by Lemma 8.3 and Lemma 8.5. Otherwise, clearly any aperiodic countable Borel equivalence relation is $\sigma_\mathbb{A}$ -structurable, hence σ -structurable; taking $F(\Gamma, 2)$ then yields that $\sigma \otimes \sigma_\Gamma \not\Rightarrow^* \sigma_{sm}$. \square

Working towards (i) in Theorem 8.1, which asserts the existence of a formula with certain properties, we now encode the WDP into formulas, using the following combinatorial notion.

Let X be a set and $1 \leq n \in \mathbb{N}$. An n -ary **intersecting family** on X is a nonempty collection \mathcal{F} of subsets of X of size n such that every pair $A, B \in \mathcal{F}$ has $A \cap B \neq \emptyset$.

Lemma 8.7. *Let L be a language. There are $L_{\omega_1\omega}$ -formulas $\phi_n(x_0, \dots, x_{n-1})$ for each $1 \leq n \in \mathbb{N}$, such that for any countable L -structure $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ without the WDP, there is some n such that*

$$\{\{x_0, \dots, x_{n-1}\} \mid \phi_n^{\mathbb{A}}(x_0, \dots, x_{n-1})\}$$

is an n -ary intersecting family on X .

Proof. Let $((L_k, n_k, \mathbb{F}_k))_k$ enumerate all countably many triples where $L_k \subseteq L$ is a finite sublanguage, $1 \leq n_k \in \mathbb{N}$, and $\mathbb{F}_k \in \text{Mod}_{n_k}(L_k)$ is an L_k -structure with universe $n_k = \{0, \dots, n_k - 1\}$. For each k , let $\psi_k(x_0, \dots, x_{n_k-1})$ be an L_k -formula asserting that x_0, \dots, x_{n_k-1} (are pairwise distinct and) form an L_k -substructure isomorphic to \mathbb{F}_k , which is not disjoint from any other such substructure. Thus \mathbb{A} does not have the WDP iff some ψ_k holds for some tuple in \mathbb{A} ; and in that case, the collection of all tuples (regarded as sets) for which ψ_k holds will form an n_k -ary intersecting family. Finally put

$$\phi_n(\bar{x}) := \bigvee_{n_k=n} (\psi_k(\bar{x}) \wedge \neg \bigvee_{k' < k} \exists \bar{y} \psi_{k'}(\bar{y})),$$

so that ϕ_{n_k} is equivalent to ψ_k for the least k which holds for some tuple. \square

Recall that (i) in Theorem 8.1 asserts the existence of a single formula defining a finite nonempty set. The following lemma, due to Clemens-Conley-Miller [CCM, 4.3], gives a way of uniformly defining a finite nonempty set from an intersecting family. For the convenience of the reader, we include its proof here.

Lemma 8.8 (Clemens-Conley-Miller). *Let \mathcal{F} be an n -ary intersecting family on X . For $1 \leq m < n$, define*

$$\mathcal{F}^{(m)} := \{A \subseteq X \mid |A| = m \wedge |\{B \in \mathcal{F} \mid A \subseteq B\}| \geq \aleph_0\}.$$

Then there exist $n > m_1 > m_2 > \dots > m_k$ such that $\mathcal{F}^{(m_1)}, \mathcal{F}^{(m_1)(m_2)}, \dots$ are (respectively m_1 -ary, m_2 -ary, etc.) intersecting families, and $\mathcal{F}^{(m_1)\dots(m_k)}$ is finite.

Proof. It suffices to show that if \mathcal{F} is infinite, then there is some $1 \leq m < n$ such that $\mathcal{F}^{(m)}$ is an m -ary intersecting family. Indeed, having shown this, we may find the desired m_1, m_2, \dots inductively; the process must terminate since a 1-ary intersecting family is necessarily a singleton.

So assume \mathcal{F} is infinite, and let $m < n$ be greatest so that $\mathcal{F}^{(m)}$ is nonempty. Let $A, B \in \mathcal{F}^{(m)}$. For each $x \in B \setminus A$, by our choice of m , there are only finitely many $C \in \mathcal{F}$ such that $A \cup \{x\} \subseteq C$. Thus by definition of $\mathcal{F}^{(m)}$, there is $C \in \mathcal{F}$ such that $A \subseteq C$ and $(B \setminus A) \cap C = \emptyset$. Similarly, there is $D \in \mathcal{F}$ such that $B \subseteq D$ and $(C \setminus B) \cap D = \emptyset$. Then $A \cap B = C \cap B = C \cap D \neq \emptyset$, as desired. \square

Corollary 8.9. *Let L be a language. There is an $L_{\omega_1\omega}$ -formula $\phi(x)$ such that for any countable L -structure \mathbb{A} without the WDP, $\phi^{\mathbb{A}}$ is a finite nonempty subset.*

Proof. This follows from Lemma 8.7 and Lemma 8.8, by a straightforward encoding of the operation $\mathcal{F} \mapsto \mathcal{F}^{(m)}$ in $L_{\omega_1\omega}$.

In more detail, for each $L_{\omega_1\omega}$ -formula $\psi(x_0, \dots, x_{n-1})$ and $m < n$, let $\psi^{(m)}(x_0, \dots, x_{m-1})$ be a formula asserting that x_0, \dots, x_{m-1} are pairwise distinct and there are infinitely many extensions (x_m, \dots, x_{n-1}) such that $\psi(x_0, \dots, x_{n-1})$ holds, so that if ψ defines (in the sense of Lemma 8.7) a family \mathcal{F} of subsets of size n , then $\psi^{(m)}$ defines $\mathcal{F}^{(m)}$. Let ϕ_n for $1 \leq n \in \mathbb{N}$ be given by Lemma 8.7. For each finite tuple $t = (n, m_1, \dots, m_k)$ such that $n > m_1 > \dots > m_k \geq 1$, let τ_t be a sentence asserting that $\phi_n^{(m_1) \dots (m_k)}$ holds for at least one but only finitely many tuples. Then letting $(t^l = (n^l, m_1^l, \dots, m_{k^l}^l))_{l \in \mathbb{N}}$ enumerate all such tuples, the desired formula ϕ can be given by

$$\phi(x) = \bigvee_l \left(\tau_{t^l} \wedge \exists \bar{x} \left(\phi_{n^l}^{(m_1^l) \dots (m_{k^l}^l)}(\bar{x}) \wedge \bigvee_i (x = x_i) \right) \wedge \neg \bigvee_{l' < l} \tau_{t^{l'}} \right).$$

By Lemmas 8.7 and 8.8, in any countable L -structure \mathbb{A} without the WDP, $\phi^{\mathbb{A}}$ will be the union of the finitely many sets in some intersecting family. \square

Corollary 8.10. *Let Γ be a countably infinite group. Let L be a language, and let σ be an $L_{\omega_1\omega}$ -sentence such that $\text{Mod}_\Gamma(\sigma) \subseteq \text{Mod}_\Gamma(L)$ is a G_δ subspace. If $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_{sm}$, then there is an $L_{\omega_1\omega}$ -formula $\phi(x)$ which defines a finite nonempty subset in any countable model of σ .*

Proof. By Corollary 8.6 and Corollary 8.9. \square

To complete the proof of Theorem 8.1, we need to remove the assumption that $\text{Mod}_\Gamma(\sigma) \subseteq \text{Mod}_\Gamma(L)$ is G_δ from Corollary 8.10. This can be done using the standard trick of **Morleyzation**, as described for example in [Hod, Section 2.6] for finitary first-order logic, or [AFP, 2.5] for $L_{\omega_1\omega}$. Given any language L and $L_{\omega_1\omega}$ -sentence σ , by adding relation symbols for each formula in a countable fragment of $L_{\omega_1\omega}$ containing the sentence σ , we obtain a new (countable) language L' and an $L'_{\omega_1\omega}$ -sentence σ' such that

- the L -reduct of every countable model of σ' is a model of σ ;
- every countable model of σ has a unique expansion to a model of σ' ;
- σ' is (logically equivalent to a formula) of the form

$$\bigwedge_i \forall \bar{x} \exists y \bigvee_j \phi_{i,j}(\bar{x}, y),$$

where each $\phi_{i,j}$ is a quantifier-free finitary L' -formula, whence $\text{Mod}_\Gamma(\sigma') \subseteq \text{Mod}_\Gamma(L')$ is G_δ .

It follows that the conditions (i) and (iv) in Theorem 8.1 for (L, σ) are equivalent to the same conditions for (L', σ') . So Corollary 8.10 holds also without the assumption that $\text{Mod}_\Gamma(\sigma) \subseteq \text{Mod}_\Gamma(L)$ is G_δ , which completes the proof of Theorem 8.1.

We conclude this section by pointing out the following analog of Lemma 8.3:

Lemma 8.11. *Let Γ be a countably infinite group and (L, σ) be a theory. Then $\sigma \otimes \sigma_\Gamma \Rightarrow^* \sigma_c$ iff $E_\Gamma^{\text{Mod}_\Gamma(\sigma)}$ is compressible.*

Proof. Exactly as in Lemma 8.3, but using probability measures instead of σ -finite measures. \square

This has the following corollaries. The first strengthens [AFP, Section 6.1.10]:

Corollary 8.12. *Let \mathcal{T}_1 denote the class of trees, and more generally, let \mathcal{T}_n denote the class of contractible n -dimensional simplicial complexes. Then for each n , there is some countably infinite group Γ such that $\text{Mod}_\Gamma(\mathcal{T}_n)$ admits no Γ -invariant measure (and thus no S_Γ -invariant measure).*

Proof. For each n , let σ_n be a sentence axiomatizing \mathcal{T}_n . For $n = 1$, take Γ to be any infinite Kazhdan group. By [AS], no free Borel action of Γ admitting an invariant probability measure is treeable, i.e., $\sigma_1 \otimes \sigma_\Gamma \Rightarrow^* \sigma_c$; thus $\text{Mod}_\Gamma(\mathcal{T}_n)$ admits no Γ -invariant measure by Lemma 8.11. For $n > 1$, take $\Gamma := \mathbb{F}_2^n \times \mathbb{Z}$. By a result of Gaboriau (see, e.g., [HK, p. 59]), no free Borel action of Γ admitting an invariant probability measure can be \mathcal{T}_n -structurable. \square

Corollary 8.13. *Let L be a language, and let σ be an $L_{\omega_1\omega}$ -sentence such that $\text{Mod}_\mathbb{N}(\sigma) \subseteq \text{Mod}_\mathbb{N}(L)$ is a closed subspace. Then for any countably infinite group Γ , there is a free Borel action of Γ which admits an invariant probability measure and is σ -structurable.*

Proof. Since $\text{Mod}_\Gamma(\sigma) \subseteq \text{Mod}_\Gamma(L)$ is closed, it is compact, so since S_Γ is amenable, $\text{Mod}_\Gamma(\sigma)$ admits a S_Γ -invariant probability measure, thus a Γ -invariant probability measure; then apply Lemma 8.11. \square

8.2 Universality of $E_{\infty\sigma}$

Several theories (L, σ) are known to axiomatize \mathcal{E} , the class of all countable Borel equivalence relations. For example, by [JKL, (proof of) 3.12], every $E \in \mathcal{E}$ is structurable via locally finite graphs. More generally, one can consider σ such that every aperiodic or compressible countable Borel equivalence relation is σ -structurable. For example, it is folklore that every aperiodic countable Borel equivalence relation can be structured via dense linear orders (this will also follow from Theorem 8.2 below), while the proof of [JKL, 3.10] shows that every compressible $E \in \mathcal{E}$ is structurable via graphs of vertex degree ≤ 3 .

A result that some particular σ axiomatizes \mathcal{E} (or all aperiodic E) shows that every (aperiodic) $E \in \mathcal{E}$ carries a certain type of structure, which can be useful in applications. A typical example is the very useful Marker Lemma (see [BK, 4.5.3]), which shows that every aperiodic E admits a decreasing sequence of Borel complete sections $A_0 \supseteq A_1 \supseteq \dots$ with empty intersection. This can be phrased as: every aperiodic countable Borel equivalence relation E is σ -structurable, where σ in the language $L = \{P_0, P_1, \dots\}$ asserts that each (unary) P_i defines a nonempty subset, $P_0 \supseteq P_1 \supseteq \dots$, and $\bigcap_i P_i = \emptyset$.

We now give the proof of Theorem 8.2, which provides a large class of examples of such theories. To do so, we first review the main result from [AFP].

Let L be a language and $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ be a countable L -structure. We say that \mathbb{A} has **trivial definable closure (TDC)** if the following equivalent conditions hold (see [AFP, 2.12–15], [Hod, 4.1.3]):

- for every finite $F \subseteq X$, the pointwise stabilizer $\text{Aut}_F(\mathbb{A})$ of the automorphism group of \mathbb{A} on F fixes no other elements of X ;
- for every finite $F \subseteq X$, the pointwise stabilizer $\text{Aut}_F(\mathbb{A})$ of the automorphism group of \mathbb{A} on F has infinite orbits on $X \setminus F$ (**trivial algebraic closure**);
- for every finite $F \subseteq X$ and $L_{\omega_1\omega}$ -formula $\phi(x)$ with parameters in F , if there is a unique $x \in X$ such that $\phi^{\mathbb{A}}(x)$ holds, then $x \in F$;

- for every finite $F \subseteq X$ and $L_{\omega_1\omega}$ -formula $\phi(x)$ with parameters in F , if there are only finitely many $x_1, \dots, x_n \in X$ such that $\phi^{\mathbb{A}}(x_i)$ holds, then $x_i \in F$ for each i .

