

# TYPE III $\sigma$ -SPECTRAL TRIPLES AND QUANTUM STATISTICAL MECHANICAL SYSTEMS

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ABSTRACT. Spectral triples and quantum statistical mechanical systems are two important constructions in noncommutative geometry. In particular, both lead to interesting reconstruction theorems for a broad range of geometric objects, including number fields, spin manifolds, graphs. There are similarities between the two structures, and we show that the notion of type III  $\sigma$ -spectral triple, introduced recently by Connes and Moscovici, provides a natural bridge between them. We investigate explicit examples, related to the Bost–Connes quantum statistical mechanical system and to Riemann surfaces and graphs.

## 1. INTRODUCTION

1.1. **Reconstruction theorems in Noncommutative Geometry.** The prototype model of a *reconstruction theorem* in Noncommutative Geometry is also the motivating origin of the very notion of noncommutative spaces, namely the Gelfand–Naimark theorem, which shows that it is possible to reconstruct (up to homeomorphism) a compact Hausdorff topological space  $X$  from the commutative  $C^*$ -algebra of continuous functions  $C(X)$  (up to  $C^*$ -isomorphism). This reconstruction theorem, which shows that considering compact Hausdorff topological space is equivalent to considering commutative  $C^*$ -algebras motivated the notion of a noncommutative topological space as the datum of a noncommutative  $C^*$ -algebra, [9].

More recently, reconstruction theorems have been proved for the analogs, in Noncommutative Geometry, of compact spin Riemannian manifolds, namely *spectral triples*, a notion we will discuss more in detail in the rest of this paper. The prototype for this kind of reconstruction theorem is the reconstruction of compact spin Riemannian manifolds from the data of a commutative spectral triple satisfying a set of axioms involving appropriate notions of orientability, Poincaré duality and smoothness, [12]. For some variants and generalizations see [30], [37]. Similar results have been proved for other classes of noncommutative spaces, such as almost-commutative geometries, [4]. With a different approach, a reconstruction theorem for Riemannian manifolds based on zeta functions of spectral triples was proved in [19], and a complete invariant in Riemannian geometry based on noncommutative methods was obtained in [11]. A reconstruction theorem for Riemann surfaces up to (anti)conformal equivalence, based on the zeta functions of a spectral triple, was obtained in [20], with an analogous result for graphs in [26].

At the same time, *quantum statistical mechanical systems*, another important construction widely used in Noncommutative Geometry, have also been used to obtain reconstruction theorems based on noncommutative methods, most notably the reconstruction result for number fields obtained in [21], and a similar result for graphs in [22].

Our purpose in this paper is to present a comparative analysis of these two constructions in Noncommutative Geometry: *spectral triples* and *quantum statistical mechanical systems*. We show that there are deep conceptual similarities between the two notions, but also important differences.

We investigate to what extent one can transform one type of structure into the other and extract from both of them similar type of information on the underlying geometric objects and the related classification problems.

The paper is organized as follows. In Sections 2 and 3 we review well known material about spectral triples and quantum statistical mechanical systems, respectively, trying to outline the differences and the apparent similarities between the two structures. Then in Section 4 we focus on the example of the Bost–Connes quantum statistical mechanical system [2] and on a related structure based on the Riemann gas with supersymmetry considered in [27], [33], [34]. We show that the data of the Bost–Connes system determine a type III  $\sigma$ -spectral triple, where the sign  $F$  of the Dirac operator and the twisting automorphism  $\sigma$  are determined by the Liouville function. A similar construction works for the Riemann gas with supersymmetry, using the Möbius function instead of the Liouville function. We discuss some properties of eta functions and KMS states. In Section we consider instead the case of limit sets of Schottky groups and of graphs, where one can use a very similar construction covering both cases, as in [20], [22], [26]. We show that the spectral triples considered in the reconstruction results of [20] and [26] for Riemann surfaces and graphs have some undesirable features, such as the fact of not admitting an interesting sign and compatible even structure. We propose then a construction of a type III  $\sigma$ -spectral triple for these cases, analogous to the construction described for the Bost–Connes case, that starts from the quantum statistical mechanical systems of [22], [31].

## 2. SPECTRAL TRIPLES

This section gives a review of well known notions about spectral triples in Noncommutative Geometry. For more details, we refer the reader to [9], [10], [13], [16].

The notion of a *spectral triple* [10] provides an abstract algebraic version of the main property satisfied by a compact Riemannian spin manifold. The main purpose of this formulation is to encode the data of the manifold and its spin and metric structures in terms that avoid the use of *local charts*, which does not have a direct analog in the noncommutative setting, and instead focuses on the use of operator relations in a Hilbert space setting. More precisely, we recall the following well known definition.

**Definition 2.1.** *A spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  consist of the following data:*

- *an involutive algebra  $\mathcal{A}$ ;*
- *a representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  in the algebra  $\mathcal{B}(\mathcal{H})$  of bounded linear operators on a separable Hilbert space  $\mathcal{H}$ ;*
- *a self adjoint operator  $D = D^*$  acting on  $\mathcal{H}$  with a dense domain  $\text{Dom}(D) \subset \mathcal{H}$ ;*
- *the condition that  $D$  has compact resolvent, namely that  $(1 + D^2)^{-1/2}$  belongs to the algebra  $\mathcal{K} \subset \mathcal{B}(\mathcal{H})$  of compact operators;*
- *the condition that all commutators  $[\pi(a), D]$  of elements in the algebra,  $a \in \mathcal{A}$ , with the Dirac operator  $D$  are bounded operators,  $[\pi(a), D] \in \mathcal{B}(\mathcal{H})$ .*

*The spectral triples is said to be even if there is a  $\mathbb{Z}/2$ -grading  $\gamma$  on  $\mathcal{H}$  with the property that*

$$[\gamma, \pi(a)] = 0, \quad \forall a \in \mathcal{A}, \quad \text{and} \quad D\gamma = -\gamma D.$$

**2.1. Spectral triples and Spin geometry.** One can see that the structure described in Definition 2.1 reflects the main properties of a compact smooth Riemannian spin manifold  $M$ , by considering the case where the involutive algebra  $\mathcal{A} = \mathcal{C}^\infty(M)$  is the algebra of smooth complex valued functions on  $M$  with the involution given by complex conjugation and where the Hilbert space is  $\mathcal{H} =$

$L^2(M, \mathbb{S})$ , the space of square-integrable sections of the spinor bundle  $\mathbb{S}$  of  $M$  (with respect to a choice of spin structure), on which elements of  $C^\infty(M)$  act as multiplication operators. One can take the operator  $D = \not{\partial}_M$  to be the Dirac operator acting on sections of the spinor bundle. One can see easily that these data satisfy all the conditions of Definition 2.1. For example, the commutator of the Dirac operator with a smooth function is given by  $[\not{\partial}_M, f] = c(df)$ , the differential  $df$  acting on spinors by Clifford multiplication. The data  $(C^\infty(M), L^2(M, \mathbb{S}), \not{\partial}_M)$  are usually referred to as the canonical spectral triple of a Riemannian spin manifold. It is well known that, for example, the Riemannian metric on  $M$  is encoded in the data of the spectral triple and can be recovered using the formula for the distance, [8],

$$d(x, y) = \sup\{|f(x) - f(y)| : f \in C^\infty(M), \|[\not{\partial}_M, f]\| \leq 1\}.$$

**2.2. Spectral metric geometry and the Dirac sign.** A Dirac operator  $D$  as in Definition 2.1 has a polar decomposition  $D = |D|F$ , into a positive part and a sign operator,  $F^2 = 1$ . In the manifold case, the sign  $F$  is non-trivial, and in fact it carries important geometric information, since it represents the fundamental class of  $M$  in  $K$ -homology. (For a detailed treatment of these topics see for instance [25].) However, in Noncommutative Geometry, one sometimes considers also a wider class of spectral triples where the sign  $F$  may be trivial. The triviality of the sign causes problems when one wishes to use the spectral triple setting to apply index theory techniques, but such spectral triples are still useful, for instance because they can still define *quantum metric spaces* in the sense of [32], and also because they can carry interesting families of zeta functions.

**2.3. Summability, dimensions, and zeta functions.** Zeta functions for spectral triples are defined under a condition of *finite summability*, which we recall in the following.

**Definition 2.2.** A spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  is *finitely summable* if for  $\beta \gg 0$  sufficiently large the operator  $|D|^{-\beta}$  is of trace class,

$$(2.1) \quad \text{Tr}(|D|^{-\beta}) < \infty.$$

The smallest  $\beta_c \geq 0$  such that (2.1) holds for all  $\beta > \beta_c$  is called the *metric dimension* of the spectral triple. A spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  is  *$\theta$ -summable* if for all  $t > 0$  the heat kernel operator  $\exp(-tD^2)$  is of trace class,

$$(2.2) \quad \text{Tr}(e^{-tD^2}) < \infty.$$

In the case of a finitely summable spectral triple, one can define a family of zeta functions associated to the data  $(\mathcal{A}, \mathcal{H}, D)$  of the spectral triple and defined as

$$(2.3) \quad \zeta_a(s) = \text{Tr}(\pi(a) |D|^{-s}),$$

where  $s$  is a complex variable, and  $a \in \mathcal{A}$ , acting on  $\mathcal{H}$  through the representation  $\pi$ . One also considers similar zeta functions of the form  $\text{Tr}(\pi(b) |D|^{-s})$ , where  $b$  is an element of the algebra  $\mathcal{B}$  generated by  $\delta^k(\pi(a))$  and  $\delta^k([D, \pi(a)])$ , for  $a \in \mathcal{A}$  and  $\delta(-) = [|D|, -]$ .

The zeta functions obtained in this way provide a refined notion of dimension for noncommutative spaces, given by the *dimension spectrum*.

**Definition 2.3.** The *dimension spectrum* of a finitely summable spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  is the subset  $\Sigma = \text{DimSp}(\mathcal{A}, \mathcal{H}, D) \subset \mathbb{C}$  given by the set of poles of the zeta functions  $\zeta_b(s)$ , for  $b \in \mathcal{B}$ .

Since points of the dimension spectrum correspond to poles of the zeta functions of the spectral triple, one can consider the associated residues,

$$(2.4) \quad \text{Res}_{s=\beta \in \Sigma} \zeta_a(s) =: \int_a D^{-\beta},$$

where the meaning of the notation on the right-hand-side of (2.4) is that one interprets these residues as integration in dimension  $\alpha \in \Sigma$ , where  $\Sigma$  is the dimension spectrum.

