

Seiberg-Witten Floer Homology and Heegaard splittings

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*Ce fut d'abord une étude...
Je finis par trouver sacré le désordre de mon esprit.*
(Rimbaud)

Abstract

The dimensional reduction of Seiberg-Witten theory defines a gauge theory of compact connected three-manifolds. Solutions of the equations modulo gauge symmetries on a three-manifold Y can be interpreted as the critical points of a functional defined on an infinite dimensional configuration space of $U(1)$ -connections and spinors. The original Seiberg-Witten equations on the infinite cylinder $Y \times \mathbb{R}$ are the downward gradient flow of the functional. Thus, it is possible to construct an infinite dimensional Morse theory. The associated Morse homology is the analogue in the context of Seiberg-Witten theory of Floer's instanton homology constructed using Yang-Mills gauge theory. The construction and the properties of this Seiberg-Witten Floer homology are essentially different according to whether the three-manifold Y is a homology sphere or has non-trivial rational homology. In this work we construct the Seiberg-Witten Floer homology for three-manifolds with $b^1(Y) > 0$. We define an associated Casson-like invariant and we prove that it satisfies the expected intersection formula under a Heegaard splitting of the three-manifold.

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1 Introduction

The Seiberg-Witten gauge theory of four-manifold has undergone a rapid and rich development in the two years following its original formulation by Witten in 1994. Following the general strategy illustrated by Atiyah [1], one is lead to investigate the relation between gauge theory of three- and of four-manifolds.

The Seiberg-Witten gauge theory on four-manifolds has a dimensional reduction that leads to equations on a three-manifold. As in the case of Donaldson theory, these equations are the gradient flow of a functional defined on a Banach manifold. The functional was originally introduced by Kronheimer and Mrowka in the proof of the Thom conjecture [19]. As pointed out by Donaldson [13], it is possible to define a Seiberg-Witten-Floer homology of a three-manifold by looking at the critical points and the gradient flow of this functional.

For a period of time after the initial introduction of Seiberg-Witten gauge theory it seemed to be generally accepted that Seiberg-Witten-Floer homology existed and had the expected properties. However, to the best of our knowledge, no detailed exposition of the construction has yet appeared in the literature. The purpose of this work was to present in complete detail, the construction and the properties of the Seiberg-Witten-Floer homology on a three-manifold with non-trivial homology and of the invariant obtained by computing its Euler characteristic. A version of this work appeared as a paper [26]. However, the published version is rather sketchy in many parts, and most of the analytical aspects are not considered. Moreover, it contains some imprecisions and a mistake that was pointed out later. The present version corrects the mistake and fills in the necessary details of the construction.

More recently, a number of results appeared that enlightened the interesting properties of the Seiberg-Witten Floer theory and of the associated Casson-like invariant, with important contributions like [11], [23], [29], [32], [33], [40], [41]. Interesting applications have been considered, as in [3], [4], [34]. Applications to contact structures have been analysed in [10], [21], [24]. Interesting conjectures arise from the Physics literature (see e.g. [5]), where a quantum field theoretic

description of the Floer homology is used. Computations with the three dimensional equations on a Seifert manifold have been worked out in [35]. The study of Seiberg-Witten theory on three-manifolds is a promising field in rapid expansion and many interesting aspects, like surgery formulae, are still to be developed.

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2 Preliminary definitions and notation

Given a compact connected, oriented 4-manifold X , the Seiberg–Witten equations are given in terms of a $U(1)$ -connection \mathbf{A} and a section Ψ of the positive spinor bundle $S^+ \otimes L$ as

$$D_{\mathbf{A}}\Psi = 0, \tag{1}$$

$$F_{\mathbf{A}}^+ = \Psi \cdot \bar{\Psi}, \tag{2}$$

where $D_{\mathbf{A}}$ is the Dirac operator twisted with the connection \mathbf{A} and $F_{\mathbf{A}}^+$ is the self-dual part of the curvature of \mathbf{A} . In the second equation $\Psi \cdot \bar{\Psi}$ represents the 2-form given in local coordinates by

$$(\Psi \cdot \bar{\Psi})_{ij} = \frac{1}{4} \langle e_i e_j \Psi, \Psi \rangle e^i \wedge e^j, \tag{3}$$

where \langle, \rangle is the inner product of sections of $S^+ \otimes L$. The e_i form a local orthonormal basis of sections of TX , which acts on Ψ via Clifford multiplication. The e^i are the dual basis elements of T^*X .

The same equations can be defined on a non-compact 4-manifold X of the form $X = Y \times \mathbb{R}$, where Y is a compact, connected, oriented 3-manifold without boundary.

Any oriented three manifold admits a *Spin*-structure, [22] 2.2.3. A choice of the metric determines a natural “trivial” *Spin*-structure with spinor bundle S . A *Spin_c*-structure is therefore obtained by twisting S with a line bundle L .

Assume a $Spin_c$ structure on Y is given, with spinor bundle $S \otimes L$. We can endow the manifold X with the $Spin_c$ structure defined by

$$S^\pm \otimes L = \pi_1^*(S \otimes L),$$

where $\pi_1 : Y \times \mathbb{R} \rightarrow Y$ is the projection $(y, t) \mapsto y$.

The positive and negative spinor bundles $S^\pm \otimes L$ are isomorphic via Clifford multiplication by dt .

Definition 2.1 *A pair (\mathbf{A}, Ψ) is in a temporal gauge if the dt component of the connection \mathbf{A} is identically zero.*

Remark 2.2 *Any pair (\mathbf{A}, Ψ) is gauge equivalent (on X) to a pair in a temporal gauge.*

An element (\mathbf{A}, Ψ) in a temporal gauge on $Y \times \mathbb{R}$ can therefore be written as a path $(A(t), \psi(t))$. We obtain the dimensional reduction of the gauge theory as follows.

Lemma 2.3 *For a pair (\mathbf{A}, Ψ) in a temporal gauge, the Seiberg–Witten equations (1) and (2) induce the following equations on Y :*

$$\frac{d}{dt}\psi(t) = -\partial_{A(t)}\psi(t), \quad (4)$$

and

$$\frac{d}{dt}A(t) = - * F_{A(t)} + \sigma(\psi(t), \psi(t)), \quad (5)$$

where the imaginary 1-form $\sigma(\psi, \psi)$ is given in local coordinates by $\frac{1}{2} \langle e_i \psi, \psi \rangle e^i$.

Proof: The first equation (4) is obtained by writing the Dirac operator on X as $D_A = \partial_t + \partial_{A(t)}$, where $\partial_{A(t)}$ is the self-adjoint Dirac operator on Y twisted with a time dependent connection $A(t)$.

To obtain the equation (5), write the equation (2) in local coordinates and consider separately the basis elements that are in $\Lambda^2 T^*Y$ and those of the form $e^i \wedge dt$, with e^i the local basis of T^*Y .

This gives equations of the form

$$\frac{1}{2}(F_{it} + \epsilon^{tijk} F_{jk})e^i \wedge dt = \frac{1}{4} \langle e_i e_t \psi, \psi \rangle e^i \wedge dt,$$

where we used that $F^+ = \frac{1}{2}(F + *F)$ on the 4-manifold X . The symbol ϵ is the sign of the permutation $\{tijk\}$ in Σ_4 . Up to composing with the $*$ -isomorphism on Y and identifying the positive and negative spinors via Clifford multiplication by dt , these are the equations (5) of the reduced gauge theory.

QED

The gauge group \mathcal{G} acting on the space of solutions on Y will be the subgroup of $\mathcal{G}(X)$, the group of gauge transformations on X , which consists of translation invariant gauge transformations, $\frac{d\lambda}{dt} = 0$.

The action of the gauge group is given by

$$(A, \psi) \mapsto (A - i\lambda^{-1}d\lambda, \lambda\psi).$$

The equations (4), (5) represent the gradient flow of a functional, defined on the space of connections on Y and sections of $S \otimes L$. This has been introduced in [25], more details have been worked out for instance in [3]. The Floer theory of this functional and its properties have been considered independently by various authors [9], [26], [40], [41].

Definition 2.4 *For a fixed connection A_0 , we define a functional*

$$\mathcal{C}(A, \psi) = \frac{-1}{2} \int_Y (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2} \int_Y \langle \psi, \partial_A \psi \rangle dv. \quad (6)$$

Upon a gauge transformation the functional (6) changes according to

$$\mathcal{C}(A - i\lambda^{-1}d\lambda, i\lambda\psi) = \mathcal{C}(A, \psi) - 2i\pi \int_Y c_1(L) \wedge \lambda^{-1}d\lambda,$$

where $c_1(L)$ is the representative of the first Chern class given by the curvature 2-form.

Let $h(\lambda) = [\frac{i}{2\pi}\lambda^{-1}d\lambda]$ be the class in $H^1(Y, \mathbb{Z})$ representing the the Cartan-Maurer form of λ . Notice that $\pi_0(\mathcal{G}) \cong H^1(Y, \mathbb{Z})$. Since the cohomology class $h(\lambda)$ represents the connected component of the gauge transformation λ in the gauge group, the functional (6) is well defined on the space $\mathcal{B}^0 = \mathcal{A}/\mathcal{G}^0$ of connections and sections, modulo the action of the trivial connected component \mathcal{G}^0 of the gauge group. In some cases, however, this configuration space turns out to be “too large”, in the sense that the corresponding moduli space can loose the compactness. This happens for instance in the case when $c_1(L) = 0$, as will be discussed later. It is therefore convenient to introduce another configuration space which is larger than \mathcal{B} but smaller than \mathcal{B}^0 , on which (6) is well defined as an \mathbb{R} -valued functional. This was first introduced by R.G. Wang [41].

Consider the subgroup of \mathcal{G}

$$\tilde{\mathcal{G}} = \{\lambda \in \mathcal{G} \mid \frac{i}{2\pi} \int_Y c_1(L) \wedge \lambda^{-1}d\lambda = 0.\}. \quad (7)$$

The subgroup $\tilde{\mathcal{G}}$ is the kernel of the homomorphism

$$\begin{aligned} \xi : \mathcal{G} &\rightarrow \mathbb{Z} \\ \xi(\lambda) &= \frac{i}{2\pi} \int_Y c_1(L) \wedge \lambda^{-1}d\lambda. \end{aligned}$$

The induced homomorphism $\xi : H^1(Y, \mathbb{Z}) \rightarrow \mathbb{Z}$ determines the subgroup

$$\tilde{H}^1(Y, \mathbb{Z}) \equiv \{h \in H^1(Y, \mathbb{Z}) \mid \langle c_1(L) \cup h, [Y] \rangle = 0\}.$$

Let $H = H^1(Y, \mathbb{Z}) / \tilde{H}^1(Y, \mathbb{Z})$. We consider the configuration space $\mathcal{B}_H = \mathcal{A} / \tilde{\mathcal{G}}$. This is a covering of \mathcal{B} with covering group H .

In particular let $\hat{\mathcal{A}}$ be the space of irreducible pairs (A, ψ) , where ψ is not identically zero. We consider this space modulo the action of $\tilde{\mathcal{G}}$,

$$\hat{\mathcal{B}}_H = \hat{\mathcal{A}} / \tilde{\mathcal{G}},$$

and restrict the functional \mathcal{C} of (6) over $\hat{\mathcal{B}}_H$.

We choose to topologize the space of connections and sections $\hat{\mathcal{A}}$ with a fixed L_k^2 Sobolev norm, with $k > 2$, and the space of gauge transformations \mathcal{G} with the L_{k+1}^2 norm. This makes $\hat{\mathcal{B}}_H$ into a Banach manifold.

The functional (6) is the analogue of the Chern–Simons functional in Donaldson theory.

3 Extremals and Morse Index

The critical points of the functional \mathcal{C} of (6) are the pairs (A, ψ) up to gauge transformation that satisfy

$$\begin{aligned} \partial_A \psi &= 0, \\ *F_A &= \sigma(\psi, \psi). \end{aligned} \tag{8}$$

In this section we prove that under a suitable perturbation the space of critical points of the functional \mathcal{C} modulo gauge transformations is an oriented compact zero dimensional manifold. We compute the Hessian of the functional and define the relative Morse index of critical points.

3.1 Deformation Complex

Let \mathcal{M}_C denote the set of critical points of \mathcal{C} in the Banach manifold $\hat{\mathcal{B}}$, i.e. the set of solutions of (8) modulo gauge. In the following we shall always write Λ^* for the complex of imaginary valued forms. The virtual tangent space of \mathcal{M}_C is given by

$$Ker(T) / Im(G),$$

where the operator

$$\begin{aligned} G : \Lambda_{L_{k+1}}^0(Y) &\rightarrow \Lambda_{L_k}^1(Y) \oplus \Gamma_{L_k}(S \otimes L), \\ G|_{(A, \psi)}(f) &= (-df, f\psi) \end{aligned} \tag{9}$$

is the infinitesimal action of the gauge group and the map T is the linearization of the equation (8) at a pair (A_0, ψ_0) ,

$$T : \Lambda_{L_k^2}^1(Y) \oplus \Gamma_{L_k^2}(S \otimes L) \rightarrow \Lambda_{L_{k-1}^2}^1(Y) \oplus \Gamma_{L_{k-1}^2}(S \otimes L),$$

$$T|_{(A,\psi)}(\alpha, \phi) = \begin{cases} *d\alpha - \sigma(\psi, \phi) - \sigma(\phi, \psi) \\ \partial_A \phi + \alpha\psi. \end{cases} \quad (10)$$

These operators fit into an elliptic complex, the *deformation complex* C^* of equation (8)

$$0 \rightarrow \Lambda_{L_k^2}^0(Y) \oplus \Lambda_{L_k^2}^1(Y) \oplus \Gamma_{L_k^2}(S \otimes L) \xrightarrow{L} \Lambda_{L_{k-1}^2}^0(Y) \oplus \Lambda_{L_{k-1}^2}^1(Y) \oplus \Gamma_{L_{k-1}^2}(S \otimes L) \rightarrow 0,$$

where the operator L is

$$L|_{(A,\psi)}(f, \alpha, \phi) = \begin{cases} T|_{(A,\psi)}(\alpha, \phi) + G|_{(A,\psi)}(f) \\ G^*|_{(A,\psi)}(\alpha, \phi) \end{cases}$$

Here the operator $G_{A,\psi}^*$ is the formal adjoint of the linearization of the group action $G_{A,\psi}$. We have

$$G^*|_{(A,\psi)}(\alpha, \phi) = -d^*\alpha + iIm \langle \psi, \phi \rangle.$$

Lemma 3.1 *The operator L has index zero.*

Proof: Up to zero order operators, L reduces to the Dirac operator (which has index zero on a three manifold), and the elliptic complex

$$0 \rightarrow \Lambda^0(Y) \oplus \Lambda^1(Y) \xrightarrow{D} \Lambda^0(Y) \oplus \Lambda^1(Y) \rightarrow 0,$$

where D is given by

$$D = \begin{pmatrix} 0 & d^* \\ d & *d \end{pmatrix}. \quad (11)$$

The index of D is the Euler characteristic that is trivial on a closed three manifold Y .

QED

The virtual dimension of the space of critical points is given by the first Betti number $h^1(C^*)$ of the short complex. Unlike the four dimensional problem, since

$$0 = Ind(L) = -\chi(C^*) = h^1(C^*) - h^2(C^*),$$

the index computation does not give enough information on the dimension $h^1(C^*)$. We shall introduce a suitable perturbation so that we can always restrict to the case with $h^1(C^*) = 0$, that is to a zero-dimensional moduli space.

3.2 Compactness

In order to prove that the set of critical points of $\tilde{\mathcal{C}}$ modulo gauge transformations is compact we use Sobolev techniques.

Theorem 3.2 *Any sequence $\{(A_j, \psi_j)\}$ of solutions of (8) has a subsequence that converges with all derivatives (up to gauge transformations) to another solution.*

It is enough to show that there exists a sequence of gauge transformations $\{\lambda_j\}$ such that all L_k^2 norms of $(A_j - i\lambda_j^{-1}d\lambda_j, \lambda_j\psi_j)$ are bounded. The result then follows by the Sobolev embedding theorem.

