Structured singular value with repeated scalar blocks

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1 Introduction

The structured singular value, μ, is an important linear algebra tool to study a class of matrix perturbation problems, [Doy]. It is useful for analyzing the robustness of stability and performance of dynamical systems [DoyWS]. This paper studies uncertainty structures involving repeated scalar parameters in more detail than in [Doy]. In [DoyP], it was shown that the frequency domain μ tests of [DoyWS] can conceptually be reduced to a **single** constant matrix μ test, but the uncertainty structure must be augmented with a large *repeated scalar block*. This paper studies the properties of μ and the upper bound with these types of uncertainty blocks, and compares the frequency domain .vs. state space μ based tests, assuming that the upper bound is what can be *reliably computed*.

2 Definitions

This section is devoted to defining the structured singular value, a matrix function denoted by $\mu(\cdot)$. We consider matrices $M \in \mathbf{C}^{n\times n}$. In the definition of $\mu(M)$, there is an underlying structure Δ, (a prescribed set of block diagonal matrices) on which everything depends. For each problem, this structure is in general different; it depends on the **uncertainty** and **performance objectives** of the problem. Defining the structure involves specifying three things; the type of each block, the total number of blocks, and their dimensions. We consider two types of blocks-*repeated scalar* and *full* blocks. Two nonnegative integers, s and f, represent the number of *repeated scalar* blocks and the number of *full* blocks, respectively. To bookkeep their dimensions, we introduce positive integers $r_1, \ldots, r_s$; $m_1, \ldots, m_f$. The i'th repeated scalar block is $r_i \times r_i$, while the j'th full block is $m_j \times m_j$. For consistency among all the dimensions, we must have $\sum_{i=1}^{s} r_i + \sum_{j=1}^{f} m_j = n$. With those integers given, define $\Delta \subset \mathbf{C}^{n\times n}$ as

$$\Delta = \{\text{diag}\ [\delta_1 I_{r_1}, \ldots, \delta_s I_{r_s}, \Delta_1, \ldots, \Delta_f] : \delta_i \in \mathbf{C}, \Delta_j \in \mathbf{C}^{m_j \times m_j}\} \tag{2.1}$$

Often, we will need norm bounded subsets of Δ, and we use the following notation, $\mathbf{B}\Delta = \{\Delta \in \Delta : \bar{\sigma}(\Delta) \leq 1\}$.

Definition 2.1 *For $M \in \mathbf{C}^{n\times n}$, $\mu_\Delta(M)$ is defined*

$$\mu_\Delta(M) := \frac{1}{\min\limits_{\Delta\in\Delta} \{\bar{\sigma}(\Delta) : \det(I + M\Delta) = 0\}} \tag{2.2}$$

unless no $\Delta \in \Delta$ makes $I + M\Delta$ singular, then $\mu_\Delta(M) = 0$.

An alternative expression follows almost immediately from the definition. $\rho(\cdot)$ denotes spectral radius. In view of this lemma, continuity of the function $\mu : \mathbf{C}^{n\times n} \to \mathbf{R}$ is apparent.

Lemma 2.2 $\mu_\Delta(M) = \max\limits_{\Delta\in\mathbf{B}\Delta} \rho(M\Delta)$

We can easily calculate $\mu_\Delta(M)$ when Δ is one of two extreme sets. If $\Delta = \{\delta I : \delta \in \mathbf{C}\}$, then $\mu_\Delta(M) = \rho(M)$. If $\Delta = \mathbf{C}^{n\times n}$, then $\mu_\Delta(M) = \bar{\sigma}(M)$. For a general Δ as in (2.1), $\{\delta I : \delta \in \mathbf{C}\} \subset \Delta \subset \mathbf{C}^{n\times n}$. Hence $\rho(M) \leq \mu_\Delta(M) \leq \bar{\sigma}(M)$. We refine these bounds by considering transformations on M that **do not affect** $\mu_\Delta(M)$, but **do affect** ρ and $\bar{\sigma}$. Define the following three subsets of $\mathbf{C}^{n\times n}$

$$\mathcal{Q} = \{Q \in \Delta : Q^* Q = I_n\} \tag{2.3}$$

$$\mathcal{D} = \left\{\text{diag}\ \left[D_1, \ldots, D_s, d_1 I_{m_1}, \ldots, d_{f-1} I_{m_{f-1}}, 0_{m_f}\right] : D_i \in \mathbf{C}^{r_i \times r_i}, D_i = D_i^*, d_j \in \mathbf{R}\right\} \tag{2.4}$$

and

$$\mathcal{D}_e = \left\{e^D : D \in \mathcal{D}\right\} \tag{2.5}$$

For all $Q \in \mathcal{Q}$ and $D \in \mathcal{D}_e$ $\mu_\Delta(MQ) = \mu_\Delta(QM) = \mu_\Delta(M) = \mu_\Delta(DMD^{-1})$. Therefore, the bounds can be tightened to

$$\max_{Q\in\mathcal{Q}} \rho(QM) \leq \mu_\Delta(M) \leq \inf_{D\in\mathcal{D}_e} \bar{\sigma}(DMD^{-1}) \tag{2.6}$$

A main result of [Doy] is that the lower bound, $\max_{Q\in\mathcal{Q}} \rho(QM)$, is always equal to $\mu_\Delta(M)$. Unfortunately, the function $l(Q) := \rho(QM)$ has local maxima which are **not** global, and computing the global maximum of such functions is, in general, impossible. In contrast to the local phenomena described above, the function $u(D) := \bar{\sigma}(DMD^{-1})$ does not have any local minima which are not global, so computing $\inf_{D\in\mathcal{D}_e} \bar{\sigma}\left(DMD^{-1}\right)$ is a reasonable task. In general though, $\mu_\Delta(M) < \inf_{D\in\mathcal{D}_e} \bar{\sigma}\left(DMD^{-1}\right)$. For certain block structures Δ, equality always holds. The situation is summarized below. The columns are the number of *full blocks*, while the rows are the number of *repeated scalar* blocks.