**2.4. Zeta functions and eta functions.** As we have seen above, for a finitely summable spectral triple, the zeta function of the Dirac operator  $D$  is defined as  $\zeta_D(s) = \text{Tr}(|D|^{-s})$ . In Riemannian geometry, another very useful series associated to a self-adjoint operator of Dirac type is the *eta function*  $\eta_D(s)$  that measures spectral asymmetry and that gives rise to the *eta invariant*, as the special value  $\eta_D(0)$ . The eta function of a Dirac operator with trivial kernel is defined as

$$(2.5) \quad \eta_D(s) = \text{Tr}(F|D|^{-s}) = \sum_{\lambda \in \text{Spec}(D)} \text{sign}(\lambda) |\lambda|^{-s}.$$

Similarly to what is usually done with zeta functions of spectral triples, one can consider a family of eta functions of the form

$$(2.6) \quad \eta_{a,D}(s) = \text{Tr}(F a |D|^{-s}), \quad \text{for } a \text{ in } \mathcal{A}^\infty.$$

**2.5. Eta functions on manifolds.** In the case of the eta function of a classical self-adjoint elliptic first order differential operator  $\mathcal{D}$  on a manifold of dimension  $n$ , the eta function can be expressed as, [23],

$$\begin{aligned} \eta_{\mathcal{D}}(s) &= \sum_{\lambda \in \text{Spec}(\mathcal{D}) \setminus \{0\}} \lambda |\lambda|^{-s-1} = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s+1}{2}-1} (\sum \lambda e^{-t\lambda^2}) dt \\ &= \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s+1}{2}-1} \text{Tr}(\mathcal{D} e^{-t\mathcal{D}^2}) dt. \end{aligned}$$

In turn, the trace  $\text{Tr}(\mathcal{D} e^{-t\mathcal{D}^2})$  has an asymptotic expansion from the heat kernel asymptotics,

$$(2.7) \quad \text{Tr}(\mathcal{D} e^{-t\mathcal{D}^2}) \sim \sum_{k=0}^N c_k(\mathcal{D}) t^{\frac{k-n-1}{2}},$$

which in turn gives

$$(2.8) \quad \eta_{\mathcal{D}}(s) \sim \frac{1}{\Gamma(\frac{s+1}{2})} \sum_{k=0}^N \frac{2c_k(\mathcal{D})}{s+k-n},$$

so that one finds poles at  $s = n - k$  with residues

$$\text{Res}_{s=n-k} \eta_{\mathcal{D}}(s) = \frac{2c_k(\mathcal{D})}{\Gamma(\frac{s+1}{2})}.$$

In the general setting of noncommutative geometry, one does not necessarily expect to always have a similar form of the heat kernel asymptotics. However, useful forms of heat kernel asymptotics exist and are related to asymptotics for the spectral action functional, [6].

**2.6. The spectral action.** It is customary to consider an *action functional* for finitely summable spectral triples, given by the *spectral action*, introduced in [6]. The functional is defined as

$$(2.9) \quad \mathrm{Tr}(f(D/\Lambda)),$$

where  $D$  is the Dirac operator of the spectral triple,  $\Lambda$  is a mass scale that makes  $D/\Lambda$  dimensionless, and  $f > 0$  is an even smooth function, which is usually taken to be a smooth approximation of a cutoff function. When written in this form, one considers the argument of this functional to be the Dirac operator  $D$ , so that the space of *fields* of the corresponding field theory would be the space of all possible Dirac operators  $D$  compatible with fixed data  $(\mathcal{A}, \mathcal{H})$ , so that  $(\mathcal{A}, \mathcal{H}, D)$  is a spectral triple, possibly with additional constraints. A variant of (2.9) consists of allowing twists of the Dirac operator by “gauge potentials”. This means replacing (2.9) with the similar expression

$$(2.10) \quad \mathrm{Tr}(f(D_A/\Lambda)),$$

where  $D_A = D + A$ , where the gauge potential  $A$  is a finite sum  $A = \sum_i \pi(a_i) [D, \pi(b_i)]$  with  $a_i, b_i \in \mathcal{A}$ .

It was shown in [6] that there is an asymptotic expansion for  $\Lambda \rightarrow \infty$  of the spectral action functional of the form

$$(2.11) \quad \mathrm{Tr}(f(D/\Lambda)) \sim \sum_{k \in \Sigma^+} f_k \Lambda^k \int |D|^{-k} + f(0) \zeta_D(0) + o(1),$$

where  $f_k = \int_0^\infty f(v) v^{k-1} dv$  are the momenta of  $f$  and where the summation runs over the positive part of the dimension spectrum  $\Sigma^+ \subset \Sigma$ . In the case of manifolds and of almost commutative geometries, the residues (2.4) that appear in this asymptotic expansion have a description in terms of local expressions in the curvature tensors of the Riemannian metric and in the curvatures of the gauge potentials, that recover some of the classical action functionals of physics, including the Einstein–Hilbert action of General Relativity, the Yang–Mills action functionals of gauge theories, and for a suitable choice of the almost commutative geometry the bosonic part of the Lagrangian of the Standard Model of elementary particle physics, see [7] and Chapter 1 of [14].

**2.7. Spectral action, summability, and spectral asymmetry.** One usually considers the spectral action in the finitely summable case, and especially for the almost commutative geometries used in models of gravity coupled to matter. However, one can consider the spectral action functional in more general cases.

For example, in the case of a  $\theta$ -summable  $D$ , one can consider the test function  $f(t) = \exp(-t^2)$  and the spectral action

$$\mathrm{Tr}(f(D/\Lambda)) = \mathrm{Tr}(e^{-\frac{1}{\Lambda^2} D^2}) < \infty$$

is then directly expressed in terms of the heat kernel.

More generally, if one considers a test function of the form  $f_+(t) = e^{-|t|}$ , then for an operator  $D = F|D|$  that satisfies a summability condition

$$(2.12) \quad \mathrm{Tr}(e^{-|D|/\Lambda}) < \infty$$

the spectral action  $\mathrm{Tr}(f_+(D/\Lambda))$  with the test function as above is well defined. Notice that, in such cases, there can be a ultraviolet divergence problem if the condition (2.12) only holds for energies below a certain scale,  $\Lambda < \Lambda_0$ . Notice that, if one imposes a cutoff  $\Lambda < \Lambda_0$  on the energy scale, then the test functions need not be smooth at the origin. In this setting, one can also consider a

“spectral asymmetry” version of the spectral action, where instead of using an even test function one uses an odd one. For instance one can consider  $f_-(t) = \text{sign}(t)e^{-|t|}$  so that one obtains

$$(2.13) \quad \text{Tr}(f_-(D/\Lambda)) = \text{Tr}(Fe^{-|D|/\Lambda}),$$

again with a cutoff  $\Lambda \leq \Lambda_0$  for which the summability condition (2.12) holds.

**2.8. Type III spectral triples.** A variant on the notion of spectral triple was introduced recently by Connes and Moscovici, [17], which uses a twisting by an automorphism in the bounded commutator condition. This works very well to extend the notion of spectral triple to certain classes of noncommutative spaces that correspond to type III von Neumann algebras, especially in the context of the geometry of foliations.

We recall here the notion of a  $\sigma$ -spectral triple from [17].

**Definition 2.4.** *Let  $\mathcal{A}$  be a unital involutive algebra, represented as bounded operators on a Hilbert space  $\mathcal{H}$  and let  $D$  be a self-adjoint operator with compact resolvent, densely defined on  $\mathcal{H}$ . The data  $(\mathcal{A}, \mathcal{H}, D)$  define a  $\sigma$ -spectral triple if there is an automorphism  $\sigma \in \text{Aut}(\mathcal{A})$  such that*

$$(2.14) \quad Da - \sigma(a)D \in \mathcal{B}(\mathcal{H})$$

*is a bounded operator, for all  $a \in \mathcal{A}$ . The  $\sigma$ -spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  is even if, moreover, there is a  $\mathbb{Z}/2\mathbb{Z}$ -grading  $\gamma$  on  $\mathcal{H}$  such that  $[\gamma, a] = 0$  for all  $a \in \mathcal{A}$  and  $D\gamma + \gamma D = 0$ . The  $\sigma$ -spectral triple is Lipschitz-regular if*

$$|D|a - \sigma(a)|D| \in \mathcal{B}(\mathcal{H})$$

*for all  $a \in \mathcal{A}$ .*

### 3. QUANTUM STATISTICAL MECHANICAL SYSTEMS

In this section we review the basic setting of Quantum Statistical Mechanical Systems in Non-commutative Geometry. For more details, we refer the reader to [3], [14].

**Definition 3.1.** *A quantum statistical mechanical system consists of the following data:*

- *A separable unital  $C^*$ -algebra  $\mathcal{A}$ ;*
- *A one-parameter family of automorphisms, namely a continuous group homomorphism*

$$(3.1) \quad \sigma : \mathbb{R} \rightarrow \text{Aut}(\mathcal{A});$$

- *A representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  of  $\mathcal{A}$  as bounded operators on a separable Hilbert space  $\mathcal{H}$*
- *A densely defined self-adjoint operator  $H$  on  $\mathcal{H}$  with  $\text{Spec}(H) \subset [0, \infty)$ , such that*

$$(3.2) \quad \pi(\sigma_t(a)) = e^{itH}\pi(a)e^{-itH}, \quad \forall a \in \mathcal{A}, \quad \forall t \in \mathbb{R}.$$

**3.1. Time evolution and Hamiltonian.** The one parameter family of automorphisms (3.1) is the *time evolution* of the quantum statistical mechanical system. The datum of the operator  $H$  implementing the time evolution as in (3.2) in general depends on the choice of the representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ , not only on the data  $(\mathcal{A}, \sigma)$ . The operator  $H = H_{(\mathcal{A}, \sigma, \pi)}$  is called the *Hamiltonian* of the quantum statistical mechanical system. It is in general an unbounded, densely defined, operator. A representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  is called *finite energy* if there is an operator  $H$  satisfying (3.2), with spectrum bounded below. This is equivalent, up to an irrelevant translation, to the condition  $\text{Spec}(H) \subset [0, \infty)$  in the definition above.

**3.2. Partition function and critical temperature.** Given data as in Definition 3.1 for a quantum statistical mechanical system, the *partition function* of the Hamiltonian  $H$  is defined as

$$(3.3) \quad Z(\beta) = \text{Tr}(e^{-\beta H}),$$

where  $\beta$  is a non-negative real variable, representing the inverse temperature (up to multiplication by the Boltzmann constant).

An analog of the summability conditions defined for spectral triples is the property that the operator  $e^{-\beta H}$  is of trace class,  $\text{Tr}(e^{-\beta H}) < \infty$ , for all sufficiently large  $\beta \gg 0$ . The *critical inverse temperature* of the quantum statistical mechanical system is then the value  $\beta_c \geq 0$  such that  $\text{Tr}(e^{-\beta H}) < \infty$ , for all  $\beta > \beta_c$ .

**3.3. KMS states.** A *state* on a unital  $C^*$ -algebra  $\mathcal{A}$  is a continuous linear functional  $\varphi : \mathcal{A} \rightarrow \mathbb{C}$  that is normalized  $\varphi(1) = 1$  and satisfies a positivity condition  $\varphi(a^*a) \geq 0$ , for all  $a \in \mathcal{A}$ . States should be thought of as the analogs of *measures* in the noncommutative world. Extremal states, namely those states that cannot be decomposed as convex combinations of other states, are an analog of *points* (Dirac measures) for noncommutative spaces.

