STATE-SPACE SOLUTIONS TO STANDARD $\mathcal{H}_2$ AND $\mathcal{H}_\infty$ CONTROL PROBLEMS

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Abstract

Simple state-space formulas are presented for a controller solving a standard $\mathcal{H}_\infty$-problem. The controller has the same state-dimension as the plant, its computation involves only two Riccati equations, and it has a separation structure reminiscent of classical LQG (i.e., $\mathcal{H}_2$) theory. This paper is also intended to be of tutorial value, so a standard $\mathcal{H}_2$-solution is developed in parallel.

1 Introduction

Two popular performance measures in optimal control theory are $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms. These are defined in the frequency-domain for closed-loop transfer matrices. The $\mathcal{H}_2$-norm of a transfer matrix $G(s)$ is

$$\|G\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}G(j\omega)^* G(j\omega) d\omega\right)^{1/2}$$

whereas the $\mathcal{H}_\infty$-norm is

$$\|G\|_\infty := \sup_\omega \sigma_{max}[G(j\omega)]$$

The former arises when the exogenous signals are either fixed or have a fixed power spectrum; the latter arises from (weighted) balls of exogenous signals. $\mathcal{H}_2$-optimal control theory was developed in the 1960's; $\mathcal{H}_\infty$-optimal control theory is continuing its development.

Zames' (1981) original formulation of $\mathcal{H}_\infty$-optimal control theory was in an input-output setting. The solution techniques involved analytic functions (Nevanlinna-Pick interpolation) or operator-theoretic methods (those of Sarason, Adamjan-Arov-Krein, Ball-Helton). For multivariable systems, however, it may be convenient to have state-space methods, especially with regards to computation. A state-space solution was presented in [Doyle, 1984], in which the steps were as follows: parametrize all internally-stabilizing controllers (via Youla *et al.*); obtain realizations of the closed-loop transfer matrix; reduce to the Nehari problem; solve the Nehari problem by the procedure of Glover (1984). See also Francis (1987) and Francis and Doyle (1987).

Controllers obtained by the above method tended to have high dimension. Moreover, the complexity of computation was substantial. However, Limebeer, Hung, and Halikias (1986, 1987) showed, for some special cases, how to reduce the state-space dimension of the controller to that of the plant. This suggested the likely existence of low dimension optimal controllers in the general case. Furthermore, Khargonekar, Petersen, Rotea, and Zhou (1987, 1988) showed that for the state-feedback problem one can choose a constant gain as a (sub)optimal controller. In addition, a formula for the state-feedback gain matrix was given in terms of an algebraic Riccati equation. These results are similar to the results in section 4 below; however, the proof techniques are entirely different. Also, these papers established interesting connections between $\mathcal{H}_\infty$-optimal control, quadratic stabilization, and linear-quadratic differential games.

To facilitate exposition, the problem chosen for treatment in this paper is the simplest special case which captures all the essential features of the general problem. The formulas for the controller in the general case, presented in Glover and Doyle (1988), were actually first obtained by the rather cumbersome method whose steps are outlined above. Their simplicity suggested the more direct derivation, in the style of Willems (1971), which is outlined here. The controller has the same dimension as the plant, its computation involves only two Riccati equations, and it has a separation structure reminiscent of classical LQG (i.e., $\mathcal{H}_2$) theory. The proofs exploit this latter feature, but because of space limitations, details are omitted.

The reader may wish to compare the solution in this paper with two other recent solutions to basically the same problem: a method by Kwakernaak (1986) using polynomial matrices and a method by Foias and Tannenbaum (1988) using operator theory. The results of Bernstein and Haddad (1988) have similarities to those in this paper and will be discussed briefly in section 2.

The notation in this paper is fairly standard. In particular, a transfer matrix in terms of state-space data is denoted