	0	1	2	3	4
0	unclear	yes easy	yes [Red]	yes [Doy]	no [MorD]
1	yes $\inf = \mu$	yes Sec. 5	no [Pac]	no	no
2	no Sec. 8	no	no	no	no

When is the upper bound, $\inf_{D\in\mathcal{D}_e} \bar{\sigma}(DMD^{-1})$, **always** equal to μ ?

The purpose of this paper is a careful study of the upper bound: its computational properties, and the relation between μ and the upper bound.

3 Facts

The next result is from [SafD] and [ChuD].

Theorem 3.1 *The function $f : \mathcal{D} \to \mathbf{R}, f(D) := \bar{\sigma}\left(e^D M e^{-D}\right)$ is convex.*

Hence, the upper bound does not have any local minimums which are not global. Therefore, steepest descent methods can be used to compute the upper bound. We calculate the first derivatives of singular values of $e^{Dt}Me^{-Dt}$ for given D in $\mathcal{D}$. The resulting formula will be used in section 4.1 to find a $D \in \mathcal{D}$ such that for $t > 0$, sufficiently small, $\bar{\sigma}\left(e^{D}Me^{-D}\right) < \bar{\sigma}(M)$, in other words, a descent direction for $\bar{\sigma}$. In general, the minimization for the upper bound will drive the top singular values together, since we are minimizing a "max" function. Therefore, the derivative calculations must be carried out for coalesced singular values.

A result from perturbation theory, ([Kat] for the theory, [FreLC] and [Doy] for this application) is that the eigenvalues of an analytic hermitian matrix are analytic, and there is a choice of orthogonal analytic eigenvectors as well. This gives:

Theorem 3.2 *Suppose $W(t)$ is of the form $e^{Dt}Me^{-Dt}$ where $D \in \mathcal{D}$ and M is given. Obviously $W(0) = M$ and $\dot{W}(0) = DM - MD$. Let*

$$W(0) = M = \sigma U_1 V_1^* + U_2 \Sigma_2 V_2^* \tag{3.1}$$

be a singular value decomposition of M; $U_1, V_1 \in \mathbf{C}^{n\times r}, U_2, V_2 \in \mathbf{C}^{n\times(n-r)}, U_1^*U_1 = V_1^*V_1 = I_r, U_2^*U_2 = V_2^*V_2 = I_{n-r}$, and $\Sigma_2 \in \mathbf{R}^{(n-r)\times(n-r)}$ is nonnegative, diagonal, and none of its diagonal entries are equal to σ. If $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the eigenvalues of $U_1^*DU_1 - V_1^*DV_1$, then for nonzero values of t, the r singular values that were σ at $t = 0$ satisfy

$$\sigma_i(t) = \sigma(1 + \lambda_i t) + g_i(t) \tag{3.2}$$

where $\lim_{t\to 0} \frac{g_i(t)}{t} = 0$.

If we can find a $D \in \mathcal{D}$ **with all the eigenvalues of** $U_1^*DU_1 - V_1^*DV_1$ **negative**, then by moving a small amount in that direction, all of the singular values in the cluster will be reduced.

4 Upper bound and μ

4.1 Finding descent directions

Our problem of finding a $D \in \mathcal{D}$ such that all the eigenvalues of $U^*DU - V^*DV$ are positive can be solved using convexity ideas [Roc]. The motivation comes from [Doy], though this section generalizes the results there. Consider square matrices, $\mathbf{C}^{n\times n}$, and a compatible block structure $\mathbf{\Delta}$, with integers $r_1, \ldots, r_s$, $m_1, \ldots, m_f$ defining the dimensions of the blocks, as outlined in section 2. Define $\mathbf{X}$ to be the following set of block diagonal, hermitian matrices:

$$\mathbf{X} := \{\ \text{diag}\ [Z_1, \ldots, Z_s, z_1, \ldots, z_{f-1}] : \\ Z_i = Z_i^* \in \mathbf{C}^{r_i\times r_i}, z_j \in \mathbf{R}\} \tag{4.1}$$

This is a real inner product space with inner product defined by $P, T \in \mathbf{X}, \langle P, T\rangle := \text{tr}(PT)$. Recall the definition for $\mathcal{D}$ in (2.4). Let $D \in \mathcal{D}$ be given. Then D looks like $\text{diag}\left[D_1, \ldots, D_s, d_1 I_{m_1}, \ldots, d_{f-1} I_{m_{f-1}}, 0_{m_f}\right]$, $D = D^*$. Associate to this $D \in \mathcal{D}$, a $\tilde{D} \in \mathbf{X}$ by setting

$$\tilde{D} = \text{diag}\ [D_1, \ldots, D_s, d_1, \ldots, d_{f-1}] \tag{4.2}$$

Now, let $M \in \mathbf{C}^{n\times n}$ be given. If the maximum singular value of M, $\bar{\sigma}$, has multiplicity equal to r, then M is