Lemma 3.3 *A section ψ of the spinor bundle $S \otimes L$ that is a solution of (8) has bounded L^2 norm on Y .*

The argument follows the line of the analogous result proved by Kronheimer and Mrowka [19] for the Seiberg-Witten equations on a compact 4 manifold. The inequality

$$|\psi| \leq \max(0, -\kappa),$$

holds, with κ the scalar curvature.

Lemma 3.4 *Suppose given a sequence $\{(A_j, \psi_j)\}$ of solutions of (8) on Y , and a fixed connection A_0 . Then there exists a sequence of gauge transformations λ_j , such that the $A_j - A_0 - i\lambda_j^{-1}d\lambda_j$ are co-closed 1-forms.*

Proof: Consider the tangent space to the set of solutions of (8). At the connection A_0 the subspace spanned by the infinitesimal action of the gauge group is the image of d . Thus we can find a sequence of gauge transformations that makes the 1-forms $A_j - A_0$ orthogonal to the image of d , i.e. in the kernel of d^* .

QED

We also need the following gauge fixing condition (this is the analogue of the four dimensional gauge fixing condition, [30] Lemma 5.3.1.

Lemma 3.5 *The gauge transformations λ_i of Lemma 3.4 can be chosen so that the sequence $\lambda_i(A_i) - A_0$ satisfies*

$$\|\lambda_i A_i - A_0\|_{L_k^2}^2 \leq C \|F_{\lambda_i A_i}\|_{L_{k-1}^2} + K.$$

Proof: We follow the analogous argument given [30] in the four-dimensional case. Consider the operator

$$(d^*, d) : \Lambda_{L_k^2}^1 \rightarrow \Lambda_{L_{k-1}^2}^2 \oplus \Lambda_{L_{k-1}^2}^0,$$

and a decomposition of the 1-form $\lambda A - A_0$ in a harmonic component and a component that is orthogonal to the harmonic forms,

$$\lambda A - A_0 = h + \beta.$$

Then β satisfies

$$\|\beta\|_{L_k^2}^2 \leq C\|(d^*(\beta), d(\beta))\|_{L_{k-1}^2}^2.$$

Since we can assume that $d^*(\beta) = 0$ by Lemma 3.4, and $d(\lambda A - A_0) = F_A - F_{A_0}$, we have

$$\|\beta\|_{L_k^2}^2 \leq C\|F_A\|_{L_{k-1}^2}^2 + C_1.$$

The harmonic component h may not be bounded, but we can rewrite $h = h_1 + h_2$, where h_2 is a harmonic one-form in the lattice Σ of imaginary valued harmonic 1-forms with periods in $2\pi i\mathbb{Z}$. The form h_1 is in the quotient $H^1(Y, i\mathbb{R})/\Sigma$, which is a compact torus. Thus, the norm of h_1 is bounded, $\|h_1\| \leq C_2$, and the term h_2 can be eliminated with a gauge transformation. In fact h_2 can be written as $h_2 = h(\lambda_2)$ for some $\lambda_2 : Y \rightarrow U(1)$, and $\lambda_2(\lambda A - A_0)$ satisfies the required estimate, with constant $K = CC_2 + C_1$.

QED

Notice that Lemma 3.5 requires the use of the full gauge group \mathcal{G} , and it no longer holds if we restrict to the identity component \mathcal{G}^0 .

Proof of theorem 3.2: For simplicity of notation we can assume that the forms $A_j - A_0$ are coclosed and satisfy the estimate of Lemma 3.5. We can write $(d + d^*)A_j$ instead of dA_j and $(dd^* + d^*d)A_j$ instead of d^*dA_j . Since $d + d^*$ and $dd^* + d^*d$ are elliptic operators, we can use elliptic estimates to bound Sobolev norms.

In fact by lemma 3.3 and the second equation of (8) we have a bound on the L^2 norm of $dA_j = F_{A_j}$. This immediately gives a bound on the L_1^2 norm of A_j , because of Lemma 3.5. The first equation in (8) and lemma 3.3 give a bound on the L_1^2 norm of the sections ψ_j , since the Dirac operator is elliptic. In fact we have an estimate

$$\|\psi_j\|_{L_1^2} \leq c(\|\partial_A \psi_j\|_{L^2} + \|\psi_j\|_{L^2}).$$

Moreover, we can obtain a bound on the L^2 norm of $d^*F_{A_j}$. This can be computed in local coordinates from the second equation of (8).

We have

$$\|d^*F\|_{L^2} \leq \sum_{i \neq j} |2 \langle \psi, e_i \nabla_j \psi \rangle|^2$$

and

$$|\langle \psi, e_i \nabla_j \psi \rangle| \leq c\|\psi\|_{L_1^2}\|\psi\|_{L^2}.$$

Therefore we obtain a bound on $\|d^*F_{A_j}\|_{L^2}$ in terms of the bounds on $\|\psi\|_{L^2}$ and $\|\psi\|_{L_1^2}$.

Now we can use a bootstrapping argument to bound higher Sobolev norms; for instance we get

$$\|A_j\|_{L^2_2} \leq c(\|d^*dA_j\|_{L^2} + \|A_j\|_{L^2}).$$

We can bound the higher Sobolev norms of ψ_j by the elliptic estimates applied to the Dirac operator (see the analogous argument in the four-dimensional case [30]).

QED

Corollary 3.6 *A consequence of theorem 3.2 is that we can improve the degree of regularity of the elements in \mathcal{M}_C . Namely, if we consider the moduli space \mathcal{M}_C^k inside $\hat{\mathcal{B}}_{L^2}^k$, with $k' > k$, then the natural map $\mathcal{M}_C^{k'} \rightarrow \mathcal{M}_C^k$ is a diffeomorphism.*

The proof of corollary 3.6 follows as in the four-dimensional case [30].

Theorem 3.2 holds when we consider critical points in the space $\hat{\mathcal{B}}$, i.e. when we identify points under the action of $H^1(Y, \mathbf{Z})$ on $\hat{\mathcal{B}}^0$. For manifolds Y with $b^1(Y) > 0$, the moduli space \mathcal{M}_C^0 in the configuration space \mathcal{B}^0 is non-compact. In fact, it is an $H^1(Y, \mathbf{Z})$ covering of \mathcal{M}_C . The moduli space \mathcal{M}_C^H inside \mathcal{B}_H is compact iff the quotient group H is finite. In fact, \mathcal{M}_C^H is an H covering of \mathcal{M}_C . Notice that the quotient H is either trivial or a copy of \mathbf{Z} , hence the moduli space \mathcal{M}_C^H is either \mathcal{M}_C or a \mathbf{Z} -covering of it, depending on the Chern class $c_1(L)$.

3.3 Perturbation

The condition $\text{Ker}(L) = 0$ in the deformation complex ensures surjectivity. Therefore, when it is satisfied at all solutions, by the implicit function theorem on Banach manifolds we have that \mathcal{M}_C is a smoothly embedded submanifold of \mathcal{B} . However, in general we also have to consider the presence of reducible solutions, i.e. solutions with $\psi \equiv 0$ and a non-trivial stabiliser of the action of the gauge group (the group of constant gauge transformations). The corresponding moduli space fails to be a smooth manifold at such points. Thus, in order to avoid reducible solutions, we consider a perturbed functional, so that the linearization at a solution of the perturbed equations will have L surjective. There is a natural choice of the perturbation, which is simply the three dimensional reduction of the perturbed Seiberg–Witten equations in four dimensions [3], [19], [43]. In four dimensions these are

$$\begin{aligned} D_A\psi &= 0, \\ F_A^+ &= \psi \cdot \bar{\psi} + \delta, \end{aligned}$$

where δ is a self-dual imaginary 2-form.

The corresponding dimensional reduction gives

$$\frac{d}{dt}\psi = -\partial_A\psi \quad (12)$$

and

$$\frac{d}{dt}A = \sigma(\psi, \psi) - *F_A + 2\rho, \quad (13)$$

where ρ is an imaginary 1-form on Y such that $\delta = dt \wedge \rho + *(dt \wedge \rho)$, as explained in [3].

These equations can be thought of as the downward gradient flow of a functional

$$\tilde{\mathcal{C}}(A, \psi) = \mathcal{C}(A, \psi) - 2 \int_Y (A - A_0) \wedge *\rho. \quad (14)$$

The functional (14) has the same behaviour as (6) under gauge transformations, since the form ρ is divergence-free. In fact, it is easy to see that the equations

$$\begin{aligned} \partial_A\psi &= 0, \\ *F_A &= \sigma(\psi, \psi) + 2i\rho \end{aligned}$$

and the expression

$$d^*\sigma(\psi, \psi) = \frac{1}{2}(\langle \partial_A\psi, \psi \rangle - \langle \psi, \partial_A\psi \rangle)$$

imply that the perturbation ρ is co-closed.

The critical points of the perturbed functional (14) satisfy the equations

$$\begin{aligned} \partial_A\psi &= 0 \\ *F_A - \sigma(\psi, \psi) - 2\rho &= 0. \end{aligned} \quad (15)$$

In order to show that this choice of perturbation has the expected property we need the following simple computation.

Lemma 3.7 *The following identity holds:*

$$\frac{1}{2} \int_Y \langle \alpha\psi, \phi \rangle dv = \int_Y \alpha \wedge *\sigma(\psi, \phi) = - \langle \alpha, \sigma(\psi, \phi) \rangle,$$

where ψ and ϕ are sections of $S \otimes L$ and α is a 1-form.

Proof: In local coordinates

$$\begin{aligned} *\sigma(\psi, \phi) &= \frac{1}{2} \langle e_i\psi, \phi \rangle e^j \wedge e^k, \\ \alpha \wedge *\sigma(\psi, \phi) &= \langle \alpha_i e_i\psi, \phi \rangle dv. \end{aligned}$$

QED

We use it to prove the following lemma.

Lemma 3.8 *Assume that the Chern class $c_1(L)$ is non-trivial. For a generic (Baire second category) set of co-closed perturbations ρ that satisfy the condition $*\rho \neq i\pi c_1(L)$, no reducible solution arises among the critical points of the functional (14). Moreover, the operator L of the deformed complex \tilde{C}^* obtained by linearization at a solution of the perturbed equations is surjective. Hence the dimension of the critical set modulo the action of the gauge group is zero, since $h^1(\tilde{C}^*) = 0$.*

Proof: Consider the operator

$$\tilde{L}|_{(A,\psi,\rho)}(\alpha, \phi, \eta) = -2\eta + L(\alpha, \phi).$$

We first prove that this operator is surjective. We know that L is a Fredholm operator, hence \tilde{L} has a closed range. Therefore it is sufficient to prove that \tilde{L} has dense range.

Let (β, ξ, g) be an element that is L^2 -orthogonal to the range of \tilde{L} . Then (β, ξ, g) is in the kernel of the adjoint, hence by elliptic regularity we can consider the L^2 pairing of L_k^2 and L_{-k}^2 ,

$$\begin{aligned} \langle \beta, - * d\alpha - df + \sigma(\psi, \phi) + \sigma(\phi, \psi) - 2\eta \rangle + \langle \xi, \partial_A \phi + \alpha\psi + f\psi \rangle + \\ + \langle g, G^*(\alpha, \phi) \rangle = 0. \end{aligned}$$

By varying η we force $\beta \equiv 0$. The vanishing of

$$\langle \xi, \partial_A \phi + \alpha\psi + f\psi \rangle + \langle g, G^*(\alpha, \phi) \rangle$$

gives an equation $\Delta g + 1/2g|\psi|^2 = 0$ which implies $g \equiv 0$ by the maximum principle. Then by varying ϕ and α we get $\partial_A \xi = 0$ and $\sigma(\xi, \psi) + \sigma(\psi, \xi) = 0$. The latter is satisfied if ξ is an imaginary multiple of ψ , $\xi = i\lambda\psi$, where neither of the two vanishes. Both ξ and ψ are in the kernel of ∂_A , thus if either of them vanishes on an open set it has to vanish identically (and we know that ψ is not identically zero). If we have $\xi = i\lambda\psi$, we obtain that ξ is identically zero as a consequence of the vanishing of the inner product $\langle \xi, f\psi \rangle$ for arbitrary smooth compactly supported functions f .

Consider the set \mathcal{W} that fibres over the co-closed 1-forms assigning to ρ the set of solutions of the corresponding perturbed equations modulo gauge. Since \tilde{L} is surjective, zero is a regular value and therefore \mathcal{W} is a smooth manifold. Now by standard Fredholm techniques we see that the projection map

$$\mathcal{W} \rightarrow Ker(d^*) \subset \Lambda^1(Y),$$

$$(A, \psi, \rho) \mapsto \rho,$$

linearizes to a Fredholm projection

$$Ker(\tilde{L}) \rightarrow \Lambda^1(Y).$$

This implies that the Morse Sard lemma applies, and the set of regular values of the projection map is a Baire second category set. At a regular value ρ the set of solutions of (8) modulo gauge is therefore a smooth submanifold of \mathcal{W} which is cut out transversally. The transversality property ensures that the operator L computed at these solutions is surjective.

The condition $*\rho \neq \pi_{c_1}(L)$, which guarantees that no reducible solution arises, comes from the analogous result in the four dimensional theory and still gives an open dense condition.

QED

Corollary 3.9 *The moduli space \mathcal{M}_C of critical points of $\tilde{\mathcal{C}}$ modulo gauge transformations is a discrete set of points. The moduli spaces \mathcal{M}_C^0 and \mathcal{M}_C^H are given by copies of \mathcal{M}_C for each point of $H^1(Y, \mathbb{Z})$ and H respectively.*

Suppose that $c_1(L)$ vanishes but we still have $b^1(Y) > 0$. In this case reducible solutions can be perturbed away adding a harmonic 1-form to the curvature equation. In the case of a homology sphere reducible solutions can not be perturbed away. In fact, for a perturbation $\rho = *d\theta$ we would have the reducible solution $(\theta, 0)$. The case of manifolds with $b^1(Y) = 0$ has been analysed in [40]. It is remarkable that for rational homology spheres the reducible solution leads to a subtle metric dependence problem which is not present in the case with $b^1(Y) > 0$.

3.4 Orientation

An orientation of the moduli space of critical points of $\tilde{\mathcal{C}}$ can be constructed following the procedure illustrated in [12]. A trivialization of the determinant line bundle $Det(L)|_{(A,\psi)}$ determines an orientation of the tangent bundle of \mathcal{M}_C if we show that the action of the gauge group is orientation preserving. We can orient $Det(\partial_A)$ with the orientation induced on the bundle of spinors by the complex structure of the bundle S^\pm on X . A choice of an orientation for the vector space $H^1(Y, \mathbb{R})$ will orient $Det(D)$, where D is the operator defined in (11). This gives an orientation of the determinant line bundle $Det(L)|_{(A,\psi)}$.

In order to have an induced orientation on the moduli space it is sufficient to show that the action of the gauge group preserves the orientation.

Lemma 3.10 *The action of the gauge group on the space of solutions of (8) preserves the orientation of the tangent space.*

Proof: The action of \mathcal{G} , $\lambda(A, \psi) = (A - i\lambda^{-1}d\lambda, \lambda\psi)$, induces an isomorphism

$$\begin{aligned} \lambda^* : Ker(L|_{(A,\psi)}) &\rightarrow Ker(L|_{\lambda(A,\psi)}), \\ (f, \alpha, \phi) &\mapsto (f, \alpha, \lambda\phi). \end{aligned}$$

This map is complex linear on the tangent space of S , hence the orientation is well defined on \mathcal{M}_C .

QED

3.5 The Hessian

The critical points of the functional \mathcal{C} (or of the perturbed one $\tilde{\mathcal{C}}$), which means the translation invariant solutions of (4) and (5) on $Y \times \mathbb{R}$, play the same role as the flat connections in Donaldson theory. Flat connections are in fact the critical points of the Chern–Simons functional.

Some technical difficulties arise in defining the Morse index of a critical point. This is defined as the number of negative eigenvalues of the Hessian. In order to compute the Hessian we can use the following results.