Remark 8.14. If \mathbb{A} is a Fraïssé structure, then TDC is further equivalent to the **strong amalgamation property (SAP)** for the age of \mathbb{A} : for any $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \text{Age}(\mathbb{A})$ living on F, G, H respectively and embeddings $f : \mathbb{H} \rightarrow \mathbb{F}$ and $g : \mathbb{H} \rightarrow \mathbb{G}$, there is $\mathbb{K} \in \text{Age}(\mathbb{A})$ and embeddings $f' : \mathbb{F} \rightarrow \mathbb{K}$ and $g' : \mathbb{G} \rightarrow \mathbb{K}$ with $f' \circ f = g' \circ g$ and $f'(F) \cap g'(G) = (f' \circ f)(H)$.

Theorem 8.15 (Ackerman-Freer-Patel [AFP, 1.1]). *Let L be a language and $\mathbb{A} = (X, R^{\mathbb{A}})_{R \in L}$ be a countably infinite L -structure. The following are equivalent:*

- (i) *The logic action of S_X on $\text{Mod}_X(\sigma_{\mathbb{A}})$ ($\sigma_{\mathbb{A}}$ the Scott sentence of \mathbb{A}) admits an invariant probability measure.*
- (ii) *\mathbb{A} has TDC.*

We will in fact need the following construction from Ackerman-Freer-Patel's proof of Theorem 8.15. Starting with a countable L -structure \mathbb{A} with TDC, they consider the Morleyization $(L', \sigma'_{\mathbb{A}})$ of the Scott sentence $\sigma_{\mathbb{A}}$ of \mathbb{A} , where

$$\sigma'_{\mathbb{A}} = \bigwedge_i \forall \bar{x} \exists y \psi_i(\bar{x}, y)$$

with each ψ_i quantifier-free, as described following Corollary 8.10. They then produce (see [AFP, Section 3.4]) a Borel L' -structure $\mathbb{A}' \models \sigma'_{\mathbb{A}}$ with universe \mathbb{R} such that for each i and $\bar{x} \in \mathbb{R}$, the corresponding subformula $\exists y \psi_i(\bar{x}, y)$ in $\sigma'_{\mathbb{A}}$ is witnessed either by some y in the tuple \bar{x} , or by all y in some nonempty open interval. Clearly then the restriction of \mathbb{A}' to any countable dense set of reals still satisfies $\sigma'_{\mathbb{A}}$, hence (its L -reduct) is isomorphic to \mathbb{A} . This shows:

Corollary 8.16 (of proof of [AFP, 1.1]). *Let L be a language and \mathbb{A} be a countable L -structure with TDC. Then there is a Borel L -structure \mathbb{A}' with universe \mathbb{R} such that for any countable dense set $A \subseteq \mathbb{R}$, $\mathbb{A}'|_A \cong \mathbb{A}$.*

Proof of Theorem 8.2. (Marks) If E is smooth, then clearly it is \mathbb{A} -structurable. So we may assume $X = 2^{\mathbb{N}}$. Let $N_s = \{x \in 2^{\mathbb{N}} \mid s \subseteq x\}$ for $s \in 2^{<\mathbb{N}}$ denote the basic clopen sets in $2^{\mathbb{N}}$. Note that the set

$$X_1 := \{x \in X \mid \exists s \in 2^{<\mathbb{N}} (|[x]_E \cap N_s| = 1)\}$$

of points whose class contains an isolated point is Borel, and $E|X_1$ is smooth (with a selector given by $x \mapsto$ the unique element of $[x]_E \cap N_s$ for the least s such that $|[x]_E \cap N_s| = 1$), hence \mathbb{A} -structurable. For $x \in X \setminus X_1$, the closure $\overline{[x]_E}$ has no isolated points, hence is homeomorphic to $2^{\mathbb{N}}$. For each such x , define $f_x(t)$ inductively for $t \in 2^{<\mathbb{N}}$ by

$$\begin{aligned} f_x(\emptyset) &:= \emptyset, \\ f_x(t \hat{\ } i) &:= s \hat{\ } i \text{ for the unique } s \supseteq f_x(t) \text{ such that } [x]_E \cap N_{f_x(t)} \subseteq N_s \text{ but } [x]_E \cap N_{f_x(t)} \not\subseteq N_{s \hat{\ } 0}, N_{s \hat{\ } 1} \end{aligned}$$

(for $i = 0, 1$), so that $f_x : 2^{\mathbb{N}} \rightarrow \overline{[x]_E}$, $f_x(y) := \bigcup_{t \subseteq y} f_x(t)$ is a homeomorphism, such that $x \in E x' \implies f_x = f_{x'}$. It is easy to see that $(x, y) \mapsto f_x(y)$ is Borel.

Now let the structure \mathbb{A}' on \mathbb{R} be given by Corollary 8.16. Let $Z = \{z_0, z_1, \dots\} \subseteq 2^{\mathbb{N}}$ be a countable set so that there is a continuous bijection $g : 2^{\mathbb{N}} \setminus Z \rightarrow \mathbb{R}$. Let

$$X_2 := \{x \in X \setminus X_1 \mid \exists x' \in [x]_E (f_x^{-1}(x') \in Z)\}.$$

Then $E|X_2$ is smooth (with selector $x \mapsto x' \in [x]_E$ such that $f_x^{-1}(x') = z_j$ with j minimal), hence \mathbb{A} -structurable. Finally, $E|(X \setminus (X_1 \cup X_2))$ is \mathbb{A} -structurable: for each $x \in X \setminus (X_1 \cup X_2)$, we have that $f_x^{-1}([x]_E) \subseteq 2^{\mathbb{N}} \setminus Z$ is dense, so $g \circ f_x^{-1}$ gives a bijection between $[x]_E$ and a dense subset of \mathbb{R} , along which we may pull back \mathbb{A}' to get a structure on $[x]_E$ isomorphic to \mathbb{A} . \square

Remark 8.17. Recently Marks [M2] found, using Baire category methods, another proof of Theorem 8.15 and of a generalization of Corollary 8.16, where \mathbb{R} can be replaced with any infinite Polish space. This allows a simplification in the above proof of Theorem 8.2 (by omitting most of the second paragraph).

Theorem 8.2 has the following immediate corollary.

Corollary 8.18. *The following Fraïssé structures can structure every aperiodic countable Borel equivalence relation: $(\mathbb{Q}, <)$, the random graph, the random K_n -free graph (where K_n is the complete graph on n vertices), the random poset, and the rational Urysohn space.*

The concept of amenability of a structure in the next result can be either the one in [JKL, 2.6(iii)] or the one in [K91, 3.4]. This result was first proved by the authors by a different method but it can also be seen as a corollary of Theorem 8.2.

Corollary 8.19. *Let \mathbb{A} be a countably infinite amenable structure. Then \mathbb{A} fails TDC.*

Proof. Since \mathbb{A} is amenable, every \mathbb{A} -structurable equivalence relation is amenable (see [JKL, 2.18] or [K91, 2.6]), thus \mathbb{A} does not structure every aperiodic countable Borel equivalence relation, and so \mathbb{A} fails TDC by Theorem 8.2. \square

We do not know of a counterexample to the converse of Theorem 8.2, i.e., of a single structure \mathbb{A} without TDC such that every aperiodic $E \in \mathcal{E}$ is \mathbb{A} -structurable. Such structures do exist which structure every compressible E , as the following simple example shows:

Proposition 8.20. *For any countable linear order $(Y, <)$, every compressible $(X, E) \in \mathcal{E}_c$ is structurable via linear orders isomorphic to $\mathbb{Q} \times (Y, <)$ with the lexicographical order.*

In particular, $\mathbb{Q} \times \mathbb{Z}$ structures every compressible equivalence relation.

Proof. By Theorem 8.2, E is structurable via linear orders isomorphic to \mathbb{Q} . Take the lexicographical order on $E \times I_Y$. \square

Concerning classes of structures (or theories) which can structure every (compressible) equivalence relation, we can provide the following examples:

Proposition 8.21. *Every $(X, E) \in \mathcal{E}$ is structurable via connected bipartite graphs.*

Proof. The finite part of E can be treed, so assume E is aperiodic. Then we may partition $X = Y \cup Z$ where Y, Z are complete sections (this is standard; see e.g., [BK, 4.5.4]). Let $f : E|Y \rightarrow Z$ and $g : E|Z \rightarrow Y$ be Borel functions such that $f(y, y') E y$ and $g(z, z') E z$ for all $(y, y') \in E|Y$ and $(z, z') \in E|Z$. Then the graph $G \subseteq E$ which connects each y, y' to $f(y, y')$ and z, z' to $g(z, z')$ (and with no other edges) works. \square

Proposition 8.22. *For every $k \geq 1$, every compressible $(X, E) \in \mathcal{E}_c$ is structurable via connected graphs in which all cycles have lengths divisible by k .*

Proof. Let $<$ be a Borel linear order on X , and let $G \subseteq E$ be any Borel graphing, e.g., $G = E \setminus \Delta_X$. Let

$$X' := X \sqcup (G \times \{1, \dots, k-1\}),$$

let E' be the equivalence relation on X' generated by E and $x E' (y, z, i)$ for $x E y E z$, $(y, z) \in G$, and $1 \leq i < k$, and let $G' \subseteq E'$ be the graph generated by, for each $(x, y) \in G$ with $x < y$,

$$x G' (x, y, 1) G' (x, y, 2) G' \cdots G' (x, y, k-1) G' y$$

(and no other edges). That is, every edge in G has been replaced by a k -length path with the same endpoints. It is clear that G' graphs E' and every cycle in G' has length divisible by k . Now since E is compressible, and the inclusion $X \subseteq X'$ is a complete section embedding $E \sqsubseteq_B E'$, we have $E \cong_B E'$, thus E is structurable via a graph isomorphic to G' . \square

Thus in contrast to the fact that not every (compressible) countable Borel equivalence relation is treeable, we have

Corollary 8.23. *Every compressible equivalence relation has graphings of arbitrarily large girth.*

We do not know if this holds for arbitrary countable Borel equivalence relations.

8.3 Classes axiomatizable by a Scott sentence

Let us say that an elementary class $\mathcal{C} \subseteq \mathcal{E}$ is **Scott axiomatizable** if it is axiomatizable by a Scott sentence $\sigma_{\mathbb{A}}$ of some structure \mathbb{A} , or equivalently by some sentence σ which is countably categorical (i.e., it has exactly one countable model up to isomorphism). Several elementary classes we have considered are naturally Scott axiomatizable: e.g., aperiodic, aperiodic smooth, aperiodic hyperfinite (by $\sigma_{\mathbb{Z}}$), free actions of a group Γ (by σ_{Γ}), and compressible (by the sentence in the language $\{R\}$ asserting that R is the graph of an injective function with infinite and coinfinite image and with no fixed points).

It is an open problem to characterize the elementary classes which are Scott axiomatizable. In fact, we do not even know if every elementary class of aperiodic equivalence relations is Scott axiomatizable. Here we describe a general construction which can be used to show that certain compressible elementary classes are Scott axiomatizable.

Let $(L, \sigma), (M, \tau)$ be theories. Let $\sigma \times \tau$ be a sentence in the language $L \sqcup M \sqcup \{R_1, R_2\}$ asserting

- (i) R_1, R_2 are equivalence relations such that the quotient maps $X \rightarrow X/R_1$ and $X \rightarrow X/R_2$ (where X is the universe) exhibit a bijection between X and $X/R_1 \times X/R_2$; and
- (ii) the L -reduct (respectively M -reduct) is an R_1 -invariant (resp., R_2 -invariant) structure which induces a model of σ (resp., τ) on the quotient X/R_1 (resp., X/R_2).

Thus, a $(\sigma \times \tau)$ -structure \mathbb{A} on a set X is essentially the same thing as a σ -structure \mathbb{B} on a set Y and a τ -structure \mathbb{C} on a set Z , together with a bijection $X \cong Y \times Z$. The following are clear:

Proposition 8.24. $E_{\infty\sigma} \times E_{\infty\tau} \models \sigma \times \tau$ (equivalently, $E_{\infty\sigma} \times E_{\infty\tau} \sqsubseteq_B^i E_{\infty(\sigma \times \tau)}$).

Remark 8.25. It is not true in general that $E_{\infty\sigma} \times E_{\infty\tau} \cong_B E_{\infty(\sigma \times \tau)}$. For example, if $\sigma = \sigma_\Gamma$ and $\tau = \sigma_\Delta$ axiomatize free actions of countable groups Γ, Δ , then it is easy to see that $\sigma_\Gamma \times \sigma_\Delta$ axiomatizes free actions of $\Gamma \times \Delta$; taking $\Gamma = \Delta = \mathbb{F}_2$, we have that $E_{\infty(\sigma \times \tau)}$ is the universal orbit equivalence of a free action of $\mathbb{F}_2 \times \mathbb{F}_2$, which does not reduce to a product of two treeables (such as $E_{\infty\sigma} \times E_{\infty\tau}$) by [HK, 8.1(iii)].

