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# $H_\infty$ OPTIMAL CONTROL AS WEIGHTED WIENER-HOPF OPTIMIZATION

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## ABSTRACT

It is shown that SISO $H_\infty$ optimal control problems are equivalent to a weighted Wiener-Hopf optimization problem. The optimal weight in this case is explicitly computed as the maximum magnitude Hankel singular vector of the system that is approximated. An interpretation of this result is discussed.

## I. INTRODUCTION

Over the last few years, $H_\infty$ design methods have been receiving considerable attention in the control community. It has been shown that many frequency domain control design problems can be meaningfully formulated as $H_\infty$ optimization problems. Examples of such problems include minimization of the sensitivity transfer function in a minimax sense and optimizing the robustness margins for unstructured uncertainty. Mathematically the $H_\infty$ optimal control problem is to minimize the weighted infinity norm of some closed-loop transfer function or combination of transfer functions over the controllers that satisfy the requirement of internal stability. A partial list of references is [1-13]. The solution of $H_\infty$ optimization problems is reduced by the so-called Youla parametrization lemma ([14]) and inner-outer factorizations ([2]) to the Nehari problem . The latter has been solved by a number of different techniques (see for example [7]).

$H_\infty$ optimization can be interpreted as a "loop-shaping" tool. In this sense the controller is selected so as to shape the magnitude of certain closed-loop transfer functions. On the other hand $H_2$ optimal or Wiener-Hopf design methods and their time-domain counterpart Linear Quadratic Guassian (LQG) optimal control have been also proposed as loop-shaping techniques ([15-17]). The $H_\infty$ approach makes a more precise tool than Wiener-Hopf or LQG, but it can be inflexible when only a frequency dependent bound on the magnitude of some transfer function must be achieved rather than an exact shape ([18]).

The purpose of this paper is to show that $H_\infty$ optimization is equivalent with $H_2$ optimization, that is a Wiener-Hopf optimal problem ([14],[18], etc.), in the sense that for any $H_\infty$ problem there exists a weighting function, such that the weighted $H_2$ problem results in the same compensator. In this paper we discuss the single-input-single-output case. We show that the weighting function is essentially unique and indeed equal to the maximum magnitude Hankel singular vector of the system that is approximated. We believe that this result offers new interpretations and insights through which a better understanding of the qualitative properties of $H_\infty$ optimal controllers, as well as of the robustness properties of Wiener-Hopf and LQG optimal controllers can be obtained. For example an immediate consequence of our results is that any $H_\infty$ controller can be reproduced as the limit of a sequence of LQG controllers. Thus, in case of a minimum phase plant LQG/LTR can provide such a sequence ([17]). Our results are also true in the multivariable case, which will be reported elsewhere since the method of proof is different than the one employed here.

The idea of solving weighted $H_2$ problems in place of $H_\infty$ problems is implicit in [8],[9],[21], and [17]. In particular, in [9] a system of polynomial equations, nonlinear in their coefficients, is solved to obtain $H_\infty$ optimal solutions. From these equations the optimal weight can also be derived, but the characterization given here is new.

The remainder of the paper is arranged as follows. In section II, we summarize our notation and some background results. In section III, the problem is formulated and then solved using a frequency domain (polynomial) approach. In section IV, our results are expressed in a state-space framework. An example is worked out in section V. Finally the paper is summarized in section VI.

