MATRIX INTERPOLATION AND H_∞ PERFORMANCE BOUNDS

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Abstract- This paper introduces a methodology for obtaining bounds on the achievable performance of a multivariable control system involving tradeoffs between potentially conflicting performance requirements.

1. Introduction

The purpose of this paper is to study the problem of achievable performance in multivariable feedback systems with H_∞ performance and robustness specifications. The problem of achievable performance is essentially one of obtaining simple bounds on the norm of the closed-loop system with the optimal controller in terms of the plant and the weights that define the optimal control problem. It is important that the bounds be simple in order to yield insight into the tradeoffs inherent in any well-posed control problem. Since it is now possible to compute the optimal controller directly for general H_∞ optimal control problems [1], it would be pointless to study complicated formulas when the optimal closed-loop system is readily available. This paper makes use of fairly elementary arguments based on the results in [1]-[6] to obtain simple bounds that can be used to give considerable insight into these tradeoffs.

The motivation for studying bounds on achievable performance comes from many sources. Although a thorough discussion of this issue is beyond the scope of this paper, it may be helpful to mention some of the author's motivation. The most obvious role of simple performance bounds is in providing a design engineer with insight into the tradeoffs between competing specifications. It is often the case that the engineers role is as much to determine the potential performance characteristics of a system as it is to design a control system to meet some prespecified and rigid performance specification. Furthermore, one of the most important roles the control engineer can play is in providing guidance to hardware designers by giving a systems level assessment of the impact of various alternative hardware configurations. Simple bounds can be useful in eliminating candidate systems with undesirable characteristics. Beyond these practical issues there is the theoretician's objective to understand in deeper and more complete terms the fundamental role of feedback in dealing with uncertainty.

One of the simplest and most useful achievable performance limitations is the well-known limit on desensitization and disturbance rejection created by right half plane (rhp) plant zeros([7],[8]). There are many approaches to studying this problem but probably the most elegant is based on using the maximum modulus property of analytic functions to show that the weighted sensitivity function is bounded below by the weight evaluated at rhp plant zeros[8]. Another well-known achievable performance limitation is that the sensitivity and so-called complementary sensitivity sum to the identity, so that at any frequency it is not possible to make both small[9]. This limitation actually involves a tradeoff between two competing objectives but is purely algebraic in nature. The rhp zero limitation exploits the analytic properties of stable closed loop transfer functions, but does not involve any tradeoff between competing objectives. Thus each result fails to capture essential features of feedback systems. This is a characteristic of most of the existing achievable performance results, and stems in part from the rather limited performance and robustness objectives that can typically be studied using standard classical ideas or their modern counterparts [10].

This paper will consider a quite general approach to multivariable control that leads to bounds on achievable performance involving tradeoffs between competing objectives and exploits the analytic properties of stable, causal transfer functions. This approach is based on a recent solution to the general H_∞ optimal control problem and the associated analysis and synthesis framework([1]-[6]). This framework and the H_∞ results will be briefly reviewed in Sections 2 and 3. In Section 4 a generalized interpolation version of the H_∞ optimal control problem is proposed which leads to a simple but useful approach to achievable performance. In some sense, this is the first truly multivariable approach to achievable performance.

In Section 5 the problem is simplified by focusing on the algebraic limitations on achievable performance. Achievable performance for μ-synthesis is also considered in Section 5. The problem of μ-synthesis arises when an H_∞ performance requirement is desired for a system with plant uncertainty in the form of perturbations, either structured or unstructured. The role of μ, the Structured Singular Value, in problems of robust performance (i.e., performance in the presence of uncertainty) and structured perturbations is briefly reviewed in Sections 2 and 3. An important consequence of the results of Section 5 is a simple scheme to compute good initial guesses for the diagonal scalings used in the μ-synthesis approach outlined in Section 3.

The paper finishes with an example to illustrate the techniques developed earlier. A simple, explicit bound is obtained on a multivariable control problem that involves a tradeoff between disturbance rejection and input signal level.

2. Analysis Review

This section will review some basic methods for analysing the performance and robustness properties of feedback systems. The particular approach taken here is from [1]-[6] which builds on results by many other researchers. In this context, analysis refers to the process of determining whether a system with a given controller has desired characteristics, whereas synthesis refers to the process of finding a controller that gives desired characteristics, usually expressed in terms of some analysis method. This is the fairly standard usage of these terms in the control community. It should be obvious that the question of analysis must be settled to some degree before a reasonable synthesis problem can be posed. The formal analysis and synthesis techniques discussed in this paper are only some of the methods that might make up the overall process of engineering design.