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI - A)^{-1}B + D$$

2 Preliminaries

2.1 Computing $\mathcal{H}_2$-norm

Consider the transfer matrix

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

with A stable (no minimality assumption). Let L_c denote the controllability gramian of (A, B) and L_o the observability gramian of (C, A). Then it is easy to show that

$$\|G\|_2^2 = \text{trace}(CL_cC') = \text{trace}(B'L_oB)$$

2.2 Computing $\mathcal{H}_\infty$-norm

For the same transfer matrix $G(s)$ define the Hamiltonian matrix

$$H := \begin{bmatrix} A & BB' \\ -C'C & -A' \end{bmatrix}$$

Lemma 1 $\|G\|_\infty < 1$ *iff H has no eigenvalues on the imaginary axis.*

How the Hamiltonian matrix arises in this $\mathcal{H}_\infty$-norm computation can be explained as follows [Willems, 1971; Boyd *et al.*, 1988]. Define $G^\sim(s) = G(-s)'$. Then

$$(I - G^\sim G)^{-1}(s) = \left[\begin{array}{cc|c} A & BB' & B \\ -C'C & -A' & 0 \\ \hline 0 & B' & I \end{array}\right]$$

So H is the A-matrix of $(I - G^\sim G)^{-1}$. It can be shown that in the above realization of $(I - G^\sim G)^{-1}$ there are no uncontrollable or unobservable modes on the imaginary axis. Thus H has no eigenvalues on the imaginary axis iff $(I - G^\sim G)^{-1}$ has no poles on the imaginary axis, i.e., $(I - G^\sim G)^{-1} \in \mathcal{RL}_\infty$. So the statement of the lemma is equivalent to the statement

$$\|G\|_\infty < 1 \iff (I - G^\sim G)^{-1} \in \mathcal{RL}_\infty$$

which is easy to prove.

The previous lemma suggests a way to compute an $\mathcal{H}_\infty$-norm: select a positive number γ; test if $\|G\|_\infty < \gamma$, i.e., if $\|\gamma^{-1}G\|_\infty < 1$, by calculating the eigenvalues of the appropriate Hamiltonian matrix; increase or decrease γ accordingly; repeat. Thus $\mathcal{H}_\infty$-norm computation requires a search. Contrast this with $\mathcal{H}_2$-norm computation, which does not. In light of this observation, we should not be surprised in the sequel if the $\mathcal{H}_\infty$-optimal control problem requires a search for the minimum cost, whereas the $\mathcal{H}_2$-problem does not.

One could use the above characterizations of the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms to obtain controllers by assuming an arbitrary controller of the same order as the plant's, and then optimizing the controller parameters. In each case one gets coupled Riccati equations, which can, in principle, be solved to obtain an optimal controller of that order. Bernstein and co-workers have explored this in a series of papers, e.g., [Bernstein and Haddad, 1988]. It is well-known that in the $\mathcal{H}_2$ case this approach yields the optimal controller. In the $\mathcal{H}_\infty$ case as well, it can be shown via some involved algebra that their 3 coupled Riccati equations can be reduced to the two obtained below. In addition, Bernstein and Haddad consider reduced order controllers, a subject not addressed here.

2.3 The Riccati operator

This subsection collects some basic material on the Riccati equation and the Riccati operator.

Let A, Q, R be real $n \times n$ matrices with Q and R symmetric. Define the $2n \times 2n$ Hamiltonian matrix

$$H := \begin{bmatrix} A & R \\ Q & -A' \end{bmatrix}$$

Assume H has no eigenvalues on the imaginary axis. Then it must have n in Re $s < 0$ and n in Re $s > 0$. Bring in the two spectral subspaces $\mathcal{X}_-(H)$ and $\mathcal{X}_+(H)$: the former is the span of all (real) generalized eigenvectors corresponding to eigenvalues in Re $s < 0$; the latter, to eigenvalues in Re $s > 0$. In this case $\mathcal{X}_-(H)$ and $\mathcal{X}_+(H)$ both have dimension n. Let's focus on $\mathcal{X}_-(H)$. Finding a basis for it, stacking the basis vectors up to form a matrix, and partitioning the matrix, we get

$$\mathcal{X}_-(H) = \operatorname{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $X_1, X_2 \in \mathbf{R}^{n\times n}$. If X_1 is nonsingular, i.e., if the two subspaces

$$\mathcal{X}_-(H), \quad \operatorname{Im} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

are complementary, we can set $X := X_2 X_1^{-1}$ to get

$$\mathcal{X}_-(H) = \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}$$

Notice that X is then uniquely determined by H, i.e., $H \mapsto X$ is a function, which will be denoted by *Ric*; thus $X = Ric(H)$.