## II. NOTATION AND BACKGROUND

| | |
|---|---|
| $R^{nxm}$ | $nxm$ matrices with real elements |
| $C^{nxm}$ | $nxm$ matrices with complex elements |
| $A^T$ | transpose of matrix $A$ |
| $A^*$ | complex-conjugate transpose of matrix $A$ |
| $\lambda(A)$ | eigenvalues of matrix $A$ |
| $Re[s]$ | real part of $s \in C$ |
| $L_\infty$ | complex valued functions of $s \in C$ which are bounded on the imaginary axis |
| $H_\infty$ | functions in $L_\infty$ which are analytic in $Re[s] > 0$ |
| $H_\infty^\perp$ | functions in $L_\infty$ which are analytic in $Re[s] < 0$ |
| $L_2$ | complex valued functions of $s \in C$ which are square integrable on the imaginary axis |
| $H_2$ | functions in $L_2$ which are analytic in $Re[s] > 0$ |
| $H_\infty^\perp$ | functions in $L_2$ which are analytic in $Re[s] < 0$ |

$\|f\|_\infty \stackrel{\text{def}}{=} sup\{|f(j\omega)|,\ \omega \in R\}$ norm on $L_\infty$, $H_\infty$, $H_\infty^\perp$

$\|f\|_2 \stackrel{\text{def}}{=} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}|f(j\omega)|^2\, d\omega\right]^{1/2}$ norm on $L_2$, $H_2$, $H_2^\perp$

| | |
|---|---|
| $\Pi_1$ | orthogonal projection from $L_2$ to $H_2^\perp$ |
| $\Pi_2$ | orthogonal projection from $L_2$ to $H_2$ |
| prefix $R$ | real rational, for example: |
| $RH_\infty$ | rational functions in $H_\infty$ (i.e. proper and stable transfer functions) |

$[g(s)]_- \stackrel{\text{def}}{=} \Pi_1 g$ for $g \in RL_\infty$.

$[g(s)]_\ominus \stackrel{\text{def}}{=} g(s) - [g(s)]_-$
$\Gamma_r$ Hankel operator of r
$\Gamma_r h = \Pi_1[r \cdot \Pi_2 h]$
$\Gamma_r^*$ adjoint of $\Gamma_r$
$\sigma_r$ largest Hankel singular value of $r$
$(f,g)$ Schmidt pair for $\Gamma_r$ corresponding to $\sigma_r$
$\Gamma_r f = \sigma_r g$
$\Gamma_r^* g = \sigma_r f$
$f$ in $(f,g)$ is a Hankel singular vector for $r$ corresponding to $\sigma_r$
$f \in L_\infty$ is allpass if $f^*(s) \cdot f(s) = 1$
$f \in H_\infty$ is inner if $f^*(s) \cdot f(s) = 1$
$f \in H_\infty$ is outer if $f^{-1} \in H_\infty$
Every $f \in H_\infty$ can be factored as $f = f_i \cdot f_o$ with $f_i$ inner and $f_o$ outer (inner-outer factorization)
$deg(p(s))$ degree of polynomial p(s)
$p(s)$ is Hurwitz if all of its roots are in $Re[s] < 0$

## III. PROBLEM FORMULATION AND SOLUTION

Consider the $H_\infty$ optimization problem

$$\min_{x(s)\in RH_\infty} \|r(s) - x(s)\|_\infty \tag{1}$$

where $r(s)$ is a given function in $RL_\infty$. It was indicated in section 1, that many interesting control problems can be reduced to problem 1. The solution of this problem has been given in many different ways. An excellent source for the novice reader is [7].

On the other hand consider the weighted $H_2$ optimization problem

$$\min_{x(s)\in RH_\infty} \|w(s)[r(s) - x(s)]\|_2 \tag{2}$$

where $w(s)$ and $r(s)$ are given functions in $RL_\infty$. Problem 2 has a particularly simple solution in view of the inner product structure of $L_2$. More precicely the solution of problem 2 is given by

$$x_{opt}^{(2)} = w_o^{-1}(s) \cdot [w_o(s)r(s)]_\ominus \tag{3}$$

where $w_o(s)$ is the outer part of $w(s)$ ([19]).

In what follows, we show that given an $H_\infty$ optimization problem 1, there exists a weight $w(s)$ so that the solution of problem 2 also solves problem 1, and furthermore we explicitly calculate such an "optimal" weight. We remark that in view of solution (3), we can assume without loss of generality (wlog) that the optimal weight $w(s)$ is an outer (i.e. stable and minimum phase) function. Our main result is stated more precisely as follows.