The general framework to be used in this paper is illustrated in the diagram in Figure 1a. Any linear interconnection of inputs, outputs, commands, perturbations, and a controller can be rearranged to match this diagram. For the purpose of analysis the controller may be thought of as just another system component and the diagram reduces to that in Figure 1b. The analysis problem involves determining whether the error e remains in a desired set for sets of inputs v and perturbations Δ. Analysis methods differ on the description of these sets and the assumptions on the interconnection structure G. In this paper G will be taken to be a linear, time-invariant, lumped system and be represented by a rational transfer function. The interconnection structure G can be partitioned so that the transfer function from v to e can be expressed as the linear fractional transformation

$$e = F_u(G,\Delta)\, v = \left[G_{22} + G_{21}\Delta(I - G_{11}\Delta)^{-1}\, G_{12}\right]\, v.$$

The performance requirements and uncertainty will be described in the frequency domain. The inputs v are assumed to lie in L_2, and the performance criteria is expressed in terms of $\|e\|_2$. The perturbations Δ are in H_∞, the "stable subspace" of L_∞ where

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\,[G(j\omega)^*\, G(j\omega)]\; d\omega\right)^{1/2}$$

$$\|G\|_\infty = \sup_\omega \bar{\sigma}\,[G(j\omega)]$$

$$\|G\|_\mu = \sup_\omega \mu\,[G(j\omega)]\,.$$

The dimensions of matrices will not be explicitly represented, but of course, all matrices and vectors are assumed to be of compatible dimension when required. The prefix B is be used to denote the unit ball in a normed space and the prefix R is used to denote the proper real-rationals.

The function $\|\bullet\|_\mu$ is not actually a norm, but it is convenient to refer to it as the μ-norm. The function μ is the Structured Singular Value ([2]-[6]) and depends not only on the matrix but also on the assumed perturbation structure. This structure can be described by a multi-index and the dependence of μ on the structure could be indicated explicitly, although to simplify notation this will not be done. A very brief review of some relevent facts about μ is given in the appendix.

Given these definitions the main analysis tools can be expressed in terms of simple theorems. Each theorem has the form of the following general analysis theorem :

General Analysis Theorem (GAT):

Performance	for all	Uncertainty
	iff	Analysis Test

The performance condition is either stability or a bound on $\|e\|_2$ expressed as $e \in BH_2$. Recall that this is analysis of the closed-loop system with controller in place, so it is assumed that the nominal system (when $\Delta = 0$) is stable, that G has no rhp poles. The uncertainty takes the form of bounds on $\|v\|_2$ and a description of the allowable Δ. Only necessary and sufficient conditions are considered in this paper, so the Analysis Test is always an exact characterisation of performance in the presence of uncertainty.

The following three theorems give the main performance/robustness methods used in this paper. Each allows successively more complex uncertainty assumptions. The first theorem simply writes the induced norm on L_2 in the form of the GAT:

Theorem P (Performance)

$e \in BH_2$ for all $v \in BH_2$
iff $\|G_{22}\|_\infty \le 1$

The simplest robustness result is a test for stability in the presence of a norm-bounded but otherwise unknown perturbation.

Theorem RSU (Robust Stability, Unstructured)

Stable for all $\Delta \in BRH_\infty$
iff $\|G_{11}\|_\infty \le 1$

Note that the test in both theorems involves the ∞-norm on different parts of the system. This is the main motivation for using this norm, in that it naturally allows for both additive noise uncertainty and plant perturbations. It also means that performance of one system is equivalent to robust stability of a related system, and vice versa. Note that to be useful for any practical problems, it is necessary that weights be used on both the external inputs and perturbations to reflect the dependence of the uncertainty on frequency and direction. Since these weights can always be absorbed into the interconnection structure, it is no loss of generality to use the unweighted L_∞ norm.