$$M = \bar{\sigma} UV^* + U_2 \Sigma_2 V_2^* \tag{4.3}$$

where $U, V \in \mathbf{C}^{n\times r}$, $U^*U = V^*V = I_r$, $U_2, V_2 \in \mathbf{C}^{n\times(n-r)}$, $U_2^*U_2 = V_2^*V_2 = I_{(n-r)}$, and $\Sigma_2 \in \mathbf{R}^{(n-r)\times(n-r)}$ is diagonal, positive semidefinite, and none of its diagonal entries are equal to $\bar{\sigma}$. Recall that we want to find a $D \in \mathcal{D}$ such that all the eigenvalues of $U^*DU - V^*DV$ are positive, or in other words, $\lambda_{\min} > 0$. For notational purposes, partition U and V compatibly with $\mathbf{\Delta}$ as

$$U = \begin{bmatrix} A_1 \\ \vdots \\ A_s \\ E_1 \\ \vdots \\ E_f \end{bmatrix} \qquad V = \begin{bmatrix} B_1 \\ \vdots \\ B_s \\ F_1 \\ \vdots \\ F_f \end{bmatrix} \tag{4.4}$$

where $A_i, B_i \in \mathbf{C}^{r_i\times r}$, $E_i, F_i \in \mathbf{C}^{m_i\times r}$. With this notation, and a bit of manipulation, we can write $\lambda_{\min}$ in terms of inner products in $\mathbf{X}$,

$$\lambda_{\min}(U^*DU - V^*DV) = \min_{\substack{\eta\in\mathbf{C}^r \\ \|\eta\|=1}} \langle \tilde{D}, P^\eta\rangle \tag{4.5}$$

where $P^\eta \in \mathbf{X}$ is defined by its block components

$$P_i^\eta := A_i\eta\eta^* A_i^* - B_i\eta\eta^* B_i^* \\ p_j^\eta := \eta^*\left(E_j^*E_j - F_j^*F_j\right)\eta. \tag{4.6}$$

Let $\nabla_M \subset \mathbf{X}$ be the set of all such P^η. That is

$$\nabla_M := \left\{\ \text{diag}\ \left[P_1^\eta, \ldots, P_s^\eta, p_1^\eta, \ldots, p_{f-1}^\eta\right] : \\ P_i^\eta, p_i^\eta \text{ as in (4.6)}, \eta \in \mathbf{C}^r, \|\eta\| = 1\right\}. \tag{4.7}$$

Recall that when $r \geq 2$, the matrices U and V are not unique, however the set ∇_M **does not** depend on the particular choice. For a given $D \in \mathcal{D}$ (and corresponding $\tilde{D} \in \mathbf{X}$) we have

$$\lambda_{\min}(U^*DU - V^*DV) = \min_{P\in\nabla_M} \langle \tilde{D}, P\rangle. \tag{4.8}$$

Hence, it is the set ∇_M that determines whether or not there is a D that gives $\lambda_{\min} > 0$. The convex hull of a set $\mathcal{V} \subset \mathbf{X}$ is denoted $\text{co}(\mathcal{V})$.

Theorem 4.1 $0 \notin \text{co}(\nabla_M)$ *if and only if there exists a $D \in \mathcal{D}$ such that $\lambda_{\min}(U^*DU - V^*DV) > 0$.*

If $0 \in \text{co}(\nabla_M)$ then for every $D \in \mathcal{D}$, $\lambda_{\min} \leq 0$ and $\lambda_{\max} \geq 0$. Hence to first order, the maximum singular value either increases or stays the same (we are at a stationary point). By convexity of $\bar{\sigma}\left(e^{D}Me^{-D}\right)$, we are at a global minimum. To summarize:

Theorem 4.2 $0 \in \text{co}(\nabla_M)$ *if and only if* $\inf_{D\in\mathcal{D}} \bar{\sigma}\left(e^{D}Me^{-D}\right) = \bar{\sigma}(M)$.

When the matrix in question, in this case M, is clear from the context, we will drop the subscript and just write ∇.

Finally, we address the problem of computing the point of minimum norm in the convex hull of ∇_M. The minimum point of the convex hull of a set $\mathcal{V}$ can be found via an iterative algorithm, due to [Gil]. Important extensions of this are found in [Wol] and [Hau]. All the algorithms have one main computational requirement: for each $x \in \mathbf{X}$, generate a point $y_x \in \mathcal{V}$ such that

$$\langle x, y_x\rangle = \min_{y\in\mathcal{V}} \langle x, y\rangle \tag{4.9}$$

Hence, for each $\tilde{D} \in \mathbf{X}$, we need to be able to find a $P_{\tilde{D}} \in \nabla_M$ that achieves

$$\langle \tilde{D}, P_{\tilde{D}}\rangle = \min_{P\in\nabla_M} \langle \tilde{D}, P\rangle. \tag{4.10}$$

Let $\tilde{D} \in \mathbf{X}$ be given, with components D_i for $i = 1, \ldots, s$ and d_j for $j = 1, \ldots, f-1$. Then $\min_{P\in\nabla_M} \langle \tilde{D}, P\rangle = \min_{\substack{\eta\in\mathbf{C}^r \\ \|\eta\|=1}} \eta^* W\eta$ where

$$W := \sum_{i=1}^{s} (A_i^* D_i A_i - B_i^* D_i B_i) + \sum_{j=1}^{f-1} d_j \left(E_j^* E_j - F_j^* F_j\right)$$

The numerical value of this is the minimum eigenvalue of the hermitian matrix W. Let $\eta_w \in \mathbf{C}^r$ be any unit length eigenvector associated with this eigenvalue, then

$$\arg \min_{P \in \nabla_M} \langle \tilde{D}, P \rangle = \operatorname{diag}\left[P_1^{\eta_w}, \ldots, P_s^{\eta_w}, p_1^{\eta_w}, \ldots, p_{f-1}^{\eta_w}\right] \in \nabla_M \tag{4.11}$$

where the P's and p's are defined as

$$\begin{aligned} P_i^{\eta_w} &:= A_i \eta_w \eta_w{}^* A_i^* - B_i \eta_w \eta_w{}^* B_i^* \\ p_j^{\eta_w} &:= \eta_w{}^* \left(E_j^* E_j - F_j^* F_j\right) \eta_w. \end{aligned} \tag{4.12}$$

for each i and j. Using this formula, and the algorithm in [Hau], we can find the minimum point in the convex hull of ∇_M as desired.