<u>Theorem 1</u>

Given $r(s) \in RL_\infty$, there exists a proper weight $w(s)$ which is stable and minimum phase, such that the solution to $\min_{x(s)\in RH_\infty} \|w(s)[r(s)-x(s)]\|_2$, also solves $\min_{x(s)\in RH_\infty} \|r(s)-x(s)\|_\infty$. This optimal weight is given by the right Hankel singular vector of $r(s)$ corresponding to its maximun Hankel singular value.

<u>Proof</u>

Let us express $r(s)$ as

$$r(s) = \frac{n_r(s)}{d_r(s)} \tag{4}$$

where $n_r(s)$ and $d_r(s)$ are polynomials. We can assume wlog that $r(s) \in RH_\infty$ and that $r(s)$ it is strictly proper. This is so because we can write $r(s) = [r(s)]_\oplus + [r(s)]_-$, and if there exists w(s) so that $x_-(s)$ is the common solution of (1) and (2) for $r(s)$ replaced by $[r(s)]_-$, then $x(s) = [r(s)]_\oplus + x_-(s)$ is a common solution of (1) and (2) for the original $r(s)$ and $w(s)$. Therefore wlog we can assume that $d_r(-s)$ is Hurwitz and that $deg(n_r(s)) < deg(d_r(s))$. Let us now assume

$$w(s) = \frac{n_w(s)}{d_w(s)} \tag{5}$$

is such that (1) and (2) have a common solution. According to the remark preceding the statement of theorem 1, $n_w(s)$ and $d_w(s)$ can be taken Hurwitz. Using (3), the error between $r(s)$ and $x_{opt}^{(2)}(s)$ is

$$r - x_{opt}^{(2)} = r - w^{-1} \cdot [wr]_\ominus = w^{-1} \cdot [wr]_- \tag{6}$$

The denominator of $[wr]_-$ is clearly $d_r(s)$. Let then

$$[wr]_- = \frac{\sigma_r \cdot n^-(s) \cdot n^-(s)}{d_r(s)} \tag{7}$$

where $n^+(s)$ and $n^-(-s)$ are Hurwitz, and $\sigma_r$ is the maximum Hankel singular value of $r(s)$ (see section II). Using (5) and (7) in (6), we easily obtain

$$r(s) - x_{opt}^{(2)}(s) = \sigma_r \cdot \frac{d_w(s)}{n_w(s)} \cdot \frac{n^+(s) \cdot n^-(s)}{d_r(s)} \tag{8}$$

A necessary and sufficient condition for $x_{opt}^{(2)}(s)$ to solve problem (1) is that

$$r(s) - x_{opt}^{(2)}(s) = \sigma_r \cdot E(s) \tag{9}$$

where $E(s)$ is allpass ([7]). This condition and (8) imply the relations

$$d_r(s) = n_1^-(s) \cdot d_w(s) \tag{10a}$$

$$n_w(s) = n_2^+(s) \cdot n^-(-s) \tag{10b}$$

$$n^-(s) = n_1^+(s) \cdot n_2^+(s) \tag{10c}$$

From (10abc) we calculate

$$w(s) = \frac{n_w(s)}{d_w(s)} = \frac{n^+(s) \cdot n^-(-s)}{d_r(-s)} \tag{11}$$

Since $n^+(s)$ is a polynomial factor of both $w(s)$ and $[wr]_-$, it must be a constant (see appendix A), which can then be incorporated in $n^-(s)$. Therefore if $w(s)$ with the required property exists it must be of the form

$$w(s) = \frac{n^-(-s)}{d_r(-s)} \tag{12}$$

In order to determine $n^-(s)$, we look again at the error function. Thus from (8) and (9) we obtain