Theorem 3.11 *Consider a parametrised family*

$$(A_s, \psi_s) = (A, \psi) + s(\alpha, \phi)$$

of connections and sections. The linear part of the increment, that is the coefficient of s in

$$\tilde{\mathcal{C}}(A_s, \psi_s) - \tilde{\mathcal{C}}(A, \psi),$$

is a 1-form on the infinite dimensional space of connections and spinors which is identified via the metric with the gradient of $\tilde{\mathcal{C}}$. The coefficient of $s^2/2$ is a 2-form which induces an operator T on the tangent space of \mathcal{B} at (A, ψ) . This operator can be identified via the metric with the linearization (10) of the equations of the gauge theory. At a critical point this will be the Hessian.

Proof: The explicit form of the increment $\tilde{\mathcal{C}}(A_s, \psi_s) - \tilde{\mathcal{C}}(A, \psi)$ is

$$\begin{aligned} & \frac{s}{2} \left(- \int_Y \alpha \wedge F_A + (A - A_0) \wedge d\alpha + \right. \\ & \int_Y (\langle \psi, \alpha\psi \rangle + \langle \phi, \partial_A \psi \rangle + \langle \psi, \partial_A \phi \rangle) dv \\ & \quad \left. + 4 \int_Y \alpha \wedge * \rho \right) + \\ & \frac{s^2}{2} \left(- \int_Y \alpha \wedge d\alpha + \int_Y (\langle \phi, \alpha\psi \rangle + \langle \psi, \alpha\phi \rangle) dv \right. \\ & \quad \left. + \int_Y \langle \phi, \partial_A \phi \rangle dv \right), \end{aligned}$$

plus higher order terms.

Using lemma 3.7 we can write the first order increment as

$$\begin{aligned} \mathcal{F}|_{(A, \psi, \rho)}(\alpha, \phi) &= - \int_Y i\alpha \wedge (F_A - 2 * \rho - * \sigma(\psi, \psi)) + \\ & \quad + \frac{1}{2} \int_Y \langle \phi, \partial_A \psi \rangle + \langle \partial_A \psi, \phi \rangle. \end{aligned} \tag{16}$$

This is the inner product of the tangent vector (α, ϕ) with the gradient flow (12), (13).

The Hessian is a quadratic form in the increment (α, ϕ) , which is a vector in the L_k^2 -tangent space. By means of lemma 3.7 the quadratic term can be rewritten as

$$\begin{aligned} & \frac{s^2}{2} \left(- \int_Y \alpha \wedge d\alpha + 2Re \int_Y \langle \phi, \alpha\psi \rangle + \int_Y \langle \phi, \partial_A \phi \rangle \right) = \\ & \frac{s^2}{2} \left(- \int_Y \alpha \wedge d\alpha + \int_Y \alpha \wedge (*\sigma(\phi, \psi) + *\sigma(\psi, \phi) + Re \int_Y \langle \phi, \alpha\psi + \partial_A \phi \rangle) \right). \end{aligned}$$

Thus we have the Hessian of the form

$$\nabla \mathcal{F}|_{(A, \psi, \rho)}(\alpha, \phi) = \langle \alpha, *d\alpha - \sigma(\psi, \phi) - \sigma(\phi, \psi) \rangle + Re \langle \phi, \partial_A \phi + \alpha\psi \rangle. \quad (17)$$

The first term is the L^2 -inner product of forms and the second is the L^2 -inner product of sections of $S \otimes L$.

The quadratic form (17) induces an operator on the L_k^2 -tangent bundle of \mathcal{B}_H , which is the same as the linearization T of the flow equations. At a critical point this is the Hessian of the functional $\tilde{\mathcal{C}}$. In fact the tangent space of \mathcal{B} at (A, ψ) is given by the pairs (α, ϕ) that satisfy $G^*(\alpha, \phi) = 0$. In fact, the following holds.

Lemma 3.12 *the following relation is satisfied for all (α, ϕ) :*

$$G_{A, \psi}^*(T_{(A, \psi)}(\alpha, \phi)) = 0.$$

Proof: We have

$$\begin{aligned} G_{A, \psi}^*(T_{(A, \psi)}(\alpha, \phi)) &= -d^*(*d\alpha - \sigma(\psi, \phi) - \sigma(\phi, \psi)) + \\ &+ iIm \langle \psi, \partial_A \phi + \alpha\psi \rangle = d^*(\sigma(\psi, \phi) + \sigma(\phi, \psi)) + iIm \langle \psi, \partial_A \phi \rangle = \\ &= -\frac{1}{2} \langle \psi, \partial_A \phi \rangle + \frac{1}{2} \langle \partial_A \phi, \psi \rangle + iIm \langle \psi, \partial_A \phi \rangle = 0, \end{aligned}$$

using the fact that $\partial_A \psi = 0$ and that

$$d^*\sigma(\psi, \psi) = \frac{1}{2}(\langle \partial_A \psi, \psi \rangle - \langle \psi, \partial_A \psi \rangle).$$

QED

Since (A, ψ) is a point in \mathcal{M}_C , by elliptic regularity we can regard the operator T at a critical point as an operator that maps the L_k^2 -tangent space to itself. This completes the proof of Theorem 3.11.

QED

4 Homology

In this section we shall assign a *Morse index* to the critical points of $\tilde{\mathcal{C}}$, upon fixing the Morse index of one particular critical point. In order to construct the Floer homology, we shall consider paths of steepest descent that connect critical points of relative Morse index equal to one. The boundary operator of the Floer homology is constructed by counting these paths with their orientation as in [15].

4.1 Gradient Flow

The analysis presented in this and the following two sections is part of a joint work with B.L. Wang and will appear in [27].

We introduce suitable moduli spaces of gradient flow lines connecting critical points. We prove in the following that, generically, these are smooth manifolds that are cut out transversely, hence with the dimension prescribed by the index theorem. This property depends on an accurate choice of a class of perturbations for the gradient flow equations.

Consider the space connections and sections (\mathbf{A}, Ψ) on $Y \times \mathbb{R}$ with the product metric $g + dt^2$, topologized with the weighted Sobolev norms [25] [15]. Here we choose the weight $e_\delta(t) = e^{\tilde{\delta}t}$, where $\tilde{\delta}$ is a smooth function with bounded derivatives, $\tilde{\delta} : \mathbb{R} \rightarrow [-\delta, \delta]$ for some fixed positive number δ , such that $\tilde{\delta}(t) \equiv -\delta$ for $t \leq -1$ and $\tilde{\delta}(t) \equiv \delta$ for $t \geq 1$. The $L_{k,\delta}^2$ norm is defined as $\|f\|_{2,k,\delta} = \|e_\delta f\|_{2,k}$; the weight e_δ imposes an exponential decay as asymptotic condition along the cylinder.

Proposition 4.1 *Let Y be a compact oriented three-manifold endowed with a fixed Riemannian metric g_0 . Consider the cylinder $Y \times \mathbb{R}$ with the metric $g_0 + dt^2$. The weighted Sobolev spaces $L_{k,\delta}^2$ on the manifold $Y \times \mathbb{R}$ satisfy the following Sobolev embeddings.*

(i) *The embedding $L_{k,\delta}^2 \hookrightarrow L_{k-1,\delta}^2$ is compact for all $k \geq 1$.*

(ii) *If $k > m + 2$ we have a continuous embedding $L_{k,\delta}^2 \hookrightarrow \mathcal{C}^m$.*

(iii) *If $k > m + 3$ the embedding $L_{k,\delta}^2 \hookrightarrow \mathcal{C}^m$ is compact.*

(iv) *If $2 < k'$ and $k \leq k'$ the multiplication map $L_{k,\delta}^2 \otimes L_{k,\delta}^2 \xrightarrow{m} L_{k,\delta/2}^2$ is continuous.*

Consider a metric $g_t + dt^2$ on the cylinder $Y \times \mathbb{R}$ such that for a fixed T we have $g_t \equiv g_0$ for $t \geq T$ and $g_t \equiv g_1$ for $t \leq -T$ and g_t varies smoothly when $t \in [-1, 1]$. The same Sobolev embedding theorems hold for the $L_{k,\delta}^2$ spaces on $(Y \times \mathbb{R}, g_t + dt^2)$.

Choose smooth representatives (A_0, ψ_0) and (A_1, ψ_1) of a and b in \mathcal{M} . Choose a smooth path $(A(t), \psi(t))$ such that for $t \leq 0$ it satisfies $(A(t), \psi(t)) \equiv (A_0, \psi_0)$ and for $t \geq 1$ it is $(A(t), \psi(t)) \equiv (A_1, \psi_1)$.

Let $\mathcal{A}_{k,\delta}(a, b)$ be the space of pairs (\mathbf{A}, Ψ) on $Y \times \mathbb{R}$ satisfying

$$(\mathbf{A}, \Psi) \in (A(t), \psi(t)) + L_{k,\delta}^2(\Lambda^1(Y \times \mathbb{R}) \oplus \Gamma(S^+ \otimes L)).$$

Consider the group $\mathcal{G}_{k+1,\delta}$ of gauge transformations in $\mathcal{G}(Y \times \mathbb{R})$, locally modelled on $L_{k+1,\delta}^2(\Lambda^0(Y \times \mathbb{R}))$, that decay to asymptotic values $\lambda_{\pm\infty}$ which satisfy condition (7) on Y . This gauge group acts on $\mathcal{A}_{k,\delta}(a, b)$ and we can form the quotient $\mathcal{B}_{k,\delta}^H(a, b)$.

We consider the perturbed gradient flow equations for a path $(A(t), \psi(t))$,

$$\frac{d}{dt}\psi(t) = -\partial_{A(t)}\psi(t) \quad (18)$$

and

$$\frac{d}{dt}A(t) = \sigma(\psi(t), \psi(t)) - *F_{A(t)} + 2i\rho + 2q_{(\mathbf{A}, \Psi)}(t). \quad (19)$$

Equations (18) and (19) can be rewritten in terms of pairs (\mathbf{A}, Ψ) in the form

$$D_{\mathbf{A}}\Psi = 0 \quad (20)$$

and

$$F_{\mathbf{A}}^+ = \Psi \cdot \bar{\Psi} + i\mu + P_{(\mathbf{A}, \Psi)}. \quad (21)$$

The perturbation $P = *q + q \wedge dt$ is a function of $\mathcal{B}^H(a, b)$ to $\pi^*(\Lambda^1(Y)) \cong \Lambda^{2+}(Y \times \mathbb{R})$, such that the corresponding equations in a temporal gauge (18) and (19) are preserved under the action of \mathbb{R} by reparametrizations of the path $(A(t), \psi(t))$. The class of such perturbations is described as follows.

Definition 4.2 *The space of perturbations \mathcal{P} is the space of maps*

$$P : \mathcal{B}_{k,\delta}^H(a, b) \rightarrow L_{k,\delta}^2\Lambda^{2+}(Y \times \mathbb{R}),$$

that satisfy the following conditions.

(1) $P_{(\mathbf{A}, \Psi)} = *q_{(\mathbf{A}, \Psi)}(t) + q_{(\mathbf{A}, \Psi)}(t) \wedge dt$, where $q_{(\mathbf{A}, \Psi)}(t)$ satisfies

$$q_{(\mathbf{A}, \Psi)^T}(t) = q_{(\mathbf{A}, \Psi)}(t + T),$$

where $(\mathbf{A}, \Psi)^T$ is the T -translate of (\mathbf{A}, Ψ) , namely the pair that is represented in a temporal gauge by $(A(t + T), \psi(t + T))$.

(2) the $L_{k,\delta}^2$ -norm of the perturbation $P_{(\mathbf{A}, \Psi)}$ is bounded uniformly with respect to (\mathbf{A}, Ψ) ;

(3) the linearization $\mathcal{D}P_{(\mathbf{A}, \Psi)}$ is a compact operator from the $L_{k,\delta}^2$ to the $L_{k-1,\delta}^2$ tangent spaces.

(4) for all $l \leq k - 1$, we have that

$$\|q_{(\mathbf{A}, \Psi)}(t)\|_{L_t^2} \leq C_l \|\nabla \tilde{\mathcal{C}}(A(t), \psi(t))\|_{L_t^2}$$

in the L^2_t -norm on $Y \times \{t\}$, for all $|t| > T$, where

$$\nabla \tilde{\mathcal{C}}(A(t), \psi(t)) = (-\partial_{A(t)}\psi(t), \sigma(\psi(t), \psi(t)) - *F_{A(t)} + 2\rho) \quad (22)$$

is the gradient flow of the functional $\tilde{\mathcal{C}}$, and $0 < C_l < 1$;

With a perturbation in the class \mathcal{P} the equations (18) and (19) are invariant with respect to the action of \mathbb{R} by translations along the gradient flow lines, that is if $(A(t), \psi(t))$ is a solution of (18) and (19), then $(A(t+T), \psi(t+T))$ is also a solution for any $T \in \mathbb{R}$.

An example of perturbation with these properties has been constructed by Froyshov [17].

Proposition 4.3 *The class of perturbations introduced by Froyshov in [17] is in our class \mathcal{P} .*

According to Froyshov's construction, for fixed smooth compactly supported functions η_1, η_2 , with $\text{supp}(\eta_1) \subset [-1, 1]$ and $\eta_2|_I(t) = t$ on an interval I containing all the critical values of $\tilde{\mathcal{C}}$, a function $h : \mathcal{B}_{k,\delta}^H(a, b) \rightarrow \mathcal{C}^m(\mathbb{R})$ is defined as

$$h_{(\mathbf{A}, \Psi)}(T) = \int_{\mathbb{R}} \eta_1(s-T) \eta_2 \left(\int_{\mathbb{R}} \eta_1(t-s) \tilde{\mathcal{C}}(A(t), \psi(t)) dt \right) ds.$$

Let $\Lambda_{\Xi}^2(Y \times \mathbb{R})$ be the set of \mathcal{C}^m 2-forms ω that are compactly supported in $Y \times \Xi$, where Ξ is the complement of a union of small intervals centered at the critical values of $\tilde{\mathcal{C}}$. Froyshov's perturbation is constructed by setting

$$P_{(\mathbf{A}, \Psi)} = (h_{(\mathbf{A}, \Psi)}^*(\omega))^+,$$

where $h_{(\mathbf{A}, \Psi)}^*(\omega)$ is the pullback of ω along the map $Id_Y \times h_{(\mathbf{A}, \Psi)} : Y \times \mathbb{R} \rightarrow Y \times \mathbb{R}$.

As shown in [17], the function $h_{(\mathbf{A}, \Psi)}$ is bounded with all derivatives, uniformly with respect to (\mathbf{A}, Ψ) . Moreover, by the choice of Ξ , the perturbation $h_{(\mathbf{A}, \Psi)}^*(\omega)$ is smooth and compactly supported, hence in $L_{k,\delta}^2$.

Condition (1) holds, since the function $h_{(\mathbf{A}, \Psi)}(t)$ satisfies

$$h_{(\mathbf{A}, \Psi)}(t + \tau) = h_{(\mathbf{A}, \Psi)^\tau}(t),$$

where $(\mathbf{A}, \Psi)^\tau$ is the τ -reparametrized solution represented in a temporal gauge by $(A(t + \tau), \psi(t + \tau))$. In fact,

$$\begin{aligned} h_{(\mathbf{A}, \Psi)^\tau}(T) &= \int_{\mathbb{R}} \eta_1(s-T) \eta_2 \left(\int_{\mathbb{R}} \eta_1(t-s) \tilde{\mathcal{C}}(A(t+\tau), \psi(t+\tau)) dt \right) ds = \\ &= \int_{\mathbb{R}} \eta_1(s-T) \eta_2 \left(\int_{\mathbb{R}} \eta_1(u-s-\tau) \tilde{\mathcal{C}}(A(u), \psi(u)) du \right) ds = \\ &= \int_{\mathbb{R}} \eta_1(v-T-\tau) \eta_2 \left(\int_{\mathbb{R}} \eta_1(u-v) \tilde{\mathcal{C}}(A(u), \psi(u)) du \right) dv = h_{(\mathbf{A}, \Psi)}(T + \tau). \end{aligned}$$

Condition (2) holds: in fact, it is shown in [17] that the function $h_{(\mathbf{A}, \Psi)}$ is bounded with all derivatives, uniformly with respect to (\mathbf{A}, Ψ) . The Sobolev embeddings of Proposition 4.1 provide the uniform bound in the $L^2_{k, \delta}$ -norms.