In the case of a quantum statistical mechanical system, an especially interesting class of states is given by the KMS equilibrium states. At a given inverse temperature  $\beta > 0$ , a state  $\varphi$  on the algebra of observables  $\mathcal{A}$  is a  $\text{KMS}_\beta$  state if there is a dense subalgebra  $\mathcal{A}^{an} \subset \mathcal{A}$  of *analytic elements*, invariant under the time evolution  $\sigma$  and such that, for all  $a, b \in \mathcal{A}^{an}$

$$(3.4) \quad \varphi(ab) = \varphi(b\sigma_{i\beta}(a)).$$

We refer the reader to [3] for more details on this and other equivalent formulations of the KMS condition, and to the discussion in Chapter 3 of [14] on how to interpret KMS states at  $\beta = \infty$  (zero temperature states).

**3.4. Gibbs states.** In particular, among KMS states, one has Gibbs states of the form

$$(3.5) \quad \varphi_\beta(a) = \frac{\text{Tr}(\pi(a)e^{-\beta H})}{\text{Tr}(e^{-\beta H})},$$

with  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  a representation of the algebra of observable with Hamiltonian  $H$  implementing the time evolution. Gibbs states (3.5) exist under the summability condition on the partition function

$$(3.6) \quad Z(\beta) = \text{Tr}(e^{-\beta H}) < \infty.$$

This typically happens in a certain range of inverse temperatures  $\beta > \beta_c$ .

Notice the similarity between this condition and the form (2.12) of the spectral action, under similar summability conditions.

#### 4. THE RIEMANN GAS AND TYPE III SPECTRAL TRIPLES

In this section we analyze a first example, where we compare quantum statistical mechanical systems and spectral triples, in the setting of the Bost–Connes system ([2], see also Chapter 3 of [14]) and the Riemann gas ([27], [33], [34]).

**4.1. The Bost–Connes system.** The Bost–Connes system is a quantum statistical mechanical system, introduced in [2], with the following data:

- The algebra of observables is the unital  $C^*$ -algebra  $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$ , with generators  $\mu_n$ ,  $n \in \mathbb{N}$  and  $e(r)$ ,  $r \in \mathbb{Q}/\mathbb{Z}$ , and relations  $\mu_n^* \mu_n = 1$ ;  $\mu_n \mu_m = \mu_m \mu_n$ ;  $\mu_n^* \mu_m = \mu_m \mu_n^*$  when  $(n, m) = 1$ ;  $e(0) = 1$ ;  $e(r + s) = e(r)e(s)$  and

$$(4.1) \quad \mu_n^* e(r) = \sigma_n(e(r)) \mu_n^*, \quad \mu_n e(r) = \rho_n(e(r)) \mu_n, \quad e(r) \tilde{\mu}_n = \tilde{\mu}_n \sigma_n(e(r)),$$

where  $\sigma_n(e(r)) = e(nr)$ , with  $\rho_n$  the partial inverse of  $\sigma_n$ ,

$$\rho_n(e(r)) = \frac{1}{n} \sum_{ns=r} e(s),$$

satisfying  $\sigma_n \rho_n = id$  and  $\rho_n \sigma_n = \pi_n$ , the projection given by the idempotent  $\pi_n = \mu_n \mu_n^* = n^{-1} \sum_{ns=0} e(s)$ .

- The time evolution is given by  $\sigma_t(e(r)) = e(r)$ , and  $\sigma_t(\mu_n) = n^{it} \mu_n$ , for all  $t \in \mathbb{R}$ .
- The Hilbert space  $\mathcal{H} = \ell^2(\mathbb{N})$ , with the standard orthonormal basis  $\{\epsilon_n\}_{n \in \mathbb{N}}$ .
- A Hilbert space representation, determined by a choice of an embedding  $\alpha$  of the roots of unity in  $\mathbb{C}$ , given by  $\mu_n \epsilon_m = \epsilon_{nm}$  (independently of  $\alpha$ ) and  $\pi_\alpha(e(r)) \epsilon_n = \zeta_r^n \epsilon_n$ , where  $\zeta_r = \alpha(e(r))$  is a root of unity.
- The Hamiltonian  $H_\alpha$  is independent of the choice of  $\alpha$  and given by  $H \epsilon_n = \log(n) \epsilon_n$  and the partition function  $\text{Tr}(e^{-\beta H}) = \zeta(\beta)$  is the Riemann zeta function.

A complete classification of the KMS equilibrium states of the system was given in [2]. We only recall here the fact that, in the low temperature range, namely for  $\beta > 1$ , the extremal KMS states are of the Gibbs form

$$(4.2) \quad \varphi_\beta(\pi_\alpha(e(r))) = \frac{\text{Tr}(\pi_\alpha(e(r))e^{-\beta H})}{\text{Tr}(e^{-\beta H})} = \frac{1}{\zeta(\beta)} \sum_{n \geq 1} \frac{\zeta_r^n}{n^\beta} = \frac{\text{Li}_\beta(\zeta_r)}{\zeta(\beta)},$$

where  $\text{Li}_s(z)$  is the polylogarithm function.

**Lemma 4.1.** *The Gibbs states (4.2) can be equivalently written as*

$$(4.3) \quad \varphi_\beta(\pi_\alpha(e(r))) = \sum_{n \geq 1} \frac{b_n}{n^\beta}, \quad \text{with} \quad b_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) \zeta_r^d,$$

where  $\zeta_r = \alpha(e(r))$ , and  $\mu$  is the Möbius function.

*Proof.* Using the relation

$$\sum \frac{a_n}{n^\beta} = \zeta(\beta) \sum \frac{b_n}{n^\beta}, \quad \text{with} \quad b_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d,$$

we rewrite the expression

$$\varphi_\beta(\pi_\alpha(e(r))) = \frac{1}{\zeta(\beta)} \sum \frac{\zeta_r^n}{n^\beta}$$

as in (4.3). □

This quantum statistical mechanical system was extended from the case of  $\mathbb{Q}$  to the case of an arbitrary number field in [15], [21], [24], [29], [36].



**4.2. A type III spectral triple for the Bost–Connes algebra.** Recall that the Liouville function is defined as

$$(4.4) \quad \lambda(n) = (-1)^{\Omega(n)},$$

where for an integer  $n \in \mathbb{N}$  the number  $\Omega(n)$  is the number of prime factors of  $n$ , counted with the multiplicity. It satisfies  $\lambda(nm) = \lambda(n)\lambda(m)$ , since  $\Omega$  is additive.

Consider the densely defined self-adjoint operator  $D = F|D|$  on  $\mathcal{H}_B = \ell^2(\mathbb{N})$ , where  $|D| = H$  is the Hamiltonian of the Bost–Connes system,  $H\epsilon_n = \log(n)\epsilon_n$ , and  $F = (-1)^\Omega = \lambda$  is the bounded linear operator on  $\mathcal{H}_B$  acting on the basis element  $\epsilon_n$  as multiplication by the Liouville function,  $F\epsilon_n = \lambda(n)\epsilon_n$ . Consider also the operator  $\tilde{D} = F|\tilde{D}|$  with the same sign  $F$  and with  $|\tilde{D}| = \exp H$ .

**Proposition 4.2.** *The operator  $\tilde{D}$  is finitely summable with zeta function  $\zeta_{\tilde{D}}(\beta) = Z(\beta) = \text{Tr}(e^{-\beta H}) = \zeta(\beta)$  equal to the partition function of the Bost–Connes system, and with eta function*

$$\eta_{\tilde{D}}(\beta) = \text{Tr}(F e^{-\beta H}) = \frac{\zeta(2\beta)}{\zeta(\beta)},$$

where  $\zeta(\beta)$  is the Riemann zeta function. The operator  $D$  is  $\theta$ -summable with  $\text{Tr}(F e^{-\beta|D|}) = \eta_{\tilde{D}}(\beta)$  and  $\text{Tr}(e^{-\beta|D|}) = \zeta_{\tilde{D}}(\beta)$ .

*Proof.* The result follows immediately, with the value of the eta function obtained through the well known formula

$$\sum_{n \geq 1} \lambda(n) n^{-\beta} = \frac{\zeta(2\beta)}{\zeta(\beta)},$$

where  $\lambda$  is the Liouville function. □

We also introduce the following involutive automorphism of the Bost–Connes  $C^*$ -algebra.

**Lemma 4.3.** *Let  $\mathcal{A}_{\mathbb{Q}}$  be the Bost–Connes  $C^*$ -algebra and let  $\mathcal{A}_{\mathbb{Q}}^{\text{alg}} \subset \mathcal{A}_{\mathbb{Q}}$  be the dense involutive subalgebra generated algebraically by the elements  $x \in \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$  and the isometries  $\mu_n$ . The transformation  $\sigma(a) = FaF$ , for  $a \in \mathcal{A}_{\mathbb{Q}}$  is an automorphism of  $\mathcal{A}_{\mathbb{Q}}$  preserving  $\mathcal{A}_{\mathbb{Q}}^{\text{alg}}$ . It satisfies  $\sigma^2 = 1$  and it acts as the identity on the abelian subalgebra  $C^*(\mathbb{Q}/\mathbb{Z})$ , while it acts by  $\sigma(\mu_n) = \lambda(n)\mu_n$  and  $\sigma(\mu_n^*) = \lambda(n)\mu_n^*$  on the other generators.*

*Proof.* It suffices to check that, if  $a \in C^*(\mathbb{Q}/\mathbb{Z})$ , then  $\sigma(a) = a$ , while for  $a = \mu_n$  or  $a = \mu_n^*$ ,  $\sigma(a) = \lambda(n)a$ . This is clear since, in any representation  $\pi_\alpha$ , the operators  $a \in C^*(\mathbb{Q}/\mathbb{Z})$  act diagonally on the basis  $\{\epsilon_n\}$ , hence they commute with  $F$ , while  $F\mu_n F\epsilon_\ell = \lambda(n\ell)\lambda(\ell)\epsilon_{n\ell} = \lambda(n)\mu_n\epsilon_\ell$ , and similarly in the case of  $\mu_n^*$ , where  $F\mu_n^* F\epsilon_\ell = \lambda(\ell/n)\lambda(\ell)\pi_n\epsilon_{\ell/n} = \lambda(n)^{-1}\pi_n\epsilon_{\ell/n} = \lambda(n)\mu_n^*\epsilon_\ell$ . □

Observe that the operator  $H = |D|$  has the property that the commutators  $[|D|, a]$  are bounded operators, for all  $a \in \mathcal{A}_{\mathbb{Q}}^{\text{alg}}$ . In fact,  $|D|$  commutes with elements  $a \in \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$ , while, for the generators  $\mu_n$  and their adjoints one has

$$[|D|, \mu_n]\epsilon_\ell = \log(n)\epsilon_{n\ell},$$

and similarly for  $\mu_n^*$ . However, this is no longer the case for  $D = F|D|$ , since, for instance, one has

$$[D, \mu_n]\epsilon_\ell = (\lambda(n\ell)\log(n\ell) - \lambda(\ell)\log(\ell))\epsilon_{n\ell},$$

which no longer gives a bounded operator when  $\lambda(n) = -1$ . However, we can correct this problem by twisting the commutator relation, according to the prescription described in [16] for type III  $\sigma$ -spectral triples. We obtain the following.