To recap, *Ric* is a function $\mathbf{R}^{2n\times 2n} \to \mathbf{R}^{2n\times 2n}$ which maps H to X where

$$\mathcal{X}_-(H) = \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}$$

The domain of *Ric*, *dom*(*Ric*), consists of Hamiltonian matrices H with two properties, namely, H has no eigenvalues on the imaginary axis and the two subspaces

$$\mathcal{X}_-(H), \quad \operatorname{Im} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

are complementary.

Some properties of X are given in the next lemma.

Lemma 2 *Suppose $H \in dom(Ric)$ and $X = Ric(H)$. Then*

(i) X is symmetric

(ii) X satisfies the algebraic Riccati equation

$$A'X + XA + XRX - Q = 0$$

(iii) $A + RX$ is stable

3 Unconstrained input

This section begins our treatment of the optimal control of linear time-invariant systems with a quadratic performance criterion.

3.1 $\mathcal{H}_2$-case

Our $\mathcal{H}_2$-version of the linear-quadratic regulator problem has the following specifications:

$$\dot{x} = Ax + B_1 w + B_2 u, \quad w = w_0 \delta$$

$$z = C_1 x + D_{12} u$$

$$\min_{u \in \mathcal{L}_2[0,\infty)} \|z\|_2^2$$

with the assumptions

(C_1, A) detectable

(A, B_2) stabilizable

$$D_{12}' \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$$

Note that the disturbance signal, w, is a fixed impulse; w_0 is a constant vector and δ is the unit impulse. The third assumption simplifies the problem; it amounts to orthogonality of C_1x and $D_{12}u$ in the output and to normalized nonsingular control weighting. The setup can be depicted as

where the transfer matrix G is

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \end{array}\right]$$

By a well-known fact, the Hamiltonian matrix

$$H_2 := \begin{bmatrix} A & -B_2B_2' \\ -C_1'C_1 & -A' \end{bmatrix}$$

belongs to $dom(Ric)$ and moreover $X_2 := Ric(H_2)$ is positive semi-definite. Define

$$F_2 := -B_2'X_2$$

$$A_{F_2} := A + B_2F_2, \quad C_{1F_2} := C_1 + D_{12}F_2$$

$$G_c(s) := \left[\begin{array}{c|c} A_{F_2} & B_1 \\ \hline C_{1F_2} & 0 \end{array}\right]$$

By lemma 2 A_{F_2} is stable, so $G_c \in \mathcal{RH}_2$.

The solution of the $\mathcal{H}_2$-LQR problem is as follows.

Theorem 1 *There exists a unique optimal control, namely $u = F_2x$. Moreover,*

$$\min_u \|z\|_2 = \|G_cw_0\|_2$$

A simple proof of the theorem uses a change of variable. Start with the system equations

$$\dot{x} = Ax + B_1w + B_2u$$

$$z = C_1x + D_{12}u$$

and define a new control variable, $v := u - F_2x$. The equations become

$$\dot{x} = A_{F_2}x + B_1w + B_2v$$

$$z = C_{1F_2}x + D_{12}v$$

or in the frequency-domain

$$z = G_cw_0 + Uv$$

where G_c is as above and U is defined to be

$$U(s) := \left[\begin{array}{c|c} A_{F_2} & B_2 \\ \hline C_{1F_2} & D_{12} \end{array}\right]$$

This matrix has two useful properties:

Lemma 3 *U is inner (i.e., $U^{\sim}U = I$) and $U^{\sim}G_c$ belongs to $\mathcal{RH}_2^{\perp}$.*

It follows that the functions G_cw_0 and Uv are orthogonal for every v in $\mathcal{H}_2$. Hence

$$\|z\|_2^2 = \|G_cw_0\|_2^2 + \|Uv\|_2^2$$

Then since U is inner we get

$$\|z\|_2^2 = \|G_cw_0\|_2^2 + \|v\|_2^2$$

This equation gives the desired conclusion immediately: the optimal v is $v = 0$ (i.e., $u = F_2x$) and the minimum norm of z equals $\|G_cw_0\|_2$.

3.2 $\mathcal{H}_\infty$-case

The $\mathcal{H}_\infty$-analog of the problem just studied has these specifications:

$$\dot{x} = Ax + B_1w + B_2u$$

$$z = C_1x + D_{12}u$$

for each w in $\mathcal{L}_2[0,\infty)$ with $\|w\|_2 \le 1$,

minimize $\|z\|_2$ over u in $\mathcal{L}_2[0,\infty)$

with the assumptions

(C_1, A) observable

(A, B_2) stabilizable

$$D_{12}'\begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$$

(The condition '(C_1, A) detectable' has been strengthened to 'observable' solely for a technical simplification below.) Note that the optimality measure,

$$\alpha := \sup_{\|w\|_2 \le 1} \inf_u \|z\|_2$$

is of the max-min type and that u is allowed to be a function of w.