Proposition 8.26. *If σ, τ are countably categorical, then so is $\sigma \times \tau$.*

Now consider the case where τ in the language $\{P_0, P_1, \dots\}$ asserts that the P_i are disjoint singleton subsets which enumerate the universe. Then clearly τ axiomatizes the aperiodic smooth countable Borel equivalence relations, i.e., $E_{\infty\tau} = \Delta_{\mathbb{R}} \times I_{\mathbb{N}}$, whence $E_{\infty\sigma} \times E_{\infty\tau} \cong_B E_{\infty\sigma} \times I_{\mathbb{N}}$.

Proposition 8.27. *For this choice of τ , $E_{\infty(\sigma \times \tau)} \cong_B E_{\infty\sigma} \times E_{\infty\tau} \cong_B E_{\infty\sigma} \times I_{\mathbb{N}}$.*

Proof. Let $E_{\infty(\sigma \times \tau)}$ live on X and let $\mathbb{E} : E_{\infty(\sigma \times \tau)} \models \sigma \times \tau$. Then from the definition of $\sigma \times \tau$, we have that (the reduct to the language of σ of) $\mathbb{E}|_{P_0^{\mathbb{E}}} : E_{\infty(\sigma \times \tau)}|_{P_0^{\mathbb{E}}} \models \sigma$ (where P_i is from the language of τ as above). Let $f : E_{\infty(\sigma \times \tau)}|_{P_0^{\mathbb{E}}} \sqsubseteq_B^i E_{\infty\sigma}$. Then it is easy to see that $g : E_{\infty(\sigma \times \tau)} \sqsubseteq_B^i E_{\infty\sigma} \times I_{\mathbb{N}}$, where $g(x) := (f(x), i)$ for the unique i such that $x \in P_i^{\mathbb{E}}$. \square

Since τ is clearly countably categorical, this yields

Corollary 8.28. *If an elementary class \mathcal{E}_E is Scott axiomatizable, then so is $\mathcal{E}_{E \times I_{\mathbb{N}}}$.*

In particular, if an elementary class \mathcal{C} is Scott axiomatizable and closed under $E \mapsto E \times I_{\mathbb{N}}$, then $\mathcal{C} \cap \mathcal{E}_c$ (i.e., the compressible elements of \mathcal{C}) is Scott axiomatizable.

Proof. If $\mathcal{E}_E = \mathcal{E}_\sigma$ where σ is countably categorical, then $E_\infty \otimes (E \times I_{\mathbb{N}}) = (E_\infty \otimes E) \times I_{\mathbb{N}} = E_{\infty\sigma} \times I_{\mathbb{N}} = E_{\infty(\sigma \times \tau)}$ (using Proposition 5.29), whence $\mathcal{E}_{E \times I_{\mathbb{N}}} = \mathcal{E}_{\sigma \times \tau}$.

For the second statement, if $\mathcal{C} = \mathcal{E}_E$ where E is universally structurable, then $\mathcal{C} \cap \mathcal{E}_c = \mathcal{E}_{E \times I_{\mathbb{N}}}$ (Proposition 5.27). \square

Corollary 8.29. *The following elementary classes are Scott axiomatizable: compressible hyperfinite, compressible treeable.*

Proof. For the compressible treeables, use that $E_{\infty\mathbb{F}_2}$ (i.e., the \sqsubseteq_B^i -universal orbit equivalence of a free action of \mathbb{F}_2) is \sqsubseteq_B -universal treeable [JKL, 3.17]; it follows that $\mathcal{E}_{\mathbb{F}_2}$ is closed under $E \mapsto E \times I_{\mathbb{N}}$, and also that $\mathcal{E}_{\mathbb{F}_2} \cap \mathcal{E}_c$ is the class of compressible treeables. \square

However, we do not know if the elementary class of aperiodic treeable equivalence relations is Scott axiomatizable.

9 Some open problems

9.1 General questions

At the end of Section 4.4 we asked:

Problem 9.1. Is $E \otimes E$ universally structurable (or equivalently, isomorphic to $E_\infty \otimes E$) for every aperiodic E ?

The following question (Remark 5.28) concerns the structure of universally structurable \sim_B -classes:

Problem 9.2. Is $E \times I_{\mathbb{N}} \sqsubseteq_B E$ for every aperiodic universally structurable E ? Equivalently, is the compressible element of every universally structurable \sim_B -class the \sqsubseteq_B^i -least of the aperiodic elements?

By Theorem 6.21, we know that there are many incomparable elementary reducibility classes, or equivalently, many \leq_B -incomparable universally structurable E . However, these were produced using the results in [AK], which use rigidity theory for measure-preserving group actions. One hope for the theory of structurability is the possibility of producing \leq_B -incomparable equivalence relations using other methods, e.g., using model theory.

Problem 9.3. Show that there are \leq_B -incomparable $E_{\infty\sigma}, E_{\infty\tau}$ without using ergodic theory.

9.2 Order-theoretic questions

We turn now to the order-theoretic structure of the lattice $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$ (equivalently, the poset of elementary classes) and the lattice $(\mathcal{E}_{\infty}/\sim_B, \leq_B)$ (equivalently, the poset of elementary reducibility classes). The following questions, posed in Section 6.2 (near end), are natural from an abstract order-theoretic perspective, though perhaps not so approachable:

Problem 9.4. Is either $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$ or $(\mathcal{E}_{\infty}/\sim_B, \leq_B)$ a complete lattice? If so, is it completely distributive?

Problem 9.5. Is either $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$ or $(\mathcal{E}_{\infty}/\sim_B, \leq_B)$ a zero-dimensional ω_1 -complete lattice, in that it embeds into 2^X for some set X ?

We noted above (Corollary 6.16) that the recent work of Marks [M] gives some examples of ω_1 -prime filters on $(\mathcal{E}_{\infty}/\sim_B, \leq_B)$, and also (Proposition 6.19) that these filters cannot separate elements of $(\mathcal{E}_{\infty}/\sim_B, \leq_B)$ below the universal treeable equivalence relation $E_{\infty T}$.

Regarding Problem 9.4, a natural attempt at a negative answer would be to show that some “sufficiently complicated” collection of universally structurable equivalence relations does not have a join. For example, one could try to find the join of a strictly increasing ω_1 -sequence.

Problem 9.6. Is there an “explicit” strictly increasing ω_1 -sequence in $(\mathcal{E}_{\infty}/\cong_B, \sqsubseteq_B^i)$? Similarly for $(\mathcal{E}_{\infty}/\sim_B, \leq_B)$.

Note that by Theorem 6.21, such a sequence does exist, abstractly; the problem is thus to find a sequence which is in some sense “definable”, preferably corresponding to some “natural” hierarchy of countable structures. For example, a long-standing open problem (implicit in e.g., [JKL, Section 2.4]) asks whether the sequence of elementary classes $(\mathcal{E}_{\alpha})_{\alpha < \omega_1}$, where

$$\begin{aligned} \mathcal{E}_0 &:= \{\text{hyperfinite}\}, \\ \mathcal{E}_{\alpha} &:= \{\text{countable increasing union of } E \in \mathcal{E}_{\beta} \text{ for } \beta < \alpha\}, \end{aligned}$$

stabilizes (or indeed is constant); a negative answer would constitute a positive solution to Problem 9.6.

One possible approach to defining an ω_1 -sequence would be by iterating a “jump” operation, $E \mapsto E'$, that sends any non-universal $E \in \mathcal{E}_{\infty}$ to some non-universal $E' \in \mathcal{E}_{\infty}$ such that $E <_B E'$.

Problem 9.7. Is there an “explicit” jump operation on the non-universal elements of $(\mathcal{E}_\infty/\sim_B, \leq_B)$?

On the other hand, this would not be possible if there were a greatest non-universal element:

Problem 9.8. Is there a greatest element among the non-universal elements of $(\mathcal{E}_\infty/\cong_B, \sqsubseteq_B^i)$, or of $(\mathcal{E}_\infty/\sim_B, \leq_B)$? If so, do the non-universal equivalence relations form an elementary class, i.e., are they downward-closed under \rightarrow_B^{cb} ?

9.3 Model-theoretic questions

The general model-theoretic question concerning structurability is which properties of a theory (L, σ) (or a Borel class of structures \mathcal{K}) yield properties of the corresponding elementary class \mathcal{E}_σ (or $\mathcal{E}_\mathcal{K}$). Theorem 8.1 fits into this mold, by characterizing the σ which yield smoothness. One could seek similar results for other properties of countable Borel equivalence relations.

Problem 9.9. Find a model-theoretic characterization of the σ such that \mathcal{E}_σ consists of only hyperfinite equivalence relations, i.e., such that $\sigma \Rightarrow \sigma_{hf}$, for any sentence σ_{hf} axiomatizing hyperfiniteness.

Less ambitiously, one might look for “natural” examples of such σ , for specific classes of structures. For example, for the Borel class of locally finite graphs, we have:

- If E is structurable via locally finite trees with one end, then E is hyperfinite [DJK, 8.2].
- If E is structurable via locally finite graphs with two ends, then E is hyperfinite [Mi, 5.1].

Remark 9.10. If E is structurable via locally finite graphs with at least 3 but finitely many ends, then E is smooth; this follows from [Mi, 6.2], or simply by observing that in any such graph, a finite nonempty subset may be defined as the set of all vertices around which the removal of a ball of minimal radius leaves ≥ 3 infinite connected components.

Problem 9.11. Find “natural” examples of σ such that \mathcal{E}_σ consists of only Fréchet-amenable equivalence relations (see [JKL, 2.12]).

For example, every E structurable via countable scattered linear orders is Fréchet-amenable [JKL, 2.19] (recall that a countable linear order is **scattered** if it does not embed the rationals); note however that the scattered linear orders do not form a Borel class of structures.

Problem 9.12. Find “natural” examples of σ such that \mathcal{E}_σ consists of only compressible equivalence relations.

For example, by [Mi2], the class of E structurable via locally finite graphs whose space of ends is not perfect but has cardinality at least 3 is exactly \mathcal{E}_c .

There is also the converse problem of determining for which σ is every equivalence relation of a certain form (e.g., compressible) σ -structurable. Theorem 8.2 fits into this mold, by giving a sufficient condition for a single structure to structure every aperiodic equivalence relation.

Problem 9.13. Is there a structure \mathbb{A} without TDC which structures every aperiodic countable Borel equivalence relation? That is, does the converse of Theorem 8.2 hold?

In particular, does $\mathbb{Q} \times \mathbb{Z}$ structure every aperiodic equivalence relation? We noted above that it structures every compressible equivalence relation, thus the analogous question for the compressibles has a negative answer.

Problem 9.14. Find a model-theoretic characterization of the structures \mathbb{A} such that every compressible equivalence relation is \mathbb{A} -structurable, i.e., $\mathcal{E}_c \subseteq \mathcal{E}_{\mathbb{A}}$.

We also have the corresponding questions for theories (or classes of structures):

Problem 9.15. Find a model-theoretic characterization of the σ such that $\mathcal{E}_\sigma = \mathcal{E}$, or more generally, $\mathcal{E}_c \subseteq \mathcal{E}_\sigma$.

We gave several examples in Section 8.2. Another example is the following [Mi, 4.1]: every $E \in \mathcal{E}$ is structurable via locally finite graphs with at most one end.

For a different sort of property that \mathcal{E}_σ may or may not have, recall (Section 5.3) that \mathcal{E}_σ is an elementary reducibility class, i.e., closed under \leq_B , when σ axiomatizes linear orders embeddable in \mathbb{Z} , or when σ axiomatizes trees.

Problem 9.16. Find a model-theoretic characterization of the σ such that \mathcal{E}_σ is closed under \leq_B .

We considered in Section 8.3 the question of which elementary classes are Scott axiomatizable, i.e., axiomatizable by a Scott sentence.

Problem 9.17. Find other “natural” examples of Scott axiomatizable elementary classes.

We showed above (Corollary 8.29) that the class of compressible treeable equivalence relations is Scott axiomatizable.

Problem 9.18. Is the class of aperiodic treeable (countable Borel) equivalence relations Scott axiomatizable?

Remark 9.19. The class of aperiodic treeables cannot be axiomatized by the Scott sentence of a single countable tree T . Indeed, since E_0 would have to be treeable by T , by a result of Adams (see [KM, 22.3]), T can have at most 2 ends; but then by [DJK, 8.2] and [Mi, 5.1], every E treeable by T is hyperfinite.

Problem 9.20. Find a model-theoretic characterization of the σ such that \mathcal{E}_σ is axiomatizable by a Scott sentence (possibly in some other language). In particular, is every elementary class of aperiodic, or compressible, equivalence relations axiomatizable by a Scott sentence?