$$E(s) = \frac{n^-(s)}{n^-(-s)} \cdot \frac{d_r(-s)}{d_r(s)} \tag{13}$$

It is well known that the infinity norm optimal error can expressed as $E(s) = g(s)/f(s)$, where $(f,g)$ is a Schmidt pair for $\Gamma_r$ corresponding to $\sigma_r$ ([7]). It is also well known that if $r(s)$ has finite structure $f(s) = p(-s)/d_r(-s)$, and $g(s) = p(s)/d_r(s)$, where $p(-s)$ is some Hurwitz polynomial ([20,7]). From these expressions and (13) we obtain $n^-(s) = \alpha \cdot p(s)$, and

$$w(s) = \alpha \cdot f(s) \tag{14}$$

where $\alpha$ is a nonzero constant.

In order to complete the proof we need only to verify (7), which is equivalent with

$$[rf]_- = \sigma_r \cdot g \qquad (15)$$

This is shown in lemma 2 of section IV. ∎

At this point we have the following corollary.

Corollary 1

Assuming that $\sigma$ is a Hankel singular value for $r(s)$ of multiplicity one, the optimal weight $w(s)$ of theorem 1 is essentially unique.

Proof

From the proof of theorem 1, $w(s)$ equals $\alpha \cdot f(s)$, where $f(s)$ is the Hankel singular vector of $r(s)$ corresponding to $\sigma_r$. Since $f(s)$ is unique when $\sigma_r$ is of multiplicity one, $w(s)$ is specified within a constant. Note that by taking $\alpha = 1$, that is $w(s) = f(s)$, it holds $\min_{x(s)\in RH_\infty} \|r(s) - x(s)\|_\infty = \min_{x(s)\in RH_\infty} \|w(s)[r(s) - x(s)]\|_2$ and both minima are achieved for $x(s)$ given by (3). ∎

Theorem 1 can be readily extended to the case of infinity norm minimization of the sensitivity transfer function ([13]), or similarly of the complementary sensitivity transfer function ([10]).

Theorem 2

Let $S(s) \stackrel{\text{def}}{=} (1 + p(s)c(s))^{-1}$ be the sensitivity transfer function in the feedback system of the plant $p(s)$ and the controller $c(s)$. Then there exists a proper weighting function $w(s)$ such that the solutions of the problems $\min_{S(s)\in \mathcal{S}} \|\alpha(s)S(s)\|_\infty$ and $\min_{S(s)\in \mathcal{S}} \|w(s)\alpha(s)S(s)\|_2$ are the same. Here $\alpha(s)$ is a given transfer function and $\mathcal{S}$ is the set of admissible sensitivity transfer functions (i.e. $S(s) \in \mathcal{S}$ if there exists a stabilizing controller $c(s)$ such that $S(s) = [1 + p(s)c(s)]^{-1}$).

Proof

The set $\mathcal{S}$ can be expressed as ([14], [7])

$$\mathcal{S} = \{S(s) \;/\; S(s) = A(s) - B(s)X(s),\; X(s) \in H_\infty\}$$

where $A(s)$ and $B(s)$ are transfer functions in $H_\infty$ determined by the plant $p(s)$. Define $\tilde{A}(s) \stackrel{\text{def}}{=} [\alpha(s)B(s)]_i^* \cdot A(s)$ and $\tilde{X}(s) \stackrel{\text{def}}{=} [\alpha(s)B(s)]_o \cdot X(s)$. Consider $w(s)$ as in theorem 1, such that the problems $\min_{\tilde{X}(s)\in H_\infty} \|\tilde{A}(s) - \tilde{X}(s)\|_\infty$ and $\min_{\tilde{X}(s)\in H_\infty} \|w(s)[\tilde{A}(s) - \tilde{X}(s)]\|_2$ have the same solution. Then $\min_{X(s)\in H_\infty} \|w(s)\alpha(s)[A(s) - B(s)X(s)]\|_2 \iff \min_{\tilde{X}(s)\in H_\infty} \|w(s)[\tilde{A}(s) - \tilde{X}(s)]\|_2 \iff \min_{\tilde{X}(s)\in H_\infty} \|\tilde{A}(s) - \tilde{X}(s)\|_\infty \iff \min_{X(s)\in H_\infty} \|\alpha(s)[A(s) - B(s)X(s)]\|_\infty$. ∎