The main objections to using this norm are that a single norm-bounded perturbation does not capture the natural known structure inherent in almost any control problem and that performance in the presence of a perturbation (i.e. robust performance) is not handled. Robust stability with structured uncertainty and robust performance require the μ-norm for analysis. These two issues are addressed by the following theorem:

Theorem RP (Robust Performance)

$$
\begin{array}{ll}
e \in BH_2 & \text{for all } v \in BH_2 \\
\text{and all} & \Delta \in BRH_\infty \\
 & \Delta = \mathrm{diag}(\Delta_1, \Delta_2, \cdots, \Delta_n) \\
\text{iff} & \|G\|_\mu \leq 1
\end{array}
$$

The μ in Theorem RP is computed with respect to the structure $\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \cdots, \Delta_n, \Delta_{n+1})$ where Δ_{n+1} is the "performance block" that converts the performance requirement into an equivalent robust stability problem. This theorem is the real payoff for using the μ-norm since it provides an exact (i.e. not conservative) test for performance of a system with a very rich class of allowable uncertainty structures. A simpler version of this theorem uses μ to assess just stability with structured uncertainty.

The following table summarises the above analysis theorems, where once again it is assumed throughout that the nominal system is stable and that $v \in BH_2$.

Analysis Summary

Performance / Perturbation	Stability	$\|e\|_2 \leq 1$
$\Delta = 0$	No C_+ poles	$\|G_{22}\|_\infty \leq 1$
$\Delta \in BRH_\infty$	$\|G_{11}\|_\infty \leq 1$	$\|G\|_\mu \leq 1$
$\Delta \in BRH_\infty$ $\Delta =$ diag $(\Delta_1, \Delta_2, \cdots, \Delta_n)$	$\|G_{11}\|_\mu \leq 1$	$\|G\|_\mu \leq 1$

3. Synthesis Review

The previous section on analysis showed that the synthesis problem reduces to finding a stabilising controller K so that

$$\|F_l(P,K)\|_\alpha \leq 1 \qquad \alpha = \infty \text{ or } \mu \tag{1}$$

where $F_l(P,K) = P_{11} + P_{12}\,K(I - P_{22}K)^{-1}\,P_{21}$. The interconnection structure for this synthesis problem is shown in the diagram in Figure 1c. The P here is a real-rational, proper transfer function matrix and is not necessarily stable. It will be assumed throughout that P_{12} has at least as many rows as columns, and vice versa for P_{21}.

A complete solution to the synthesis problem for the ∞-norm was recently obtained ([1],[4]), which removed the previous restrictions that P_{12} and P_{21} be square. Another feature of this solution is that it provided an efficient computational scheme using standard real matrix operations on state-space representations. This H_∞-synthesis solution can be used to provide an approach to solving the μ-norm synthesis problem, refered to as μ-synthesis. These results will be briefly reviewed before considering the achievable performance results.

The first step in the H_∞ synthesis solution involves finding J so that the substitution $K = F_l(J,Q)$ yields

$$F_l(P,K) = F_l(P, F_l(J,Q)) = R + UQV \tag{2}$$

with $F_l(P,K)$ internally stable iff $Q \in H_\infty$. This is a version of the so-called Youla parametrisation. Further, U is inner and V co-inner ($U^*U = I$ and $VV^* = I$), and there exist complementary inner factors $U_\perp$ and $V_\perp$ such that $[UU_\perp]$ and $\begin{bmatrix} V \\ V_\perp \end{bmatrix}$ are both square and inner. The U and V are obtained from coprime factorizations $P_{12} = UM_1^{-1}$ and $P_{21} = M_2^{-1}V$.

The next step involves using a rational matrix version of the Davis-Kahan-Weinberger matrix dilation results [11] to further reduce the problem to one of finding $\hat{Q} \in RH_\infty$ such that

$$\|G + \hat{Q}\|_\infty \leq 1 \tag{3}$$

where $G \in RL_\infty$. This problem can then be solved using the Hankel norm approximation methods developed by Glover [12]. The resulting optimal $\hat{Q}$ can then be used to find first the optimal Q and then the optimal K.

The μ-synthesis problem does not yet have as complete a solution as does the H_∞ synthesis problem. A reasonable approach would be to try to find a stabilising controller K and scaling D so that

$$\|DF_l(P,K)D^{-1}\|_\infty \leq 1. \tag{4}$$

One method to do this is to alternately minimize the above expression for either K and D while holding the other constant. For fixed D the left-hand side of (4) is just an H_∞ control problem and can be solved using the methods reviewed above. For fixed K, the left-hand side of (4) can be minimised at each frequency as a convex optimisation problem in D. The resulting D can be fit with a stable, rational transfer function with stable inverse (the phase of D does not affect the norm).