Condition (3) also follows from Froyshov ([17], Prop.5): for ω a \mathcal{C}^m form, the linearization of the perturbation $h^*_{(\mathbf{A}, \Psi)}(\omega)$ at the point $(\omega, \mathbf{A}, \Psi)$ is a bounded operator $K_{(\omega, \mathbf{A}, \Psi)} : L^2_{k, \delta} \rightarrow \mathcal{C}^m$ with

$$\text{supp} \left(K_{(\omega, \mathbf{A}, \Psi)}(\alpha, \Phi) \right) \subset h^{-1}_{(\mathbf{A}, \Psi)}(\Xi) \times Y.$$

Condition (4) follows from the fact that the perturbation $h^*_{(\mathbf{A}, \Psi)}(\omega)$ is compactly supported. That is, $q_{(\mathbf{A}, \Psi)}(t)$ will be identically zero for large enough T and in particular (4) is satisfied for large enough t .

Notice that in properties (3) and (4), the interval $[-T, T]$ cannot be chosen uniformly with respect of (\mathbf{A}, Ψ) .

Other suitable perturbations of the functional \mathcal{C} can be used to achieve transversality of the moduli space of flow lines. For instance one can consider perturbations by functions of the holonomy of the connection A . This kind of perturbation is used in [8].

4.2 Flow lines and transversality

Let $\mathcal{L}_{(\mathbf{A}, \Psi)}$ be the linearization of equations (20) and (21) on $\mathcal{B}^H_{k, \delta}(a, b)$.

The operator \mathcal{L} is of the form

$$\mathcal{L}_{(\mathbf{A}, \Psi, P)}(\alpha, \Phi) = \begin{cases} D_{\mathbf{A}}\Phi + \alpha\Psi \\ d^+\alpha - \frac{1}{2}Im(\Psi \cdot \bar{\Phi}) + \mathcal{D}P_{(\mathbf{A}, \Psi)}(\alpha, \Phi) \\ G^*_{(\mathbf{A}, \Psi)}(\alpha, \Phi) \end{cases}$$

mapping

$$L^2_{k, \delta}(\Lambda^1(Y \times \mathbb{R}) \oplus \Gamma(S^+ \otimes L)) \rightarrow L^2_{k-1, \delta}(\Lambda^0(Y \times \mathbb{R}) \oplus \Lambda^{2+}(Y \times \mathbb{R})).$$

The operator G^* is the adjoint of $G_{(\mathbf{A}, \Psi)}(f) = (-df, f\Psi)$.

As the following proposition shows, the operator $\mathcal{L}_{(\mathbf{A}, \Psi, P)}$ is obtained by adding the small perturbation $\mathcal{D}P_{(\mathbf{A}, \Psi)}$ to a Fredholm map from $L^2_{k, \delta}$ to $L^2_{k-1, \delta}$, hence it is Fredholm. Therefore we have a well defined relative Morse index of two critical points a and b in \mathcal{M}^H_C .

Proposition 4.4 *Suppose a and b are irreducible critical points for the functional $\tilde{\mathcal{C}}$. Let $\{\lambda_a\}$ and $\{\lambda_b\}$ be the eigenvalues of the Hessian T at the points a and b . Assume that the positive number δ satisfies $\delta < \min\{|\lambda_a|, |\lambda_b|\}$. Let (\mathbf{A}, Ψ) be a solution of (20) and (21) in $\mathcal{B}_{k, \delta}(a, b)$. Then the linearization $\mathcal{L}_{(\mathbf{A}, \Psi)}$ is a Fredholm operator of index*

$$\text{Ind}(\mathcal{L}_{(\mathbf{A}, \Psi)}) = \sigma(a, b).$$

The right hand side $\sigma(a, b)$ is the spectral flow of the operator $\nabla\mathcal{F}$ along a path $(A(t), \psi(t))$ in \mathcal{A} that corresponds to (\mathbf{A}, Ψ) under π^* . The quantity $\sigma(a, b)$ is independent of the path, hence

$$\sigma(a, b) = \mu(a) - \mu(b)$$

defines a relative Morse index of a and b , where $\mu(a)$ is the spectral flow of $\nabla\mathcal{F}$ on a path joining a to a fixed $[A_0, \psi_0]$ in \mathcal{M}_G^H .

The Fredholm property follows from [25] theorem 1.3. The result about the spectral flow requires the following lemma proven by R.G. Wang [41].

Lemma 4.5 *Suppose given a path $(A(t), \psi(t))$ in \mathcal{A} that decays exponentially fast to asymptotic values (A, ψ) and $\lambda(A, \psi)$ in the same gauge class, with $\lambda \in \mathcal{G}$. Then the index of the linearization*

$$Ind(\mathcal{L}_{(\mathbf{A}, \Psi)}) = \frac{i}{2\pi} \int_Y c_1(L) \wedge \lambda^{-1} d\lambda.$$

We report here the proof given in [41]. Namely, $\lambda : Y \rightarrow U(1)$ determines a $U(1)$ bundle over $Y \times S^1$, by identifying the ends of the cylinder; the connection $A(t)$ gives rise to a connection on this line bundle \hat{L} over $Y \times S^1$. The index of the linearization $\mathcal{L}_{(\mathbf{A}, \Psi)}$ is therefore given by

$$Ind(\mathcal{L}_{(\mathbf{A}, \Psi)}) = -\frac{1}{16\pi^2} \int_{Y \times S^1} c_1(\hat{L})^2 - \frac{2\chi + 3\sigma}{4},$$

since $\mathcal{L}_{(\mathbf{A}, \Psi)}$ is, up to compact perturbations, the linearization of the four-dimensional Seiberg-Witten equations on $Y \times S^1$ with the $Spin_c$ structure specified by the line bundle \hat{L}^2 . The term $2\chi + 3\sigma = 0$ on a manifold of the form $Y \times S^1$. As for the first term, notice that we can write

$$F_{A(t)} \wedge F_{A(t)} = F_{A(t)} \wedge \frac{dA(t)}{dt} \wedge dt,$$

and therefore we get

$$\begin{aligned} \frac{-1}{8\pi^2} \int_{Y \times S^1} F_{A(t)} \wedge \frac{dA(t)}{dt} \wedge dt &= \frac{i}{2\pi} \int_Y c_1(L) \int_{S^1} \frac{dA}{dt} = \\ &= \frac{i}{2\pi} \int_Y c_1(L) \wedge \lambda^{-1} d\lambda, \end{aligned}$$

since $A(+\infty) - A(-\infty) = \lambda^{-1} d\lambda$.

The spectral flow of the family of operators $T_t = T_{(A(t), \psi(t))}$, which is the index of the operator $\frac{\partial}{\partial t} + T_{(A(t), \psi(t))}$ by [2], p.57, p.95. Suppose given a path

$(A(t), \psi(t))$ in \mathcal{A} with endpoints that are gauge equivalent through a gauge transformation in $\tilde{\mathcal{G}}$. Then, by the previous Lemma, the spectral flow is given by

$$\text{Ind}\left(\frac{\partial}{\partial t} + T_{(A(t), \psi(t))}\right) = 0,$$

that is, the spectral flow around a loop in \mathcal{B}^H is trivial.

The additivity of spectral flows implies that

$$\sigma(a, c) = \mu(a) - \mu(c) = \sigma(a, b) + \sigma(b, c).$$

This means that the relative Morse index of critical points in \mathcal{M}_C^H is well defined and equal to the spectral flow. We can consider the same construction of moduli spaces of flow lines in $\mathcal{B}_{k, \delta}(a, b)$, that is modulo the action of the full gauge group $\mathcal{G}(Y \times \mathbb{R})$. This determines a relative Morse index for points in \mathcal{M}_C given by the spectral flow. However, in this case the relative Morse index is only defined up to an integer multiple of l , where

$$l = g.c.d.\{\langle c_1(L) \cup h, [Y] \rangle \mid h \in H\}. \quad (23)$$

This follows from the spectral formula 4.5. Notice that l is an even number.

Thus we have an important difference between Seiberg–Witten and Donaldson Floer homology: the relative index is well defined and there is no ambiguity coming from loops in $\hat{\mathcal{B}}^0$. Thus in our case the Floer groups will be \mathbb{Z} -graded. This makes the Seiberg–Witten–Floer homology more similar to the finite dimensional case [42]. In fact, as proved in [32], for three-manifolds of the form $\Sigma \times S^1$, with Σ a Riemann surface, the Floer groups are just the ordinary homology of a symmetric product of Σ . Similar results for the mapping cylinder of a surface Σ have been obtained by A.L. Carey and B.L. Wang [9].

The ambiguity mod 8 in Donaldson Floer homology is related to the possibility of rescaling and gluing instantons at different $t \in \mathbb{R}$ on $Y \times \mathbb{R}$, as explained in [15]. There is a form of periodicity in the Seiberg–Witten–Floer groups as well if we consider the moduli space \mathcal{M}_C instead of \mathcal{M}_C^H , as we discuss later. However, this depends only upon the nature of the covering $\mathcal{B}^H \rightarrow \mathcal{B}$. In fact, the non-rescaling property of the solutions of the monopole equations imply the absence of “bubbling” phenomena.

Consider the moduli space $\mathcal{M}^H(a, b)$ of solutions of the equations (20) and (21) in $\mathcal{B}_{k, \delta}^H(a, b)$.

Proposition 4.6 *Given a and b , two critical points of $\tilde{\mathcal{C}}$, for a generic choice of the perturbation $P \in \mathcal{P}$, the moduli space $\mathcal{M}^H(a, b)$ of gradient flow lines is a smooth manifold, cut out transversely by the equations, of dimension*

$$\dim(\mathcal{M}^H(a, b)) = \mu(a) - \mu(b),$$

where $\mu(a) - \mu(b)$ is the relative Morse index of the critical points.

Proof: It is first necessary to prove that there are no reducible flow lines connecting the critical points a and b . This is proven later in the next section. Once it is known that $\mathcal{M}^H(a, b)$ lies in the irreducible component $\hat{\mathcal{B}}_{k, \delta}^H(a, b)$, the statement follows via the implicit function theorem, upon showing that, for a generic choice of the perturbation P , the linearization \mathcal{L} is surjective.

Consider the operator

$$\hat{\mathcal{L}}_{(\mathbf{A}, \Psi, P)}(\alpha, \Phi, p) = \mathcal{L}_{(\mathbf{A}, \Psi, P)}(\alpha, \Phi) + p_{(\mathbf{A}, \Psi, P)}(\alpha, \Phi),$$

where we vary the perturbation by an element $p_{(\mathbf{A}, \Psi, P)}$ of $T_P\mathcal{P}$. This corresponds to varying the parameter $\omega \in \Lambda_{\Xi}^2(Y \times \mathbb{R})$ in Froyshov's class of perturbations.

The operator \mathcal{L} is Fredholm, therefore $\hat{\mathcal{L}}$ has a closed range. We show that $\hat{\mathcal{L}}$ is surjective by proving that it has dense range.

Suppose given an element (β, ξ, g) in $L_{-k-1, -\delta}^2$ that is L^2 -orthogonal to the range of the operator $\hat{\mathcal{L}}$. Here β is an element in $\Lambda^{2+}(Y \times \mathbb{R}, i\mathbb{R})$, ξ is a spinor, and g is a zero-form. The element (β, ξ, g) is in the kernel of the adjoint $\hat{\mathcal{L}}^*$, which is an elliptic operator with $L_{-k, -\delta}^2$ coefficients, thus (β, ξ) lives in $L_{-k, -\delta}^2$ by elliptic regularity. For the same reason, g lives in $L_{k-1, \delta}^2$, since the operator G^* is the adjoint of G with respect to the $L_{k-1, \delta}^2$ inner product. If we consider the L^2 -pairing of $L_{k, \delta}^2$ and $L_{-k, -\delta}^2$, we get

$$\begin{aligned} \langle \beta, d^+ \alpha - \frac{1}{2} \text{Im}(\Psi \cdot \bar{\Phi}) \rangle + \mathcal{D}P_{(\mathbf{A}, \Psi)}(\alpha, \Phi) + p_{(\mathbf{A}, \Psi, P)}(\alpha, \Phi) + \\ + \langle \xi, D_{\mathbf{A}} \Phi + \alpha \Psi \rangle + \langle g, G_{(\mathbf{A}, \Psi)}^*(\alpha, \Phi) \rangle = 0. \end{aligned}$$

By varying $p \in \mathcal{P}$ we force $\beta \equiv 0$. The remaining inner product

$$\langle \xi, D_{\mathbf{A}} \Phi + \alpha \Psi \rangle + \langle g, G_{(\mathbf{A}, \Psi)}^*(\alpha, \Phi) \rangle = 0$$

gives the following equations

- (a) $(e_{-\delta} de_{\delta})g = \frac{1}{2} \xi \cdot \bar{\Psi}$ and
- (b) $D_{\mathbf{A}} \xi - g \Psi = 0$.

We assume that Ψ is not identically zero. Applying d^* to (a) and using (b) we obtain $d^*(e_{-\delta} de_{\delta} g) + g |\Psi|^2 = 0$. Equivalently, we get

$$(e_{\delta/2} d^* e_{-\delta/2})(e_{-\delta/2} de_{\delta/2}) e_{\delta/2} g + |\Psi|^2 e_{\delta/2} g = 0.$$

The equation

$$\Delta_{\delta/2} g + e_{\delta/2} g |\Psi|^2 = 0,$$

with

$$\Delta_{\delta} = e_{-\delta} \Delta e_{\delta},$$

implies that $g \equiv 0$, since g decays at $\pm\infty$ and the maximum principle applies. Then, by varying α alone in $\langle \xi, D_{\mathbf{A}} \Phi + \alpha \Psi \rangle = 0$, we force ξ to vanish on some

open set. The pair (β, ξ) is in the kernel of the operator $T_{(\mathbf{A}, \Psi)}^*$, hence by analytic continuation $(\tilde{\beta}, \tilde{\xi}) \equiv 0$.

Thus the operator $\tilde{\mathcal{L}}$ is surjective. This implies that zero is a regular value for the map defined by the equations (20) and (21). Therefore the moduli space $\mathcal{M}od$ of triples $([\mathbf{A}, \Psi], P)$ in $\hat{\mathcal{B}}_{L^2_{k,\delta}}^H(a, b) \oplus \mathcal{P}$ that satisfy the equations is a smooth (infinite dimensional) manifold with virtual tangent space $Ker(\tilde{\mathcal{L}})$.

The projection $\Pi : \mathcal{M}od \rightarrow \mathcal{P}$ given by $\Pi([\mathbf{A}, \Psi], P) = P$ linearizes to a surjective Fredholm operator

$$\mathcal{D}\Pi : Ker(\tilde{\mathcal{L}}) \rightarrow T_P\mathcal{P}.$$

The kernel of $\mathcal{D}\Pi$ is $Ker(\mathcal{D}\Pi_{(\mathbf{A}, \Psi, P)}) = Ker(\mathcal{L}_{(\mathbf{A}, \Psi, P)})$. The infinite dimensional Sard theorem implies that the moduli space $\mathcal{M}^H(a, b)$, for a generic perturbation $P \in \mathcal{P}$, is the inverse image under the projection map from $\mathcal{M}od$ to \mathcal{P} of a regular value. Thus $\mathcal{M}^H(a, b)$ is a smooth manifold which is cut out transversely by the equations. Equivalently, the linearization \mathcal{L} with a fixed generic q is surjective.

As we shall prove in the following, the virtual dimension of the moduli space $\mathcal{M}^H(a, b)$ equals the index of the Fredholm operator \mathcal{L} . According to Proposition 4.4, this is the relative Morse index $\mu(a) - \mu(b)$.

QED

The case of points a and b in \mathcal{M}_C is analogous. In this case we obtain a moduli space $\mathcal{M}(a, b)$ with components of dimension $\mu(a) - \mu(b) + \xi(h)$ with $h \in H$, according to the spectral formula 4.5.

4.3 Flow equations

It is clear that moduli spaces of flow lines can be defined either using equations (18) and (19), for pairs $(A(t), \psi(t))$ in a temporal gauge with gauge transformations in \mathcal{G} , or using the corresponding equations (20) and (21) for pairs (\mathbf{A}, Ψ) with gauge action of $\mathcal{G}(a, b)$, as we did in the previous section.