We conclude by stating two very general (and ambitious) questions concerning the relationship between structurability and model theory. For the first, note that by (i) \iff (ii) in Theorem 8.1, the condition $\sigma \Rightarrow^* \sigma_{sm}$ is equivalent to the existence of a formula(s) in the language of σ with some definable properties which are logically implied by σ . Our question is whether a similar equivalence continues to hold when σ_{sm} is replaced by an arbitrary sentence τ .

Problem 9.21. Is there, for any τ , a sentence $\tau'(R_1, R_2, \dots)$ in a language consisting of relation symbols R_1, R_2, \dots (thought of as “predicate variables”), such that for any σ , we have $\sigma \Rightarrow^* \tau$ iff there are formulas ϕ_1, ϕ_2, \dots in the language of σ such that σ logically implies $\tau'(\phi_1, \phi_2, \dots)$ (the result of “substituting” ϕ_i for R_i in τ')?

Finally, there is the question of completely characterizing containment between elementary classes:

Problem 9.22. Find a model-theoretic characterization of the pairs (σ, τ) such that $\sigma \Rightarrow^* \tau$.

A Appendix: Fiber spaces

In this appendix, we discuss fiber spaces on countable Borel equivalence relations, which provide a more general context for structurability and related notions. The application of fiber spaces to structurability was previously considered in [G] and [HK, Appendix D].

In both this appendix and the next, we will use categorical terminology somewhat more liberally than in the body of this paper.

A.1 Fiber spaces

Let $(X, E) \in \mathcal{E}$ be a countable Borel equivalence relation. A **fiber space over E** consists of a countable Borel equivalence relation (U, P) , together with a surjective countable-to-1 class-bijective homomorphism $p : P \rightarrow_B^{cb} E$. We refer to the fiber space by (U, P, p) , by (P, p, E) , by (P, p) , or (ambiguously) by P . We call (U, P) the **total space**, (X, E) the **base space**, and p the **projection**. For $x \in X$, the **fiber over x** is the set $p^{-1}(x) \subseteq U$. For $x, x' \in X$ such that $x E x'$, we let

$$p^{-1}(x, x') : p^{-1}(x) \rightarrow p^{-1}(x')$$

denote the **fiber transport map**, where for $u \in p^{-1}(x)$, $p^{-1}(x, x')(y)$ is the unique $u' \in p^{-1}(x')$ such that $u P u'$.

For two fiber spaces $(U, P, p), (V, Q, q)$ over (X, E) , a **fiberwise map between them over E** , denoted $\tilde{f} : (P, p) \rightarrow_E (Q, q)$ (we use letters like f, \tilde{g} for maps between total spaces), is a homomorphism $\tilde{f} : P \rightarrow_B Q$ such that $p = q \circ \tilde{f}$ (note that this implies that \tilde{f} is class-bijective):

$$\begin{array}{ccc} (U, P) & \xrightarrow{\tilde{f}} & (V, Q) \\ & \searrow p & \swarrow q \\ & & (X, E) \end{array}$$

For a fiber space (U, P, p) over (X, E) and a fiber space (V, Q, q) over (Y, F) , a **fiber space homomorphism** from (P, p, E) to (Q, q, F) , denoted $f : (P, p, E) \rightarrow (Q, q, F)$, consists of two homomorphisms $f : E \rightarrow_B F$ and $\tilde{f} : P \rightarrow_B Q$ such that $f \circ p = q \circ \tilde{f}$:

$$\begin{array}{ccc} (U, P) & \xrightarrow{\tilde{f}} & (V, Q) \\ p \downarrow & & \downarrow q \\ (X, E) & \xrightarrow{f} & (Y, F) \end{array}$$

We sometimes refer to \tilde{f} as the fiber space homomorphism; note that f is determined by \tilde{f} (since p is surjective). We say that \tilde{f} is a fiber space homomorphism **over f** . Note that a fiberwise map over E is the same thing as a fiber space homomorphism over the identity function on E .

A fiber space homomorphism $f : (P, p, E) \rightarrow (Q, q, F)$ is **fiber-bijective** if $\tilde{f}|_{p^{-1}(x)} : p^{-1}(x) \rightarrow q^{-1}(f(x))$ is a bijection for each $x \in X$ (where E lives on X); **fiber-injective**, **fiber-surjective** are defined similarly.

Let (U, P, p) be a fiber space over (X, E) , and let $(Y, F) \in \mathcal{E}$ be a countable Borel equivalence relation with a homomorphism $f : F \rightarrow_B E$. Recall (Section 2.6) that we have the fiber product

equivalence relation $(Y \times_X U, F \times_E P)$ with respect to f and p , which comes equipped with the canonical projections $\pi_1 : F \times_E P \rightarrow_B F$ and $\pi_2 : F \times_E P \rightarrow_B P$ obeying $f \circ \pi_1 = p \circ \pi_2$. It is easy to check that π_1 is class-bijective, surjective, and countable-to-1 (because p is). In this situation, we also use the notation

$$(f^{-1}(U), f^{-1}(P), f^{-1}(p)) = f^{-1}(U, P, p) := (Y \times_X U, F \times_E P, \pi_1).$$

Note that $\tilde{f} := \pi_2 : f^{-1}(P) \rightarrow_B P$ is then a fiber space homomorphism over f . We refer to $f^{-1}(U, P, p)$ as the **pullback of (U, P, p) along f** . Here is a diagram:

$$\begin{array}{ccc} f^{-1}(U, P) & \xrightarrow{\tilde{f}} & (U, P) \\ f^{-1}(p) \downarrow & & \downarrow p \\ (Y, F) & \xrightarrow{f} & (X, E) \end{array}$$

Let $\mathbf{Fib}(E)$ denote the category of fiber spaces and fiberwise maps over E , and let $\int_{\mathcal{E}} \mathbf{Fib}$ denote the category of fiber spaces and fiber space homomorphisms. For a homomorphism $f : E \rightarrow_B F$, pullback along f gives a functor $f^{-1} : \mathbf{Fib}(F) \rightarrow \mathbf{Fib}(E)$ (with the obvious action on fiberwise maps). The assignment $f \mapsto f^{-1}$ is itself functorial, and turns \mathbf{Fib} into a contravariant functor from the category $(\mathcal{E}, \rightarrow_B)$ to the category of (essentially small) categories. (Technically $f \mapsto f^{-1}$ is only *pseudofunctorial*, i.e., $f^{-1}(g^{-1}(P))$ is naturally isomorphic, not equal, to $(g \circ f)^{-1}(P)$; we will not bother to make this distinction.)

A.2 Fiber spaces and cocycles

Let (U, P, p) be a fiber space over $(X, E) \in \mathcal{E}$. By Lusin-Novikov uniformization, we may Borel partition X according to the cardinalities of the fibers. Suppose for simplicity that each fiber is countably infinite. Again by Lusin-Novikov uniformization, there is a Borel map $T : X \rightarrow U^{\mathbb{N}}$ such that each $T(x)$ is a bijection $\mathbb{N} \rightarrow p^{-1}(x)$. Let $\alpha_T : E \rightarrow S_{\infty}$ be the cocycle given by

$$\alpha_T(x, x') := T(x')^{-1} \circ p^{-1}(x, x') \circ T(x)$$

(where $p^{-1}(x, x')$ is the fiber transport map; compare Remark 4.3). We then have a (fiberwise) isomorphism of fiber spaces over E , between (U, P, p) and the skew product $E \times_{\alpha_T} \mathbb{N}$ (with its canonical projection $q : E \times_{\alpha_T} \mathbb{N} \rightarrow_B^c E$):

$$\begin{aligned} (U, P, p) &\longleftrightarrow (X \times \mathbb{N}, E \times_{\alpha_T} \mathbb{N}, q) \\ u &\longmapsto (p(u), T(p(u))^{-1}(u)) \\ T(x)(n) &\longleftarrow (x, n). \end{aligned}$$

Recall that two cocycles $\alpha, \beta : E \rightarrow S_{\infty}$ are **cohomologous** if there is a Borel map $\phi : X \rightarrow S_{\infty}$ such that $\phi(x')\alpha(x, x') = \beta(x, x')\phi(x)$, for all $(x, x') \in E$. It is easy to see that in the above, changing the map $T : X \rightarrow U^{\mathbb{N}}$ results in a cohomologous cocycle $\alpha_T : E \rightarrow S_{\infty}$; so we get a well-defined map from (isomorphism classes of) fiber spaces over E with countably infinite fibers to S_{∞} -valued cohomology classes on E . Conversely, given any cocycle $\alpha : E \rightarrow S_{\infty}$, the skew product $E \times_{\alpha} \mathbb{N}$ yields a fiber space over E with countably infinite fibers. These two operations are inverse to each other, so we have a bijection

$$\{\text{iso. classes of fiber spaces over } E \text{ with } \aleph_0\text{-sized fibers}\} \cong \{S_{\infty}\text{-valued cohomology classes on } E\}.$$

Remark A.1. In fact, we have the following more refined correspondence, which also smoothly handles the case with finite fibers. Let \mathbf{C} denote the category whose objects are $1, 2, \dots, \mathbb{N}$ and morphisms are maps between them (where as usual, $n = \{0, \dots, n-1\}$ for $n \in \mathbb{N}$). Then \mathbf{C} is a “standard Borel category”. Regarding E as the groupoid on X with a single morphism between any two related points, we have a **Borel functor category** \mathbf{C}_B^E , whose objects are Borel functors $E \rightarrow \mathbf{C}$ and morphisms are Borel natural transformations. We then have a functor

$$\mathbf{C}_B^E \longrightarrow \mathbf{Fib}(E)$$

which takes a Borel functor $\alpha : E \rightarrow \mathbf{C}$ to the obvious generalization of the skew product of E with respect to α (but where the fibers are no longer uniformly \mathbb{N} , but vary from point to point according to α); and this functor is an equivalence of categories. We leave the details to the reader.

Using this correspondence between fiber spaces and cocycles, we obtain

Proposition A.2. *There is a fiber space $(U_\infty, P_\infty, p_\infty)$ over E_∞ , which is universal with respect to fiber-bijective invariant embeddings: for any other fiber space (U, P, p) over E , there is a fiber-bijective homomorphism $\tilde{f} : P \rightarrow P_\infty$ over an invariant embedding $f : E \sqsubseteq_B^i E_\infty$.*

Proof. For simplicity, we restrict again to the case where P has countably infinite fibers. Let σ be a sentence over the language $L = \{R_{ij}\}_{i,j \in \mathbb{N}}$, where each R_{ij} is binary, asserting that

$$\alpha(x, y)(i) = j \iff R_{ij}(x, y)$$

defines a cocycle $\alpha : I_X \rightarrow S_\infty$, where X is the universe of the structure (and I_X is the indiscrete equivalence relation on X). Then the canonical σ -structure on $E_{\infty\sigma}$ corresponds to a cocycle $\alpha_\infty : E_{\infty\sigma} \rightarrow S_\infty$. We will in fact define the universal fiber space P_∞ over $E_{\infty\sigma}$, since clearly $E_\infty \sqsubseteq_B^i E_{\infty\sigma}$ (by giving E_∞ the trivial cocycle). Let $P_\infty := E_{\infty\sigma} \times_{\alpha_\infty} \mathbb{N}$, with $p_\infty : P_\infty \rightarrow_B^{cb} E_{\infty\sigma}$ the canonical projection. For another fiber space (U, P, p) over E with countably infinite fibers, by the above remarks, P is isomorphic (over E) to a skew product $E \times_\alpha \mathbb{N}$, for some cocycle $\alpha : E \rightarrow S_\infty$. This α corresponds to a σ -structure on E , which yields an invariant embedding $f : E \sqsubseteq_B^i E_{\infty\sigma}$ such that α is the restriction of α_∞ along f , giving the desired fiber-bijective homomorphism $\tilde{f} := f \times \mathbb{N} : E \times_\alpha \mathbb{N} \rightarrow E_{\infty\sigma} \times_{\alpha_\infty} \mathbb{N}$ over f . \square

There is a different kind of universality one could ask for, which we do not know how to obtain. Namely, for each $E \in \mathcal{E}$, is there a fiber space $(U_\infty, P_\infty, p_\infty)$ over E which is universal with respect to fiberwise injective maps?

A.3 Equivalence relations as fiber spaces

Let $(X, E) \in \mathcal{E}$ be a countable Borel equivalence relation. The **tautological fiber space** over E is (E, \widehat{E}, π_1) , where \widehat{E} is the equivalence relation on the set $E \subseteq X^2$ given by

$$(x, x') \widehat{E} (y, y') \iff x' = y',$$

and $\pi_1 : (E, \widehat{E}) \rightarrow_B^{cb} (X, E)$ is the first coordinate projection (i.e., $\pi(x, x') = x$). In other words, the \widehat{E} -fiber over each E -class $C \in X/E$ consists of the elements of C .

Note that \widehat{E} is the kernel of the second coordinate projection $\pi_2 : E \rightarrow X$; thus \widehat{E} is smooth, and in fact E/\widehat{E} is isomorphic to X (via π_2). Now let (U, P, p) be any smooth fiber space over (X, E) , and let F be the (countable Borel) equivalence relation on $Y := U/P$ given by

$$[u]_P F [u']_P \iff p(u) E p(u').$$

By Lusin-Novikov uniformization, there is a Borel map $X \rightarrow U$ which is a section of p , which when composed with the projection $U \rightarrow Y$ gives a reduction $f : (X, E) \leq_B (Y, F)$ whose image is a complete section.