This approach to μ-synthesis has been successfully applied to several example problems. In principle, it could be used to obtain controllers that are arbitrarily close to μ-optimal in the case of 3 or fewer blocks (see appendix) and provide nearly optimal controllers for the general case. This would depend on the suggested iterative scheme converging to the global optimal K and D. Unfortunately, individual convexity in the two parameters of an optimisation problem does not imply joint convexity, and this scheme is not always guaranteed to converge globally to the best K and D. This issue will be addressed further in Section 5.

4. Interpolation and Achievable Performance

The H_∞-synthesis approach reviewed in the previous section has as an intermediate step the following generalized approximation problem, stated here as a minimization problem instead of an inequality.

Problem A (Approximation):

$$\min_{Q \in RH_\infty} \|R + UQV\|_\infty \tag{4}$$

Here $R, U, V \in RH_\infty$ and it assumed that U is inner and has as many rows as columns and V is co-inner and has as many columns as rows. The optimal norm in Problem A is also the minimum achievable norm in the original problem in (1). The remainder of this paper will exploit interpolation versions of Problem A to obtain simple achievable performance bounds on the minimum.

Some simplification is obtained when all the quantities in Problem A are scalars. Since this case has been thoroughly studied and is well-understood, it will be considered first in order to motivate the general results. Since scalars commute, V can be absorbed into U which can be written as

$$U = \prod \frac{(1 - s_i/z_i)}{(1 + s_i/z_i)}. \tag{5}$$

For simplicity, assume that all the z_i are distinct. Then it is a standard and easily verified result that Problem A is equivalent to

Problem DI (Discrete Interpolation):

$$\min_{T \in RH_\infty} \left\{ \|T\|_\infty \;\middle|\; T(z_i) = R(z_i) \right\} \tag{6}$$

and a solution to one problem yields a solution to the other through the formulas $T_0 = R + UQ_0$ and $Q_0 = (T_0 - R)/U$.

Since T is analytic in $\mathbf{C}_+$, it has the maximum modulus property there so

$$\|T_0\|_\infty \geq \sup_{s \in \mathbf{C}_+} |T_0(s)| \geq \max_i |R(z_i)|. \tag{7}$$

The quality of this bound obviously depends on the number and location of the interpolation points and the interpolating values. When there is only one interpolation point the bound is achieved, but when there are two or more it can be arbitrarily far off. This is because the bound in (7) treats each interpolation condition as independent, whereas the interaction between interpolation conditions can lead to much larger T than would any of them individually. A well-known example of this is caused by nearby rhp poles and zeros in minimal sensitivity or robust stabilization problems [15]. Nevertheless, the bound in (7) has proven quite useful.

The simplest matrix generalisation of the bound in (7) is obtained by noting that both the Euclidean norm $\|\bullet\|$ and the maximum singular value $\bar\sigma$ satisfy the maximum modulus principle when the vectors and matrices are analytic. Suppose $\exists x \in \mathbf{C}^p$ with $\|x\| = 1$ such that $x^* U(s_0) = 0$ for some $s_0 \in \mathbf{C}_+$. Since $T = R + UQV$ is analytic in $\mathbf{C}_+$

$$\|T\|_\infty \geq \bar\sigma(T(s_0)) \geq \|x^* R(s_0)\|. \tag{8}$$

This provides a bound similar to the one in (7). The same thing could be done with V. The maximum modulus property used here is one consequence of the fact that $\|\bullet\|$, $\bar\sigma$, and μ are subharmonic when their arguments are analytic. For an interesting exposition of the role of subharmonic functions in bounds on achievable performance see [13].

In order to understand what factors impact the accuracy of the bound in (8) it would be useful to have an interpolation version of the general case of Problem A similar to Problem DI for the scalar case. In the matrix case, the interpolation problem has a direct generalisation only in the case where U and V are square. The case of particular practical interest is when at least one is not square. This also turns out to be of the most mathematical interest.

Assume for now that both U and V are nonsquare but have no infinite or rhp zeros (i.e., they do not lose rank at any point in $\mathbf{C}_+$). This may seem like a restrictive assumption but it actually holds for all practically motivated problems. Discussion of this point is beyond the scope of this paper. In any case, the generalisation of what follows to the case where U and V have rhp zeros is straightforward.