The two descriptions are equivalent, however, the Fredholm analysis is somewhat simpler in the latter case. In fact, in order to set up a good Fredholm analysis with equations (18) and (19), one has to add a correction term $\gamma(A, \psi)$ in the flow equations in order to make the flow tangent to a fixed slice of the gauge action at a point $[A_0, \psi_0]$. This corresponds to the analysis worked out in [31] for the case of Donaldson Floer theory. One difficulty arises in this case, since the correction term $\gamma(A, \psi)$ does not preserve the temporal gauge condition. This problem can be overcome by replacing the temporal gauge condition with the condition of *standard form* introduced in [31] and allowing time-dependent gauge transformations.

The linearization of the equations contains the extra term L_γ that linearizes

$\gamma(A, \psi)$,

$$\mathcal{L}_{(A(t), \psi(t), q)}(\alpha, \phi) = \frac{\partial}{\partial t} + L_{A(t), \psi(t)} + \mathcal{D}q_{(A(t), \psi(t))} + L_\gamma.$$

Anisotropic Sobolev norms $L^2_{(k,m)}$ can be introduced on the spaces of connections and sections and gauge transformations, as analysed in [31]. The linearization L_γ is a compact operator with respect to these norms. We have the equivalent of Proposition 4.4 within this formulation: $\mathcal{L}_{(A(t), \psi(t), q)}$ is a Fredholm operator whose index is the spectral flow of the family of operators $L_{(A(t), \psi(t))}$. The transversality result can also be proven in this context. This formulation, however, will not be worked out in this paper.

The key to the equivalence of these two formulations is the elliptic regularity and a decay estimate that will be proven later. In fact, if we choose a smooth perturbation in \mathcal{P} , by elliptic regularity it is possible to represent any solution of (20) and (21) in $\mathcal{B}_{k,\delta}(a, b)$ by a smooth representative. This defines a solution of (18) and (19) in $L^2_{(k,m)}$ in a standard form. Conversely solutions of (18) and (19) give rise to solutions of (20) and (21) in $\mathcal{B}_{k,\delta}(a, b)$ due to the exponential decay towards the endpoints that will be proven in the next section.

4.4 Decay estimate

In this subsection we show that the moduli space $\mathcal{M}^H(a, b)$ only contains irreducible flow lines. Given that a or b are irreducible points, the result follows directly from the decay estimate below. The analysis is based on [32] and [40].

Theorem 4.7 *Let a and b be non-degenerate critical points in \mathcal{B}^H . There exists a weight $\delta > 0$ such that the following holds. Suppose given any solution $[\mathbf{A}, \Psi]$ of (20) and (21) that is represented by a smooth pair $(A(t), \psi(t))$ in a temporal gauge, with asymptotic values (A_a, ψ_a) and (A_b, ψ_b) representing the elements a and b . Then there exists a constant K such that, for t outside an interval $[-T, T]$, the distance in any fixed \mathcal{C}^l -topology of $(A(t), \psi(t))$ from the endpoints is*

$$\text{dist}_{\mathcal{C}^l}((A(t), \psi(t)), (A_i, \psi_i)) < K \exp(-\delta|t|),$$

with $i = a$ if $t < -T$ and $i = b$ if $t > T$.

Proof: The proof consists of a few steps, mainly based on the analysis worked out in [32]. Let us consider the decaying as $t \rightarrow \infty$; the other case is analogous.

Claim 1: Let $(A(t), \psi(t))$ be a solution of the flow equations (18) and (19) with finite energy and with limit $\lim_{t \rightarrow \infty} (A(t), \psi(t)) = (A_b, \psi_b)$. Then there is a $T \gg 0$ and a constant K_b , such that the inequality

$$\tilde{\mathcal{C}}(A(t), \psi(t)) - \tilde{\mathcal{C}}(A_b, \psi_b) \leq K_b \|\nabla \tilde{\mathcal{C}}(A(t), \psi(t))\|_{L^2}^2$$

holds for $t \geq T$.

Lojasiewicz inequality [38] shows that there exists $T > 0$ and an exponent $0 < \theta < 1/2$ such that

$$|\tilde{\mathcal{C}}(A(t), \psi(t)) - \tilde{\mathcal{C}}(A_b, \psi_b)|^{1-\theta} \leq K_b \|\nabla \tilde{\mathcal{C}}(A(t), \psi(t))\|_{L^2},$$

for $t \geq T$. Under the assumption of non-degenerate Hessian at the point b , the exponent can be improved to $\theta = \frac{1}{2}$.

Claim 2: For a solution $(A(t), \psi(t))$ of (18) and (19), the inequality

$$\begin{aligned} \frac{1}{2} \int_t^\infty \|\nabla \tilde{\mathcal{C}}(A(s), \psi(s))\|^2 ds &\leq \tilde{\mathcal{C}}(A(t), \psi(t)) - \\ &- \tilde{\mathcal{C}}(A_b, \psi_b) \leq \frac{3}{2} \int_t^\infty \|\nabla \tilde{\mathcal{C}}(A(s), \psi(s))\|^2 ds \end{aligned}$$

holds for large t .

Without loss of generality we can assume that the perturbation in \mathcal{P} satisfies Condition (4) of 4.2 with $C_0 < 1/2$, so that

$$\|q_{(\mathbf{A}, \Psi)}(t)\|_{L^2} < \frac{1}{2} \|\nabla \tilde{\mathcal{C}}(A(t), \psi(t))\|_{L^2}.$$

Thus, we can replace the equality

$$\begin{aligned} \tilde{\mathcal{C}}(A(t), \psi(t)) - \tilde{\mathcal{C}}(A_b, \psi_b) &= \int_t^\infty -\frac{d}{ds} \tilde{\mathcal{C}}(A(s), \psi(s)) ds \\ &= \int_t^\infty -\left\langle \frac{d}{ds}(A(s), \psi(s)), \nabla \tilde{\mathcal{C}}(A(s), \psi(s)) \right\rangle ds = \\ &= \int_t^\infty \|\nabla \tilde{\mathcal{C}}(A(s), \psi(s))\|^2 ds, \end{aligned}$$

that holds for solutions of the unperturbed equations with the inequality of Claim 2 for solutions of the perturbed equations.

Claim 3: The quantity

$$E(t) = \int_t^\infty \|\nabla \tilde{\mathcal{C}}(A(s), \psi(s))\|^2 ds$$

decays exponentially as $t \rightarrow \infty$.

In fact, the inequality of Claim 2 gives the first inequality in the following estimate:

$$\begin{aligned} E(t) &\leq 2(\tilde{\mathcal{C}}(A(t), \psi(t)) - \tilde{\mathcal{C}}(A_b, \psi_b)) \leq \\ &\leq K_b \|\nabla \tilde{\mathcal{C}}(A(t), \psi(t))\|^2 = -K_b \frac{d}{dt} E(t). \end{aligned}$$

The second inequality follows from Claim 1 with the best exponent $\theta = 1/2$.

Claim 4: For large t we have the inequality

$$\text{dist}_{L_1^2}((A(t), \psi(t)), (A_b, \psi_b)) \leq K \int_{t-1}^{\infty} \|\nabla \tilde{\mathcal{C}}(A(s), \psi(s))\|_{L_2^2}^2 ds.$$

In fact, the perturbation in \mathcal{P} satisfies

$$\|q_{(A(t), \psi(t))}(t)\|_{L_1^2} \leq C_1 \|\nabla \tilde{\mathcal{C}}(A(t), \psi(t))\|_{L_1^2},$$

with $0 < C_1 < 1$ for large t . Thus we have

$$\text{dist}_{L_1^2}((A(t), \psi(t)), (A_b, \psi_b)) \leq K \int_t^{\infty} \|\nabla \tilde{\mathcal{C}}(A(s), \psi(s))\|_{L_1^2}^2 ds.$$

Lemma 6.14 of [32] implies that the latter is bounded by

$$K \int_{t-1}^{\infty} \|\nabla \tilde{\mathcal{C}}(A(s), \psi(s))\|_{L_2^2}^2 ds,$$

thus proving the inequality.

The exponential decay of $E(t)$ proves the claim of the theorem for the case of L_1^2 -topology. Smooth estimates then follow by a bootstrapping argument and elliptic regularity.

QED

Analogous exponential decay estimates have been proven in [40]. An immediate corollary of Theorem 4.7 is the following.

Corollary 4.8 *No reducible solution arises among the flow lines in $\mathcal{M}^H(a, b)$.*

In fact, (A_a, ψ_a) and (A_b, ψ_b) have non-trivial spinor. Thus, by the exponential decay estimate, $\psi(t)$ is forced to be non-trivial.

4.5 The Boundary Operator

We have described all the ingredients that are needed in order to construct the Floer homology. Following [15] we can define the boundary operator.

Let $\hat{\mathcal{M}}^H(a, b)$ and $\hat{\mathcal{M}}(a, b)$ be the quotients of $\mathcal{M}^H(a, b)$ and of $\mathcal{M}(a, b)$ by the action of \mathbb{R} by translations.

Definition 4.9 *Let a and b be two critical points in \mathcal{M}_C^H of relative Morse index $\mu(a) - \mu(b) = 1$. Define the boundary ∂ to be the operator with matrix elements*

$$\langle \partial^H a, b \rangle = \epsilon^H(a, b),$$

where $\epsilon^H(a, b)$ is the algebraic sum over the paths joining a and b of the signs given by the orientation,

$$\epsilon^H(a, b) = \sum_{\gamma \in \hat{\mathcal{M}}^H(a, b)} \epsilon_{\gamma}.$$

For points a and b in \mathcal{M}_C , with $\mu(a) - \mu(b) = 1 \pmod{l}$, define the boundary components by considering the one dimensional component $\mathcal{M}^1(a, b)$ of the moduli space $\mathcal{M}(a, b)$,

$$\langle \partial a, b \rangle = \epsilon(a, b),$$

where

$$\epsilon(a, b) = \sum_{\gamma \in \hat{\mathcal{M}}^1(a, b)} \epsilon_\gamma.$$

The construction of the Seiberg–Witten Floer homology is the result of the following theorem.

Theorem 4.10 *The boundary operators of definition 4.9*

$$\partial^H a = \sum_{b | \mu(a) - \mu(b) = 1} \epsilon^H(a, b) b \quad (24)$$

and

$$\partial a = \sum_{b | \mu(a) - \mu(b) = 1 \pmod{l}} \epsilon(a, b) b \quad (25)$$

satisfy $\partial^H \circ \partial^H = 0$ and $\partial \circ \partial = 0$. Upon fixing the Morse index of a critical point, it is possible to construct chain complexes with

$$C_q^H = \{b \in \mathcal{M}_C^H \mid \mu(b) = q\},$$

$$C_q = \{b \in \mathcal{M}_C \mid \mu(b) = q \pmod{l}\},$$

and with the boundary operators described above. The \mathbf{Z} -graded Seiberg-Witten Floer homology is defined as

$$SWH_q(Y) := H_q(C_*^H, \partial^H) = \frac{\text{Ker}(\partial_{q-1}^H)}{\text{Im}(\partial_q^H)}.$$

The \mathbf{Z}_l -graded Seiberg-Witten Floer homology is the homology

$$SWH_q^l(Y) := H_q(C_*, \partial).$$

The fact that the boundary square is zero can be shown following the analogous argument for the Donaldson case given in [15]. In fact, since $\langle \partial^H a, b \rangle$ is the algebraic sum over the zero dimensional manifold $\hat{\mathcal{M}}^H(a, b)$, the matrix elements for $\partial^H \circ \partial^H$ are given by

$$\langle \partial^H \partial^H a, c \rangle = \sum_b \langle \partial^H a, b \rangle \langle \partial^H b, c \rangle.$$

This is the algebraic sum of the points in the the zero dimensional manifold

$$\mathcal{M}^2(a, c) := \cup_b \hat{\mathcal{M}}^H(a, b) \times \hat{\mathcal{M}}^H(b, c),$$

with the product orientation $\epsilon^2(A, B) = \epsilon(A)\epsilon(B)$.

Now the claim follows if we prove that the manifold $\mathcal{M}^2(a, c)$ with the above orientation is the oriented boundary of the 1-dimensional manifold $\hat{\mathcal{M}}^H(a, c)$. This depends on a gluing formula like the one proven in [15]. The gluing formula will be proven in the following section.

The same argument applies to prove that $\partial^2 = 0$, according to the spectral formula 4.5.

The compactness of the moduli space of solutions of (8) implies that there is just a finite number of critical points of the functional $\tilde{\mathcal{C}}$, and therefore only finitely many of the Floer homology groups $SWH_*^l(Y)$ are non-trivial and they are finitely generated. In the case of the Floer groups $SWH_*(Y)$, if H is infinite, there are infinitely many non-trivial groups. Each one is finitely generated, but, if the group H is non-trivial, the \mathbb{Z} -graded complex has generators in infinitely many degrees. In fact we have $C_k \cong C_{k+l}$, so that the same groups appear with an l -periodicity. Notice that if H is trivial then \mathcal{M}_C^H is just \mathcal{M}_C , the Floer homology $SWH_*^l(Y)$ is \mathbb{Z} -graded and finitely generated and coincides with $SWH_*(Y)$. Consider the homomorphism $\xi : H \rightarrow \mathbb{Z}$ given by

$$\xi(h) = \langle c_1(L) \cup h, [X] \rangle.$$

The map ξ is injective since $H = H^1(X, \mathbb{Z}) / \tilde{H}^1(X, \mathbb{Z})$, where $\tilde{H}^1(X, \mathbb{Z}) = \ker(\xi)$. In particular this means that H is either trivial or isomorphic to \mathbb{Z} via the map ξ .

4.6 Convergence and Gluing

The analysis presented in this section is part of a joint work with B.L. Wang and will appear in [27].

We show that the moduli spaces $\hat{\mathcal{M}}(a, b)$ of unparametrized flow lines have a compactification with boundary strata consisting of trajectories that break through other critical orbits. We discuss convergence in the unparametrized moduli spaces $\hat{\mathcal{M}}(a, b)$ and a gluing formula that describes the boundary strata.

We give the following preliminary definition.

Definition 4.11 *A smooth path $(A(t), \psi(t))$ in \mathcal{A} is of finite energy if the integral*

$$\int_{-\infty}^{\infty} \|\nabla \tilde{\mathcal{C}}(A(t), \psi(t))\|_{L^2}^2 dt < \infty \tag{26}$$

is finite.

Notice that any solution of (18) and (19) with asymptotic values a and b is of finite energy, in fact in this case the total variation of the functional $\tilde{\mathcal{C}}$ along the path $(A(t), \psi(t))$ is finite and (26) satisfies

$$\int_{-\infty}^{\infty} \|\nabla \tilde{\mathcal{C}}(A(t), \psi(t))\|_{L^2}^2 dt \leq C(\tilde{\mathcal{C}}(a) - \tilde{\mathcal{C}}(b)),$$

because of the assumptions on the perturbation $q_{(\mathbf{A}, \Psi)}$. Finite energy solutions of the flow equations have nice properties: they necessarily decay to asymptotic values that are critical points of $\tilde{\mathcal{C}}$ as we prove in Proposition 4.13. We begin by introducing some analytic properties of the functional $\tilde{\mathcal{C}}$ (see also [17], [32], [40]).