**Theorem 4.4.** *The data  $(\mathcal{A}_{\mathbb{Q}}^{alg}, \mathcal{H}_B, D)$  determine a  $\theta$ -summable type III  $\sigma$ -spectral triple, with respect to the automorphism  $\sigma$  of Lemma 4.3. This  $\sigma$ -spectral triple does not satisfy Lipschitz-regularity.*

*Proof.* The operator  $D$  is self-adjoint with compact resolvent by construction. To see that it is  $\theta$ -summable, notice that the convergence of the series

$$\mathrm{Tr}(e^{-\beta D^2}) = \sum_{n \geq 1} e^{-\beta(\log n)^2}$$

is controlled by the integral

$$\int_1^\infty e^{-\beta(\log x)^2} dx = \int_0^\infty e^{-\beta u^2 + u} du \leq \int_{\mathbb{R}} e^{-t^2 + \beta^{-1/2}t} dt = \sqrt{\pi} e^{1/(4\beta)}.$$

We then only need to check that the twisted commutator condition (2.14) of Definition 2.4 is satisfied. We have  $Da - \sigma(a)D = 0$  for  $a \in \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$  and

$$(D\mu_n - \sigma(\mu_n)D)\epsilon_\ell = (\lambda(n\ell) \log(n\ell) - \lambda(n)\lambda(\ell) \log(\ell))\epsilon_{n\ell} = \lambda(n\ell) \log(n)\mu_n\epsilon_\ell,$$

which is a bounded operator. One shows similarly that, for a generating monomial  $a = x\mu_n\mu_m^*$ , with  $x \in \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$  and  $n, m \in \mathbb{N}$ , the twisted commutator is bounded. Lipschitz-regularity fails because, for example,

$$(|D|\mu_n - \sigma(\mu_n)|D|)\epsilon_\ell = (\log(n\ell) - \lambda(n) \log(\ell))\epsilon_{n\ell},$$

which is not a bounded operator when  $\lambda(n) = -1$ . □

Notice that, in general, when one sets  $|D| = H$ , the convergence of the partition function of the quantum statistical mechanical system may not always imply (nor be implied by) the  $\theta$ -summable condition for the operator  $|D|$ . In fact, in general the convergence of the partition function  $\mathrm{Tr}(e^{-\beta H})$  will only happen for sufficiently low temperature  $\beta > \beta_c$ , where the critical inverse temperature  $\beta_c$  can be strictly positive, while the  $\theta$ -summable condition requires that  $\mathrm{Tr}(e^{-tD^2}) < \infty$  for all  $t > 0$ .

We also have the following simple observation.

**Proposition 4.5.** *The functional  $\varphi_\beta(a) = \mathrm{Tr}(Fae^{-\beta|D|})$  satisfies*

$$\varphi_\beta(ab) = \varphi_\beta(b\sigma_{i\beta}(\sigma(a))),$$

for all  $a, b \in \mathcal{A}_{\mathbb{Q}}^{an}$ , the analytic subalgebra of the Bost–Connes system, with  $\sigma_t$  the time evolution of the Bost–Connes system and  $\sigma$  the automorphism of Lemma 4.3.

*Proof.* For  $a \in \mathcal{A}_{\mathbb{Q}}^{an}$  we have  $e^{-\beta|D|}ae^{\beta|D|} = \sigma_{i\beta}(a)$ , the analytic continuation to the strip of width  $\beta$  of the time evolution  $\sigma_t$ . Moreover, we have  $\sigma_t \circ \sigma = \sigma \circ \sigma_t$ , since the sign  $F$  commutes with  $|D|$ . We then have

$$\varphi_\beta(ab) = \mathrm{Tr}(F ab e^{-\beta|D|}) = \mathrm{Tr}(F b(e^{-\beta|D|}F a F e^{\beta|D|}) e^{-\beta|D|}) = \varphi_\beta(b\sigma_{i\beta}(\sigma(a))).$$

□

**4.3. The Riemann gas with supersymmetry.** Recall that the Möbius function is defined as

$$(4.5) \quad \mu(n) = \begin{cases} +1 & n \text{ squarefree with an even number of prime factors} \\ -1 & n \text{ squarefree with an odd number of prime factors} \\ 0 & n \text{ not squarefree.} \end{cases}$$

The vanishing for integers that contain squares,  $p^2|n$ , can be seen, from a physical perspective, as a Pauli exclusion principle (see [33], [34]), where one thinks of the primes as fermionic particles, where no two particles of the same kind can occupy the same state.

More precisely, the *bosonic* Fock space generated by the primes, as described in [2], is the tensor product  $\mathcal{H}_B = \mathcal{S}\mathcal{H}_P$ , where  $\mathcal{H}_P = \ell^2(\mathcal{P})$ , with  $\mathcal{P}$  the set of primes, and  $\mathcal{S}\mathcal{H}_P = \bigoplus_{n \geq 0} S^n \mathcal{H}_P$  the Hilbert space completion of the sum of the symmetric powers. This can be identified with  $\mathcal{H}_B = \ell^2(\mathbb{N})$ . The *fermionic* Fock space, on the other hand, is defined as  $\mathcal{H}_F = \Lambda \mathcal{H}_P = \bigoplus_n \Lambda^n \mathcal{H}_P$ , the Hilbert space completion of the sum of the alternating powers.

The Fermionic fock space has a canonical orthonormal basis given by the elements  $\{p_1 \wedge \cdots \wedge p_n \mid p_i \in \mathcal{P}\}$ . Among these basis elements, those containing an even number of primes ( $n$  even) are bosons, in the same way in which, in physics, a helium nucleus is a boson, while the elements consisting of an odd number of primes ( $n$  odd) are fermions.

As a Hilbert space, one can identify  $\mathcal{H}_F$  with the quotient of  $\mathcal{H}_B$  by the kernel of the continuous linear functional  $\mu : \mathcal{H}_B \rightarrow \mathbb{C}$  defined on the canonical orthonormal basis  $\{\epsilon_n\}_{n \in \mathbb{N}}$  of  $\mathcal{H}_B = \ell^2(\mathbb{N})$  by the values of the Möbius function  $\epsilon_n \mapsto \mu(n)$ . The induced functional  $\bar{\mu} : \mathcal{H}_F \rightarrow \mathbb{C}$  on the quotient is a sign function  $\bar{\mu} = (-1)^F$ , which has values  $+1$  on the bosons and  $-1$  on the fermions, hence it is the sign function that implements Supersymmetry on  $\mathcal{H}_F$ , as explained in [33]. Notice that here one is implicitly using the fact that there is a natural choice of ordering of the primes, namely the order in which they occur inside the natural numbers, see [33] for explicit and more detailed comments on this point.

**4.4. A type III spectral triple for the Riemann gas.** As an example of how to relate the structures of spectral triples and quantum statistical mechanical systems, we show here how to use the data of the Bost–Connes system and construct out of them a spectral triple related to the supersymmetric Riemann gas.

By the above description of the Hilbert spaces  $\mathcal{H}_B$  and  $\mathcal{H}_F$ , we see that we can identify  $\mathcal{H}_F$  with the subspace of  $\mathcal{H}_B$  spanned by those elements  $\{\epsilon_n\}$  of the orthonormal basis of  $\mathcal{H}_B$  where  $n$  is a squarefree integer. We denote by  $\Pi_F$  the orthogonal projection of  $\mathcal{H}_B$  onto this subspace,  $\Pi_F : \mathcal{H}_B \rightarrow \mathcal{H}_F$ .

In the following, we denote by  $\mathcal{A}_{\mathbb{Q}}$  the algebra of observables of the Bost–Connes system described above. We then consider the compression of  $\mathcal{A}_{\mathbb{Q}}$  by the projection  $\Pi_F$ . By this we mean the algebra  $\mathcal{A}_{\mathbb{Q},F} \subset \mathcal{B}(\mathcal{H}_B)$  generated by the operators  $\Pi_F \pi_\alpha(a) \Pi_F$ , for all  $a \in \mathcal{A}_{\mathbb{Q}}$ . This descends by construction to an algebra  $\mathcal{A}_{\mathbb{Q},F} \subset \mathcal{B}(\mathcal{H}_F)$  acting on the fermionic Hilbert space  $\mathcal{H}_F$ .

**Theorem 4.6.** *The algebra  $\mathcal{A}_{\mathbb{Q},F}$  is isomorphic to the  $C^*$ -algebra with generators  $e(r)$ ,  $r \in \mathbb{Q}/\mathbb{Z}$ , and  $\tilde{\mu}_n$ , for  $n \in \mathbb{N}$  squarefree, where the  $e(r)$  satisfy the relations  $e(s+r) = e(s)e(r)$  and  $e(0) = 1$ , and generate  $C^*(\mathbb{Q}/\mathbb{Z})$  and the  $\tilde{\mu}_n$  satisfy  $\tilde{\mu}_n^* \tilde{\mu}_n = P_n$  and  $\tilde{\mu}_n \tilde{\mu}_n^* = Q_n$ , with  $P_n$  and  $Q_n$  projections with  $P_p + Q_p = 1$ , for  $p$  a prime, and with  $\tilde{\mu}_n \tilde{\mu}_m = \tilde{\mu}_m \tilde{\mu}_n = 0$  if  $(n, m) \neq 1$  and  $\tilde{\mu}_{nm}$  if  $(n, m) = 1$ . They also satisfy  $\tilde{\mu}_m^* \tilde{\mu}_n = \tilde{\mu}_n \tilde{\mu}_m^*$  if  $(n, m) = 1$ , and*

$$(4.6) \quad \tilde{\mu}_n^* e(r) = \sigma_n(e(r)) \tilde{\mu}_n^*, \quad \tilde{\mu}_n e(r) = \rho_n(e(r)) \tilde{\mu}_n, \quad e(r) \tilde{\mu}_n = \tilde{\mu}_n \sigma_n(e(r)),$$

with  $\sigma_n$  and  $\rho_n$  as in (4.1).