Let $\tilde U \in RH_\infty (\tilde V \in RH_\infty)$ be a left (right) inverse for U (V). Let $\tilde U_\perp$ $(\tilde V_\perp)$ be a matrix-valued function of $s \in \mathbf{C}_+$ such that $\tilde U_\perp U = 0$ $(V \tilde V_\perp = 0)$ and $\begin{bmatrix} \tilde U \\ \tilde U_\perp \end{bmatrix}$ $\left(\begin{bmatrix} \tilde V & \tilde V_\perp \end{bmatrix}\right)$ is square and nonsingular in $\mathbf{C}_+$. $\tilde U$ and $\tilde V$ are easily constructed using the algorithms in [1]. Note that $\tilde U_\perp$ and $\tilde V_\perp$ need not be in RH_∞ or even H_∞, although it is possible to construct them to be in RH_∞.

Using the above definitions, the following problem is easily shown to be equivalent to Problem A:

Problem CI (Continuous Interpolation):

$$\min_{T \in RH_\infty} \{\|T\|_\infty \mid \tilde U_\perp (T - R) = 0\,,\ (T - R)\tilde V_\perp = 0 \text{ in } \mathbf{C}_+\}$$

Solutions to one problem yield solutions to the other through the formulas $T_0 = R + UQ_0V$ and $Q_0 = \tilde U(T_0 - R)\tilde V$. Note that both problems may have nonunique solutions.

Problem CI may be thought of as a "continuous interpolation" version of Problem A. Here the interpolation conditions involve subspaces at all points in $\mathbf{C}_+$, whereas the classical interpolation problem requires equality at isolated points. To reinforce this distinction, consider the following alternative statement of Problem CI:

Problem CI′:

$$\min_{T \in RH_\infty} \{\|T\|_\infty \mid \text{range}(T - R) \subset \text{range}(U),$$
$$\ker(T - R) \supset \ker(V) \text{ in } \mathbf{C}_+\}$$

If U and V had $\mathbf{C}_+$ zeros, Problems A and CI′ would still be equivalent, but Problem CI would require additional discrete interpolation conditions.

Problem CI gives us a way to interpret the bound in (8). By analogy with the scalar interpolation problem, bounds of the type in (8) essentially ignore the interaction between the continuous interpolation conditions of Problem CI just as the bounds in (7) ignore the interaction between the discrete interpolation conditions of Problem DI. They can be used, however, to obtain useful achieveable performance bounds. In the next section the problem of finding the best possible bounds using these methods is considered by studying the constant matrix version of Problem A.

5. Constant Matrix Bounds

A useful way to compute achievable performance bounds is to temporarily remove the restriction that the controller be causal. This is equivalent to dropping the stability requirement and essentially turns the synthesis problem into one of solving constant matrix problems at each frequency. Obviously, any achievable performance (lower) bounds obtained for the acausal controller problem will also be lower bounds for the causal controller problem. More importantly, the constant matrix solution may be applied at any point in $\mathbf{C}_+$ and bounds on the causal controller problem obtained by applying the maximum modulus principle, as discussed in the last section. For the remainder of this section, all matrices will be assumed to be constant.

The key result for the constant matrix case is the following version of the well-known matrix dilation result of Davis-Kahan-Weinberger. For constant R, U, V with $U^*U = 1$ and $VV^* = 1$

$$\min_Q \bar{\sigma}(R + UQV) = \max(\bar{\sigma}(U_\perp^* R)\ ,\ \bar{\sigma}(RV_\perp^*)) \qquad (9)$$

where $U_\perp^*$ and $V_\perp^*$ are chosen so that $[U\ U_\perp]$ and $\left[\begin{smallmatrix} V \\ V_\perp \end{smallmatrix}\right]$ are both square and unitary. All of these quantities are easily computed using standard SVD routines.

The constant matrix case with the D-scalings for μ is

$$\min_{D,Q} \bar{\sigma}\left(D(R + UQV)D^{-1}\right). \qquad (10)$$

It is known that this problem is convex in either D or Q individually when the other is held fixed, but is not convex in both variables jointly. This means that the iterative scheme suggested as a possible approach to μ-synthesis is not guaranteed to converge even in the constant matrix case. It is possible, however, to compute the desired D in (10) directly.