Lemma 4.12 *Let \mathcal{M}_C be the moduli space of critical points of $\tilde{\mathcal{C}}$, with ρ a sufficiently small perturbation. For any $\epsilon > 0$ there is a $\lambda > 0$ such that if the L_1^2 -distance from a point $[A, \psi]$ of \mathcal{B} to all the points in \mathcal{M}_C is at least ϵ , then*

$$\|\nabla \tilde{\mathcal{C}}(A, \psi)\|_{L^2} > \lambda.$$

Proof. For a sequence $[A_i, \psi_i]$ of elements of \mathcal{B} with a distance at least ϵ from all the critical points, such that

$$\|\nabla \tilde{\mathcal{C}}(A_i, \psi_i)\|_{L^2} \rightarrow 0,$$

as $i \rightarrow \infty$, we would have

$$\|*F_{A_i} - \sigma(\psi_i, \psi_i) - i\rho\| + \|\partial_{A_i} \psi_i\| \rightarrow 0.$$

Thus, there is a constant C such that

$$\int_Y |*F_{A_i} - \sigma(\psi_i, \psi_i) - i\rho|^2 + |\partial_{A_i} \psi_i|^2 dv < C.$$

If the perturbation ρ is sufficiently small, the Weizenböck implies that

$$\int_Y |F_{A_i}|^2 + |\sigma(\psi_i, \psi_i)|^2 + \frac{\kappa}{2} |\psi_i|^2 + 2|\nabla_{A_i} \psi_i|^2 dv < C.$$

Thus we have a uniform bound on the norms $\|\psi_i\|_{L^4}$, $\|F_{A_i}\|_{L^2}$, and $\|\nabla_{A_i} \psi_i\|_{L^2}$. An elliptic estimate shows that there is a subsequence that converges in the L_1^2 norm at a solution of the critical point equations (15), and this contradicts the assumption.

QED

Corollary 4.13 *Let $(A(t), \psi(t))$ be a smooth finite energy solution of equations (18) and (19) with a smooth perturbation q . Then there exist critical points a and b of $\tilde{\mathcal{C}}$, such that the $\lim_{t \rightarrow \pm\infty} (A(t), \psi(t))$ are in the gauge classes of a and b .*

Proof: The finite energy condition (26) implies that

$$\|\nabla\tilde{\mathcal{C}}(A(t), \psi(t))\| \rightarrow 0$$

as $t \rightarrow \pm\infty$. The Palais-Smale condition of Lemma 4.12 implies that there exist T large, such that for $|t| > T$, $(A(t), \psi(t))$ lies in a very small ϵ -neighbourhood of critical points of $\tilde{\mathcal{C}}$.

QED

Now we can analyse convergence in the moduli space $\hat{\mathcal{M}}(a, b)$, see [9], [41].

Theorem 4.14 *The space $\hat{\mathcal{M}}(a, b)$ is precompact. Namely, any sequence x_i of elements in $\hat{\mathcal{M}}(a, b)$ has a subsequence of smooth representatives that converges with all derivatives to a solution x of the flow equations which lies in some $\hat{\mathcal{M}}(c, d)$ with $\mu(a) > \mu(c) > \mu(d) > \mu(b)$.*

Proof: We choose a lift of the elements of $\hat{\mathcal{M}}(a, b)$ to $\mathcal{M}(a, b)$ such that the gradient flow $[A(t), \psi(t)]$ has equal energy on $(-\infty, 0]$ and on $[0, \infty)$:

$$\int_{-\infty}^0 \|\nabla\tilde{\mathcal{C}}([A(t), \psi(t)])\|_{L^2(Y)} dt = \int_0^{\infty} \|\nabla\tilde{\mathcal{C}}([A(t), \psi(t)])\|_{L^2(Y)} dt. \quad (27)$$

This lift is unique, since in the family of gradient flows $\{[\mathbf{A}, \Psi]^T, T \in \mathbb{R}\}$ there is a unique element satisfying the equal energy condition (27).

Suppose $x_i = (\mathbf{A}_i, \Psi_i) \in \mathcal{A}_{k, \delta}(a, b)$ ($i = 1, 2, \dots, \infty$) is a sequence of solutions to the equations (20) and (21) which are represented in a temporal gauge by the “equal energy” lifts $(A_i(t), \psi_i(t))$ of the sequence x_i in $\hat{\mathcal{M}}(a, b)$. This implies that the (\mathbf{A}_i, Ψ_i) have a uniformly finite energy E . By the Palais-Smale condition (Lemma 4.12), we can find $T \gg 1$ (choose $T > E/\lambda$ where λ is the constant appearing in Lemma 4.12), such that for $|t| > T$, the $[A_i(t), \psi_i(t)]$ lie in a very small ϵ -neighbourhood of a or b .

Therefore, the $[A_i(t), \psi_i(t)]$ have a uniform exponential decay over $(-\infty, -T]$ and $[T, \infty)$. On $Y \times [-T-1, T+1]$ we proceed with an argument that is analogous to the usual proof of the compactness for the Seiberg-Witten moduli space on a compact 4-manifold [19], [30]. Namely, we have the following.

A uniform L^2 bound on the spinors Ψ_i follows from the Weizenböck formula

$$D_{\mathbf{A}} D_{\mathbf{A}}^* \Psi = \nabla_{\mathbf{A}} \nabla_{\mathbf{A}} \Psi + \frac{1}{2} F_{\mathbf{A}}^+ \Psi + \frac{\kappa}{4} \Psi,$$

where κ is the scalar curvature. We have

$$\|\Psi_i\| \leq \max_{Y \times [-T-1, T+1]} (-\kappa, 0) + 2\|P_{(\mathbf{A}_i, \Psi_i)}\|.$$

The perturbations $P_{(\mathbf{A}_i, \Psi_i)}$ are bounded uniformly with respect to (\mathbf{A}_i, Ψ_i) by assumption, 4.2. This also give a uniform bound on $\|\nabla_A \Psi_i\|_{L^2}$, as in [30] Lemma

5.1.7(?). The presence of the perturbation does not affect the estimates, because of the boundedness of $P_{(\mathbf{A}_i, \Psi_i)}$.

We have the following gauge fixing condition: up to gauge transformations λ_i in the group $\mathcal{G}_{k+1}(Y \times [-T-1, T+1])$, it is possible to make $\mathbf{A}_i - \mathbf{A}_0$ into a sequence of co-closed 1-forms, with the property that

$$\|\mathbf{A}_i - \mathbf{A}_0\|_{L_k^2}^2 \leq C \|F_{\mathbf{A}_i}^+\|_{L_{k-1}^2} + K.$$

This is proven in Lemma 5.3.1 of [30].

From the curvature equation we have

$$\|F_{\mathbf{A}_i}^+\| \leq \|\Psi_i \cdot \bar{\Psi}_i\| + \|i\mu + P_{(\mathbf{A}_i, \Psi_i)}\|,$$

which gives a uniform L^2 -bound on $\|F_{\mathbf{A}_i}^+\|$. By the gauge fixing condition this provides an L_k^2 -bound on the $\mathbf{A}_i - \mathbf{A}_0$.

The elliptic estimates

$$\|\mathbf{A}_i - \mathbf{A}_0\|_{L_k^2} \leq C \left(\|(d^* + d^+)(\mathbf{A}_i - \mathbf{A}_0)\|_{L_{k-1}^2} + \|\mathbf{A}_i - \mathbf{A}_0\|_{L_{k-1}^2} \right)$$

and

$$\|\Psi_i\|_{L_k^2} \leq C \left(\|\nabla_A \Psi_i\|_{L_{k-1}^2} + \|\Psi_i\|_{L_{k-1}^2} \right)$$

provide a uniform bound on the L_k^2 -norms of the (\mathbf{A}_i, Ψ_i) on $Y \times [-T-1, T+1]$. By the Sobolev embeddings, this implies that on $Y \times [-T-1, T+1]$ there is a subsequence $(\mathbf{A}_{i'}, \Psi_{i'})$ that converges uniformly with all derivatives.

Thus we can show that (\mathbf{A}_i, Ψ_i) has subsequence that converges to a solution of (20) and (21) on $Y \times R$. In fact, by the uniform exponential decay, there is a subsequence converging strongly on $Y \times ((-\infty, -T] \cup [T, \infty))$. On $Y \times [-T-1, T+1]$, as we have seen, after passing to a further subsequence, there exist gauge transformations $u_i \in \mathcal{G}_{k+1}^0(Y \times [-T-1, T+1])$ such that the transformed solutions converge strongly on $Y \times [-T-1, T+1]$. We need to merge $\{u_i\}$ on the overlap $K = Y \times ([-T-1, -T] \cup [T, T+1])$. This can be done by choosing a cut-off function c equal to 1 on $Y \times [-T, T]$ and to 0 on $Y \times ((-\infty, -T-1] \cup [T+1, \infty))$. Over K , there exists a subsequence of $\{u_i\}$ converging strongly to a gauge transformation u . For a sufficiently large N and for all $i > N$, we have the C^0 -bound $|u_i - u_N| < 1/2$. Then for all $i > N$, $u_i = u_N \exp(2\pi i \theta)$ for a unique function θ_i on K satisfying $|\theta_i| < 1/2$. Now define gauge transformations $\{v_i\}$ on $Y \times R$ by

$$v_i = \begin{cases} u_i & \text{on } Y \times [-T, T], \\ u_N \exp(2\pi i c \theta_i) & \text{on } Y \times ([-T-1, -T] \cup [T, T+1]), \\ u_N & \text{on } Y \times ((-\infty, -T-1] \cup [T+1, \infty)). \end{cases}$$

Then the sequence $v_i(\mathbf{A}_i, \Psi_i)$ converges strongly on $Y \times \mathbb{R}$. We denote with $(\mathbf{A}_\infty, \Psi_\infty)$ the limit. This satisfies equations (20) and (21) on $Y \times \mathbb{R}$ and is of

finite energy. Thus by Corollary 4.13 it determines an element of $\hat{\mathcal{M}}(c, d)$. For dimensional reasons we must have $\mu(a) > \mu(c) > \mu(d) > \mu(b)$.

QED

The same argument applies to the moduli spaces $\hat{\mathcal{M}}^H(a, b)$, see [41].

Notice that there can be at most finitely many distinct orbits c, d in \mathcal{B}^H that appear as endpoints of limits of sequences of solutions in $\hat{\mathcal{M}}^H(a, b)$. This happens because we impose the condition $\int_Y c_1(L) \wedge [\lambda] = 0$ for the gauge transformations on Y . In fact, with this condition no two gauge equivalent points b and $\lambda \cdot b$ can satisfy $\mu(a) - \mu(b) = \mu(a) - \mu(\lambda \cdot b)$. Thus there can only be finitely many possible $\hat{\mathcal{M}}^H(c, d)$ in the boundary of $\hat{\mathcal{M}}^H(a, b)$.

If one considers the identity component in the group of gauge transformations on Y and the critical orbits in the space \mathcal{B}^0 of connections and sections modulo the action of the connected component of the based gauge group, then there may be infinitely many orbits $\lambda \cdot b$ that have the same index. In fact, suppose the group

$$H = \{h \in H^1(Y, \mathbf{Z}) \mid \langle c_1(L) \cup h, [Y] \rangle = 0\}$$

is infinite. (This is obviously the case for instance when $c_1(L) = 0$ rationally.) Then there are infinitely many distinct critical orbits $\lambda \cdot b$ in \mathcal{B}^0 with $\mu(a) - \mu(b) = \mu(a) - \mu(\lambda \cdot b)$. Thus, all the moduli spaces $\hat{\mathcal{M}}(a, \lambda \cdot b)$ can appear in the boundary of $\hat{\mathcal{M}}(a, b)$. In this case $\hat{\mathcal{M}}(a, b)$ does not have a nice compactification.

Theorem 4.14 proves that lower dimensional moduli spaces appear naturally in the compactification of the spaces $\hat{\mathcal{M}}^H(a, b)$. In the rest of this section we describe a gluing formula, thus proving that the boundary strata consist precisely of broken trajectories that live in lower dimensional moduli spaces.

Theorem 4.15 *Suppose given a, b and c in \mathcal{M}_C^H with $\mu(a) > \mu(b) > \mu(c)$. Assume that b is irreducible. Then, given a compact set*

$$K \subset \mathcal{M}^H(a, b) \times \mathcal{M}^H(b, c),$$

there are a lower bound $T(K) > 0$ and a smooth map

$$\# : K \times [T(K), \infty) \rightarrow \mathcal{M}^H(a, c)$$

$$((\mathbf{A}_1, \Psi_1), (\mathbf{A}_2, \Psi_2), T) \mapsto (\mathbf{A}_1 \#_T \mathbf{A}_2, \Psi_1 \#_T \Psi_2),$$

such that $\#_T$ is an embedding for all $T > T(K)$. The gluing map $\#$ induces a smooth embedding

$$\hat{\#} : \hat{K} \times [T(K), \infty) \rightarrow \hat{\mathcal{M}}^H(a, c),$$

where

$$\hat{K} \subset \hat{\mathcal{M}}^H(a, b) \times \hat{\mathcal{M}}^H(b, c).$$

This gives the compactification of the space $\hat{\mathcal{M}}^H(a, b)$. The same result applies to the moduli space $\mathcal{M}(a, b)$ and gives an analogous compactification.

Corollary 4.16 *For a generic choice of the metric and of the perturbation, $\hat{\mathcal{M}}^H(a, b)$ has a compactification obtained by adding boundary strata of the form*

$$\bigcup_{c_1, \dots, c_k} \hat{\mathcal{M}}^H(a, c_1) \times \hat{\mathcal{M}}^H(c_1, c_2) \times \dots \times \hat{\mathcal{M}}^H(c_k, b).$$

Here the union is over all possible sequences of the critical points c_1, \dots, c_k with decreasing indices.

Proof of Theorem 4.15: The proof consists of several steps. We first define a pre-gluing map $\#_T^0$ which provides an approximate solution, and then we prove that this can be perturbed to an actual solution. We follow the similar argument presented in [37].

The pre-gluing map is defined via the following construction. Let $x_1(t) = (A_1(t), \psi_1(t))$ and $x_2(t) = (A_2(t), \psi_2(t))$ be elements in the moduli spaces $\mathcal{M}^H(a, b)$ and $\mathcal{M}^H(b, c)$ respectively.

We can write $x_1(t) = b + (\alpha_1(t), \phi_1(t))$ and $x_2(-t) = b + (\alpha_2(t), \phi_2(t))$, where the elements (α_i, ϕ_i) have exponentially decaying behaviour, as in Proposition 4.7. We construct an approximate solution $x(t) = (A_1 \#_T^0 A_2(t), \psi_1 \#_T^0 \psi_2(t))$ as in [26], [40], of the form

$$x(t) = \begin{cases} (A_1(t+2T), \psi_1(t+2T)) & t \leq -1 \\ b + \rho^-(t)(\alpha_1(t+2T), \phi_1(t+2T)) + \\ \rho^+(t)(\alpha_2(t-2T), \phi_2(t-2T)) & -1 \leq t \leq 1 \\ (A_2(t-2T), \psi_2(t-2T)) & t \geq 1. \end{cases}$$

Here $\rho^\pm(t)$ are smooth cutoff functions with bounded derivative, such that $\rho^-(t)$ is equal to one for $t \leq -1$ and to zero for $t \leq 0$ and $\rho^+(t)$ is equal to zero for $t \leq 0$ and to one for $t \geq 1$.

Consider the Hilbert bundles \mathcal{T}_1 and \mathcal{T}_0 that are defined respectively as pullbacks via the map $\#_T^0$ of the $L_{1,\delta}^2$ and of the $L_{0,\delta}^2$ tangent bundles of $\mathcal{B}^H(a, c)$, on the base space $K \times [T_0, \infty)$.

The flow

$$\frac{d}{dt} \psi_1 \#_T^0 \psi_2 = -\partial_{A_1 \#_T^0 A_2} \psi_1 \#_T^0 \psi_2,$$

$$\frac{d}{dt} A_1 \#_T^0 A_2 = \sigma(\psi_1 \#_T^0 \psi_2, \psi_1 \#_T^0 \psi_2) - *F_{A_1 \#_T^0 A_2} + 2i\rho + 2q_{(A_1 \#_T^0 A_2, \psi_1 \#_T^0 \psi_2)}$$

defines the fibre restriction of a bundle map from \mathcal{T}_1 to \mathcal{T}_2 defined on a neighbourhood of the zero section in \mathcal{T}_1 . The linearization \mathcal{L}_x at the approximate solution x is the fibre derivative of the above bundle map.

Since the linearizations $\mathcal{L}_{\mathbf{A}_1, \Psi_1}$ and $\mathcal{L}_{\mathbf{A}_2, \Psi_2}$ are surjective, then

$$\mathcal{K} = \bigcup_{K \times [T_0, \infty)} \text{Ker}(\mathcal{L}_{\mathbf{A}_1, \Psi_1}) \times \text{Ker}(\mathcal{L}_{\mathbf{A}_2, \Psi_2})$$

is a subbundle of \mathcal{T}_1 . Thus we can define a space \mathcal{T}_χ^\perp for $\chi \in K \times [T_0, \infty)$ given by all elements of \mathcal{T}_1 that are orthogonal to the image of $\text{Ker}(\mathcal{L}_{\mathbf{A}_1, \Psi_1}) \times \text{Ker}(\mathcal{L}_{\mathbf{A}_2, \Psi_2})$ under the linearization $L_\#$ of the pre-gluing $\#_T^0$.