The solution involves a different Hamiltonian matrix,

$$H_\infty := \begin{bmatrix} A & B_1B_1' - B_2B_2' \\ -C_1'C_1 & -A' \end{bmatrix}$$

The important difference here is that the (1,2)-block is sign indefinite. This may be interpreted in a number of interesting ways, which will be explored in the full version of this paper. The connection with differential games is discussed in [Khargonekar *et al.*, 1988] and the connection with risk-sensitive LQG stochastic control, as formulated by Whittle (1981), in [Glover and Doyle, 1988].

Theorem 2 *The bound*

$$\alpha < 1 \tag{1}$$

holds iff

$$H_\infty \in dom(Ric), \quad Ric(H_\infty) > 0 \tag{2}$$

Moreover, if (2) holds and the control law

$$u = F_\infty x$$

$$F_\infty := -B_2'X_\infty$$

$$X_\infty := Ric(H_\infty)$$

is applied, then the $\mathcal{H}_\infty$-norm of the transfer matrix from w to z is less than 1.

Regarding the second part of the theorem statement, when a stabilizing state-feedback control law, say $u = Fx$, is applied, the system from w to z is linear, the transfer matrix T_{zw} belongs to $\mathcal{RH}_\infty$, and

$$\alpha \le \sup_{\|w\|_2 \le 1} \|z\|_2 = \|T_{zw}\|_\infty$$

The theorem does not give an explicit formula for α. However, it can be computed as closely as desired by a search technique involving successive scaling.

The proof of this theorem is harder than that of theorem 1. As before, one changes control variables to arrive at the equation

$$z = G_c w + Uv$$

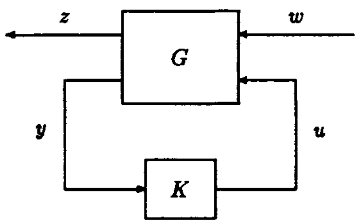
Whereas before w was a constant vector, w_0, now it belongs to $\mathcal{H}_2$. The two functions $G_c w$ and Uv are no longer orthogonal. The control function v can be freely selected from $\mathcal{H}_2$, so the range of influence of Uv is the subspace $\mathcal{U} := U\mathcal{H}_2$. The first step in the proof of theorem 2 is to show that α, which equals

$$\sup_{\|w\|_2 \le 1} \inf_u \|G_c w + Uv\|_2$$

can be expressed as the norm of a certain operator Ξ. The definition of Ξ is that it maps $\mathcal{H}_2$ to $\mathcal{U}^\perp$; it takes f in $\mathcal{H}_2$ to the projection of $G_c f$ onto $\mathcal{U}^\perp$. Thus $\alpha < 1$ iff Ξ is a strict contraction. The second step is to show that Ξ is a strict contraction iff (2) holds.

4 State feedback controllers

This section considers the system described by the following block diagram:



This setup differs from the one in the previous section in that now u is generated by a controller K processing a measured vector y. Both G and K are real-rational proper transfer matrices. In the first subsection, K is to minimize the $\mathcal{H}_2$-norm of T_{zw}, the transfer matrix from w to z; in the second subsection, the $\mathcal{H}_\infty$-norm. In both cases K is constrained to provide internal stability (usual meaning).

In this section we assume the controller has full information about the state x of G. Thus $y = x$. The realization of the transfer matrix G is taken to be of the form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right]$$

Notice that the parts from w and u to z are exactly as in the previous section. Also, as in the previous section, the following assumptions are made:

(i) (A, B_1) is stabilizable and (C_1, A) is detectable

(ii) (A, B_2) is stabilizable

(iii) $D_{12}' \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

4.1 $\mathcal{H}_2$-case

The problem is this: find a proper, real-rational, internally-stabilizing controller K which minimizes $\|T_{zw}\|_2$.

Since the state x is available, the solution is the same as in the unconstrained input case.