Let us say that a **presentation** of the quotient space X/E consists of a countable Borel equivalence relation $(Y, F) \in \mathcal{E}$ together with a bijection $X/E \cong Y/F$ which admits a Borel lifting $X \rightarrow_B Y$ (which is then a reduction $E \leq_B F$ with image a complete section). By the above, every smooth fiber space over E gives rise to a presentation of X/E . Conversely, given any presentation (Y, F) of X/E , letting $f : E \leq_B F$ with image a complete section, the pullback $f^{-1}(\widehat{F})$ is a fiber space over E , which is smooth (because $f^{-1}(\widehat{F})$ reduces to \widehat{F} , via the map f coming from the pullback). It is easily seen that the two operations we have just described are mutually inverse up to isomorphism, yielding a bijection

$$\{\text{iso. classes of smooth fiber spaces over } E\} \cong \{\text{iso. classes of presentations of } X/E\}.$$

Remark A.3. This correspondence between smooth fiber spaces and presentations of the same quotient space is essentially the proof of [HK, D.1].

We now describe the correspondence between homomorphisms of equivalence relations and fiber space homomorphisms. Let $(X, E), (Y, F) \in \mathcal{E}$. A homomorphism $f : E \rightarrow_B F$ induces a fiber space homomorphism $\widehat{f} : \widehat{E} \rightarrow \widehat{F}$ over f , given by

$$\widehat{f}(x, x') := (f(x), f(x')).$$

Conversely, let $\widetilde{g} : \widehat{E} \rightarrow \widehat{F}$ be any fiber space homomorphism over $g : E \rightarrow_B F$. Then \widetilde{g} must be given by $\widetilde{g}(x, x') = (g(x), f(x'))$ for some $f : X \rightarrow Y$ such that $g(x) F f(x)$ for each $x \in X$; in particular, f is a homomorphism $E \rightarrow_B F$.

Let us say that two homomorphisms $f, g : E \rightarrow_B F$ are **equivalent**, denoted $f \simeq g$, if $f(x) F g(x)$ for each $x \in X$; equivalently, they induce the same map on the quotient spaces $X/E \rightarrow Y/F$. The above yield mutually inverse bijections

$$\{\text{homomorphisms } E \rightarrow_B F\} \cong \{\simeq\text{-classes of fiber space homomorphisms } \widehat{E} \rightarrow \widehat{F}\}.$$

Class-injectivity on the left translates to fiber-injectivity on the right, etc.

A.4 Countable Borel quotient spaces

We discuss here an alternative point of view on fiber spaces and equivalence relations. The idea is that the tautological fiber space \widehat{E} over an equivalence relation (X, E) allows a clean distinction to be made between the quotient space X/E and the presentation (X, E) .

A **countable Borel quotient space** is, formally, the same thing as a countable Borel equivalence relation (X, E) , except that we denote it by X/E . A **Borel map** between countable Borel quotient spaces X/E and Y/F , denoted $f : X/E \rightarrow_B Y/F$, is a map which admits a Borel lifting

$X \rightarrow Y$, or equivalently an \simeq -class of Borel homomorphisms $E \rightarrow_B F$. Let $(\mathcal{Q}, \rightarrow_B)$ denote the category of countable Borel quotient spaces and Borel maps. (Note that $X/E, Y/F$ are isomorphic in $(\mathcal{Q}, \rightarrow_B)$ iff they are *bireducible* as countable Borel equivalence relations.)

Let \mathcal{B} denote the class of standard Borel spaces. By identifying $X \in \mathcal{B}$ with $X/\Delta_X \in \mathcal{Q}$, we regard $(\mathcal{B}, \rightarrow_B)$ as a full subcategory of $(\mathcal{Q}, \rightarrow_B)$. By regarding Borel maps in \mathcal{Q} as \simeq -classes of homomorphisms, we have that $(\mathcal{Q}, \rightarrow_B)$ is the quotient category of $(\mathcal{E}, \rightarrow_B)$ (with the same objects) by the congruence \simeq .

A **(quotient) fiber space** over a quotient space $X/E \in \mathcal{Q}$ is a quotient space $U/P \in \mathcal{Q}$ together with a countable-to-1 surjection $p : U/P \rightarrow_B X/E$. This definition agrees with the previous notion of fiber space over (X, E) , in that fiber spaces over X/E are in natural bijection with fiber spaces over (X, E) , up to isomorphism. Indeed, by Proposition 5.8, we may factor any lifting $(U, P) \rightarrow_B (X, E)$ of p into a reduction with image a complete section, followed by a class-bijective homomorphism; the former map becomes an isomorphism when we pass to the quotient, so U/P is isomorphic to a fiber space with class-bijective projection.

We have obvious versions of the notions of **fiberwise map over X/E** , **fiber space homomorphism**, and **fiber-bijective homomorphism** for quotient fiber spaces. Let $\mathbf{Fib}(X/E)$ denote the category of fiber spaces over X/E ; in light of the above remarks, $\mathbf{Fib}(X/E)$ is equivalent to $\mathbf{Fib}(E)$. Let $\int_{\mathcal{Q}} \mathbf{Fib}$ denote the category of quotient fiber spaces and homomorphisms ($\int_{\mathcal{Q}} \mathbf{Fib}$ is then the quotient of $\int_{\mathcal{E}} \mathbf{Fib}$ by \simeq). We now have a full embedding

$$(\mathcal{E}, \rightarrow_B) \longrightarrow \int_{\mathcal{Q}} \mathbf{Fib},$$

that sends an equivalence relation (X, E) to its tautological fiber space (E, \widehat{E}) but regarded as the quotient fiber space $\widehat{E}/E \cong X$ over X/E , and sends a homomorphism f to the corresponding fiber space homomorphism \widehat{f} given above. Thus, we may regard equivalence relations as special cases of fiber spaces over quotient spaces.

To summarize, here is a (non-commuting) diagram of several relevant categories and functors:

$$\begin{array}{ccc}
 \int_{\mathcal{E}} \mathbf{Fib} & & \\
 \downarrow & \searrow & \\
 (\mathcal{E}, \rightarrow_B) & \longleftrightarrow & \int_{\mathcal{Q}} \mathbf{Fib} \\
 \downarrow & \searrow & \downarrow \\
 (\mathcal{B}, \rightarrow_B) & \longleftrightarrow & (\mathcal{Q}, \rightarrow_B)
 \end{array}$$

The horizontal arrows are full embeddings, the diagonal arrows are quotients by \simeq , and the vertical arrows are forgetful functors that send a fiber space to its base space.

A.5 Factorizations of fiber space homomorphisms

Let $(U, P, p), (V, Q, q)$ be fiber spaces over $(X, E), (Y, F)$ respectively, and $f : E \rightarrow_B F$. A fiber space homomorphism $\widetilde{f} : P \rightarrow Q$ over f corresponds, via the universal property of the pullback

$f^{-1}(V, Q, q)$, to a fiberwise map $\tilde{f}' : P \rightarrow_E f^{-1}(Q)$ over E :

$$\begin{array}{ccccc}
 P & & & & \\
 \searrow & \tilde{f} & \searrow & & \\
 & & & & Q \\
 \tilde{f}' \nearrow & f^{-1}(Q) & \longrightarrow & & \\
 & \downarrow f^{-1}(q) & & & \downarrow q \\
 p \searrow & E & \xrightarrow{f} & & F
 \end{array}$$

Note that \tilde{f} is fiber-bijective iff \tilde{f}' is an isomorphism. In general, since \tilde{f}' is countable-to-1, we may further factor it into the surjection onto its image ($\tilde{f}'(U), \tilde{f}'(P)$) followed by an inclusion:

$$\begin{array}{ccccc}
 P & & & & \\
 \searrow & \tilde{f} & \searrow & & \\
 & & & & Q \\
 \tilde{f}' \nearrow & \tilde{f}'(P) \hookrightarrow f^{-1}(Q) & \longrightarrow & & \\
 & \downarrow f^{-1}(q) & & & \downarrow q \\
 p \searrow & E & \xrightarrow{f} & & F
 \end{array}$$

So we have a canonical factorization of any fiber space homomorphism \tilde{f} into a fiberwise surjection over E , followed by a fiberwise injection over E , followed by a fiber-bijective homomorphism.

In the case where $P = \hat{E}$, $Q = \hat{F}$, and $\tilde{f} = \hat{f} : \hat{E} \rightarrow \hat{F}$ is the fiber space homomorphism induced by f , the fiber space $f^{-1}(V, Q, q) = f^{-1}(F, \hat{F}, \pi_1)$ is given by

$$\begin{aligned}
 f^{-1}(F) &= \{(x, (y_1, y_2)) \in F \mid f(x) = y_1\} \cong \{(x, y) \in F \mid f(x) = y\}, \\
 (x, y) f^{-1}(\hat{F})(x', y') &\iff y = y', \\
 f^{-1}(\pi_1)(x, y) &= x,
 \end{aligned}$$

while the map $\tilde{f}' : (E, \hat{E}) = (U, P) \rightarrow f^{-1}(V, Q) = f^{-1}(F, \hat{F})$ is given by

$$\tilde{f}'(x, x') = (x, f(x')).$$

Comparing with the proofs of Propositions 5.3 and 5.8 reveals that when $f : E \rightarrow_B F$ is smooth, the above factorization of \hat{f} corresponds (via the correspondence between smooth fiber spaces and presentations from Appendix A.3) to the factorization of f produced by Proposition 5.8. In particular, we obtain a characterization of smooth homomorphisms in terms of fiber spaces:

Proposition A.4. $f : E \rightarrow_B F$ is smooth iff the fiber space $f^{-1}(\hat{F})$ over E is smooth.

Remark A.5. In fact, the proof of Proposition 5.3 is essentially just the above correspondence, plus the observation that $\tilde{f}'(\hat{E}) \rightarrow_B^{\text{ci}} \hat{F}$ and smoothness of \hat{F} imply that $\tilde{f}'(\hat{E})$ is smooth (compare also [HK, D.2]).

A.6 Structures on fiber spaces

Let L be a language and (U, P, p) be a fiber space over $(X, E) \in \mathcal{E}$. A **Borel L -structure on (U, P, p)** is a Borel L -structure $\mathbb{A} = (U, R^{\mathbb{A}})_{R \in L}$ with universe U which only relates elements within the same fiber, i.e.,

$$R^{\mathbb{A}}(u_1, \dots, u_n) \implies p(u_1) = \dots = p(u_n),$$

such that structures on fibers over the same E -class are related via fiber transport, i.e.,

$$x E x' \implies p^{-1}(x, x')(\mathbb{A}|_{p^{-1}(x)}) = \mathbb{A}|_{p^{-1}(x')}.$$

For an $L_{\omega_1\omega}$ -sentence σ , we say that \mathbb{A} is a **Borel σ -structure on (U, P, p)** , denoted

$$\mathbb{A} : (U, P, p) \models \sigma,$$

if $\mathbb{A}|_{p^{-1}(x)}$ satisfies σ for each $x \in X$.

For $(X, E) \in \mathcal{E}$, σ -structures on E are in bijection with σ -structures on the tautological fiber space (\widehat{E}, π_1) over E , where $\mathbb{A} : E \models \sigma$ corresponds to $\widehat{\mathbb{A}} : (\widehat{E}, \pi_1) \models \sigma$ given by

$$R^{\widehat{\mathbb{A}}}((x, x_1), \dots, (x, x_n)) \iff R^{\mathbb{A}}(x_1, \dots, x_n).$$

In other words, for each $x \in X$, $\mathbb{A}|_{[x]_E}$ and $\widehat{\mathbb{A}}|_{\pi_1^{-1}(x)}$ are isomorphic via the canonical bijection $x' \mapsto (x, x')$ between $[x]_E$ and $\pi_1^{-1}(x)$.