Proposition 4.17 *There exist a bound $T(K) \geq T_0$ such that, for all $T \geq T(K)$ and for all broken trajectories*

$$((A_1(t), \psi_1(t)), (A_2(t), \psi_2(t))) \in K,$$

the Fredholm operator \mathcal{L}_x

$$\mathcal{L}_x : \mathcal{T}_{1x} \rightarrow \mathcal{T}_{0x}$$

is surjective, where $x(t)$ is the approximate solution. Moreover, composition of the pre-gluing map $\#_T^0$ with the orthogonal projection on $\text{Ker} \mathcal{L}_x$ gives an isomorphism

$$\text{Ker}(\mathcal{L}_{\mathbf{A}_1, \Psi_1}) \times \text{Ker}(\mathcal{L}_{\mathbf{A}_2, \Psi_2}) \xrightarrow{\cong} \text{Ker} \mathcal{L}_x.$$

Proof of Proposition 4.17: We know that \mathcal{L}_x is Fredholm of index $\mu(a) - \mu(c)$. We also know that $\dim \text{Ker}(\mathcal{L}_{x_1}) = \mu(a) - \mu(b)$ and $\dim \text{Ker}(\mathcal{L}_{x_2}) = \mu(b) - \mu(c)$.

We need to show that for any pair $((\mathbf{A}_1, \Psi_1), (\mathbf{A}_2, \Psi_2))$ there is a bound $T_0 = T(x_1, x_2)$ such that \mathcal{L}_x is surjective for $T \geq T_0$. The compactness of K will ensure that there is a uniform such bound $T(K)$.

It is therefore enough to prove the following crucial step.

Lemma 4.18 *There exist T_0 and a constant $C > 0$ such that*

$$\|\mathcal{L}_x \xi\|_{L_{0,\delta}^2} \geq C \|\xi\|_{L_{1,\delta}^2}.$$

for all $T \geq T_0$ and ξ in \mathcal{T}_χ^\perp , where in our notation $x = \#_T^0(\chi)$.

Proof of Lemma 4.18: Suppose there are sequences $T_k \rightarrow \infty$ and $\xi_k \in \mathcal{T}_\chi^\perp$ such that $\|\xi_k\| = 1$ and $\|\mathcal{L}_x \xi_k\| \rightarrow 0$.

We first show that the supports of the ξ_k become more and more concentrated at the asymptotic ends as $k \rightarrow \infty$. We consider the operator $\mathcal{L}_b = \frac{\partial}{\partial t} + L_b$. If $\zeta : \mathbb{R} \rightarrow [0, 1]$ is a smooth function which is equal to 1 on $[-1/2, 1/2]$ and equal to zero outside $(-1, 1)$, let $\zeta_k(t) = \zeta(\frac{t}{2T_k})$. Then we have

$$\|\mathcal{L}_b \zeta_k \xi_k\| \leq \|\zeta_k' \xi_k\| + \|\zeta_k \mathcal{L}_b \xi_k\| \leq$$

$$\begin{aligned}
&\leq \frac{1}{2T_k} \max |\zeta'| + \|(\mathcal{L}_x - \mathcal{L}_b)\xi_k\| + \|\mathcal{L}_x \xi_k\| \leq \\
&\leq \frac{1}{T_k} \max |\zeta'| + \sup_{t \in [-T_k, T-k]} \|L_{x(t)} - L_b\| \|\xi_k\| + \|\mathcal{L}_x \xi_k\|.
\end{aligned}$$

Thus, $\|\mathcal{L}_b \zeta_k \xi_k\| \rightarrow 0$ as $k \rightarrow \infty$. In fact, the term $\sup_{t \in [-2T_k, 2T-k]} \|L_{x(t)} - L_b\|$ is bounded by

$$\sup_{t \in [-1, 1]} \|L_{x(t)} - L_b\| + \sup_{t \in [-2T_k, -1]} \|L_{x_1(t+2T_k)} - L_b\| + \sup_{t \in [1, 2T_k]} \|L_{x_2(t-2T_k)} - L_b\|.$$

All these terms tend to zero because of the exponential decay to the critical point b of the trajectories $(A_1(t), \psi_1(t))$ and $(A_2(t), \psi_2(t))$. The operator \mathcal{L}_b is an isomorphism between the spaces $L_{1,\delta}^2(\mathbb{R}, T(\mathcal{B}^H))$ and $L_{0,\delta}^2(\mathbb{R}, T(\mathcal{B}^H))$, hence we have $\xi_k \rightarrow 0$ in the L_1^2 norm over $Y \times [-2T_k, 2T_k]$.

This result allows us to rephrase the convergence condition $\|\mathcal{L}_x \xi_k\| \rightarrow 0$ in terms of the Fredholm operators \mathcal{L}_{x_1} and \mathcal{L}_{x_2} :

$$\begin{aligned}
\|\mathcal{L}_{x_1}(\rho_{1-T_k}^- \xi_k^{-T_k})\| &\leq \|\rho'^- \xi_k\| + \|\rho^- \mathcal{L}_x \xi_k\| \leq \\
&\leq C \|\xi_k\|_{Y \times [-1, 1]} + \|\mathcal{L}_x \xi_k\| \rightarrow 0,
\end{aligned}$$

where $\rho_{1-T_k}^-(t) = \rho(t+1-T_k)$ and $\xi_k^{-T_k}(t) = \xi_k(t-T_k)$. This implies $\rho_{1-T_k}^- \xi_k^{-T_k} \rightarrow v$ where $v \in \text{Ker}(\mathcal{L}_{x_1})$, since \mathcal{L}_{x_1} is a Fredholm operator. Thus, $\|\rho_1^- \xi_k - v^{T_k}\| \rightarrow 0$. Similary we obtain an element u in $\text{Ker}(\mathcal{L}_{x_2})$ such that $\|\rho_{-1}^+ \xi_k - u^{-T_k}\| \rightarrow 0$.

We now use these estimates to derive a contradiction with the assumption that $\|\xi_k\| = 1$ and $\xi_k \in \mathcal{T}_X^\perp$. We have

$$1 = \lim_k \|\xi_k\| = \lim_k \langle \rho_1^- \xi_k, \xi_k \rangle + \langle \rho_{-1}^+ \xi_k, \xi_k \rangle,$$

since the remaining term satisfies

$$\langle (1 - \rho_1^- - \rho_{-1}^+) \xi_k, \xi_k \rangle = 0$$

for large k because $(1 - \rho_1^- - \rho_{-1}^+)$ is supported in $[-2, 2]$. Thus the equality can be rewritten as

$$1 = \lim_k \langle \rho^- v, \xi_k \rangle + \lim_k \langle \rho^+ u, \xi_k \rangle = \lim_k \langle L_\#(u, v), \xi_k \rangle = 0.$$

The last equality holds since, by construction, $\xi_k \in \mathcal{T}^\perp$ is orthogonal to the image $L_\#$ of the linearization of the pre-gluing map.

This completes the proof of Lemma 4.18. Now the rest of Proposition 4.17 follows, since we obtain

$$\dim \text{Ker}(\mathcal{L}_x) = \dim \text{Ker}(\mathcal{L}_{\mathbf{A}_1, \Psi_1}) + \dim \text{Ker}(\mathcal{L}_{\mathbf{A}_2, \Psi_2}).$$

QED

This means, as we are going to discuss, that \mathcal{T}^\perp is the normal bundle for the gluing construction.

Now we want to define the actual gluing map $\#$ that provides a solution of the flow equations in $\mathcal{M}^H(a, c)$. This means that we want to obtain a section σ of \mathcal{T}_1 such that the image under the bundle homomorphism given by the flow equation is zero in \mathcal{T}_0 . Moreover, we want this element $\sigma(\mathbf{A}_1, \Psi_1, \mathbf{A}_2, \Psi_2, T)$ to converge to zero sufficiently rapidly as $T \rightarrow \infty$, so that the glued solution will converge to the broken trajectory in the limit $T \rightarrow \infty$.

The result is obtained as a fixed point theorem in Banach spaces. Consider the right inverse map of \mathcal{L} restricted to \mathcal{T}^\perp ,

$$G : \mathcal{T}_0 \rightarrow \mathcal{T}^\perp.$$

There is a $T(K)$ and a constant $C > 0$ such that

$$\|G_\chi \xi\| \leq C \|\xi\|$$

for $\chi \in K \times [T(K), \infty)$. Some care is needed in obtaining the uniformity of the constant C with respect to χ . We refer to [37] for further details.

We aim at using the contraction principle. Namely, suppose given a smooth map $f : E \rightarrow F$ between Banach spaces of the form

$$f(x) = f(0) + Df(0)x + N(x),$$

with $\text{Ker}(Df(0))$ finite dimensional, with a right inverse $Df(0) \circ G = \text{Id}_F$, and with the nonlinear part $N(x)$ satisfying the estimate

$$\|GN(x) - GN(y)\| \leq C(\|x\| + \|y\|)\|x - y\| \quad (28)$$

for some constant $C > 0$ and x and y in a small neighbourhood $B_{\epsilon(C)}(0)$. Then, with the initial condition $\|G(f(0))\| \leq \epsilon/2$, there is a unique zero x_0 of the map f in $B_\epsilon(0) \cap G(F)$. This satisfies $\|x_0\| \leq \epsilon$.

The map f is given in our case by the flow equation, viewed as a bundle homomorphism $\mathcal{T}_1 \mapsto \mathcal{T}_0$. We write f as a sum of a linear and a non-linear term, where the linear term is \mathcal{L} and the nonlinear term is

$$N_{(A(t), \psi(t))}(\alpha, \phi) = (\sigma(\phi, \phi) + \mathcal{N}q_{(A(t), \psi(t))}(\alpha, \phi), \alpha \cdot \phi).$$

Here we write the perturbation q of equation (19) as sum of a linear and a non-linear term, $2q = \mathcal{D}q + \mathcal{N}q$.

It is clear that the quadratic form $(\sigma(\phi, \phi), \alpha \cdot \phi)$ satisfies an estimate of the form (28). The perturbation term also satisfies a similar estimate for large enough T because of the assumptions on the perturbation space \mathcal{P} . This implies the estimate (28) for $G \circ N$.

We have to verify the initial condition. This is provided by the exponential decay. In fact, we have

$$\|f(A_1\#_T^0 A_2, \psi_1\#_T^0 \psi_2)\| \leq C(\|(\alpha_1, \phi_1)^{2T}\|_{Y \times [-1,0]} + \|(\alpha_2, \phi_2)^{-2T}\|_{Y \times [0,1]}).$$

The exponential decay of $(A_1(t), \psi_1(t))$ and $(A_2(t), \psi_2(t))$ to the endpoints implies a decay

$$\|f(A_1\#_T^0 A_2, \psi_1\#_T^0 \psi_2)\| \leq C e^{-\delta T} \quad (29)$$

For all $T \geq T_0$. The constant C and the lower bound T_0 can be chosen uniformly due to the compactness of K .

This provides the existence of a unique correction term

$$\sigma(\mathbf{A}_1, \Psi_1, \mathbf{A}_2, \Psi_2, T) \in B_\epsilon(0) \cap \mathcal{T}^\perp$$

satisfying $f(\sigma) = 0$. The implicit function theorem ensures that σ is smooth. The exponential decay (29) ensures an analogous decay for σ , hence the glued trajectory approaches the broken trajectory when T is very large. The gluing map is given by

$$(A_1\#_T A_2, \psi_1\#_T \psi_2) = (A_1\#_T^0 A_2, \psi_1\#_T^0 \psi_2) + \sigma(\mathbf{A}_1, \Psi_1, \mathbf{A}_2, \Psi_2, T).$$

QED

5 A Casson-like Invariant

One can define an invariant by taking the Euler characteristic of the SWH_* groups. I have learned from B.L. Wang that the construction of this invariant have been worked out independently in [7], which is a more recent version of [6] where the invariant was introduced in terms of the partition function of a Topological Quantum Field Theory.

In the case of Donaldson-Floer theory, there is a nice identification of flat connections (critical points of the Chern-Simons functional) with representations of the fundamental group in $SU(2)$. Therefore it is possible to describe the invariant in purely geometric terms, and it turns out to be the Casson invariant of homology 3-spheres, as shown in [39].

In our case there is no immediate geometric interpretation of the solutions of (8) in non-gauge theoretic terms.

However some of the results of [39] can be carried over to the present case, with some minor modifications in the arguments. We shall present these results in the next section.

The following result makes it, in principle, easier to compute this invariant in the case when \mathcal{M}_C , the moduli space of critical points, is zero dimensional.

Theorem 5.1 *The invariant $\chi(SWH_*(Y))$ is just the sum over points in \mathcal{M}_C of the signs given by the orientations defined in section 2.4,*

$$\chi(SWH_*(Y)) = \sum_{a \in \mathcal{M}_C} \epsilon_a.$$

Proof: Since the dimension of the set of critical points is zero, the operator L has trivial kernel. This implies $Ker(T) = 0$ at the critical points. Given a path γ between two critical points, consider the set $\Omega = \gamma \times \mathbf{C}$. Since $Ker(T_0) = Ker(T_1) = 0$ (see [2]), the spectral flow of

$$T_t = T |_{(A(t), \psi(t))}$$

along γ can be thought of as the algebraic sum of the intersections in Ω of the set

$$\mathcal{S} = \cup_{t \in \gamma} Spec(T_t)$$

with the line $\{(t, 0) \mid t \in \gamma\}$. This counts the points where the discrete spectrum of the operator T crosses zero, with the appropriate sign. Up to perturbations we can make these crossings transverse.

We can express the same procedure in a different, yet equivalent, way. Consider T_t as a section over \mathcal{B} of the bundle of index zero Fredholm operators, Fr_0 . There is a first Stiefel-Whitney class w_1 in $H^1(Fr_0; \mathbf{Z}_2)$ that classifies the determinant line bundle of Fr_0 (see [39], [18]).

The submanifold of codimension one in Fr_0 that represents the class w_1 is given by Fredholm operators of nonempty kernel, $Fr_0^1 \subset Fr_0$. This submanifold can be thought of as the zero set of a generic section of the determinant line bundle.

Given a path $\gamma : I \rightarrow \mathcal{B}$ joining two points a and b , the image of γ composed with a generic section σ of Fr_0 will meet Fr_0^1 transversally.

We call this intersection number $\delta_\sigma(a, b)$. This counts the points in $Det(\sigma \circ \gamma)^{-1}(0)$ with the orientation. If we take the section $\sigma \circ \gamma$ to be T_t what we get is exactly the spectral flow between the critical points a and b . This number is defined modulo l , but the mod 2 spectral flow does not depend on the choice of the path. Now we want to show that this same intersection number measures the change of orientation between the two points a and b of \mathcal{M}_C .

The orientation of the tangent space at a critical point is given by a trivialization of $Det(L)$ or equivalently of $Det(T)$ at that point.

Up to a perturbation, we can assume that $Coker(\tilde{T}_t)$ is trivial along the path γ , see lemma 3.8. Therefore we can think of $Det(T_t)$ as $\Lambda^{top} Ker(T_t)$, which specifies the orientation of the space $Ker(T_t)$. Thus we obtain that the change of orientation between two critical points is measured by the mod 2 spectral flow.

This means that the sign attached to the point in the grading of the Floer complex is the same as the sign that comes from the orientation of \mathcal{M}_C .

QED

A definition of the Seiberg-Witten invariant of three-manifolds as the number obtained by counting points in \mathcal{M}_C with the orientation was given in [4].