Theorem 3 *The unique optimal controller is $K_{opt}(s) := F_2$. Moreover,*

$$\min \|T_{zw}\|_2 = \|G_c\|_2$$

4.2 $\mathcal{H}_\infty$-case

Now the problem is: find a proper, real-rational, internally-stabilizing controller K which minimizes $\|T_{zw}\|_\infty$. Again, we shall strengthen the assumption of detectability of (C_1, A) to observability.

Theorem 4 *There exists an internally-stabilizing controller such that $\|T_{zw}\|_\infty < 1$ iff $H_\infty \in dom(Ric)$ and $Ric(H_\infty)$ is positive definite. When these conditions hold, one such controller is $K(s) = F_\infty$.*

5 Output feedback

This section considers the same block diagram as in the previous one, except that y is no longer assumed to contain full information about x; y is just some linear combination of x and w. Thus, the realization of the transfer matrix G is taken to be of the form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

The following assumptions are made throughout this section:

(i) (A, B_1) is stabilizable and (C_1, A) is detectable

(ii) (A, B_2) is stabilizable and (C_2, A) is detectable

(iii) $D_{12}' \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}$

The new assumption, the fourth, is dual to the third and concerns how the exogenous signal w enters G: the plant disturbance and the sensor noise are orthogonal, and the sensor noise weighting is normalized and nonsingular.

5.1 $\mathcal{H}_2$-case

Again, the problem is to find a proper, real-rational, internally-stabilizing controller K which minimizes $\|T_{zw}\|_2$. The solution uses a second Hamiltonian matrix,

$$J_2 := \begin{bmatrix} A' & -C_2'C_2 \\ -B_1B_1' & -A \end{bmatrix}$$

Define in addition

$$Y_2 := Ric(J_2), \quad L_2 := -Y_2 C_2'$$

$$A_{L_2} := A + L_2 C_2, \quad B_{1L_2} := B_1 + L_2 D_{21}$$

$$G_f(s) := \left[\begin{array}{c|c} A_{L_2} & B_{1L_2} \\ \hline F_2 & 0 \end{array}\right]$$

Theorem 5 *The unique optimal controller is*

$$K_{opt}(s) := \left[\begin{array}{c|c} A + B_2F_2 + L_2C_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right]$$

Moreover,

$$\min \|T_{zw}\|_2^2 = \|G_c\|_2^2 + \|G_f\|_2^2$$

The first term in the minimum cost, $\|G_c\|_2^2$, is associated with optimal control with state feedback, while the second, $\|G_f\|_2^2$, with optimal filtering.

A sketch of the proof of theorem 5 goes like this. Let K be any internally-stabilizing controller. Setting $v := u - F_2x$, we get as in the proof of theorem 1

$$z = G_cw + Uv$$

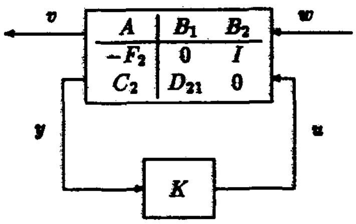
Let T_{vw} denote the transfer matrix from w to v. Then

$$T_{zw} = G_c + UT_{vw}$$

and hence from lemma 3

$$\|T_{zw}\|_2^2 = \|G_c\|_2^2 + \|T_{vw}\|_2^2 \tag{3}$$

Now look at how v is generated:



Note that K stabilizes G iff K stabilizes the above system (the two closed-loop systems have the same A-matrix). So from (3)

$$\min \|T_{zw}\|_2^2 = \|G_c\|_2^2 + \min \|T_{vw}\|_2^2$$

and the problem reduces to minimization of $\|T_{vw}\|_2$. For this latter problem it can be shown that the unique optimal controller is

$$\left[\begin{array}{c|c} A + B_2F_2 + L_2C_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right]$$

and also that

$$\min \|T_{vw}\|_2 = \|G_f\|_2$$

The controller displayed in theorem 5 has a well-known separation structure: the controller equations can be written as

$$\dot{\hat{x}} = A\hat{x} + B_2u + L_2(C_2\hat{x} - y)$$

$$u = F_2\hat{x}$$

F_2 is the optimal feedback gain were x directly measured; L_2 is the optimal filter gain; $\hat{x}$ is the optimal estimate of x.