*Proof.* Elements of the algebra  $\mathcal{A}_{\mathbb{Q},F}$ , seen as a subalgebra of  $\mathcal{B}(\mathcal{H}_B)$ , are by construction of the form  $\Pi_F \pi_\alpha(a) \Pi_F$ , with  $a \in \mathcal{A}_{\mathbb{Q}}$ . The algebra  $\mathcal{A}_{\mathbb{Q}}$  is the closure of the linear span of monomials of the form  $\mu_n x \mu_m^*$ , with  $x \in C^*(\mathbb{Q}/\mathbb{Z})$  and  $n, m \in \mathbb{N}$ , where  $\mu_1 = \mu_1^* = 1$ . First observe that, in the representations  $\pi_\alpha \mathcal{A}_{\mathbb{Q}} \rightarrow \mathcal{B}(\mathcal{H}_B)$  described above, the operators  $\pi_\alpha(e(r))$  are diagonal, hence they satisfy  $\Pi_F \pi_\alpha(e(r)) = \pi_\alpha(e(r)) \Pi_F$ . The relations (4.1) for the algebra  $\mathcal{A}_{\mathbb{Q}}$  then imply that  $\Pi_F \mu_n \pi_\alpha(x) \mu_m^* \Pi_F = \Pi_F \pi_\alpha(\rho_n(x)) \mu_n \mu_m^* \Pi_F = \pi_\alpha(\rho_n(x)) \Pi_F \mu_n \mu_m^* \Pi_F$ . Thus,  $\mathcal{A}_{\mathbb{Q},F}$  is generated linearly by elements of the form  $\pi_\alpha(x) \Pi_F \mu_n \mu_m^* \Pi_F$ , with  $x \in C^*(\mathbb{Q}/\mathbb{Z})$  and  $n, m \in \mathbb{N}$ . The operator  $\Pi_F \mu_n \mu_m^* \Pi_F$  acts on a basis element  $\epsilon_\ell$  of  $\mathcal{H}_B$  as zero, unless  $\ell$  is squarefree,  $m|\ell$  (and in particular  $m$  is itself square free), and  $n\ell/m$  is squarefree. If all these conditions are met, then  $\Pi_F \mu_n \mu_m^* \Pi_F \epsilon_\ell = \epsilon_{n\ell/m}$ . Consider the operator  $\Pi_F \mu_n \Pi_F \mu_m^* \Pi_F$ . This acts on  $\epsilon_\ell$  as zero, unless  $\ell$  is squarefree,  $m|\ell$  and  $\ell/m$  is squarefree,  $(n, \ell/m) = 1$  and  $n$  is squarefree. If these conditions are met then  $\Pi_F \mu_n \Pi_F \mu_m^* \Pi_F \epsilon_\ell = \epsilon_{n\ell/m}$ . The condition that  $\ell/m$  is squarefree is automatically satisfied since we are already assuming that  $\ell$  is. The condition  $(n, \ell/m) = 1$  with  $n$  squarefree then implies that  $n\ell/m$  is also squarefree, since  $\ell/m$  is squarefree. Conversely, if we know that  $n\ell/m$  is squarefree, knowing that  $\ell$  is squarefree, it follows that also  $n$  and  $\ell/m$  are square free and it also follows that  $(n, \ell/m) = 1$ . Thus,  $\Pi_F \mu_n \mu_m^* \Pi_F = \Pi_F \mu_n \Pi_F \mu_m^* \Pi_F$ , hence  $\mathcal{A}_{\mathbb{Q},F}$  is generated linearly by elements of the form  $\pi_\alpha(x) \Pi_F \tilde{\mu}_n \tilde{\mu}_m^* \Pi_F$ , where  $\tilde{\mu}_n = \Pi_F \mu_n \Pi_F$  and  $\tilde{\mu}_m^* = \Pi_F \mu_m^* \Pi_F$ . The relations (4.1) imply that the elements  $x \in C^*(\mathbb{Q}/\mathbb{Z})$  and the generators  $\tilde{\mu}_n$  satisfy the relations (4.6), since we have

$$\begin{aligned} \tilde{\mu}_n^* \pi_\alpha(e(r)) &= \Pi_F \tilde{\mu}_n^* \Pi_F \pi_\alpha(e(r)) = \Pi_F \tilde{\mu}_n^* \pi_\alpha(e(r)) \Pi_F = \Pi_F \pi_\alpha(\sigma_n(e(r))) \mu_n^* \Pi_F \\ &= \pi_\alpha(\sigma_n(e(r))) \Pi_F \mu_n^* \Pi_F = \pi_\alpha(\sigma_n(e(r))) \tilde{\mu}_n^*, \end{aligned}$$

and similarly for the other two relations in (4.6). Moreover, we have  $\tilde{\mu}_n \tilde{\mu}_m \epsilon_\ell = 0$  unless  $nml$  is squarefree and equal to  $\epsilon_{nml}$  in that case. The condition  $nml$  squarefree is verified when so are  $\ell$ ,  $n$ , and  $m$  individually, with  $(n, \ell) = 1$ ,  $(n, m) = 1$  and  $(m, \ell) = 1$ . Thus, when  $(n, m) \neq 1$  we have  $\tilde{\mu}_n \tilde{\mu}_m = 0$  and when  $(n, m) = 1$ , we have  $\tilde{\mu}_n \tilde{\mu}_m = \tilde{\mu}_{nm} = \tilde{\mu}_m \tilde{\mu}_n$ , since the latter acts on  $\epsilon_\ell$  as zero unless  $nml$  is square free and as  $\epsilon_{nml}$  when that is the case. The case of  $\tilde{\mu}_m^* \tilde{\mu}_n = \tilde{\mu}_n \tilde{\mu}_m^*$  for  $(n, m) = 1$  is similar. Observe then that  $\tilde{\mu}_n^* \tilde{\mu}_n$  acts on  $\epsilon_\ell$  as zero unless  $\ell$  is square free and  $(n, \ell) = 1$ , in which case it acts as the identity. Thus  $P_n$  is the projection onto the subspace of  $\mathcal{H}_F$  spanned by the basis elements  $\epsilon_\ell$  with such that none of the prime factors of  $n$  divides  $\ell$ . This is the same as the projection  $\prod_{p|n} (1 - \mu_p \mu_p^*) \Pi_F$ , where the projections  $\Pi_F$  and  $\prod_{p|n} (1 - \mu_p \mu_p^*)$  commute, since the latter is an element of  $C^*(\mathbb{Q}/\mathbb{Z})$ . The element  $\tilde{\mu}_n \tilde{\mu}_n^*$  acts on  $\epsilon_\ell$  as zero unless  $\ell$  is square free and  $n$  divides  $\ell$ , in which case it is the identity. Thus, for a single prime  $p$ , we have  $Q_p = 1 - P_p$ .  $\square$

Consider then the operator  $\tilde{D} = F|\tilde{D}|$  with  $F = \mu$  the operator on  $\mathcal{H}_B$  defined by the Möbius function and  $|\tilde{D}| = \exp H$ , with  $H$  the Hamiltonian of the Bost-Connes system. Consider also the operator  $D = F|D|$ , with the same  $F$  and with  $|D| = H$ .

**Proposition 4.7.** *The operators  $F$  and  $|\tilde{D}|$  commute and the operator  $\tilde{D}$  defines a densely defined unbounded self-adjoint operator on  $\mathcal{H}_F$  with trivial kernel and with compact resolvent, which is finitely summable. The zeta function satisfies*

$$(4.7) \quad \zeta_{\tilde{D}}(\beta) = \text{Tr}_{\mathcal{H}_F} (|\tilde{D}|^{-\beta}) = \frac{\zeta(\beta)}{\zeta(2\beta)},$$

while the eta function satisfies

$$(4.8) \quad \eta_{\tilde{D}}(\beta) = \text{Tr}_{\mathcal{H}_F} (F|\tilde{D}|^{-\beta}) = \zeta(\beta)^{-1},$$

where  $\zeta(\beta)$  is the Riemann zeta function. The operator  $D$  also defines an unbounded self-adjoint operator on  $\mathcal{H}_F$ , which is  $\theta$ -summable, with  $\mathrm{Tr}_{\mathcal{H}_F}(F e^{-\beta|D|}) = \mathrm{Tr}_{\mathcal{H}_F}(F|\tilde{D}|^{-\beta})$  and equal to the Witten index.

*Proof.* We have  $F\epsilon_\ell = \mu(\ell)\epsilon_\ell$ , with  $\mu(\ell)$  the value at the integer  $\ell$  of the Möbius function, and  $|\tilde{D}|\epsilon_\ell = \ell\epsilon_\ell$ . These satisfy  $[F, |\tilde{D}|] = 0$ . Since  $\tilde{D}$  preserves  $\mathcal{H}_F$ , identified as a subspace of  $\mathcal{H}_B$  as above, it induces an unbounded self-adjoint linear operator on this space, which satisfies the compact resolvent condition, since  $(1 + \tilde{D}^2)^{-1/2}\epsilon_\ell = (1 + \ell^2)^{-1/2}\epsilon_\ell$ . The zeta function is given by

$$\zeta_{\tilde{D}}(\beta) = \mathrm{Tr}_{\mathcal{H}_F}(|\tilde{D}|^{-\beta}) = \sum_{n \geq 1} |\mu(n)| n^{-\beta} = \frac{\zeta(\beta)}{\zeta(2\beta)},$$

where the absolute value  $|\mu(n)|$  of the Möbius function is the Dirichlet inverse of the Liouville function (4.4), while, by the Möbius inversion formula, we have

$$\eta_{\tilde{D}}(\beta) = \mathrm{Tr}_{\mathcal{H}_F}(F|\tilde{D}|^{-\beta}) = \sum_{n \geq 1} \mu(n) n^{-\beta} = \frac{1}{\zeta(\beta)}.$$

The operator  $D$  is theta summable and, as observed in [33], the zeta function  $\mathrm{Tr}_{\mathcal{H}_F}(F|\tilde{D}|^{-\beta})$ , which is the same as  $\mathrm{Tr}_{\mathcal{H}_F}(F e^{-\beta|D|})$ , computes the Witten index [35]

$$(4.9) \quad \Delta = \mathrm{Tr}((-1)^F e^{-\beta H}),$$

where  $(-1)^F$  is the sign on  $\mathcal{H}_F$  that implements the supersymmetry (see also [1], [5]).  $\square$

In particular, by Proposition 4.7 the eta function (4.8) is equal to the reciprocal of the Riemann zeta function, hence poles of the eta function  $\eta_{\tilde{D}}(s)$  occur at zeros of the Riemann zeta function. This suggests interpreting the zeros as phase transitions in this model. An interpretation of zeros of zeta and  $L$ -functions as phase transitions was also recently given in [28].

We then obtain the following.

**Theorem 4.8.** *The data  $(\mathcal{A}_{\mathbb{Q},F}, \mathcal{H}_F, D)$ , with  $D$  as above, determine a  $\theta$ -summable type III spectral triple, with respect to the automorphism  $\sigma \in \mathrm{Aut}(\mathcal{A}_{\mathbb{Q},F})$  given by  $\sigma(a) = FaF$ .*