For a fiber space homomorphism $f : (P, p, E) \rightarrow (Q, q, F)$ and a σ -structure $\mathbb{A} : (Q, q) \models \sigma$, the **fiberwise pullback structure** $f_{(P,p)}^{-1}(\mathbb{A}) : (P, p) \models \sigma$ is defined in the obvious way, i.e.,

$$R^{f_{(P,p)}^{-1}(\mathbb{A})}(u_1, \dots, u_n) \iff R^{\mathbb{A}}(\widetilde{f}(u_1), \dots, \widetilde{f}(u_n)) \wedge p(u_1) = \dots = p(u_n).$$

We have the following generalization of Theorem 4.1:

Proposition A.6. *Let (U, P, p) be a fiber space over $(X, E) \in \mathcal{E}$ and (L, σ) be a theory. There is a fiber space $(U, P, p) \times \sigma = (U \times_p \sigma, P \times_p \sigma, p \times \sigma)$ over an equivalence relation $E \times_p \sigma \in \mathcal{E}$, together with a fiber-bijective homomorphism $\pi : (P, p, E) \times \sigma \rightarrow (P, p, E)$ and a σ -structure $\mathbb{E} : (P, p) \times \sigma \models \sigma$, such that the triple $((U, P, p) \times \sigma, \pi, \mathbb{E})$ is universal: for any other fiber space (V, Q, q) over $(Y, F) \in \mathcal{E}$ with a fiber-bijective homomorphism $f : (Q, q, F) \rightarrow (P, p, E)$ and a structure $\mathbb{A} : (Q, q) \models \sigma$, there is a unique fiber-bijective $g : (Q, q, F) \rightarrow (P, p, E) \times \sigma$ such that $f = \pi \circ g$ and $\mathbb{A} = g_{(Q,q)}^{-1}(\mathbb{E})$.*

Proof sketch. This is a straightforward generalization of Theorem 4.1 (despite the excessive notation). The equivalence relation $E \times_p \sigma$ lives on

$$\{(x, \mathbb{B}) \mid x \in X, \mathbb{B} \in \text{Mod}_{p^{-1}(x)}(\sigma)\},$$

and is given by

$$(x, \mathbb{B}) (E \times_p \sigma) (x', \mathbb{B}') \iff x E x' \wedge p^{-1}(x, x^{-1})(\mathbb{B}) = \mathbb{B}'.$$

As usual, the Borel structure on $E \times_p \sigma$ is given by uniformly enumerating each $p^{-1}(x)$. The base space part of π is given by $\pi(x, \mathbb{B}) := x$, the fiber space $(U, P, p) \times \sigma$ is given by the pullback $\pi^{-1}(U, P, p)$, and the structure \mathbb{E} is given by

$$R^{\mathbb{E}}((x, \mathbb{B}, u_1), \dots, (x, \mathbb{B}, u_n)) \iff R^{\mathbb{A}}(u_1, \dots, u_n)$$

for $x \in X$, $\mathbb{B} \in \text{Mod}_{p^{-1}(x)}(\sigma)$, and $u_1, \dots, u_n \in p^{-1}(x)$. The universal property is straightforward. \square

Remark A.7. However, our other basic universal construction for structuring equivalence relations, the “Scott sentence” (Theorem 4.7), fails to generalize in a straightforward fashion to fiber spaces; this is essentially because we require languages to be countable, whereas the invariant Borel σ -algebra of a nonsmooth fiber space is not countably generated.

Remark A.8. Nonetheless, we may define the **fiber-bijective product** $(P, p, E) \otimes (Q, q, F)$ of two fiber spaces $(U, P, p), (V, Q, q)$ over $(X, E), (Y, F) \in \mathcal{E}$ respectively, by generalizing Remark 4.18, yielding their categorical product in the category of fiber spaces and fiber-bijective homomorphisms; we leave the details to the reader. In particular, by taking (Q, q, F) to be the universal fiber space $(P_\infty, p_\infty, E_\infty)$ from Proposition A.2, we obtain

Proposition A.9. *For every fiber space (P, p) over $E \in \mathcal{E}$, there is a fiber space $(P_\infty, p_\infty, E_\infty) \otimes (P, p, E)$ admitting a fiber-bijective homomorphism to (P, p, E) and which is universal among such fiber spaces with respect to fiber-bijective embeddings.*

We conclude by noting that restricting attention to smooth fiber spaces and applying the correspondence with presentations gives a different perspective on some results from Section 5:

- [HK, D.1] If $(X, E) \in \mathcal{E}$ admits a smooth fiber space (P, p) , and $\mathbb{A} : (P, p) \models \sigma$, then (P, p) corresponds to a presentation (Y, F) of X/E , and \mathbb{A} corresponds to a structure $(\widehat{F}, \pi_1) \models \sigma$, i.e., a structure $F \models \sigma$; hence E is bireducible with a σ -structurable equivalence relation.
- In particular, if $f : E \rightarrow_B^{sm} F$ and $\mathbb{A} : F \models \sigma$, then pulling back $\widehat{\mathbb{A}} : (\widehat{F}, \pi_1) \models \sigma$ along f gives a smooth σ -structured fiber space (namely $f^{-1}(\widehat{F})$) over E , whence E is bireducible with a σ -structurable equivalence relation. So \mathcal{E}_σ^r is closed under \rightarrow_B^{sm} (Theorem 5.2).
- If $f : E \rightarrow_B^{ci} F$, then the induced $\widehat{f} : \widehat{E} \rightarrow \widehat{F}$ is fiber-injective, which yields a fiberwise injection $\widehat{E} \rightarrow_E f^{-1}(\widehat{F})$ over E , whence E embeds into the σ -structurable presentation corresponding to $f^{-1}(\widehat{F})$; this similarly re-proves part of Theorem 5.1.

B Appendix: The category of theories

We discuss here a categorical structure on the class of all theories, which interacts well with several of the constructions we have considered.

B.1 Interpretations

Let \mathcal{T} denote the class of all theories. Let $(L, \sigma), (M, \tau), (N, \nu) \in \mathcal{T}$ be theories. We denote logical equivalence of $L_{\omega_1\omega}$ -formulas by \equiv , and logical equivalence modulo σ by \equiv_σ .

By an **interpretation** of L in (M, τ) , written

$$\alpha : L \rightarrow_I (M, \tau),$$

we mean a function $\alpha : L \rightarrow M_{\omega_1\omega}/\equiv_\tau$ mapping each n -ary relation $R \in L$ to a τ -equivalence class of $M_{\omega_1\omega}$ -formulas with free variables from x_1, \dots, x_n . We generally abuse notation by identifying equivalence classes of formulas with individual formulas. Thus, for $R \in L$, we denote by $\alpha(R) = \alpha(R)(x_1, \dots, x_n)$ any formula in the equivalence class $\alpha(R)$.

Given an interpretation $\alpha : L \rightarrow_I (M, \tau)$ and an $L_{\omega_1\omega}$ -formula $\phi(x_1, \dots, x_n)$, we define the $M_{\omega_1\omega}$ -formula $\alpha(\phi)(x_1, \dots, x_n)$ (modulo \equiv_τ) by “substituting” $\alpha(R)$ for R in ϕ , for each $R \in L$; formally, this is defined by induction on ϕ in the obvious manner. (In general this will require renaming bound variables in ϕ ; hence $\alpha(\phi)$ is only well-defined modulo \equiv_τ .)

Given an interpretation $\alpha : L \rightarrow_I (M, \tau)$ and a model $\mathbb{A} = (X, S^\mathbb{A})_{S \in M}$ of τ , the α -**reduct** of \mathbb{A} is the L -structure $\alpha^*\mathbb{A} = (X, R^{\alpha^*\mathbb{A}})_{R \in L}$ given by

$$R^{\alpha^*\mathbb{A}} := \alpha(R)^\mathbb{A}.$$

It follows by induction that for any $L_{\omega_1\omega}$ -formula ϕ , we have $\phi^{\alpha^*\mathbb{A}} = \alpha(\phi)^\mathbb{A}$. When α is the inclusion of a sublanguage $L \subseteq M$, $\alpha^*\mathbb{A}$ is the L -reduct in the usual sense.

Now by an **interpretation** of (L, σ) in (M, τ) , written

$$\alpha : (L, \sigma) \rightarrow_I (M, \tau) \quad (\text{or } \alpha : \sigma \rightarrow_I \tau),$$

we mean an interpretation α of L in (M, τ) such that the $M_{\omega_1\omega}$ -sentence $\alpha(\sigma)$ is logically implied by τ ; equivalently, for any (countable) model $\mathbb{A} \models \tau$, we have $\alpha^*\mathbb{A} \models \sigma$.

Remark B.1. This notion of “interpretation” is more restrictive than the usual notion considered in model theory (see [Hod, Section 5.3]).

Given interpretations $\alpha : (L, \sigma) \rightarrow_I (M, \tau)$ and $\beta : (M, \tau) \rightarrow_I (N, \nu)$, we may compose them in the obvious manner to get $\beta \circ \alpha : (L, \sigma) \rightarrow_I (N, \nu)$, where $(\beta \circ \alpha)(R) := \beta(\alpha(R))$ for $R \in L$. We also have the identity interpretation $1_\sigma : (L, \sigma) \rightarrow_I (L, \sigma)$. Thus, we have a **category** $(\mathcal{T}, \rightarrow_I)$ of **theories and interpretations**.

B.2 Products and coproducts of theories

The category $(\mathcal{T}, \rightarrow_I)$ has an initial object (\emptyset, \top) where \top is a tautology, as well as a terminal object (\emptyset, \perp) where \perp is a contradictory sentence. It also has countable coproducts and products, given by the operations \otimes and \oplus respectively (*not* vice-versa) from Section 6.2; we sketch here the verification of the universal properties.

Let $((L_i, \sigma_i))_i$ be a countable family of theories. We verify that their coproduct is

$$\otimes_i (L_i, \sigma_i) = (\bigsqcup_i L_i, \bigwedge_i \sigma_i),$$

with the canonical injections $\iota_i : (L_i, \sigma_i) \rightarrow_I \otimes_j (L_j, \sigma_j)$ given by the inclusions $L_i \rightarrow \bigsqcup_j L_j$. Let (M, τ) be another theory, and let $\alpha_i : \sigma_i \rightarrow_I \tau$ be interpretations for each i ; we must find a unique interpretation $\alpha : \otimes_i \sigma_i \rightarrow_I \tau$ such that $\alpha \circ \iota_i = \alpha_i$ for each i . This condition means precisely that $\alpha(R) \equiv_\tau \alpha_i(R)$ for each $R \in L_i$, which determines α uniquely as an interpretation $\bigsqcup_i L_i \rightarrow_I (M, \tau)$; and since τ logically implies each $\alpha_i(\sigma_i)$, τ also logically implies $\bigwedge_i \alpha_i(\sigma_i) = \alpha(\bigwedge_i \sigma_i)$, whence α is an interpretation $\otimes_i \sigma_i \rightarrow_I \tau$, as desired.

Now we verify that the product is

$$\begin{aligned} \oplus_i (L_i, \sigma_i) &= (\oplus_i L_i, \oplus_i \sigma_i) \\ &= (\bigsqcup_i (L_i \sqcup \{P_i\}), \bigvee_i ((\forall x P_i(x)) \wedge \sigma_i \wedge \bigwedge_{j \neq i} \bigwedge_{R \in L_i \sqcup \{P_i\}} \forall \bar{x} \neg R(\bar{x}))), \end{aligned}$$

with the canonical projections $\pi_i : \bigoplus_j (L_j, \sigma_j) \rightarrow_I (L_i, \sigma_i)$ given by

$$\pi_i(R) := \begin{cases} R & \text{if } R \in L_i, \\ \top & \text{if } R = P_i, \\ \perp & \text{otherwise;} \end{cases}$$

a computation shows that $\pi_i(\bigoplus_j \sigma_j)$ is logically equivalent to (hence logically implied by) σ_i , whence π_i is an interpretation $\bigoplus_j \sigma_j \rightarrow_I \sigma_i$. By a straightforward induction,

$$(*) \text{ for each } (\bigoplus_i L_i)_{\omega_1\omega} \text{-formula } \phi, \text{ we have } \phi \equiv_{\bigoplus_i \sigma_i} \bigvee_i (\pi_i(\phi) \wedge \forall x P_i(x)).$$

Now let (M, τ) be another theory, and let $\alpha_i : \tau \rightarrow_I \sigma_i$ for each i ; we must find a unique $\alpha : \tau \rightarrow_I \bigoplus_i \sigma_i$ such that $\pi_i \circ \alpha = \alpha_i$ for each i . This condition means that for each $S \in M$ and each i , we have $\pi_i(\alpha(S)) \equiv_{\bigoplus_j \sigma_j} \alpha_i(S)$. So by $(*)$, we must have

$$\alpha(S) \equiv_{\bigoplus_i \sigma_i} \bigvee_i (\alpha_i(S) \wedge \forall x P_i(x)).$$

This determines α uniquely. To check that $\bigoplus_i \sigma_i$ logically implies $\alpha(\tau)$, use $(*)$ on $\alpha(\tau)$, together with $\pi_i(\alpha(\tau)) \equiv \alpha_i(\tau)$ (by definition of α and π_i); we omit the details.

In Appendix B.4 we will give a more abstract description of countable coproducts (and other countable colimits) in $(\mathcal{T}, \rightarrow_I)$.

B.3 Interpretations and structurability

We now relate interpretations to structurability.

For an interpretation $\alpha : (L, \sigma) \rightarrow_I (M, \tau)$ and a countable Borel equivalence relation $(X, E) \in \mathcal{E}$ with a τ -structure $\mathbb{A} : E \models \tau$, the **classwise α -reduct** of \mathbb{A} is the σ -structure $\alpha_E^* \mathbb{A} : E \models \sigma$ where

$$R^{\alpha_E^* \mathbb{A}}(\bar{x}) \iff \alpha(R)^{\mathbb{A}|_{[x_1]_E}(\bar{x})}$$

for n -ary $R \in L$ and $\bar{x} = (x_1, \dots, x_n) \in X^n$ with $x_1 E \dots E x_n$. In other words, for each E -class $C \in X/E$, we have

$$\alpha_E^* \mathbb{A}|_C = \alpha^*(\mathbb{A}|_C).$$

(Note that in general, $\alpha_E^* \mathbb{A} \neq \alpha^* \mathbb{A}$.) Clearly, $\alpha \mapsto \alpha_E^*$ preserves identity and reverses composition: $(1_\sigma)_E^* \mathbb{A} = \mathbb{A}$, and $(\beta \circ \alpha)_E^* \mathbb{A} = \alpha_E^* \beta_E^* \mathbb{A}$ (for $\alpha : \sigma \rightarrow_I \tau$, $\beta : \tau \rightarrow_I \nu$, and $\mathbb{A} : E \models \nu$).