If the matrix M is real, then the minimum point in the convex hull of ∇ is real. The implication this has is that, roughly speaking, each block of the optimal $D \in \mathcal{D}$ can be chosen to be real, symmetric.

Theorem 4.3 *Let $\mathcal{D}_R$ be the set of real, symmetric members of $\mathcal{D}$. If M is real, and the infimum $\inf_{D_R \in \mathcal{D}_R} \bar{\sigma}\left(e^{D_R} M e^{-D_R}\right)$ is achieved, then in fact*

$$\inf_{D \in \mathcal{D}} \bar{\sigma}\left(e^{D} M e^{-D}\right) = \inf_{D_R \in \mathcal{D}_R} \bar{\sigma}\left(e^{D_R} M e^{-D_R}\right). \tag{4.13}$$

4.2 When $\mu = \bar{\sigma}$

The results of this section relate the upper bound to μ. As usual, let $\mathbf{\Delta}$ be a given structure, and M be a given complex matrix. In the last section we showed that $\bar{\sigma}(M) = \inf \bar{\sigma}\left(e^{D} M e^{-D}\right)$ if and only if $0 \in \operatorname{co}(\nabla_M)$. A natural question is: "When does $\bar{\sigma}(M) = \mu_{\mathbf{\Delta}}(M)$?". The answer links the upper bound and μ together. Again, the set ∇ plays a crucial role.

Theorem 4.4 *$\bar{\sigma}(M) = \mu_{\mathbf{\Delta}}(M)$ if and only if $0 \in \nabla_M$.*

Remark: This is exactly the result obtained in [Doy]. [Doy] only considers structures with full blocks ($s = 0$). This generalizes that result.

Proof: The following 4 statements are easily shown equivalent:

1. $0 \in \nabla_M$
2. There exists $\eta \in \mathbf{C}^r$, $\|\eta\| = 1$ and $Q \in \mathcal{Q}$ with $QU\eta = V\eta$
3. There exists $\xi \in \mathbf{C}^n$, $\|\xi\| = 1$ and $Q \in \mathcal{Q}$ with $QM\xi = \bar{\sigma}\xi$
4. $\bar{\sigma}(M) = \mu_{\mathbf{\Delta}}(M)$. ∎

Theorem 4.4 helps determine when the upper bound is μ.

Theorem 4.5 *If the block structure $\mathbf{\Delta}$ has the property that $0 \in \operatorname{co}(\nabla)$ always implies $0 \in \nabla$, then*

$$\mu_{\mathbf{\Delta}}(M) = \inf_{D \in \mathcal{D}} \bar{\sigma}\left(e^{D} M e^{-D}\right)$$

Corollary 4.6 *If, at the minimum of $\bar{\sigma}\left(e^{D} M e^{-D}\right)$, the maximum singular value has multiplicity of 1, then*

$$\mu(M) = \min_{D \in \mathcal{D}} \bar{\sigma}\left(e^{D} M e^{-D}\right)$$

5 Properties of the set ∇_M

As is well known, [Doy] and [FanT], when $s = 0$ and $f \leq 3$ (3 or less full blocks), the set ∇_M is itself convex. In addition, there also exist 4 block examples, [MorD], where $0 \in \operatorname{co}(\nabla)$ but $0 \notin \nabla$. Until now, the case of *repeated scalars* ($s \neq 0$) blocks has not been investigated. In this section, we consider a block structure of *one* repeated scalar block, and *one* full block. Recall the definition of ∇_M, equation (4.7). With this structure, the set ∇_M will always be of the form

$$\nabla = \{A\eta\eta^* A^* - B\eta\eta^* B^* : \eta \in \mathbf{C}^r, \|\eta\| = 1\} \tag{5.1}$$

for some given $r > 0$ and $A, B \in \mathbf{C}^{r_1 \times r}$. It is easy to see that in general, ∇ is not convex. For instance, take $A = I$ and $B = 0$. However the following (which is all we need) is always true.

Theorem 5.1 *Let ∇ be defined as in (5.1). If $0 \in \operatorname{co}(\nabla)$, then $0 \in \nabla$.*

Proof: Suppose that $0 \in \operatorname{co}(\nabla)$. Then, for some integer p, there exist nonnegative α_i with $\sum_{i=1}^{p} \alpha_i = 1$ and vectors $\eta_i \in \mathbf{C}^r$ with $\|\eta_i\| = 1$ such that

$$\sum_{i=1}^{p} \alpha_i \left(A\eta_i\eta_i{}^* A^* - B\eta_i\eta_i{}^* B^*\right) = 0 \tag{5.2}$$

which is rewritten as

$$A\left(\sum_{i=1}^{p} \alpha_i \eta_i \eta_i{}^*\right) A^* = B\left(\sum_{i=1}^{p} \alpha_i \eta_i \eta_i{}^*\right) B^* \tag{5.3}$$