The invariant $\chi(SWH_*(Y))$ vanishes for all Y which admit a metric of positive scalar curvature. This is a consequence of the Weitzenböck formula, [43]. However the invariant is non-trivial. In fact consider the case of a 3-manifold $Y = \Sigma \times S^1$, with the $Spin_c$ structure determined by the pullback of a line bundle L of degree d on the surface Σ . As proved in [32] (see also [13] and [34]), the Seiberg-Witten-Floer homology of Y is the ordinary homology of the symmetric product $s^r(\Sigma)$, with $r = (2g - 2) - d > 0$. But the Euler characteristic of $s^r(\Sigma)$ is

$$\chi(s^r(\Sigma)) = \chi(SWH_*(\Sigma \times S^1)) = \binom{2g - 2}{r}.$$

Notice, however, that in this particular case the invariant is computed with the unperturbed equations, which give rise to a positive dimensional moduli space \mathcal{M}_C . This is therefore an analogue of a degenerate Morse theory.

5.1 Invariance

We prove here that the Euler characteristic of the Floer homology is a topological invariant of the 3-manifold Y , if the manifold has Betti number $b^1(Y) > 1$. This means that the Euler characteristic of the Floer homology is independent of the choice of the metric and of the perturbation. We give an argument which is similar to the one constructed for Donaldson theory in [14], pg. 140-146.

Suppose we are given two Riemannian metrics on Y , g_0, g_1 . Consider a path of metrics $g_t, t \in [0, 1]$, connecting them. Take the infinite dimensional manifold $\mathcal{B} \times [0, 1]$. We want to construct a “universal moduli space” for the equation (8). Consider the product bundle over $\mathcal{B} \times [0, 1]$ with fibre $\Lambda^1(Y) \oplus \Gamma(S \otimes L)$. We have a section given by

$$s(A, \psi, t) = (\partial_A^t \psi, *_t F_A - 2\rho_t - \sigma(\psi, \psi)). \quad (30)$$

Note that in (30) both the Dirac operator and the Hodge $*$ -operator depend on the metric, hence on the parameter t . Here $\{\rho_t\}$ is any family of 1-forms that are co-closed with respect to $*_t$ and away from the wall, i.e. $*_t \rho_t \neq \pi c_1(L)$. The universal moduli space is the zero set of the section s , $\mathcal{MU} = s^{-1}(0)$.

We need the following lemma. Recall that a map of Banach manifolds is Fredholm if its linearization is a Fredholm operator.

Lemma 5.2 *The section s is a Fredholm section. The operator that linearizes (30) is onto.*

Proof: The linearization of (30) is given by

$$ds_{(A, \psi, \rho, t)}(\alpha, \phi, \eta, \epsilon) = \epsilon \frac{\partial}{\partial t} s(A, \psi, \rho, t) + \tilde{T}|_{(A, \psi, \rho, t)}(\alpha, \phi, \eta),$$

where (α, ϕ) are coordinates in the tangent space of \mathcal{B} , η is a 1-form, and $\epsilon \in \mathbb{R}$.

The operator \tilde{T} is the perturbed Hessian,

$$\tilde{T}|_{(A, \psi, \rho, t)}(\alpha, \phi, \eta) = -2\eta + T|_{(A, \psi, \rho, t)}(\alpha, \phi).$$

We follow the same argument used in lemma 3.8 to show that \tilde{T} is surjective for a generic choice of the perturbation.

QED

Now we can apply again the implicit function theorem on Banach manifolds, and we get that s is transverse to the zero section. Therefore the universal moduli space is a smooth manifold, provided the perturbation ρ_t is chosen in such a way that the moduli space corresponding to each value of t does not contain reducibles ($\psi \equiv 0$). It is sufficient to add the constraint that $*\rho_t \neq \pi c_1(L)$. The dimension of the universal moduli space is $\text{Ind}(T) + 1 = 1$. The proofs of the compactness and orientability of the universal moduli space are analogous to the proofs given in section 2.

The independence of the metric now follows from the fact that the moduli spaces corresponding to the metrics g_0 and g_1 form the boundary of a compact oriented 1-manifold, and the total oriented boundary of such a manifold is zero.

Remark 5.3 *The condition that $*\rho_t \neq \pi c_1(L)$ is satisfied for a generic choice of the perturbation if the manifold Y has $b^1(Y) > 1$. If Y has $b^1(Y) = 1$ then the condition is satisfied by sufficiently small 1-form ρ_t , hence in this case the metric and the perturbation cannot be both chosen arbitrarily. However, given g_0 and g_1 it is possible to find sufficiently small perturbations such that the invariant does not change along the path of metrics g_t . This has been pointed out already in [4].*

5.2 Heegaard Splittings

In this section we extend to the case of Seiberg-Witten-Floer homology a result of Taubes' [39] concerning the behaviour of the Euler characteristic of Donaldson-Floer homology under a Heegaard splitting of a homology 3-sphere. This proves that the invariant $\chi(SWH_*(Y))$ behaves very much like the Casson invariant. In fact it was conjectured by Kronheimer [20], and recently proved by Chen [11] and Lim [23] independently, that in the case of a homology sphere the Euler characteristic of the Seiberg-Witten Floer homology differs from the Casson invariant for a correction term that depends on the index of the Dirac operator and on the signature of a *Spin*-four manifold that bounds the homology sphere [20].

We follow the same argument and the notation used in [39]. We do not write out the analysis in full details in the proof of the Heegaard splitting formula, since we prefer to concentrate on the topological result and refer for the analysis to the case of the instanton Floer homology and the Casson invariant as worked out by Taubes [39].

Let us introduce some preliminary definitions. Let \mathcal{F} denote the 1-form on $\hat{\mathcal{A}}$ obtained as the differential of the functional $\tilde{\mathcal{C}}$ as in (16):

$$\mathcal{F}|_{(A,\psi)}(\alpha, \phi) = - \int_Y \alpha \wedge (F_A - 2 * \rho - * \sigma(\psi, \psi)) + \int_Y \langle \phi, \partial_A \psi \rangle .$$

Consider the open sets Y_1, Y_2 and their intersection Y_0 , which is of the form $Y_0 = \Sigma \times I$ for some interval I and some closed surface Σ of genus g .

Consider the pullback of the spinor bundle under the inclusion maps $i_l : Y_l \hookrightarrow Y$ and $j_l : Y_0 \hookrightarrow Y_l, l = 1, 2$. We shall indicate by $\mathcal{A}_l, l = 0, 1, 2$, the space of pullback connections and sections of the pullback bundles. Consider the maps

$$\mathcal{A} \xrightarrow{i_1^* \times i_2^*} \mathcal{A}_1 \times \mathcal{A}_2 \xrightarrow{j_1^* \times j_2^*} \mathcal{A}_0 \times \mathcal{A}_0 .$$

We shall denote by $\hat{\mathcal{A}}_l$ the set of irreducible pairs in \mathcal{A} (which means ψ not identically zero) that map to irreducible pairs in \mathcal{A}_0 . Note that if a pair is a critical point of the functional $\tilde{\mathcal{C}}$ on Y then it maps to an irreducible pair. In fact the section ψ satisfies the equation $\partial_A \psi = 0$. Therefore if it vanishes on an open set it has to be identically zero on all of Y . Consider the induced maps on the quotient space $\hat{\mathcal{B}}_l$ of $\hat{\mathcal{A}}_l$ with respect to the action of the gauge group. We define 1-forms \mathcal{F}_l on the tangent spaces of \mathcal{A}_l and induced forms on $\hat{\mathcal{B}}_l$ using the same expression given in theorem 3.11. We shall use the notation \mathcal{M}_l to denote the set $\mathcal{F}_l^{-1}(0)$ in $\hat{\mathcal{B}}_l$.

Theorem 5.4 *Consider a Heegaard splitting of a closed oriented 3-manifold $Y, Y = Y_1 \cup_\Sigma Y_2$. The Euler characteristic*

$$\chi = \chi(SWH_*(Y))$$

of the Floer homology is the intersection number of the manifolds $j_1^ \mathcal{M}_1$ and $j_2^* \mathcal{M}_2$ inside \mathcal{M}_0 .*

Proof: To prove the theorem we need some preliminary steps.

The L_k^2 -tangent spaces to the gauge orbits are

$$\mathcal{T}_l = \{(\alpha, \phi) \in \Lambda^1(Y_l) \oplus \Gamma(\tilde{S} \otimes L|_{Y_l}), G^*(\alpha, \phi) = 0, i_{\partial Y_l}^*(\alpha, \phi) = 0\},$$

where we impose the vanishing condition on the boundary. We shall also consider the Banach bundle \mathcal{L}_l given by

$$\mathcal{L}_l = \{(\alpha, \phi) \in \Lambda^1(Y_l) \oplus \Gamma(\tilde{S} \otimes L|_{Y_l}), G^*(\alpha, \phi) = 0\},$$

with the L_k^2 -norm and no boundary conditions.

Lemma 5.5 *In the sequence of maps*

$$\mathcal{A} \xrightarrow{I} \mathcal{A}_1 \times \mathcal{A}_2 \xrightarrow{J} \mathcal{A}_0 \times \mathcal{A}_0,$$

with $I = i_1^* \times i_2^*$ and $J = j_1^* \times j_2^*$, I is an embedding and J is a submersion. Moreover, $\text{Im}(I) = J^{-1}(\Delta)$, i.e. the image of I is the inverse image of the diagonal under the map J . Moreover the same result holds for the induced sequence on the spaces $\hat{\mathcal{B}}_l$.

Proof: We need to prove that the induced maps have the required properties on the level of tangent spaces. This result can be obtained by the same proof as that given in [39]. In fact the tangent maps induced by I and J can be written as

$$I_*(\alpha, \phi) = (i_1^*(\alpha, \phi) + G(f_1), i_2^*(\alpha, \phi) + G(f_2))$$

and

$$J_*(\alpha_1, \phi_1, \alpha_2, \phi_2) = (j_1^*(\alpha_1, \phi_1) + G(g_1), j_2^*(\alpha_2, \phi_2) + G(g_2)),$$

subject to the condition $G^*G(f_i) = G^*G(g_i) = 0$ and the vanishing condition on the boundary. The gauge maps are chosen so as to guarantee that I_* and J_* map tangent spaces to tangent spaces.

But the only solution to the equation $G^*Gf = 0$ on Y_l and $f = 0$ on ∂Y_l is the trivial one. Hence I_* is injective, J_* is surjective and $\text{Im}(I_*) = \text{Ker}((j_1^*)_* - (j_2^*)_*)$.

QED

Define the forms $\nabla \mathcal{F}_l = \frac{d}{ds} \mathcal{F}_l((A, \psi) + s(\alpha, \phi)) |_{s=0}$, as in (17). These quadratic forms define operators T_l on the tangent space to $\hat{\mathcal{B}}_l$ that coincide with the Hessian of theorem 3.11 when (A, ψ) is a critical point of $\tilde{\mathcal{C}}$, i.e. when $\mathcal{F}_l |_{(A, \psi)} \equiv 0$.

Lemma 5.6 *The operator T_l is a Fredholm operator from*

$$\Lambda^0(Y_l) \oplus \Lambda^1(Y_l) \oplus \Gamma(S \otimes L |_{Y_l}),$$

completed in the norm L_k^2 with vanishing conditions on the boundary, to the same space completed in the L_{k-1}^2 norm.

Proof: Up to a compact perturbation we have the operator D of (11) and the Dirac operator. The first is Fredholm from $\Lambda^0(Y_l) \oplus \Lambda^1(Y_l)$, completed in the L_k^2 norm with vanishing conditions, to $\Lambda^0(Y_l) \oplus \Lambda^1(Y_l)$, completed in the L_{k-1}^2 norm, and the latter is Fredholm between the corresponding spaces of sections $\Gamma(S \otimes L |_{Y_l})$.

QED

As a consequence of these lemmata we have the following result.

Theorem 5.7 *The sets $\mathcal{F}_l^{-1}(0)$ obtained from the perturbed equations on Y_l are embedded submanifolds $\mathcal{M}_{C,1}$ of $\hat{\mathcal{B}}_l$. Moreover the intersection of $\mathcal{M}_{C,1}$ and $\mathcal{M}_{C,2}$ in $\mathcal{M}_{C,0}$ is transverse.*

Proof: The first assertion of the theorem is an application of the implicit function theorem for Banach manifolds and Fredholm operators. It is sufficient to prove that $\dim Ker(T_l) = Ind(T_l)$. Since we are assuming vanishing conditions on the boundary, surjectivity of T_l for a suitable choice of the perturbation follows from the argument used in lemma 3.8.

For the trasversality property, we can prove the following claim. The intersection is transverse at a point $(A, \psi) \in \mathcal{M}_C$ if and only if the kernel of T at that point is trivial. But at a critical point this follows if we have that $Ker(L)$ is trivial. We proved in lemma 3.8 that this is the case for a generic perturbation.

In order to prove the claim we follow [39] and introduce the operator

$$\begin{aligned} H|_{(A, \psi)}: \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{L}_0 &\rightarrow \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{T}_0, \\ H|_{(A, \psi)}(\alpha_l, \phi_l) &= (T_1(\alpha_1, \phi_1), T_2(\alpha_2, \phi_2), \\ (j_1^*)_*(\alpha_1, \phi_1) - (j_2^*)_*(\alpha_2, \phi_2)) - T_0^*(\alpha_0, \phi_0)). \end{aligned}$$

The kernel of the operator H measures the lack of transversality of the intersection. In fact $Ker(H|_{(A, \psi)})$ is given by the pairs $(\alpha_1, \phi_1) \in Ker(T_1)$ and $(\alpha_2, \phi_2) \in Ker(T_2)$ such that the vector

$$(j_1^*)_*(\alpha_1, \phi_1) - (j_2^*)_*(\alpha_2, \phi_2)$$

is orthogonal to $Ker(T_0)$. This means exactly that the tangent spaces of $\mathcal{M}_{C,1}$ and $\mathcal{M}_{C,2}$ meet in a non trivial subspace in order to sum up to the dimension of the tangent space of $\mathcal{M}_{C,0}$.

Now suppose (A, ψ) is a point in \mathcal{M}_C and (α_l, ϕ_l) is in the kernel of T_l for $l = 1, 2$. It is not hard to check that

$$(j_1^*)_*(\alpha_1, \phi_1) - (j_2^*)_*(\alpha_2, \phi_2)$$

is in $Ker(T_0)$. So elements in the kernel of $H|_{(A, \psi)}$ have

$$(j_1^*)_*(\alpha_1, \phi_1) - (j_2^*)_*(\alpha_2, \phi_2) = 0.$$

This determines an element (α, ϕ) in $Ker(T)$. Thus at a critical point the intersection is transverse if and only if $Ker(T)$ is trivial.

QED

We can see with an index computation that the dimensions of the intersecting manifolds match properly.

Lemma 5.8 *The sum of the indices of the operators T_l over the manifolds Y_l with $l = 1, 2$ gives exactly the index of T_0 on Y_0 .*

Proof: The metric on Y is such that it is a cylinder on Y_0 and in a neighbourhood of the boundary on Y_l , $l = 1, 2$. Thus on $\Sigma \times I$ we have $Ind(D_0) = -\chi(\Sigma) = 2g - 2$ where g is the genus of Σ . On Y_l with $l = 1, 2$ we have $H^1(Y_l; \mathbf{Z}) = \mathbf{Z}^g$

and $H^1(Y_l, \partial Y_l; \mathbf{Z}) = 0$ since Y_l is a handlebody. Therefore $H^2(Y_l, \mathbf{Z}) = 0$ and $g - 1 = -\chi(Y_l) = \text{Ind}(D_l)$. The index of the Dirac operator is just the Atiyah-Patodi-Singer boundary term on ∂Y_l , $l = 1, 2$. These sum up to give exactly the boundary term on ∂Y_0 .

QED

From the previous results we have the following.

Lemma 5.9 *Up to a choice of the orientation, we can write the set of critical points as*

$$\mathcal{F}^{-1}(0) = I^{-1}((\mathcal{F}_1^{-1}(0) \times \mathcal{F}_2^{-1}(0)) \cap J^{-1}(\Delta)).$$

The last step of the proof is to show that the sign difference of two points in the oriented intersection is given exactly by the spectral flow that defines the relative Morse index in the grading of the Floer groups, up to an overall sign that comes from fixing the Morse index of one particular solution.

The argument is analogous to the proof given in [39] adapted to the present case along the line of the proof of theorem 5.1.

QED

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