To make things more explicit, let **Set** denote the category of sets, and let

$$\text{Str}(E, \sigma) := \{\mathbb{A} \mid \mathbb{A} : E \models \sigma\}$$

denote the set of σ -structures on E . Then for $\alpha : \sigma \rightarrow_I \tau$, and fixed $E \in \mathcal{E}$, we get a function $\alpha_E^* : \text{Str}(E, \tau) \rightarrow \text{Str}(E, \sigma)$; and the assignment $\alpha \mapsto \alpha_E^*$ yields a functor

$$\text{Str}(E, \cdot) : (\mathcal{T}, \rightarrow_I)^{\text{op}} \rightarrow \mathbf{Set}$$

(here $(\cdot)^{\text{op}}$ means opposite category). On the other hand, for fixed σ , and $f : E \rightarrow_B^{\text{cb}} F$, recall that we have the classwise pullback operation $f_E^{-1} : \text{Str}(F, \sigma) \rightarrow \text{Str}(E, \sigma)$; the assignment $f \mapsto f_E^{-1}$

is also (contravariantly) functorial, and furthermore f_E^{-1} commutes with α_E^* . So we in fact get a bifunctor

$$\text{Str} : (\mathcal{E}, \rightarrow_B^{cb})^{\text{op}} \times (\mathcal{T}, \rightarrow_I)^{\text{op}} \rightarrow \mathbf{Set}$$

(here \times denotes the product category), where for $f : E \rightarrow_B^{cb} F$ and $\alpha : \sigma \rightarrow_I \tau$, $\text{Str}(f, \alpha)$ is the function $f_E^{-1} \circ \alpha_F^* = \alpha_E^* \circ f_E^{-1} : \text{Str}(F, \tau) \rightarrow \text{Str}(E, \sigma)$.

The following proposition says that the bifunctor Str is “representable in the second coordinate”: for every E , there is a canonical theory, namely σ_E (the “Scott sentence” of E), such that σ -structures on E are in natural bijection with interpretations $\sigma \rightarrow \sigma_E$.

Proposition B.2. *Let $(X, E) \in \mathcal{E}$ be a countable Borel equivalence relation. Let σ_E be the “Scott sentence”, and let $\mathbb{H} : E \models \sigma_E$ be the canonical structure given by Theorem 4.7. Then for any other theory (L, σ) and structure $\mathbb{A} : E \models \sigma$, there is a unique interpretation $\alpha : \sigma \rightarrow_I \sigma_E$ such that $\alpha_E^* \mathbb{H} = \mathbb{A}$. This is illustrated by the following diagram:*

$$\begin{array}{ccc} E & \xrightarrow{\mathbb{H}} & \sigma_E \\ & \searrow \mathbb{A} & \uparrow \alpha \\ & & \sigma \end{array}$$

Proof. We recall the definition of σ_E from the proof of Theorem 4.7. Let $M = \{R_0, R_1, \dots\}$ be the language of σ_E . Recall that we regard X as a subspace of $2^{\mathbb{N}}$, and that the structure \mathbb{H} is given by $R_i^{\mathbb{H}}(x) \iff x(i) = 1$.

Now in order to have $\alpha_E^* \mathbb{H} = \mathbb{A}$, we must have, for each n -ary $R \in L$ and $C \in X/E$, that

$$\alpha(R)^{\mathbb{H}|C} = R^{\alpha^*(\mathbb{H}|C)} = R^{\alpha_E^* \mathbb{H}|C} = R^{\mathbb{A}|C} = R^{\mathbb{A}}|C. \quad (*)$$

This can be achieved by letting $\alpha(R)(x_1, \dots, x_n)$ be a quantifier-free $M_{\omega_1\omega}$ -formula such that $\alpha(R)^{\mathbb{H}} = R^{\mathbb{A}} \subseteq X^n$; such $\alpha(R)$ exists, as in the remarks following the proof of Theorem 4.7. Since $\alpha(R)$ is quantifier-free, we have $\alpha(R)^{\mathbb{H}|C} = \alpha(R)^{\mathbb{H}}|C$, whence $(*)$ holds. Furthermore, α is an interpretation $\sigma \rightarrow_I \sigma_E$, since (from the proof of Theorem 4.7) every countable model of σ_E is isomorphic to $\mathbb{H}|C$ for some $C \in X/E$, and for such models we have $\alpha^*(\mathbb{H}|C) = \mathbb{A}|C \models \sigma$.

To check that α is unique, suppose $\alpha' : \sigma \rightarrow_I \sigma_E$ is another interpretation such that $\alpha_E'^* \mathbb{H} = \mathbb{A}$. Then α' obeys $(*)$, so for every $R \in L$ and $C \in X/E$, we have $\alpha'(R)^{\mathbb{H}|C} = R^{\mathbb{A}|C} = \alpha(R)^{\mathbb{H}|C}$. Since every countable model of σ_E is isomorphic to some $\mathbb{H}|C$, this means that $\alpha'(R) \equiv_{\sigma_E} \alpha(R)$ for every $R \in L$, i.e., that $\alpha' = \alpha$ as interpretations $\sigma \rightarrow_I \sigma_E$. \square

As usual with representable bifunctors, we may now extend $E \mapsto \sigma_E$ to a functor

$$S : (\mathcal{E}, \rightarrow_B^{cb})^{\text{op}} \rightarrow (\mathcal{T}, \rightarrow_I),$$

where $S(E) := \sigma_E$, and for a class-bijective homomorphism $f : E \rightarrow_B^{cb} F$, $S(f) : \sigma_F \rightarrow_I \sigma_E$ is the unique interpretation (given by Proposition B.2) such that $S(f)_E^* \mathbb{H}_E = f_E^{-1}(\mathbb{H}_F)$, as illustrated by the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{\mathbb{H}_E} & \sigma_E \\ & \searrow f_E^{-1}(\mathbb{H}_F) & \uparrow S(f) \\ & & \sigma_F \\ & \swarrow f & \\ F & \xrightarrow{\mathbb{H}_F} & \sigma_F \end{array}$$

Then Theorem 4.7 says precisely that this functor is full and faithful, whence

Corollary B.3. *The functor S is a contravariant equivalence of categories between $(\mathcal{E}, \rightarrow_B^{cb})$ and a full subcategory of $(\mathcal{T}, \rightarrow_I)$.*

Remark B.4. For $f : E \rightarrow_B^{cb} F$, unravelling the definitions and proofs reveals that $S(f) : \sigma_F \rightarrow_I \sigma_E$ maps each R_i in the language of σ_F coding the i th subbasic clopen subset of $2^{\mathbb{N}}$ to a quantifier-free formula coding the preimage of that subset under f .

It would be interesting to characterize the essential image of S , i.e., those theories which are isomorphic (in the sense of \rightarrow_I) to σ_E for some countable Borel equivalence relation E .

We note that the operation $(E, \sigma) \mapsto E \times \sigma$, being characterized by a universal property, can now be extended to a bifunctor $(\mathcal{E}, \rightarrow_B^{cb}) \times (\mathcal{T}, \rightarrow_I)^{\text{op}} \rightarrow (\mathcal{E}, \rightarrow_B^{cb})$ (satisfying the obvious compatibility conditions).

Finally, we note that the notion of interpretation gives us a convenient way of restating Theorem 8.1. Let σ_{fns} be a sentence in the language $L_{fns} := \{R\}$, R unary, asserting that R defines a finite nonempty subset. Clearly σ_{fns} axiomatizes the smooth countable Borel equivalence relations. For another theory (L, σ) , an interpretation $\alpha : (L_{fns}, \sigma_{fns}) \rightarrow_I (L, \sigma)$ is the same thing as a choice of an $L_{\omega_1\omega}$ -formula $\alpha(R)(x)$ such that σ logically implies $\alpha(\sigma_{fns})$, i.e., σ logically implies that $\alpha(R)$ defines a finite nonempty subset. Thus (i) \iff (ii) in Theorem 8.1 can be restated simply as

$$\sigma_{fns} \rightarrow_I \sigma \iff \sigma \Rightarrow^* \sigma_{fns}.$$

Similarly, Problem 9.21 above asks whether for any τ , there is a $\tau' \Leftrightarrow^* \tau$ such that for any σ , $\tau' \rightarrow_I \sigma$ iff $\sigma \Rightarrow^* \tau'$.

B.4 $\omega_1\omega$ -Boolean algebras

There is a more abstract viewpoint on theories and interpretations, explaining much of the categorical structure in $(\mathcal{T}, \rightarrow_I)$, which we now sketch. Roughly, a theory (L, σ) can be seen as a “presentation” of a more intrinsic algebraic structure, where L is the set of generators and σ is the relations (combined into one). An interpretation between theories is then simply a homomorphism of algebras; and general universal-algebraic constructions may be used to combine theories.

Remark B.5. The notion of “ $\omega_1\omega$ -Boolean algebra” we are about to introduce is a special case of a much more general notion, *hyperdoctrines*, in categorical logic (see [Law]). Similar notions have also appeared in model theory (often in dualized forms; see [Hru, Section 3], [Ben, Section 2.2]). However, for the sake of being self-contained, and because our point of view is different from both categorical logic and traditional model theory, we will not use these preexisting terminologies.

Let \mathbf{N} denote the category of natural numbers and functions between them (where as usual, $n = \{0, \dots, n-1\}$ for $n \in \mathbb{N}$).

Recall that a **Boolean σ -algebra** is an ω_1 -complete Boolean algebra; to avoid confusion with sentences σ , we will use the term **ω_1 -Boolean algebra** throughout this section. Let $\omega_1\mathbf{Bool}$ denote the category of ω_1 -Boolean algebras and ω_1 -homomorphisms (i.e., Boolean algebra homomorphisms preserving countable joins).

An **$\omega_1\omega$ -Boolean algebra** is a functor

$$T : \mathbf{N} \rightarrow \omega_1\mathbf{Bool}$$

satisfying certain conditions, as follows. For a morphism $f : m \rightarrow n$ in \mathbf{N} , the ω_1 -homomorphism $T(f) : T(m) \rightarrow T(n)$ is required to have a **left adjoint**, denoted $\exists_f : T(n) \rightarrow T(m)$, i.e., a function (not necessarily an ω_1 -homomorphism) satisfying

$$\exists_f(\psi) \leq \phi \iff \psi \leq T(f)(\phi), \quad (*)$$

for all $\psi \in T(n)$ and $\phi \in T(m)$ (such \exists_f is uniquely determined by $T(f)$). The left adjoints are required to satisfy the **Beck-Chevalley condition**: for any **pushout** square

$$\begin{array}{ccc} m_0 & \xrightarrow{f_2} & m_2 \\ f_1 \downarrow & & \downarrow g_2 \\ m_1 & \xrightarrow{g_1} & n \end{array}$$

in \mathbf{N} (i.e., the square commutes, and exhibits n as (an isomorphic copy of) the quotient of $m_1 \sqcup m_2$ by the smallest equivalence relation identifying $f_1(i)$ with $f_2(i)$ for each $i \in m_0$), we have

$$T(f_1) \circ \exists_{f_2} = \exists_{g_1} \circ T(g_2) \quad (**)$$

as functions $T(m_2) \rightarrow T(m_1)$. (Note: the \geq inequality is automatic.)

Remark B.6. In fact, $(**)$ is symmetric (under exchanging 1 and 2 in subscripts); this can be seen by verifying that it is equivalent to:

$$\exists_{f_1}(\phi) \wedge \exists_{f_2}(\psi) = \exists_{g_1 \circ f_1}(T(g_1)(\phi) \wedge T(g_2)(\psi))$$

for all $\phi \in T(m_1)$ and $\psi \in T(m_2)$ (again, the \geq inequality is automatic).

A **homomorphism** between $\omega_1\omega$ -Boolean algebras T, U is a natural transformation of functors $\alpha : T \rightarrow U$, i.e., a system of ω_1 -Boolean homomorphisms $(\alpha_n : T(n) \rightarrow U(n))_{n \in \mathbf{N}}$ intertwining $T(f), U(f)$ for $f : m \rightarrow n$ in \mathbf{N} , which furthermore commutes with the operations \exists_f , i.e.,

$$\alpha_m \circ \exists_f^T = \exists_f^U \circ \alpha_n : T(n) \rightarrow U(m)$$

for all $f : m \rightarrow n$ in \mathbf{N} , where \exists_f^T (respectively \exists_f^U) denotes the operation \exists_f in T (respectively U). Let $\omega_1\omega\mathbf{Bool}$ denote the category of $\omega_1\omega$ -Boolean algebras and homomorphisms.