Since the α_i are nonnegative, and not all 0, the dyad summation in (5.3) is a positive semidefinite matrix that is not zero. Let $X^{\frac{1}{2}}$ be its hermitian, positive semidefinite square root. Therefore $AX^{\frac{1}{2}}X^{\frac{1}{2}}A^* = BX^{\frac{1}{2}}X^{\frac{1}{2}}B^*$. Hence, there is a unitary matrix V such that $AX^{\frac{1}{2}} = BX^{\frac{1}{2}}V$. Let v be an eigenvector of V (with eigenvalue $e^{j\theta}$) such that $X^{\frac{1}{2}}v \neq 0$, and define $u := X^{\frac{1}{2}}v$. Note that u is nonzero. This gives $Au = e^{j\theta}Bu$, which implies that $0 \in \nabla$. ∎

6 Linear Fractional Transformations

6.1 Introduction

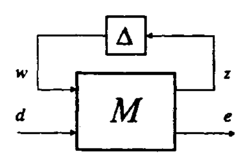
Let M be a complex matrix M partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{6.1}$$

and suppose there is a defined block structure $\mathbf{\Delta}$ which is compatible in size with M_{11}. The **linear fractional transformation**, $F_u(M, \Delta)$ is well posed if $I - M_{11}\Delta$ is invertible, and is then defined as

$$F_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} \tag{6.2}$$

In a feedback diagram, $F_u(M, \Delta)$ appears as:



From a system point of view, we interpret vector d as the "disturbance", and e is the "error", whereas vectors z and w are internal variables. M_{22} is the nominal map between the disturbance and error, and Δ represents unknown quantities, called perturbations, which affect the map in a known way-namely through M_{12}, M_{21}, M_{11}, and the formula F_u. The subscript u on F_u pertains to the "upper" loop of M is closed by Δ. An analogous formula describes $F_l(M,\Delta)$, which is the resulting matrix obtained by closing the "lower" loop of M.

Suppose there are two defined block structures $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ which are compatible in size with M_{11} and M_{22} respectively. Define a third structure $\tilde{\mathbf{\Delta}}$ as

$$\tilde{\mathbf{\Delta}} = \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \Delta_1 \in \mathbf{\Delta}_1, \Delta_2 \in \mathbf{\Delta}_2 \right\} \tag{6.3}$$

Now we have three structures with which we may compute μ with respect to. The notation we will use to keep track of this is as follows: $\mu_1(\cdot)$ is with respect to $\mathbf{\Delta}_1$, $\mu_2(\cdot)$ is with respect to $\mathbf{\Delta}_2$, : $\mu_{1,2}(\cdot)$ is with respect to $\tilde{\mathbf{\Delta}}$. In view of this, $\mu_1(M_{11})$, $\mu_2(M_{22})$ and $\mu_{1,2}(M)$ all make sense, though for instance, $\mu_1(M)$ does not. The first theorem is nothing more than a restatement of the definition of μ.

Theorem 6.1 *Let $\beta > 0$. The LFT is well posed for all $\Delta_1 \in \frac{1}{\beta}\mathbf{B\Delta}$ if and only if $\mu_1(M_{11}) < \beta$.*

As the "perturbation" Δ_1 deviates from zero, the matrix relating d to e deviates from M_{22}. Using the quantity $\mu_{1,2}(M)$, we can bookkeep what happens to $\mu_2(F_u(M,\Delta_1))$ as follows:

Theorem 6.2 (Robust Performance: constant) *Let $\beta > 0$. Then $\mu_{1,2}(M) < \beta$ if and only if $\mu_1(M_{11}) < \beta$, and for all $\Delta_1 \in \frac{1}{\beta}\mathbf{B\Delta}_1$, $\mu_2(F_u(M,\Delta_1)) < \beta$.*

We have a test that determines if for all $\bar{\sigma}(\Delta_1) \leq \frac{1}{\beta}$, the quantity $\mu_2(F_u(M,\Delta_1))$ stays bounded by β. Since both $\rho(\cdot)$ and $\bar{\sigma}(\cdot)$ are special cases of μ, by the appropriate choice of the set $\mathbf{\Delta}_2$, either $\rho(F_u(M,\Delta_1))$ or $\bar{\sigma}(F_u(M,\Delta_1))$ could be "watched". Of course for different choices of $\mathbf{\Delta}_2$, the theorem gives information about $\mu_2(F_u(M,\Delta_1))$. Note that in this test, *the bound on the performance is dependent on the bound on the perturbation*, namely they are reciprocals. For other values, we must scale M and recompute. Specifically, for $\alpha \geq 0$, define $\mathbf{M}_\alpha$ as

$$\mathbf{M}_\alpha = \begin{bmatrix} M_{11} & M_{12} \\ \alpha M_{21} & \alpha M_{22} \end{bmatrix} \tag{6.4}$$

For $\gamma > \mu_1(M_{11})$, define $\alpha_\gamma = \max_{\alpha > 0} \{\alpha : \mu_{1,2}(\mathbf{M}_\alpha) = \gamma\}$. This leads to the following variant of Theorem 6.2;

Theorem 6.3 (Worst Case: constant) *Let $\gamma > \mu_1(M_{11})$ be given, and α_γ be computed as above. Then*

$$\sup_{\Delta_1 \in \frac{1}{\gamma}\mathbf{B\Delta}_1} \mu_2(F_u(M,\Delta_1)) = \frac{\gamma}{\alpha_\gamma} \tag{6.5}$$

Remark: The basic idea of the theorem is this: find the largest α such that for all $\Delta_1 \in \frac{1}{\beta}\mathbf{B\Delta}$, $\mu_2(F_u(M,\Delta_1)) < \frac{\beta}{\alpha}$. This is the same as: find the largest α such that for all $\bar{\sigma}(\Delta_1) \leq \frac{1}{\beta}$, $\mu_2(\mathbf{M}_\alpha) < \beta$. **This** test we can do, by applying Theorem 6.2 on $\mathbf{M}_\alpha$, which then gives the result.