*Proof.* As a dense involutive subalgebra  $\mathcal{A}_{\mathbb{Q},F}^{alg}$  of the  $C^*$ -algebra  $\mathcal{A}_{\mathbb{Q},F}$  we can take the involutive algebra generated algebraically by the group ring  $\mathbb{C}[\mathbb{Q}/\mathbb{Z}]$  and the operators  $\tilde{\mu}_n$ , with the same relations as in Theorem 4.6. The representation of  $\mathcal{A}_{\mathbb{Q},F}$  on the Hilbert space  $\mathcal{H}_F$  is induced by the choice of a representation  $\pi_\alpha$  of the algebra  $\mathcal{A}_{\mathbb{Q}}$  on  $\mathcal{H}_B$ , together with the projection  $\Pi_F$ . The operator  $D$  satisfies  $[D, x] = 0$  for all  $x \in \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$ , as both operators are diagonal in the basis  $\epsilon_\ell$ , but we have

$$[D, \tilde{\mu}_n]\epsilon_\ell = (\mu(n\ell) \log(n\ell) - \mu(\ell) \log(\ell))\epsilon_\ell = (\mu(n)\mu(\ell) \log(n) + (\mu(n) - 1)\mu(\ell) \log(\ell))\epsilon_{n\ell},$$

where we used the fact that  $(n, \ell) = 1$  (otherwise  $\mu_n\epsilon_\ell = 0$ ) and that for coprime integers the Möbius function is multiplicative,  $\mu(n\ell) = \mu(n)\mu(\ell)$ . Thus, we see that the commutators  $[D, a]$  are not always bounded, so the data  $(\mathcal{A}_{\mathbb{Q},F}, \mathcal{H}_F, D)$  do not define an ordinary spectral triple. However, they do define a type III spectral triple, in the sense of the  $\sigma$ -spectral triples defined in [17]. In fact, consider the automorphism  $\sigma \in \mathrm{Aut}(\mathcal{A}_{\mathbb{Q},F})$  given by  $\sigma(a) = FaF$ . It is clearly an automorphism since  $F^2 = 1$  and  $F^* = F$ , and it satisfies  $\sigma^2 = 1$ . For all elements  $x \in C^*(\mathbb{Q}/\mathbb{Z})$  this gives  $\sigma(x) = x$ , while for the generators  $\tilde{\mu}_n$  one has  $\sigma(\tilde{\mu}_n)\epsilon_\ell = F\tilde{\mu}_n\mu(\ell)\epsilon_\ell = \mu(n)\mu(\ell)^2\epsilon_{n\ell}$  for  $(n, \ell) = 1$  and  $n$  and  $\ell$  squarefree, and zero otherwise, so that we have  $\sigma(\tilde{\mu}_n) = \mu(n)\tilde{\mu}_n$ , multiplication by the value of

the Möbius function at  $n$ . Similarly we have  $\sigma(\tilde{\mu}_n^*) = \mu(n)^{-1}\tilde{\mu}_n^* = \mu(n)\tilde{\mu}_n^*$ . Thus, we obtain that, for instance,

$$(D\tilde{\mu}_n - \sigma(\tilde{\mu}_n)D)\epsilon_\ell = \mu(n)\mu(\ell)\log(n)\epsilon_{n\ell},$$

(or zero if  $(n, \ell) \neq 1$  or  $n$  is not squarefree), which gives a bounded operator. Similarly, for all elements  $a \in \mathcal{A}_{\mathbb{Q}, F}^{alg}$  we obtain that  $Da - \sigma(a)D$  is bounded, using the fact that elements of this algebra are finite linear combinations of elements of the form  $x\tilde{\mu}_n\tilde{\mu}_m^*$ . The boundedness for these elements is checked as in the case of  $\tilde{\mu}_n$  above. The  $\theta$ -summable condition follows as in Theorem 4.4.  $\square$

**Remark 4.9.** The  $\sigma$ -spectral triple obtained in this way does not satisfy the Lipschitz-regularity condition of [17], because the operators  $|D|a - \sigma(a)|D|$  are not always bounded. For instance

$$(|D|\tilde{\mu}_n - \sigma(\tilde{\mu}_n)|D|)\epsilon_\ell = (\log(n) + \log(\ell)(1 - \mu(n)))\epsilon_{n\ell},$$

for  $(n, \ell) = 1$  and  $n$  and  $\ell$  squarefree.

**4.5. Eta functions and polylogarithms.** We consider here the functionals of the form

$$(4.10) \quad \varphi_\beta(a) = \text{Tr}(Fae^{-\beta|D|}) = \eta_{\alpha, \tilde{D}}(\beta),$$

for  $a \in \mathcal{A}_{\mathbb{Q}, F}$ , with  $F$  and  $D = F|D|$  as in Theorem 4.8 above and  $\tilde{D}$  as in Proposition 4.7. We write

$$\varphi_\beta(a) = \text{Tr}(Fae^{-\beta|D|}) = \sum_{n \geq 1} \mu(n)n^{-\beta} \langle \epsilon_n | \pi_\alpha(a) \epsilon_n \rangle.$$

By the previous considerations, it suffices to evaluate it on monomials of the form  $e(r)\tilde{\mu}_n\tilde{\mu}_m^*$  in  $\mathcal{A}_{\mathbb{Q}, F}$ . We see that non-trivial values occur for elements of the form  $e(r)\tilde{\mu}_n\tilde{\mu}_n^*$ . This means that we can restrict to evaluating (4.10) for  $a \in \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$ . We have

$$(4.11) \quad \varphi_\beta(e(r)) = \eta_{e(r), \tilde{D}}(\beta) = \sum_{n \geq 1} \mu(n) \frac{\zeta_r^n}{n^\beta},$$

which is a ‘‘Möbius inversion’’ of the polylogarithm function

$$(4.12) \quad \text{Li}_\beta(\zeta) = \sum_{n \geq 1} \frac{\zeta_r^n}{n^\beta}.$$

at a root of unity  $\zeta_r = \pi_\alpha(e(r))$ .

## 5. LIMIT SETS OF SCHOTTKY GROUPS AND GRAPHS: FROM SPECTRAL TRIPLES FOR QSM

We discuss in this section another example where one can relate the notions of spectral triple and of quantum statistical mechanical system. We refer here to a construction of spectral triples on the limit set of Schottky groups, used in the reconstruction result for Riemann surfaces in [20] and to a similar version for graphs constructed in [26], which in turn has applications to Mumford curves with  $p$ -adic uniformization.

**5.1. Spectral triples for limit sets and for graphs.** We recall here briefly the construction of spectral triples for limit sets of Schottky groups and for graphs, as in [20], [26]. We let  $\Lambda$  denote either the limit set in  $\mathbb{P}^1(\mathbb{C})$  of a Schottky group  $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ , or the boundary  $\partial X$  of the universal covering tree of a finite, connected, unoriented graph with Betti number  $g \geq 2$  and vertex valencies  $\geq 3$ . In the graph case we denote by  $\Gamma$  the free group of rank  $g$  acting freely on the covering tree, with quotient the graph.

We consider the unital commutative  $C^*$ -algebra  $\mathcal{A} = C(\Lambda)$ . This is an  $AF$   $C^*$ -algebra, since  $\Lambda$  is topologically a Cantor set. We also consider the involutive dense subalgebra  $\mathcal{A}^\infty = C(\Lambda, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ , the algebra of locally constant functions. The Hilbert space  $\mathcal{H}$  is the completion of  $\mathcal{A}^\infty$  (as a vector space) in the norm  $\|f\|^2 = \int_{\Lambda} |f|^2 d\mu$ , where  $\mu$  is the Patterson–Sullivan measure on  $\Lambda$ . There is a natural filtration on  $\mathcal{A}^\infty$  by  $\{\mathcal{H}_n^\infty\}_{n \geq 0}$ , where  $\mathcal{H}_n^\infty$  is the finite dimensional linear span of the characteristic functions  $1_w = 1_{\Lambda(w)}$  of the clopen subset  $\Lambda(w) \subset \Lambda$  of infinite reduced words in the generators of  $\Gamma$  that start with the finite word  $w$  of length  $\leq n$ . For the rest of this section we use the notation  $P_n : \mathcal{H} \rightarrow \mathcal{H}_n^\infty$ , the orthogonal projection. The Dirac operator is given by

$$(5.1) \quad |D| = \lambda_0 P_0 + \sum_{n \geq 1} \lambda_n Q_n,$$

where  $Q_n = P_n - P_{n-1}$  and  $\lambda_n = (\dim \mathcal{H}_n^\infty)^3$ . The operator  $|D|$  is finitely summable (in fact 1-summable).

Since the main aspect of these spectral triples considered in [20] and [26] is the associated family of zeta functions  $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$ , the Dirac operator only matters up to a sign, hence one considers only  $|D|$ . We will discuss the problem of the sign in the following subsection.

**5.2. Spectral triples and the sign problem.** The discussion presented here on the sign of the Dirac operator applies to both the spectral triples for limit sets of Schottky groups in [20] and the spectral triples of graphs of [26]. In both cases the  $C^*$ -algebra  $\mathcal{A}$  of the spectral triple is equal to the set of continuous functions on what is topologically a Cantor set.

We consider here the question of the existence of a  $\mathbb{Z}/2\mathbb{Z}$ -grading, compatible with the spectral triple, giving it an even structure. We show that, for these spectral triples, one cannot find a compatible choice of a non-trivial grading and a non-trivial sign  $F$  for the Dirac operator.

**Lemma 5.1.** *Any grading operator  $\epsilon$  for the spectral triples on limits sets of Schottky groups or on graphs is a multiplication operator by a  $\pm 1$ -valued function.*

*Proof.* We write here  $\Lambda$  for either the limit set of the Schottky group or the boundary  $\partial X$  of the covering tree of the graph.

A grading operator  $\epsilon$  compatible with the spectral triple commutes with the action of  $\mathcal{A}$  by multiplication on the Hilbert space  $\mathcal{H} = L^2(\Lambda, d\mu)$ , with  $\mu$  the Patterson–Sullivan measure. It can be easily shown that any such operator  $\epsilon$  must itself be a multiplication operator, and since  $\epsilon^2 = 1$ , that  $\epsilon$  is multiplication by a  $\pm 1$ -valued function. That is,  $\epsilon = 1_E - 1_{E^c}$ , for some measurable set  $E \subset \Lambda$ .

For a word/open set  $\Lambda(w)$ , let  $a \in C(\Lambda)$  be the continuous function  $1_w = 1_{\Lambda(w)}$  and let  $f \in L^2(\Lambda, d\mu)$  also be  $1_w$ . Then

$$\epsilon a f(x) = \epsilon(1_w)(x)$$

and

$$a \epsilon f(x) = 1_w(x) \cdot \epsilon(1_w)(x).$$

Since  $\epsilon$  commutes with  $a$  this implies that  $\epsilon(1_w)(x) = 0$  whenever  $x \notin w$ . Then it follows easily that for any two words  $w, v$ , containing  $x$ , that

$$\epsilon(1_w)(x) = \epsilon(1_v)(x).$$

Therefore we may define  $g(x)$  to be equal to  $\epsilon(1_w)(x)$  for any word  $w$  containing  $x$ . Now let  $v$  be any finite word. Given  $x \in \Lambda$ , either  $x$  is in  $v$  or it is not. In either case, one can see that  $\epsilon(1_v)(x) = g(x)1_v(x)$ . Since the linear span of the characteristic functions  $1_w$  are dense in  $\mathcal{H}$ , this proves that  $\epsilon$  is equal to multiplication by  $g(x)$ . Since  $\epsilon^2 = \text{id}$ , we see that  $g(x)$  must be  $\pm 1$ -valued.  $\square$

With respect to this grading, the  $\mathbb{Z}/2\mathbb{Z}$ -decomposition of  $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$  is given by

$$\mathcal{H}^+ = L^2(E, d\mu)$$

$$\mathcal{H}^- = L^2(E^c, d\mu)$$

We then show that neither  $E$  nor  $E^c$  contains an open set of  $\Lambda$ .

**Lemma 5.2.** *Suppose  $\epsilon = 1_E - 1_{E^c}$  is a grading defining the even structure on the spectral triple. For no open set  $U \subset \Lambda$  does  $E$  or  $E^c$  contain almost every point of  $U$ .*

*Proof.* We may restrict our attention to the open sets corresponding to words  $w$ , of finite length in the generators, as these open sets form a basis for the topology. We denote these open sets by  $\Lambda(w)$ . Suppose  $E$  contained almost every point of the open set,  $\Lambda(w)$ . Now, for even spectral triples, given an idempotent  $e \in \mathcal{A}$ ,  $eP : e\mathcal{H}^+ \rightarrow e\mathcal{H}^-$  defines a Fredholm operator, where  $P$  denotes the restriction of the sign operator  $F$  to  $\mathcal{H}^+$ .