The result in (9) may be applied to (10) to obtain

$$\min_Q \bar{\sigma}\left(D(R + UQV)D^{-1}\right) =$$

$$\max\left\{\bar{\sigma}\left((DU)_\perp^* DR\right)\ ,\ \bar{\sigma}\left(RD^{-1}(VD^{-1})_\perp^*\right)\right\} \quad (11)$$

where

$$(DU)_\perp \triangleq D^{-1}U_\perp(U_\perp^* D^{-2} U_\perp)^{-1/2} \qquad (12)$$

and $(VD^{-1})_\perp$ is defined similarly. Note that $\left[DU(U^*D^2U)^{-1/2}\ \ (DU)_\perp\right]$ is unitary. It can be shown (the existing proof is straightforward but too long for inclusion) that the right hand side of (11) is convex in D so that the "optimal" scaling for (10) may be computed by search in advance. This gives a tight lower bound for (10) and the resulting D scaling may be used to compute the optimal Q.

A simple example will illustrate all the essential features of this possibly confusing sequence of ideas. Consider the problem

$$\min_q \mu\left(\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} q[1\ 0]\right)$$

$$= \min_{q,d} \bar{\sigma}\left(\begin{bmatrix} -1 & d \\ q/d & 1 \end{bmatrix}\right). \qquad (13)$$

The μ-optimal q is $q = 0$ which gives $\mu = 1$. For fixed d the $\bar{\sigma}$-optimal $q = d^2$ and for fixed $q > 0$ the $\bar{\sigma}$-optimal d id $d = \sqrt{q}$. Thus, iteratively solving for either q or d will immediately converge to the curve $q = d^2$. For example, with the initial guess of $q = d = 1$, the iterative scheme will not change either q or d and will thus fail to find the global optimum.

On the other hand,

$$\min_q \bar{\sigma}\begin{bmatrix} -1 & d \\ q/d & 1 \end{bmatrix} = \max\left(\bar{\sigma}([-1\ d])\ ,\ \bar{\sigma}\left(\left[\begin{smallmatrix} d \\ 1 \end{smallmatrix}\right]\right)\right)$$

$$= \sqrt{1 + d^2}. \qquad (14)$$

Thus,

$$\min_d \left(\min_q \bar{\sigma}\begin{bmatrix} -1 & d \\ q/d & 1 \end{bmatrix}\right) = \min_d \sqrt{1 + d^2} \qquad (15)$$

which is clearly convex and achieves its minimum as $d \to 0$. If the expression in (11) were used to compute the d in advance, it would be possible to find the optimal achievable level for (10). This example also illustrates why, strictly speaking, inf, not min must be used for the D scalings as in the appendix. This issue will not be taken up in this paper. It turns out to be of little significance anyway.

The simplest application of these ideas to the selection of the D scalings for the μ-synthesis problem is to compute an initial guess for D at each frequency using (11). This would be the optimal D for an acausal controller, and should provide a good initial guess for the optimal D for the causal controller problem. The resulting bound for the acausal controller could also be used as a lower bound for the causal problem. This could also be applied to any point in $\mathbf{C}_+$ and by the maximum modulus property of μ be used to obtain a possibly less conservative lower bound. A deeper question is whether some generalisation of (11) and its convexity properties applies to the rational case. While this seems likely, the details have not been worked out and the practical implications are uncertain. For some additional results on μ-synthesis, see [14].

6. Example

This section will apply the results of the previous sections to the problem Figure 2. This is one of the simplest nontrivial problems that can be studied. It involves a tradeoff between disturbance rejection and input signal level and is genuinely multivariable. Assume that the plant G is open-loop stable and has more inputs than outputs. The closed loop system is

$$F_l(P,K) = \begin{bmatrix} 0 \\ W_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 G \end{bmatrix} K(I-GK)^{-1}. \quad (6.1)$$