Finally, we state a maximum modulus like result for μ. The proof uses the fact that the lower bound achieves μ, along with ideas similar to the ones here.

Theorem 6.4 (Maximum modulus: LFT) *Let M be given as in (6.1), along with two block structures $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$. Suppose that $\mu_1(M_{11}) < 1$. Then*

$$\max_{\Delta_1 \in \mathbf{B\Delta}_1} \mu_2(F_u(M,\Delta_1)) = \max_{Q_1 \in \mathcal{Q}_1} \mu_2(F_u(M,Q_1)) \tag{6.6}$$

Remarks: In light of this, any μ test with at least one repeated scalar block can always be reduced to a one dimensional search of μ tests without that block. This is similar to a theorem in [BoyD]. They show for that any H bounded and analytic on $|z| \leq 1$, the function $k(z) := \mu(H(z))$ is subharmonic.

In this section, all of the results were stated and proven for $F_u(M,\Delta)$. Of course, analogous results hold for $F_l(M,\Delta)$.

6.2 Transfer functions as LFT's

Consider a stable, discrete time, linear system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{6.7}$$

with transfer function $G(z) = D + C(zI - A)^{-1}B$ (n states, and for simplicity, we assume that this has m inputs and outputs, though everything that follows holds for nonsquare plants also). The infinity norm of G is defined as

$$\|G\|_\infty = \sup_{\substack{z \in \mathbf{C} \\ |z| \geq 1}} \bar{\sigma}(G(z))$$

Define $\mathbf{\Delta}_1 = \{\delta I_n : \delta \in \mathbf{C}\}$, $\mathbf{\Delta}_2 = \mathbf{C}^{m \times m}$ and

$$M := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbf{R}^{(n+m)\times(n+m)}. \tag{6.8}$$

In μ notation, we can write (6.2) as

$$\|G\|_\infty = \sup_{\Delta_1 \in \mathbf{B\Delta}_1} \mu_2(F_u(M,\Delta_1)) \tag{6.9}$$

Applying theorem 6.2, gives $\|G\|_\infty < 1$ iff $\mu_{1,2}(M) < 1$. In view of Theorem 5.1, actually $\|G\|_\infty < 1$ if and only if there exists a coordinate transformation $T \in \mathbf{C}^{n \times n}$ such that

$$\bar{\sigma}\left(\begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix}\right) < 1.$$

Hence, we have an algorithm for *generating all stable rational transfer functions that have* $\|\cdot\|_\infty < 1$. Simply choose any matrix M so that $\bar{\sigma}(M) < 1$ and partition M as shown above. Then G will be stable, and have norm less than one, **and** all stable rational $G(z)$, with $\|G\|_\infty < 1$ can be generated in this fashion. This result can also be shown using results from dissipative systems, and linear quadratic optimal control theory (with nondefinite cost functions). In fact, if $\|G\|_\infty \leq 1$, then solving one Riccati equation yields a $T \in \mathbf{C}^{n \times n}$ such that

$$\bar{\sigma}\left(\begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix}\right) = 1.$$

The details of this calculation are interesting, and follow directly from the results in [Wil]. We do not include them here because the Riccati solution has the undesirable property that n of the singular values will be coalesced at $\bar{\sigma} = 1$. This seems to limit the usefulness of the Riccati solution as a viable computational alternative to gradient searching along the "full" D directions.

6.3 Upper bound LFT results

Theorems 6.1 and 6.2 give necessary and sufficient conditions for a constant matrix performance/robustness characteristic in terms of a μ evaluation. The μ test always looks like "Is $\mu(M) < \beta$?" (or $\leq$). In this section, we will concentrate on the additional information that is obtained in using the $\bar{\sigma}(DMD^{-1})$ upper bound. As usual, let $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ be two given structures, and let $\tilde{\mathbf{\Delta}} = \{\text{diag}[\Delta_1, \Delta_2] : \Delta_i \in \mathbf{\Delta}_i\}$. Similarly, let $\mathcal{D}_i$ be the appropriate exponentiated D scaling sets for the two structures, (equation (2.5)) and denote $\tilde{\mathcal{D}}$ as the obvious diagonal augmentation of these two sets.

Lemma 6.5 (Constant D lemma) *Let M be given as in the* **robust performance** *theorem, 6.2. Suppose there is a $\tilde{D} \in \tilde{\mathcal{D}}$ such that $\bar{\sigma}\left(\tilde{D}M\tilde{D}^{-1}\right) < \beta$. Then there exists a $D_2 \in \mathcal{D}_2$ such that*

$$\max_{\Delta_1 \in \frac{1}{\beta}\mathbf{B}\boldsymbol{\Delta}_1} \bar{\sigma}\left(D_2 F_u(M,\Delta_1) D_2^{-1}\right) < \beta$$

Remark: Initially, one might guess that if we replace μ by the $\bar{\sigma}(DMD^{-1})$ upper bound in the robust performance theorem hypothesis, the resulting claim would just have μ replaced by $\bar{\sigma}(DMD^{-1})$. This lemma shows that we get quite a bit more: this is indeed so, **but using only a single $D_2 \in \mathcal{D}_2$.**

Proof: The easist method of proof is just to track the norms of the various vectors in the loop equations for the LFT's $F_u(DMD^{-1}, \Delta_1)$ and $D_2 F_u(M, \Delta_1) D_2^{-1}$.