Let  $e$  be the characteristic function  $1_w = 1_{\Lambda w}$ , which is an idempotent in  $\mathcal{A}$ . Then  $e\mathcal{H}^- = \{0\}$ , and therefore,  $eP$  has infinite-dimensional kernel which contradicts the fact that it is Fredholm. Similarly if  $E^c$  contains almost every point of the open set,  $w$ , then  $e\mathcal{H}^+ = \{0\}$ , and so  $eP$  would have infinite-dimensional cokernel which is again a contradiction.  $\square$

As we recalled above, the Dirac operator, in both the Schottky group and the graph spectral triple, is given by the formula

$$|D| = \sum \lambda_n Q_n,$$

where  $Q_n$  is the orthogonal projection onto the finite-dimensional subspace  $\mathcal{A}_n \ominus \mathcal{A}_{n-1}$ , with  $Q_0 = P_0$ . The following is an easy yet important observation for our discussion

**Lemma 5.3.** *A function  $\psi \in \mathcal{H}$  is an eigenvector for  $|D|$  if and only if  $\psi$  is in the range of  $Q_n$  for some  $n \in \mathbb{N}$ . A vector,  $\psi$ , is an eigenvector of  $D$ , (whatever the sign  $F$  may be) implies that  $\psi$  is an eigenvector of  $|D|$  and hence is in the range of  $Q_n$  for some  $n$ . In particular, each eigenvector of  $D$  is locally constant.*

Now we arrive at the main conclusion.

**Theorem 5.4.** *There is no choice of grading  $\epsilon$  and non-trivial sign  $F$  giving the graph spectral triple an even structure.*

*Proof.* Let  $\psi$  be an eigenvector of  $D$  with eigenvalue  $\lambda$ . Then since  $D$  anticommutes with  $\epsilon$ ,  $\epsilon\psi$  is an eigenvector with eigenvalue  $-\lambda$ . However, for no  $\psi \neq 0$  in the range of  $Q_n$  is  $\epsilon\psi$  locally constant,



which contradicts lemma 5.3. The easiest way to see this is to look at a basis vector for the range of  $Q_n$ ,

$$\psi = c_k \mathbf{1}_{w_1} - c_1 \mathbf{1}_{w_k},$$

where  $w_1, w_k$  are distinct words of length  $n$  and hence are disjoint. Then

$$\epsilon\psi = c_k \mathbf{1}_{w_1 \cap E} - c_1 \mathbf{1}_{w_k \cap E} + c_1 \mathbf{1}_{w_k \cap E^c} - c_k \mathbf{1}_{w_1 \cap E^c}$$

By lemma 5.2,  $\epsilon\psi$  is not locally constant on any neighborhood of any point contained inside  $w_k$ . This argument holds for any non-zero linear combination of such basis vectors and therefore we have shown that  $\epsilon$  sends every non-zero vector in the range of  $Q_n$  outside of  $Q_n$ .  $\square$

This has the following immediate consequence.

**Corollary 5.5.** *For no choice of sign  $F$  is the Chern character of the graph spectral triple non-trivial.*

*Proof.* All  $AF$   $C^*$ -algebras have trivial  $K_1$ -group. Therefore, for there to be any hope of obtaining a non-trivial Chern character for the spectral triple, one must exhibit a  $\mathbb{Z}/2\mathbb{Z}$ -grading compatible with the spectral triple, giving it an even structure, so that its Chern character pairs with  $K_0(A)$ , but Theorem 5.4 shows that there is no non-trivial compatible choice of a  $\mathbb{Z}/2\mathbb{Z}$ -grading and a sign  $F$  for the spectral triple.  $\square$

**5.3. From spectral geometry to quantum statistical system.** In comparing the structures of spectral triple and of quantum statistical mechanical system recalled at the beginning of the paper, it seems natural to expect that the operator  $|D|$  of the spectral triple should be related to the Hamiltonian generating the time evolution in the quantum statistical mechanical system. We have seen this, in the reverse direction, in the case of the Riemann gas in the previous section, where we used  $|D| = H$  to obtain a  $\theta$  summable  $\sigma$ -spectral triple from a quantum statistical mechanical system. Here we start, instead, from the operator  $|D|$  of the spectral triple. If we follow the same intuition, we would expect to be able to take  $|D|$  itself as generator of the time evolution. However, this requires enlarging the algebra  $\mathcal{A}$  of the spectral triple, as explained below.

**5.4. Enlarging the algebra and time evolution.** Given a spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  as above, let  $\mathcal{A}_D$  be the  $C^*$ -subalgebra of  $\mathcal{H}$  generated by the elements of the  $C^*$ -algebra  $\mathcal{A}$  and the spectral projections of the Dirac operator  $D$ .

**Lemma 5.6.** *For a spectral triple  $(\mathcal{A}, \mathcal{H}, D)$ , where  $D$  has trivial kernel, the time one-parameter family of automorphisms  $\sigma_t(a) = |D|^{it} a |D|^{-it}$  define a time evolution on the algebra  $\mathcal{A}_D$ . This time evolution is by inner automorphisms.*

*Proof.* By the spectral theorem for self-adjoint operators, for every  $t \in \mathbb{R}$ , the unitary operator  $|D|^{it}$  is a bounded operator contained in the  $C^*$ -algebra generated by the spectral projections of  $D$ . Thus, for any  $a \in \mathcal{A}_D$ , the element  $|D|^{it} a |D|^{-it}$  is again in  $\mathcal{A}_D$  and  $t \mapsto \sigma_t(a)$  defines a time evolution by inner automorphisms.  $\square$

In the case of the spectral triples of [20], [26], the algebra  $\mathcal{A}_D$  is generated by the functions  $f \in C(\Lambda)$  and the projections  $Q_n$ . Passing from the algebra  $\mathcal{A} = C(\Lambda)$  of the spectral triple to the algebras  $\mathcal{A}_D$  ensures that the time evolution  $\sigma_t(a) = |D|^{it} a |D|^{-it}$  maps the algebra to itself.

Let  $I_m$  denote an inductively constructed orthonormal basis of  $\mathcal{H}_m^\infty \ominus \mathcal{H}_{m-1}^\infty$ , obtained by Gram-Schmidt orthonormalization. Elements  $\phi \in I_m$  are by construction elements in  $\mathcal{A} = C(\Lambda)$ .

**Proposition 5.7.** *In the case of the spectral triples of limit sets of Schottky groups and of graphs, the time evolution  $\sigma_t(a) = |D|^{it} a |D|^{-it}$  on  $\mathcal{A}_D$  is given explicitly by*

$$(5.2) \quad \sigma_t(a)\psi = \sum_{m \leq \max\{n, \ell\}} \left(\frac{\lambda_m}{\lambda_n}\right)^{it} Q_m \phi \cdot \psi,$$

for elements  $a = \phi \in I_n - I_{n-1}$  and  $\psi \in I_\ell - I_{\ell-1}$ , with  $\lambda_n = (\dim \mathcal{H}_n^\infty)^3$ .

*Proof.* The time evolution is determined by its values on elements of the form  $a = \phi \in I_n - I_{n-1}$  for all  $n \geq 0$ . In turn  $\sigma_t(\phi)$  is determined by its values on the orthonormal basis of elements in  $I_n - I_{n-1}$  for  $n \geq 0$ . For such elements we have

$$\begin{aligned} \sigma_t(\phi)\psi &= |D|^{it} \phi |D|^{-it} \psi = \lambda_n^{-it} |D|^{it} \phi \psi \\ &= \sum_{m \leq \max\{n, \ell\}} \lambda_m^{it} \lambda_n^{-it} \sum_{\eta \in I_m - I_{m-1}} \langle \eta, \phi \psi \rangle \eta = \sum_{m \leq \max\{n, \ell\}} \left(\frac{\lambda_m}{\lambda_n}\right)^{it} Q_m \phi \psi. \end{aligned}$$

□

Thus, one obtains a quantum statistical mechanical system where the algebra of observables is  $\mathcal{A}_D$ , the Hilbert space is the same as for the spectral triple, and the Hamiltonian is  $H = \log |D|$ . However, the fact that the time evolution constructed in this way is inner is not a desirable feature: we will describe a different relation between spectral triples and quantum statistical mechanical systems for limit sets and graphs that avoids this problem in §6 below.

**5.5. Partition function, zeta functions and Gibbs states.** The quantum statistical mechanical system constructed above from the data of a finitely summable spectral triple identifies zeta functions and Gibbs states in the following way.

**Proposition 5.8.** *Let  $(\mathcal{A}, \mathcal{H}, D)$  be the finitely summable spectral triple for limit sets of Schottky groups or for graphs, as described above, and let  $(\mathcal{A}_D, \mathcal{H}, H = |D|)$  be the quantum statistical mechanical system obtained as in Lemma 5.6. The partition function of the QSM system is equal to the zeta function of the Dirac operator of the spectral triple and, more generally, the family of zeta functions  $\zeta_{a,D}(\beta)$ , normalized by  $\zeta_D(\beta)$ , correspond to the values  $\varphi_\beta(a)$  of a Gibbs KMS state at inverse temperature  $\beta$ .*

*Proof.* This is a direct consequence of the identification  $H = \log |D|$ , with

$$\varphi_\beta(a) = \frac{\text{Tr}(a e^{-\beta H})}{\text{Tr}(e^{-\beta H})} = \frac{\text{Tr}(a |D|^{-\beta})}{\text{Tr}(|D|^{-\beta})} = \frac{\zeta_{a,D}(\beta)}{\zeta_D(\beta)}.$$

□

## 6. LIMIT SETS OF SCHOTTKY GROUPS AND GRAPHS: FROM QSM TO SPECTRAL TRIPLES

We present here a different way of relating spectral triples and quantum statistical mechanical systems for limit sets of Schottky groups and for graphs, which works more similarly to the case of the Riemann gas discussed earlier in the paper, namely by proceeding from quantum statistical mechanical systems to (type III) spectral triples.

**6.1. Quantum statistical mechanical systems for Schottky groups and graphs.** We recall from [22] that one can construct quantum statistical mechanical systems associated to graphs. A very similar construction can be done in the case of limit sets of Schottky groups as we describe briefly here (see also [18], [31]).

The  $C^*$ -algebra of observables of the quantum statistical mechanical system is in this case the noncommutative algebra  $\mathcal{A} = C(\Lambda) \rtimes \Gamma$ , where  $\Lambda$  is either the limit set in  $P^1(\mathbb{C})$  of the Schottky group  $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ , or else the boundary  $\Lambda = \partial T_X$  of the universal covering tree  $T_X$  of the graph  $X$ , acted upon by the free group  $\Gamma$ , the fundamental group of the graph. Every element in the algebra can be obtained as a limit of elements of the form  $\sum_{\gamma} f_{\gamma} U_{\gamma}$ , with  $f_{\gamma} \in C(\Lambda)$  and  $U_{\gamma}$  the unitary operators corresponding to the group elements  $\gamma \in \Gamma$ .