5.2 $\mathcal{H}_\infty$-case

Strengthen assumption (i) above to read

(i) (A, B_1) is controllable and (C_1, A) is observable

Also, bring in a second Hamiltonian matrix:

$$J_\infty := \begin{bmatrix} A' & C_1'C_1 - C_2'C_2 \\ -B_1B_1' & -A \end{bmatrix}$$

Theorem 6 *There exists an internally-stabilizing controller such that $\|T_{zw}\|_\infty < 1$ iff the following three conditions hold:*

(i) H_∞ and J_∞ belong to dom(Ric)

(ii) $X_\infty := Ric(H_\infty)$ and $Y_\infty := Ric(J_\infty)$ are positive definite

(iii) $Y_\infty^{-1} > X_\infty$ (equivalently, $\rho(X_\infty Y_\infty) < 1$)

Moreover, when these conditions hold, one solution is

$$K_{sub}(s) := \left[\begin{array}{c|c} A + B_1B_1'X_\infty + B_2F_\infty + L_\infty C_2 & -L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$L_\infty := -(Y_\infty^{-1} - X_\infty)^{-1}C_2'$$

The main idea of the proof is to change variables and reduce the problem to a simpler one. Let K be any internally-stabilizing controller. Write the equations for G:

$$\dot{x} = Ax + B_1w + B_2u$$

$$z = C_1x + D_{12}u$$

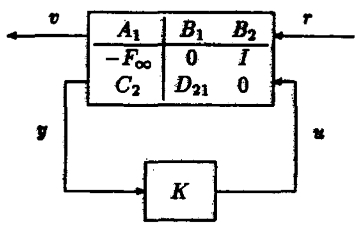
Now change variables by defining

$$r := w - B_1'X_\infty x, \quad v := u - F_\infty x$$

Differentiate $x(t)'X_\infty x(t)$ along solutions, use the Riccati equation for X_∞, and integrate from 0 to ∞ to get

$$\|z\|_2^2 - \|w\|_2^2 = \|v\|_2^2 - \|r\|_2^2$$

Thus $\|T_{zw}\|_\infty < 1$ iff $\|T_{vr}\|_\infty < 1$. The setup for the latter optimization problem is



where $A_1 := A + B_1B_1'X_\infty$. Solution of this latter problem yields the theorem.

The controller displayed in theorem 6 also has an interesting separation structure. Write the equations for G,

$$\dot{x} = Ax + B_1w + B_2u$$

$$z = C_1x + D_{12}u$$

$$y = C_2x + D_{21}w$$

and now those for K_{sub},

$$\dot{\hat{x}} = A\hat{x} + B_1\hat{w}_{worst} + B_2u + L_\infty(\hat{y} - y)$$

$$\hat{y} = C_2\hat{x}$$

$$u = F_\infty\hat{x}$$

$$\hat{w}_{worst} = B_1'X_\infty\hat{x}$$

These latter equations have the structure of an observer-based compensator. Note that F_∞ is a suboptimal state-feedback gain, i.e., a suboptimal controller when the state x is directly available; see theorem 4. Likewise, L_∞ is a suboptimal filter gain. Finally, $w_{worst} := B_1'X_\infty x$ can be interpreted as the worst disturbance input in the sense that it maximizes the quantity $\|z\|_2^2 - \|w\|_2^2$, and $\hat{w}_{worst}$ is its estimate.

6 Conclusion

This paper considers a standard $\mathcal{H}_\infty$ control problem that mimics a standard $\mathcal{H}_2$ problem. It is the simplest possible problem that captures all the essential features of the general case. There are a number of obvious generalizations, many of which have been completed and will appear elsewhere. Assumptions (i) to (iv) in section 5, and the assumptions that $D_{11} = 0, D_{22} = 0$, can be relaxed and similar results obtained using essentially the same methods [Glover and Doyle, 1988], but the formulas are substantially more complicated.

This paper gives necessary and sufficient conditions for the existence of a controller that achieves a given $\mathcal{H}_\infty$-norm bound, and a formula for such a controller when it exists. This controller is the maximum entropy solution [Mustafa and Glover, 1988]. It is relatively easy to use the same methods to parametrize all controllers achieving the norm bound as a linear fractional transformation on a stable contraction. Details will be given in the extended version of this paper. The formulas for all controllers in the general case can be found in [Glover and Doyle, 1988].

Other extensions that we are way too cool to write up include the discrete-time and time-varying cases. Generalizations to infinite dimensional state-space models are the domain of the Mighty Thor.

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