Suppose $\mu_1(M_{11}) < 1$. Therefore, for all $\Delta_1 \in \mathbf{B}\boldsymbol{\Delta}_1$, the linear fractional transformation $F_u(M, \Delta_1)$ is defined. Can we compute the value of

$$\inf_{D_2 \in \mathcal{D}_2} \max_{\Delta_1 \in \mathbf{B}\boldsymbol{\Delta}_1} \bar{\sigma}\left(D_2 F_u(M,\Delta_1) D_2^{-1}\right) \tag{6.10}$$

and also find a D_2 that achieves it? Yes. Suppose the dimension of the structure $\boldsymbol{\Delta}_2$ is $m \times m$. Define an additional structure

$$\hat{\boldsymbol{\Delta}} := \{\mathrm{diag}[\Delta_1, \Delta] : \Delta_1 \in \boldsymbol{\Delta}_1, \Delta \in \mathbf{C}^{m \times m}\} \tag{6.11}$$

Theorem 6.6 *Let M, $\boldsymbol{\Delta}_1$, $\boldsymbol{\Delta}_2$, $\mathcal{D}_1$, $\mathcal{D}_2$, and $\hat{\boldsymbol{\Delta}}$ be given as above. Suppose that $\mu_1(M_{11}) < 1$. Define γ by*

$$\gamma = \sup_{\alpha > 0}\left\{\alpha : \inf_{D_2 \in \mathcal{D}_2} \mu_{\hat{\boldsymbol{\Delta}}}\begin{pmatrix} M_{11} & M_{12}D_2^{-1} \\ \alpha D_2 M_{21} & \alpha D_2 M_{22} D_2^{-1}\end{pmatrix} < 1\right\} \tag{6.12}$$

Then

$$\inf_{D_2 \in \mathcal{D}_2} \max_{\Delta_1 \in \mathbf{B}\boldsymbol{\Delta}_1} \bar{\sigma}\left(D_2 F_u(M,\Delta_1) D_2^{-1}\right) = \frac{1}{\gamma} \tag{6.13}$$

This is an interesting result. Note that the structure which we need to compute μ with respect to does not depend on $\boldsymbol{\Delta}_2$. If $\mu_{\hat{\boldsymbol{\Delta}}}$ can be computed, then, modulo the necessary search over the D_2 and α this is a useful theorem.

7 Optimal Constant D scalings

This section combines two results from previous sections, to yield a method for sub-optimal and optimal scaling of multivariable transfer functions using constant, diagonal D matrices. Let $G(z)$ be a given, stable, transfer function, with m inputs, and m outputs, and state space realization

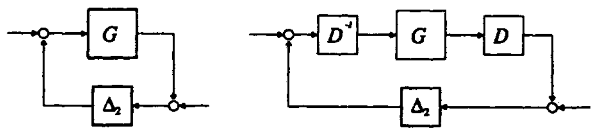
$$G(z) = D + C(zI - A)^{-1} B \tag{7.1}$$

where $A \in \mathbf{C}^{n \times n}, B \in \mathbf{C}^{n \times m}, C \in \mathbf{C}^{m \times n}$, and $D \in \mathbf{C}^{m \times m}$. Since G is stable, $\rho(A) < 1$. Suppose a perturbation structure $\boldsymbol{\Delta}_2$ is given, and is compatible with $G(z)$. That is, $\boldsymbol{\Delta}_2 \subset \mathbf{C}^{m \times m}$. As usual, let $\mathcal{D}_2$ denote the set of diagonal scalings that commute with all elements of $\boldsymbol{\Delta}_2$.

Optimal constant scaling is the constant D matrix that achieves the following infimum (if it exists, otherwise, a scaling that gets arbitrarily close)

$$\inf_{D_2 \in \mathcal{D}_2} \sup_{\substack{z \in \mathbf{C} \\ |z| \geq 1}} \bar{\sigma}\left(D_2 G(z) D_2^{-1}\right) \tag{7.2}$$

Remark: This is useful because any linear perturbation, **even a time varying perturbation**, with the appropriate block diagonal structure as defined by $\boldsymbol{\Delta}_2$, commutes with these **constant** D scales. Therefore, for every constant $D \in \mathcal{D}$ and every operator Δ_2, with the correct block diagonal structure, the operators $D\Delta_2 D^{-1}$ and Δ_2 are the same. Therefore, for any operator G, the following systems are equivalent.



Simple application of the small gain theorem, ([Zam] and [DesV]), on the right figure gives that if Δ_2 is a stable operator mapping $l_2 \to l_2$, and the induced norm of Δ_2, $\|\Delta_2\|$, satisfies

$$\|\Delta_2\| < \frac{1}{\|DGD^{-1}\|_\infty}$$

then the loop is stable. This calls for a minimization of the form $\inf_{D \in \mathcal{D}} \|DGD^{-1}\|_\infty$. An important point to reiterate is that the D's are constant. If they were frequency varying, then in general they would not commute with time varying Δ's, and hence the equivalence of the two figures would be invalid.

Conceptually, Theorem 6.6 gives the value of the infimum in equation (7.2). Here we capitalize on the additional structure that is present in this specific problem, and use the result for block structures with $f = s = 1$ which was obtained in section 5.