In the case of the Schottky group, let  $T = T_{\Gamma}$  denote the Cayley graph of the Schottky group, embedded in the hyperbolic space  $\mathbb{H}^3$ . Then the Schottky group  $\Gamma$  acts freely on its Cayley graph and the limit set  $\Lambda$  of  $\Gamma$  is also the boundary  $\Lambda = \partial T_{\Gamma}$  of the tree  $T_{\Gamma}$ , mirroring what happens in the graph case.

The time evolution is constructed in [22] as the one-parameter family of automorphisms of  $\mathcal{A}$  given by

$$(6.1) \quad \sigma_t \left( \sum_{\gamma} f_{\gamma}(\xi) U_{\gamma} \right) = \sum_{\gamma} e^{it B(x_0, \gamma x_0, \xi)} f_{\gamma}(\xi) U_{\gamma},$$

where the points  $x, y$  are vertices on the tree  $T_X$  of the graph (respectively the Cayley graph  $T_{\Gamma}$ ), with  $x_0$  a base point, and  $B(x_0, \gamma x_0, \xi)$  is the Busemann function

$$(6.2) \quad B(x_0, \gamma x_0, \xi) = \lim_{x \rightarrow \xi} d(x_0, x) - d(\gamma x_0, x).$$

**6.2. KMS state and Patterson–Sullivan measure.** It is also known (see [22] and [18], [31]) that the quantum statistical mechanical system with algebra  $C(\Lambda) \rtimes \Gamma$  and time evolution (6.1) has a unique inverse temperature  $\beta = \delta(\Lambda)$ , which is the critical exponent for the Poincaré series of  $\Gamma$  (the Hausdorff dimension of the limit set  $\Lambda$ ), for which the set of  $\mathrm{KMS}_{\beta}$  states is non-empty. This set consists of a unique KMS state, given by integration with respect to the normalized Patterson–Sullivan measure  $\mu_{x_0}$

$$(6.3) \quad \varphi_{\beta, x_0} \left( \sum_{\gamma} f_{\gamma}(\xi) U_{\gamma} \right) = \int_{\Lambda} f_1(\xi) d\mu_{x_0}(\xi).$$

**6.3. Hamiltonian and time evolution.** We reformulate the time evolution (6.1) in terms of an explicit covariant realization given by a Hilbert space representation of the algebra of observables and a Hamiltonian.

**Theorem 6.1.** *The  $C^*$ -algebra  $\mathcal{A} = C(\Lambda) \rtimes \Gamma$  acts by bounded operators on the Hilbert space  $\mathcal{H} = L^2(\Lambda, d\mu) \otimes \ell^2(\Gamma)$ , where  $\mu = \mu_{x_0}$  is the Patterson–Sullivan measure, with the action given by*

$$(6.4) \quad f U_{\gamma} h \otimes \gamma' = f v_{\gamma}(h) \otimes \gamma \gamma',$$

where  $v$  denotes the action of  $\Gamma$  on  $L^2(\Lambda, \mu)$  induced by the action of  $\Gamma$  on  $\Lambda$ . In this representation, the time evolution is implemented by the Hamiltonian

$$(6.5) \quad H h(\xi) \otimes \gamma = B(x_0, \gamma x_0, \xi) h(\xi) \otimes \gamma.$$

*Proof.* First we check that (6.4) indeed determines a representation of  $\mathcal{A}$  on the Hilbert space  $\mathcal{H}$ . We have

$$f_1 U_{\gamma_1} f_2 U_{\gamma_2} h \otimes \gamma' = f_1 U_{\gamma_1} f_2 v_{\gamma_2}(h) \otimes \gamma_2 \gamma' = f_1 v_{\gamma_1}(f_2 v_{\gamma_2}(h)) \otimes \gamma_1 \gamma_2 \gamma',$$

where the action  $v_\gamma$  is given by

$$v_\gamma(h)(\xi) = \rho_{\gamma,\mu}^{-1/2} h(\gamma^{-1}\xi) = \rho_{\gamma,\mu}^{-1/2} \alpha_\gamma(h),$$

where  $\rho_\gamma(\mu)$  is the Radon–Nikodym derivative

$$\rho_{\gamma,\mu} = \frac{d\gamma^*\mu}{d\mu},$$

and we use the notation  $\alpha_\gamma(h) = h(\gamma^{-1}\xi)$ . Thus we obtain

$$\begin{aligned} f_1 U_{\gamma_1} f_2 U_{\gamma_2} h \otimes \gamma' &= f_1 v_{\gamma_1}(f_2 v_{\gamma_2}(h)) \otimes \gamma_1 \gamma_2 \gamma' = f_1 \alpha_{\gamma_1}(f_2) \rho_{\gamma_1 \gamma_2, \mu}^{-1/2} \alpha_{\gamma_1 \gamma_2}(h) \otimes \gamma_1 \gamma_2 \gamma' \\ &= f_1 \alpha_{\gamma_1}(f_2) v_{\gamma_1 \gamma_2}(h) \otimes \gamma_1 \gamma_2 \gamma' = f_1 \alpha_{\gamma_1}(f_2) U_{\gamma_1 \gamma_2} h \otimes \gamma', \end{aligned}$$

where we have used the fact that, for all  $\gamma, \tilde{\gamma} \in \Gamma$ ,

$$\frac{d\tilde{\gamma}^*\mu}{d\gamma^*\mu} \frac{d\gamma^*\mu}{d\mu} = \frac{d\tilde{\gamma}^*\mu}{d\mu}.$$

We consider then the operators  $e^{itH} f U_\gamma e^{-itH}$ , with  $H$  as in (6.5). These act on elements  $h \otimes \gamma'$  of the Hilbert space  $\mathcal{H}$  as

$$\begin{aligned} e^{itH} f U_\gamma e^{-itH} h \otimes \gamma' &= e^{itH} f v_\gamma(e^{-itB(x_0, \gamma'x_0, \xi)} h(\xi)) \otimes \gamma' \\ &= e^{itH} f(\xi) e^{-itB(x_0, \gamma'x_0, \gamma^{-1}\xi)} v_\gamma(h)(\xi) \otimes \gamma' = e^{it(B(x_0, \gamma\gamma'x_0, \xi) - B(x_0, \gamma'x_0, \gamma^{-1}\xi))} f(\xi) v_\gamma(h)(\xi) \otimes \gamma'. \end{aligned}$$

We then use the fact that the Busemann function satisfies

$$B(x, y, \xi) = -B(y, x, \xi), \quad B(x, y, \xi) + B(y, z, \xi) = B(x, z, \xi), \quad B(\gamma x, \gamma y, \gamma \xi) = B(x, y, \xi)$$

to transform the above into

$$\begin{aligned} e^{itH} f U_\gamma e^{-itH} h \otimes \gamma' &= e^{it(B(x_0, \gamma\gamma'x_0, \xi) + B(\gamma\gamma'x_0, \gamma x_0, \xi))} f U_\gamma h \otimes \gamma' \\ &= e^{itB(x_0, \gamma x_0, \xi)} f U_\gamma h \otimes \gamma' = \sigma_t(f U_\gamma) h \otimes \gamma'. \end{aligned}$$

□

**6.4. A type III spectral triple for limit sets and graphs.** We first introduce a suitable sign operator  $F$  for the Dirac operator of the spectral triple. We proceed by analogy to what we did in the case of the Riemann gas and we consider an operator of the form

$$(6.6) \quad F h \otimes \gamma = (-1)^{\ell(\gamma)} h \otimes \gamma,$$

where  $\ell : \Gamma \rightarrow \mathbb{Z}$  is a group homomorphism.

**Lemma 6.2.** *The operator  $F$  of (6.6) is a sign operator for  $D = F|D|$  with  $|D| = H$  of (6.5).*

*Proof.* This follows by checking that  $[|D|, F] = 0$ . □

We now show that the quantum statistical mechanical system described above determines a type III  $\sigma$ -spectral triple, in the sense of [17].

**Theorem 6.3.** *Consider the automorphism  $\sigma(a) = FaF$  of the algebra  $\mathcal{A} = C(\Lambda) \rtimes \Gamma$ , with  $F$  as in (6.6), and the representation of  $\mathcal{A}$  on  $\mathcal{H} = L^2(\Lambda, d\mu) \otimes \ell^2(\Gamma)$  as in Theorem 6.1 above. We also consider the dense subalgebra  $\mathcal{A}^\infty = \mathcal{A}^{alg} = C(\Lambda) \rtimes^{alg} \Gamma$ , algebraically generated by functions in  $C(\Lambda)$  and the unitaries  $U_\gamma$ ,  $\gamma \in \Gamma$ . We set  $D = F|D|$  with  $F$  as in (6.6) and  $|D| = H$ , as in (6.5). Then the data  $(\mathcal{A}^\infty, \mathcal{H}, D)$  define a  $\sigma$ -spectral triple.*

*Proof.* The map  $a \mapsto \sigma(a)$  is an automorphism since  $\ell : \Gamma \rightarrow \mathbb{Z}$  is a group automorphism. In fact,  $\sigma(a) = FaF$  acts as

$$\sigma(fU_\gamma) h \otimes \gamma' = F fU_\gamma F h \otimes \gamma' = (-1)^{\ell(\gamma')} (-1)^{\ell(\gamma\gamma')} fU_\gamma h \otimes \gamma',$$

where  $\ell(\gamma\gamma') = \ell(\gamma) + \ell(\gamma')$ , so that we have  $\sigma(ab) = \sigma(a)\sigma(b)$  and  $\sigma(f) = f$  for  $f \in C(\Lambda)$  and  $\sigma(U_\gamma) = (-1)^{\ell(\gamma)} U_\gamma$ . Elements in the dense involutive subalgebra  $\mathcal{A}^\infty = \mathcal{A}^{alg}$  are finite sums  $\sum_\gamma f_\gamma U_\gamma$ . Thus, it suffices to check the commutator properties for monomials of the form  $fU_\gamma$ . We have, for  $|D| = H$ ,

$$(|D| fU_\gamma - fU_\gamma |D|) h \otimes \gamma' = (B(x_0, \gamma\gamma'x_0, \xi) - B(x_0, \gamma'x_0, \gamma^{-1}\xi)) fU_\gamma h \otimes \gamma' = B(x_0, \gamma x_0, \xi) fU_\gamma h \otimes \gamma',$$

while for  $D = F|D|$  we have

$$(D fU_\gamma - fU_\gamma D) h \otimes \gamma' = (-1)^{\ell(\gamma')} ((-1)^{\ell(\gamma)} B(x_0, \gamma\gamma'x_0, \xi) - B(x_0, \gamma'x_0, \gamma^{-1}\xi)) fU_\gamma h \otimes \gamma'.$$

The twisted commutator satisfies

$$(D fU_\gamma - \sigma(fU_\gamma) D) h \otimes \gamma' = (-1)^{\ell(\gamma\gamma')} B(x_0, \gamma x_0, \xi) fU_\gamma h \otimes \gamma'.$$

□

Notice that, for the purpose of reconstruction of graphs from the associated quantum statistical mechanical systems, in [22] one considers, instead of  $\mathcal{A}^{alg}$ , a non-involutive subalgebra of  $\mathcal{A}$ , as in the case of the quantum statistical mechanical systems of number fields in [21].

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