Theorem 2 allows an interesting interpretation of the optimal weighting function $w(s)$. Assume that disturbances are injected at the plant output that belong to

$$\mathcal{B}_\alpha \stackrel{\text{def}}{=} \{f(s)/f(s) = \alpha(s) \cdot g(s),\; \|g(s)\|_2 = 1\}$$

It is a well known fact ([13], [4], [7]) that if one wants to minimize the effect of the disturbances at the output of the plant for the worst possible signal in $\mathcal{B}_\alpha$, one must minimize $\min_{S(s)\in \mathcal{S}} \|\alpha(s)S(s)\|_\infty$. This is so since

$$\|\alpha(s)S(s)\|_\infty = \sup\{\|S(s)f(s)\|_2,\; f(s) \in \mathcal{B}_\alpha\}$$

Theorem 2 and corollary 2 show that the optimal weight, that is the first Hankel singular vector of $[\alpha(s)B(s)]_i^* \cdot A(s)$, is a worst case signal in $\mathcal{B}_\alpha$. This result provides additional justification for the minimization of the infinity norm of $S(s)$, since it shows that the worst case is not only achieved by limits of sequences of pure sinusoids.

In the remainder of this section we relate the previous results to other works. From the identity $[wr]_- + [wr]_\oplus = wr$, we obtain

$$w^{-1}[wr]_- + w^{-1}[wr]_\oplus = r \qquad (16)$$

Since $r(s)$ is assumed antistable,

$$[wr]_\oplus = \frac{\pi(s)}{d_r(-s)} \qquad (17)$$

for some polynomial $\pi(s)$ with $deg(\pi(s)) \leq deg(d_r(s))$. Then by substituting (4),(12),(13), and (17) in (16), we obtain

$$\sigma_r \frac{n^-(s)}{n^-(-s)} \frac{d_r(-s)}{d_r(s)} + \frac{d_r(-s)}{n^-(-s)} \frac{\pi(s)}{d_r(-s)} = \frac{n_r(s)}{d_r(s)}$$

or

$$\sigma_r \cdot n^-(s) \cdot d_r(-s) + \pi(s) \cdot d_r(s) = n^-(-s) \cdot n_r(s) \qquad (18)$$

Thus the optimal weight and the $H_\infty$ solution can be computed by solving (18) for $\sigma$, $n^-(s)$ and $\pi(s)$. We remark that (18) appears in [20] in the context of the optimal Hankel norm model reduction problem. It also appears in [9] in minimizing the infinity norm of the sensitivity transfer function. In both [20] and [9], (18) is shown to be equivalent with a generalized eigenvalue problem. Here we remark that (18) is also equivalent to a system of linear equations in the coefficients of $n^-(s)$ and $\pi(s)$, after $\sigma_r$ is found as the largest root of a polynomial equation.

## IV. STATE SPACE SOLUTION

In this section we summarize the state space expressions for the Schmidt pair of $\Gamma_r$, which in view of theorem 1 also provide a state space description for the optimal weight. These expressions are then used to prove (15), and thus complete the proof of theorem 1.

Let

$$r(s) = c(sI - A)^{-1}b \qquad (19)$$

be a minimal state space realization of $r(s)$. We maintain the assumption that $r(s)$ is completely unstable, i.e. $Re[\lambda(A)] \geq 0$. Let $L_c$ and $L_o$ be the controllability and observability grammians, that is $L_c$ and $L_o$ solve the Lyapunov equations

$$AL_c + L_cA^T = bb^T \qquad (20a)$$

$$A^TL_o + L_oA = c^Tc \qquad (20b)$$

Then the largest eigenvalue of $L_cL_o$ is $\sigma_r^2$. Let $u$ be the corresponding eigenvector, i.e.

$$L_cL_ou = \sigma_r^2 u \qquad (21a)$$

and define

$$v = \frac{1}{\sigma_r} L_0 u \qquad (21b)$$

It obviously holds

$$L_cv = \sigma_r u \qquad (22)$$

The following lemma is true. Its proof can be found for example in [7].