Theorem 7.1 *Let $G(z)$ and $\boldsymbol{\Delta}_2$ be given as in the beginning of this section. Define $\gamma \in \mathbf{R}$ by*

$$\gamma := \sup_{\alpha > 0}\left\{\alpha : \inf_{\substack{D_1 \text{ invertible} \\ D_2 \in \mathcal{D}_2}} \bar{\sigma}\begin{pmatrix} D_1 A D_1^{-1} & D_1 B D_2^{-1} \\ \alpha D_2 C D_1^{-1} & \alpha D_2 D D_2^{-1}\end{pmatrix} < 1\right\}. \tag{7.3}$$

Then

$$\inf_{D_2 \in \mathcal{D}_2} \sup_{\substack{z \in \mathbf{C} \\ |z| \geq 1}} \bar{\sigma}\left(D_2 G(z) D_2^{-1}\right) = \frac{1}{\gamma}. \ \blacksquare \tag{7.4}$$

8 Counterexample

This section shows, via a detailed example, that $\mu(M)$ is not always equal to the $\bar{\sigma}(DMD^{-1})$ upper bound. An appealing aspect of this example is its simplicity.

8.1.a Let $a \in (0,1)$ and $\gamma \in (0,1)$ be given. Define the matrix $M \in \mathbf{R}^{4 \times 4}$ by

$$M := \begin{bmatrix} 0 & 1 & 0 & 1 \\ \gamma & 0 & \gamma & 0 \\ 2a & 0 & a & 0 \\ 0 & -2a & 0 & -a \end{bmatrix} \tag{8.1}$$

Define a block structure $\boldsymbol{\Delta} := \{\delta I_{2 \times 2} : \delta \in \mathbf{C}\}$.

8.1.b For all $\Delta \in \mathbf{B}\boldsymbol{\Delta}$ the LFT $F_l(M, \Delta)$ is well defined, and appears as

$$F_l(M, \Delta) = \begin{bmatrix} 0 & \dfrac{1 - a\delta}{1 + a\delta} \\ \gamma\dfrac{1 + a\delta}{1 - a\delta} & 0 \end{bmatrix}. \tag{8.2}$$

Note that for each such Δ, the spectral radius of $F_l(M, \Delta)$ is simply $\sqrt{\gamma}$, which by assumption is less than 1. With respect to the structure

$$\tilde{\Delta} := \{\text{diag}[\delta I_{2\times 2}, \Delta] : \delta \in \mathbf{C}, \Delta \in \Delta\},$$

Theorem 6.2 implies that $\mu_{\tilde{\Delta}}(M) < 1$.

8.1.c Consider the product of two linear fractional transformations with different Δ's in $\mathbf{B\Delta}$.

$$F_l(M, -I_{2\times 2})\ F_l(M, I_{2\times 2}) = \begin{bmatrix} \gamma \frac{(1+a)^2}{(1-a)^2} & 0 \\ 0 & \gamma \frac{(1-a)^2}{(1+a)} \end{bmatrix}$$

For any $\gamma \in (0,1)$, it is easy to chose $a \in (0,1)$ so that the spectral radius of the above product is greater than 1. For such choices, then, we must have $\inf_{\tilde{D}\in\tilde{\mathcal{D}}} \bar{\sigma}\left(\tilde{D}M\tilde{D}^{-1}\right) \geq 1$, otherwise, by Lemma 6.5, the spectral radius of any product of these LFT's would be less than 1.

Remark: A bit more analysis can show that by proper choice of γ and a, the value of $\inf_{\tilde{D}\in\tilde{\mathcal{D}}} \bar{\sigma}\left(\tilde{D}M\tilde{D}^{-1}\right)$ can be made arbitrarily close to $1+\sqrt{2}$ while $\mu_{\tilde{\Delta}}(M) < 1$.

In light of this example, it appears that the upper bound can be quite far from the actual value of μ, especially when $s \neq 0$. For instance, in this example, the upper bound (in the limit) equals $\left(1+\sqrt{2}\right) \times \mu$. Limited computing experience with uncertainty structures having $s \neq 0$ indicates that there is often a gap, though usually not as large. For block structures with no repeated scalar blocks, $s = 0$, this contrasts directly with our computational experience. In that case, the worst known ratio of upper bound to μ is 1.14, [MorD], and usually, it is much closer to 1. Given that the upper bound can be computed, and in general, it is impossible to verify that a lower bound is indeed μ, how should this all be interpreted?

Suppose an uncertainty structure has only full blocks, and the perturbations are modeled as linear, time invariant. Using the constant, state space μ test in [DoyP] requires that the actual uncertainty structure be augmented with a large (size of state dimension) repeated scalar block. In view of the counterexample, it is likely that the upper bound will not equal μ, and the conclusions will be conservative. In this situation, a frequency domain upper bound test, [DoyWS], is appropriate, since it scales (a peak > 1 **does** give useful information), and with this block structure, we always have found μ and the upper bound very close. It is important to realize that the frequency domain test only gives conclusions about linear, time invariant perturbations.

If the perturbations are time varying and/or nonlinear, then, in general the frequency domain tests are not valid, though [Saf] derives conditions on the frequency dependent scalings which allow for conclusions about slope bounded nonlinearities. The upper bound approaches based on **constant matrix operations** (for example, the optimal constant scaling, section 7), handle this type of uncertainty, and the motivation which led to their development was the relationship between μ and the upper bound, and the role this difference plays in the behavior of linear fractional transformations.

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