To apply the methods of the previous section, suppose that the constant matrix solution is to be taken at some specified point $s_0 \in \mathbf{C}_+$. Let $\tilde{U}_\perp = M^{-1/2}N$ be a constant matrix where $N = \left[GW_1^{-1} \;\; -W_2^{-1}\right]\Big|_{s=s_0}$ and $M = NN^*$. Since by construction $\tilde{U}_\perp P_{12}(s_0) = 0$, for any stabilizing K

$$\|F_l(P,K)\|_\infty \geq \bar{\sigma}\left(\tilde{U}_\perp \begin{bmatrix} 0 \\ W_2(s_0) \end{bmatrix}\right) = \bar{\sigma}\left(M^{-1/2}\right)$$
$$= \frac{1}{\underline{\sigma}(N)} = \frac{1}{\underline{\sigma}\left[GW_1^{-1}\, W_2^{-1}\right]}\Bigg|_{s=s_0} \quad (6.2)$$

The bound in (6.2) is intuitively pleasing since it shows that that if GW_1^{-1} is "small" where W_2 is large (i.e.,where disturbance rejection is desired) then poor performance relative to these weightings is obtained. To be more precise, suppose for simplicity that $W_1 = I$ and $W_2 = w_2 I$, a scalar times identity. The bound in (6.2) may be further simplified to

$$\|F_l(P,K)\|_\infty \geq \frac{1}{1/w_2(s_0) + \underline{\sigma}(G(s_0))}. \quad (6.3)$$

If G has a small minimum singular value $\underline{\sigma}$ at any point in $s_0 \in \mathbf{C}_+$ where the weight w_2 is large, the bound will also be large. Note that G need not have a rhp zero for poor performance to result. In fact, consider the case where G is nonsquare and has no finite transmission zeros. The minimum sensitivity problem ($W_1 = 0$) can be made arbitrarily small, but the resulting controller must have large input signal level if G has a small $\underline{\sigma}$ at a location in $\mathbf{C}_+$ where w_2 is large.

A small rhp $\underline{\sigma}(G)$ may be thought of as limiting the performance of the system in (6.1) in a similar way to the effect of rhp zeros on weighted sensitivity. One could (somewhat whimsically) speak of "rhp small σ's" as a more flexible notion than "rhp zeros". Note that a "rhp small σ" would have meaning only relative to the weightings used to define the problem in (6.1). This appears to be the first truly multivariable achievable performance bound, in that it is not a direct generalization of some standard SISO result. It also appears to be the first one that involves a tradeoff between competing performance objectives while making use of the analytic properties of closed-loop transfer functions. Similar bounds can be obtained for a great variety of interesting multivariable control problems using the results of this paper.

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Appendix

Given the sets

$$\mathcal{X} = \{\Delta = \text{diag}(\Delta_1, \Delta_2, \ldots, \Delta_n) \mid \Delta \in \mathbf{C}_+^{k_i \times k_i}\}$$
$$\mathcal{D} = \{D = \text{diag}(d_1 I_{k_1}, d_2 I_{k_2}, \ldots, d_n I_{k_n}) \mid d_i \in R_+\}$$

the positive real-valued function μ satisfies the property

$$\det(I - M\Delta) \neq 0 \text{ for } \forall\, \Delta \in \mathcal{X}, \bar{\sigma}(\Delta) < \gamma$$
$$\text{iff } \gamma\mu(M) \leq 1.$$

Note that μ is a function of M that depends on the structure of the Δ's in $\mathcal{X}$. This dependency is typically not represented explicitly. It is also possible to consider more general structure such as nonsquare and repeated blocks. An important bound on μ is

$$\mu(M) \leq \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$$

which is an equality for three or fewer blocks, independent of block size. This bound also appears to be quite tight for more blocks, and the implied minimization problem is convex in the D scalings. This makes this bound quite useful in compute μ and estimates for μ. For more details, see ([2],[5],[6]).

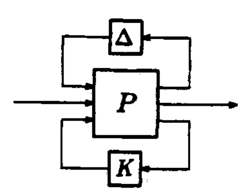

Figure 1a

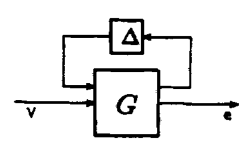

Figure 1b

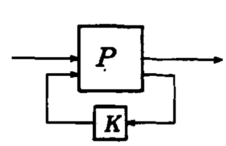

Figure 1c

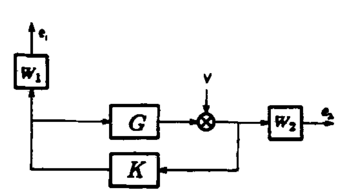

Figure 2