From its definition, the notion of $\omega_1\omega$ -Boolean algebra is clearly (multisorted) **algebraic**, in the sense that an $\omega_1\omega$ -Boolean algebra is a system of sets $(T(n))_{n \in \mathbf{N}}$ equipped with various (infinitary) operations between them which are required to satisfy some universal equational axioms. Namely, the operations are: the Boolean operations \neg , (countable) \bigvee, \bigwedge on each $T(n)$, and the unary operations $T(f) : T(m) \rightarrow T(n)$ and $\exists_f : T(n) \rightarrow T(m)$ for each morphism $f : m \rightarrow n$ in \mathbf{N} ; and the equational axioms are: the axioms of an ω_1 -Boolean algebra for each $T(n)$, the statements that each $T(f)$ is an ω_1 -homomorphism, and the axioms $(*)$ and $(**)$ for each choice of parameters f, f_1, f_2, g_1, g_2 (where $(*)$ is replaced with equivalent equations using \wedge , say). Furthermore, a homomorphism of $\omega_1\omega$ -Boolean algebras is precisely a system of functions which are required to preserve all of the operations.

It follows from universal algebra that the category $\omega_1\omega\mathbf{Bool}$ admits all sorts of constructions. We describe some of these, while at the same time relating $\omega_1\omega$ -Boolean algebras to theories.

Let L be a language. For $n \in \mathbb{N}$, let $L(n) \subseteq L$ denote the n -ary relation symbols in L ; we may thus regard L as an \mathbb{N} -graded set $(L(n))_{n \in \mathbb{N}}$. Similarly, let $L_{\omega_1\omega}(n)$ denote the $L_{\omega_1\omega}$ -formulas with free variables from x_0, \dots, x_{n-1} , and let $(L_{\omega_1\omega}/\equiv)(n) := L_{\omega_1\omega}(n)/\equiv$ denote the logical equivalence classes of such formulas. We identify each relation $R \in L(n)$ with the atomic formula $R(x_0, \dots, x_{n-1}) \in L_{\omega_1\omega}(n)$. Each $(L_{\omega_1\omega}/\equiv)(n)$ is an ω_1 -Boolean algebra under the logical connectives (the **Lindenbaum-Tarski algebra** with n free variables). For $f : m \rightarrow n$ in \mathbf{N} , let

$$\begin{aligned} (L_{\omega_1\omega}/\equiv)(f) : (L_{\omega_1\omega}/\equiv)(m) &\longrightarrow (L_{\omega_1\omega}/\equiv)(n) \\ \phi(x_0, \dots, x_{m-1}) &\longmapsto \phi(x_{f(0)}, \dots, x_{f(m-1)}) \end{aligned}$$

be the variable substitution map. Then

Proposition B.7. $L_{\omega_1\omega}/\equiv : \mathbf{N} \rightarrow \omega_1\mathbf{Bool}$ is the free $\omega_1\omega$ -Boolean algebra generated by L , i.e., by the atomic formulas $R(x_0, \dots, x_{n-1}) \in (L_{\omega_1\omega}/\equiv)(n)$ for $R \in L(n)$.

Proof sketch. One verifies that a complete proof system for $L_{\omega_1\omega}$, e.g., the Gentzen system in [LE], corresponds to the axioms of $\omega_1\omega$ -Boolean algebras. The details are tedious but straightforward; we only comment here on the treatment of quantifiers and equality in $\omega_1\omega$ -Boolean algebras. Both are encoded into the left adjoints \exists_f in an $\omega_1\omega$ -Boolean algebra. Namely, when $f : m \rightarrow n$ is an injective morphism in \mathbf{N} , say (for notational simplicity) $f : m \rightarrow m+1$ is the inclusion, then \exists_f corresponds to existential quantification over the free variables not in the image of f :

$$\begin{aligned} \exists_f : (L_{\omega_1\omega}/\equiv)(m+1) &\longrightarrow (L_{\omega_1\omega}/\equiv)(m) \\ \psi(x_0, \dots, x_m) &\longmapsto \exists x_m \psi(x_0, \dots, x_m). \end{aligned}$$

The axiom (**) above (with $f_2 = f$) then says that substitution of variables along f_1 commutes with the quantifier $\exists x_m$, while the axiom (*) corresponds to the logical rule of inference characterizing the existential quantifier:

$$(\exists x_m \psi(x_0, \dots, x_m)) \rightarrow \phi(x_0, \dots, x_{m-1}) \iff \psi(x_0, \dots, x_m) \rightarrow \phi(x_0, \dots, x_{m-1}).$$

When $f : m \rightarrow n$ is surjective, say $f : n+1 \rightarrow n$ is the identity on n and $f(n) = n-1 = f(n-1)$, then \exists_f corresponds to equating the variables in the kernel of f :

$$\begin{aligned} \exists_f : (L_{\omega_1\omega}/\equiv)(n) &\longrightarrow (L_{\omega_1\omega}/\equiv)(n+1) \\ \phi(x_0, \dots, x_{n-1}) &\longmapsto (x_n = x_{n-1}) \wedge \phi(x_0, \dots, x_{n-1}). \end{aligned}$$

Again (**) says that substitution preserves equality, while (*) is the Leibniz rule for equality. We leave the details to the reader. \square

Similarly, one can verify that for an $L_{\omega_1\omega}$ -sentence σ , $L_{\omega_1\omega}/\equiv_\sigma : \mathbf{N} \rightarrow \omega_1\omega\mathbf{Bool}$ is the free $\omega_1\omega$ -Boolean algebra generated by L , subject to the relation $\sigma = \top$ (where \top is a tautology). But these are precisely the countably presented $\omega_1\omega$ -Boolean algebras, since countably many relations may be conjuncted into one; hence

Proposition B.8. *The functor*

$$\begin{aligned} (\mathcal{T}, \rightarrow_I) &\longrightarrow \omega_1\omega\mathbf{Bool} \\ (L, \sigma) &\longmapsto L_{\omega_1\omega}/\equiv_\sigma \end{aligned}$$

(sending interpretations to homomorphisms) is an equivalence between the category $(\mathcal{T}, \rightarrow_I)$ of theories and the full subcategory of $\omega_1\omega\mathbf{Bool}$ consisting of the countably presented $\omega_1\omega$ -Boolean algebras.

The category $\omega_1\omega\mathbf{Bool}$, being (multisorted) algebraic, has all (small) limits and colimits. We have already seen (Appendix B.2) that the subcategory of countably presented algebras, or equivalently $(\mathcal{T}, \rightarrow_I)$, has countable products and coproducts. The latter can be seen as an instance of the standard fact that countable colimits of countably presented algebras are countably presented (see e.g., [Joh, VI 2.1] for the analogous statement for finitely presented algebras):

Proposition B.9. *The category $(\mathcal{T}, \rightarrow_I)$ has all colimits of countable diagrams.*

For example, to construct the colimit of the diagram consisting of two theories (L, σ) and (M, τ) and two interpretations $\alpha : \sigma \rightarrow_I \tau$ and $\beta : \sigma \rightarrow_I \tau$ (the **coequalizer** of α, β), one takes the “quotient” of (M, τ) by the “relations” $\alpha(R) \leftrightarrow \beta(R)$ for each $R \in L$, i.e., the theory

$$(M, \tau \wedge \bigwedge_{R \in L} \forall \bar{x} (\alpha(R)(\bar{x}) \leftrightarrow \beta(R)(\bar{x}))).$$

C Appendix: Representation of ω_1 -distributive lattices

Recall the definition of **ω_1 -distributive lattice** from Section 6.2. A **Boolean σ -algebra** is an ω_1 -complete Boolean algebra, which is then automatically ω_1 -distributive. The **Loomis-Sikorski representation theorem** states that every Boolean σ -algebra is a quotient of a sub-Boolean σ -algebra of 2^X for some set X . One proof (see [Sik, 29.1]) is via the Stone representation theorem.

By replacing the Stone representation theorem with the Priestley representation theorem for distributive lattices, we may prove an analogous result for ω_1 -distributive lattices. We could not find this result in the literature, although it is quite possibly folklore. We give the proof here for the sake of completeness.

Theorem C.1. *Let P be an ω_1 -distributive lattice. Then P is isomorphic to a quotient ω_1 -complete lattice of a sub- ω_1 -complete lattice of 2^X for some set X .*

Proof. We recall the Priestley representation theorem; see [Joh, II 4.5–8, VII 1.1]. Let P be a distributive lattice. A **prime filter** on P is an upward-closed subset $F \subseteq P$ which is closed under finite meets (including the top element) and whose complement is closed under finite joins. Let X be the set of prime filters on P . We order X by \subseteq . For $x \in P$, let

$$\eta(x) := \{F \in X \mid x \in F\}.$$

We equip X with the topology generated by the sets $\eta(x)$ and their complements.

Theorem C.2 (Priestley). *(i) X is a compact Hausdorff zero-dimensional space, and the order \subseteq on X is closed as a subset of X^2 .*

(ii) The map η is an order-isomorphism between P and the poset of clopen (\subseteq) -upward-closed subsets of X under inclusion.

Now let P be an ω_1 -distributive lattice, X be as above, $I \subseteq 2^X$ be the σ -ideal of meager subsets of X , and $p : 2^X \rightarrow 2^X/I$ be the quotient map. Then $p \circ \eta : P \rightarrow 2^X/I$ is a lattice homomorphism. It is injective, since if $x, y \in P$ with $x \neq y$ then $\eta(x) \triangle \eta(y) \notin I$ by the Baire category theorem.

Lemma C.3. For $x_0, x_1, \dots \in P$, $\eta(\bigvee_i x_i) = \overline{\bigcup_i \eta(x_i)}$ (the closure of $\bigcup_i \eta(x_i)$).

Proof. Clearly $\eta(\bigvee_i x_i)$ contains each $\eta(x_i)$ and is closed; thus $\eta(\bigvee_i x_i) \supseteq \overline{\bigcup_i \eta(x_i)}$. Suppose there were some $F \in \eta(\bigvee_i x_i) \setminus \overline{\bigcup_i \eta(x_i)}$. So $\bigvee_i x_i \in F$. Let

$$\mathcal{G} := \{G \in \overline{\bigcup_i \eta(x_i)} \mid G \subseteq F\}.$$

For each $G \in \mathcal{G}$, we have some $y_G \in F \setminus G$. Then $\mathcal{G} \cap \bigcap_{G \in \mathcal{G}} \eta(y_G) = \emptyset$, so since \mathcal{G} is compact, there are $G_1, \dots, G_k \in \mathcal{G}$ such that $\mathcal{G} \cap \eta(\bigwedge_j y_{G_j}) = \mathcal{G} \cap \bigcap_j \eta(y_{G_j}) = \emptyset$. Put $y := \bigwedge_j y_{G_j}$, so that $\mathcal{G} \cap \eta(y) = \emptyset$. Then for all $G \in \overline{\bigcup_i \eta(y \wedge x_i)} = \eta(y) \cap \overline{\bigcup_i \eta(x_i)} \subseteq \eta(y) \cap \bigcup_i \eta(x_i)$, we have $G \notin \mathcal{G}$, so by definition of \mathcal{G} , $G \not\subseteq F$. Also, we have $\bigvee_i \eta(y \wedge x_i) = \bigwedge_j y_{G_j} \wedge \bigvee_i x_i \in F$ since F is a filter.

Now for each $G \in \overline{\bigcup_i \eta(y \wedge x_i)}$, we find $z_G \in G \setminus F$; then the union of $\eta(z_G)$ for all these G covers $\overline{\bigcup_i \eta(y \wedge x_i)}$, so by compactness of the latter, by taking the join of finitely many z_G 's, we get a $z \notin F$ (because F is prime) such that $\overline{\bigcup_i \eta(y \wedge x_i)} \subseteq \eta(z)$. But then $\eta(y \wedge x_i) \subseteq \eta(z)$, i.e., $y \wedge x_i \leq z$, for every i , while $\bigvee_i (y \wedge x_i) \not\leq z$ because F separates them, a contradiction. \square

Similarly (or simply by reversing the order), we have

Lemma C.4. For $x_0, x_1, \dots \in P$, $\eta(\bigwedge_i x_i) = (\bigcap_i \eta(x_i))^\circ$ (the interior of $\bigcap_i \eta(x_i)$).

It follows from these lemmas that $p \circ \eta : P \rightarrow 2^X/I$ is an ω_1 -complete lattice homomorphism (i.e., preserves countable meets and joins), so P is isomorphic to its image $p(\eta(P))$. But $p(\eta(P))$ is a sub- ω_1 -complete lattice of a quotient of 2^X , hence (as with any kind of algebra) also a quotient of a sub- ω_1 -complete lattice of 2^X (namely $p^{-1}(p(\eta(P)))/I$). \square

Corollary C.5. If an equation between ω_1 -complete lattice terms (i.e., formal expressions built from variables, \wedge , and \vee) holds in the ω_1 -complete lattice 2 , then it holds in every ω_1 -distributive lattice.

Remark C.6. Another proof of Theorem C.1 may be given using the $L_{\omega_1\omega}$ completeness theorem for the Gentzen system in [LE].

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