Lemma 1

Consider $r(s)$ given by (19) and $u$ and $v$ defined by (21a) and (21b) respectively. Define the rational functions

$$g(s) \stackrel{\text{def}}{=} c(sI - A)^{-1}u \qquad (23a)$$

$$f(s) \stackrel{\text{def}}{=} b^T(sI - A^T)^{-1}v \qquad (23b)$$

Then $(f, g)$ is a Schmidt pair for $\Gamma_r$ corresponding to $\sigma_r$. ▪

In the next lemma we prove (15).

Lemma 2

For $r(s)$ given by (19), and $f(s)$ and $g(s)$ defined by (23ab), it holds

$$[r(s)f(s)]_- = \sigma_r \cdot g(s) \qquad (24)$$

Proof

From (19) and (23a) we obtain

$$r(s)f(s) = \hat{c}(sI - \hat{A})^{-1}\hat{b} \qquad (25)$$

where

$$\hat{A} = \begin{pmatrix} A & bb^T \\ O & -A^T \end{pmatrix}, \ \hat{b} = \begin{pmatrix} 0 \\ v \end{pmatrix}, \ \hat{c} = \begin{pmatrix} c & 0 \end{pmatrix} \qquad (26)$$

Through the similarity transformation $T = \begin{pmatrix} I & L_c \\ O & I \end{pmatrix}$, (26) tranforms to

$$T\hat{A}T^{-1} = \begin{pmatrix} A & O \\ O & -A^T \end{pmatrix}, \ T\hat{b} = \begin{pmatrix} L_c v \\ v \end{pmatrix}, \ \hat{c}T^{-1} = \begin{pmatrix} c & -cL_c \end{pmatrix} \qquad (27)$$

From (27) we readily obtain

$$[r(s)f(s)]_- = c(sI - A)^{-1}L_c v \qquad (28)$$

Using (23a) and (22) in (28), we immediately obtain (24) and the proof is complete. ▪

Corollary 2

For $r(s) = c(sI - A)^{-1}b$ and antistable, the optimal weight of theorem 1 is given by $w(s) = b^T(sI - A^T)^{-1}v$, where $v$ is defined by (20) through (22). ▪

We remark that in place of (20ab), one can solve

$$AL - LA - bc = 0 \qquad (29)$$

and obtain $v$ as the eigenvector of the largest eigenvalue of $L^T$, since it holds

$$L^2 = L_c L_o \qquad (30)$$

Note that (30) is valid only in the SISO case. The proof of (30) is done first for $(A, b, c)$ balanced and by using the relations $b^T = Rc$ and $A^T = RAR$ in (20ab), where $R$ is an appropriate diagonal sign matrix, which hold for SISO balanced state-space realizations ([20]).

## V. EXAMPLE

Consider the simple SISO plant

$$p(s) = \frac{s - 0.2}{s - 1}$$

and suppose that we want to minimize the peak of the sensitivity transfer function $S(s) \stackrel{\text{def}}{=} (1 + p(s)c(s))^{-1}$ over all stabilizing controllers $c(s)$. It is easily verified that all admissible S(s) can be parametrized as

$$S(s) = \frac{2.5(s - 0.2)}{s + 1} - \frac{(s - 1)(s - 0.2)}{(s - 1)^2} q(s)$$

for $q(s) \in RH_\infty$ The problem is then equivalent with

$$\min_{x(s) \in RH_\infty} \left\| \frac{2.5(s + 0.2)}{s - 1} - x(s) \right\|_\infty \qquad (31)$$

A minimal state space realization for

$$r(s) = \frac{2.5(s - 0.2)}{s - 1}$$

is $A = 1$, $b = 1$, $c = 3$, $d = 2.5$. This gives $L_c = 0.5$, $L_o = 3$, $u = v = 1$, $\sigma = 1.5$, and optimal weight

$$w(s) = \frac{1}{s + 1} \qquad (32)$$

Therefore the optimal sensitivity can be found from

$$\min \| w(s)S(s) \|_2 \qquad (33)$$

where minimization is over all stabilizing controllers. Note that according to the interpretation of the optimal weight following theorem 2, (32) provides a worst case output disturbance of bounded energy for the given plant. For the above example, the fact, that the solution of (33) with $w(s)$ given by (32) results in a flat sensitivity transfer function, has been observed in [17].

## VI. CONCLUSION

In this paper it was shown that SISO $H_\infty$ optimization problems are equivalent with weighted $H_2$ optimization problems. This result is true also in the MIMO case which will be reported elsewhere. The optimal weight was explicitly computed to be the outer function in the Schmidt pair corresponding to the largest Hankel singular value of the system that is approximated. A summary of a state space procedure to calculate the optimal weight was given. We remark that the proof of theorem 1 and lemma 2 can be extended to show that optimal Hankel norm approximation problems are equivalent to a weighted $H_2$ approximation problem. In this case the optimal weight is the $(n-1)th$ Hankel singular vector, where $n$ is the degree of the reduced order system.

Our result establishes a direct link between $H_\infty$ optimal control and Wiener-Hopf and, therefore, LQG optimal control theories. Such a link is expected to offer new insights and interpretations for the robustness properties of LQG designs, as well as the qualitative properties of $H_\infty$ optimal controllers. These issues are currently investigated by the authors.


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## APPENDIX A

In this appendix the following lemma is proved

Lemma A.1

Let $R(s)$ be a proper rational function and assume that $\rho(s)$ is a polynomial factor of both $R(s)$ and $[R(s)]_-$. Then $\rho(s)$ is a constant.

Proof

By assumption it holds

$$R(s) = \rho(s)P(s) \qquad (A.1)$$

and

$$[R(s)]_- = \rho(s)P_1(s) \qquad (A.2)$$

Then it also holds

$$[R(s)]_\oplus = \rho(s)P_2(s) \qquad (A.3)$$

From the factorization $P(s) = [P(s)]_\ominus + [P(s)]_-$, we obtain

$$\rho(s) \cdot [P(s)]_\ominus + \rho(s) \cdot [P(s)]_- = R(s) = [R(s)]_\ominus + [R(s)]_-$$

The above relation yields

$$-[R(s)]_- + \rho(s) \cdot [P(s)]_- = [R(s)]_\ominus - \rho(s) \cdot [P(s)]_\ominus \quad (A.4)$$

The left hand side of (A.4) is antistable while the right hand side of (A.4) is stable. Therefore both sides are equal to a constant, which in view of (A.2) must be zero, or $\rho(s)$ is constant. We then have

$$[R(s)]_- = [\rho(s)P(s)]_- = \rho(s)[P(s)]_- \quad (A.5)$$

Now let

$$P(s) = \sum_{i=1}^{n} \frac{P_i^+}{s - p_i} + \sum_{j=1}^{m} \frac{P_j^-}{s - \pi_j} + P_o \quad (A.6)$$

Then

$$[\rho(s)P(s)]_- = \sum_{i=1}^{n} \frac{P_i^+ \rho(p_i)}{s - p_i} \quad (A.7)$$

and from (A.5), (A.6), and (A.7) we obtain

$$\sum_{i=1}^{n} \frac{P_i^+ \rho(p_i)}{s - p_i} = \rho(s) \cdot \sum_{i=1}^{n} \frac{P_i^+}{s - p_i} \quad (A.8)$$

Clearing out denominators in (A.8), we observe that the resulting polynomial equation can be satisfied only if $\rho(s)$ is constant. ∎

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