Finite-Size Scaling of Out-of-Time-Ordered Correlators at Late Times

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Chaotic dynamics in quantum many-body systems scrambles local information so that at late times it can no longer be accessed locally. This is reflected quantitatively in the out-of-time-ordered correlator of local operators, which is expected to decay to 0 with time. However, for systems of finite size, out-of-time-ordered correlators do not decay exactly to 0 and in this paper we show that the residual value can provide useful insights into the chaotic dynamics. When energy is conserved, the late-time saturation value of the out-of-time-ordered correlator of generic traceless local operators scales as an inverse polynomial in the system size. This is in contrast to the inverse exponential scaling expected for chaotic dynamics without energy conservation. We provide both analytical arguments and numerical simulations to support this conclusion.

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Nonintegrable quantum many-body systems are expected to exhibit chaotic dynamics, which not only leads to thermalization but also scrambles local information into a nonlocal form. In the Heisenberg picture, the support of $A(t) \coloneqq e^{iHt}Ae^{-iHt}$ for a local operator A should grow with time under the chaotic dynamics. This growth is reflected in the noncommutativity of $A(t)$ and another local operator B at a different site, which leads to the decay of the out-of-time-ordered correlator (OTOC) $\mathrm{Re}\langle A^\dagger(t)B^\dagger A(t)B\rangle$ [1–15]. Assume for simplicity that A and B are unitary. Then,

$$\mathrm{Re}\langle A^\dagger(t)B^\dagger A(t)B\rangle = 1 - \langle [A(t),B]^\dagger [A(t),B]\rangle/2 \qquad (1)$$

so that when the commutator $[\cdots]$ grows, OTOC decays. The chaotic nature of the dynamics is reflected in the fast decay of OTOC away from 1 in a relatively short time period and the approaching of OTOC to 0 at late times.

Why does chaotic dynamics lead to such decaying behavior of OTOC? While it is not possible to solve exactly the dynamics of nonintegrable systems in general, we might be able to extract some universal features, at least in certain limits. In a large class of chaotic systems without spatial locality (e.g., large-N theories), OTOC at early time t is given by $1 - \epsilon e^{\lambda_L t}$, where ϵ is a small prefactor and λ_L is a constant. Such an exponential deviation from the initial value is reminiscent of the so-called sensitive dependence on initial conditions in classical chaos. Thus, λ_L may be interpreted as the Lyapunov exponent for quantum systems [3]. In chaotic systems with spatial locality, OTOC of two local operators starts to decay only after a delay that is proportional to the distance between the operators [1,2,9,10,15]. This is a consequence of the Lieb-Robinson bound [16–18].

In this paper, we study the behavior of OTOC at late times. For simplicity, consider a system of n qubits at infinite temperature so that $\langle\cdots\rangle = \mathrm{tr}(\cdots)/2^n$. In the limit $t \to \infty$, a naive understanding of why OTOC approaches 0 is as follows. We expand the time-evolved operator in the n-qubit Pauli basis $\{\sigma_0 = I, \sigma_x, \sigma_y, \sigma_z\}^{\otimes n}$,

$$A(t) = \sum_{(k_1,k_2,\ldots,k_n)\in\{0,x,y,z\}^n} a_{k_1k_2\cdots k_n}\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_n}. \qquad (2)$$

The unitarity of $A(t)$ implies $\sum_{k_1,k_2,\ldots,k_n}|a_{k_1k_2\cdots k_n}|^2 = 1$. After undergoing chaotic evolution for a sufficiently long time, the support of $A(t)$ should be the whole system, and one might expect that the coefficients $a_{k_1k_2\cdots k_n}$ behave like random variables due to the chaotic nature of the dynamics. If we choose B to be the Pauli operator σ_x of qubit 1, then half of the terms in the expansion (2) of $A(t)$ commute with B and half of them do not. Thus,

$$\langle [A(t),B]^\dagger [A(t),B]\rangle = 4\sum_{k_2,k_3,\ldots,k_n}|a_{yk_2k_3\cdots k_n}|^2 + |a_{zk_2k_3\cdots k_n}|^2$$
$$\approx 4\times 0.5 = 2. \qquad (3)$$

The approximation step follows from the fact that we sum over half of the random variables. Substituting (3) into (1), we see that OTOC approaches 0 at late times.

Equation (2) with random coefficients is a very simple way to approximate $A(t)$ for large t in chaotic systems and it is oversimplified in some respects. For example, one major difference between this approximation and the exact evolution $A(t) = e^{iHt}Ae^{-iHt}$ is that the latter preserves the spectrum of A while the former does not. How does this discrepancy affect our understanding of the late-time behavior of OTOC? Is it necessary to use more refined and sophisticated approximations in order to fully capture the essence of chaotic dynamics at late times?

We focus on the scaling of late-time OTOC with system size. In finite-size systems, OTOC may converge to a small but finite value, which goes to 0 when the system size goes to infinity. One might expect this residual value to be exponentially small in the system size because we sum over an exponential number of random variables in (3). However, using a more refined approximation we show that the finite-size scaling of generic late-time OTOC should be inverse polynomial. In fact, the power-law scaling is closely related to energy conservation during the time evolution, which is not captured by simply setting the coefficients in the expansion (2) to be random.

Results.—We introduce basic definitions and provide a summary of results.

Throughout this Letter, asymptotic notations are used extensively. Let $f, g: \mathbb{R}^+ \to \mathbb{R}^+$ be two positive functions. One writes $f(x) = O(g(x))$ if and only if there exist positive numbers M, x_0 such that $f(x) \le Mg(x)$ for all $x > x_0$; $f(x) = \Omega(g(x))$ if and only if there exist positive numbers M, x_0 such that $f(x) \ge Mg(x)$ for all $x > x_0$; $f(x) = \Theta(g(x))$ if and only if there exist positive numbers M_1, M_2, x_0 such that $M_1 g(x) \le f(x) \le M_2 g(x)$ for all $x > x_0$. To simplify the notation, we use a tilde to hide a polylogarithmic factor, e.g., $\tilde{O}(f(x)) := O(f(x) \mathrm{poly} \log f(x))$.

For concreteness, consider a chain of n qubits or spin-1/2's with total Hilbert space dimension $d = 2^n$ governed by a translationally invariant Hamiltonian $H = \sum_{i=1}^n H_i$, where H_i acts on spins i, $i+1$ (nearest-neighbor interaction). While our discussion is based on a one-dimensional spin system, our results do not rely on the dimensionality of the system or the degrees of freedom being spins. A minor modification of our method leads to similar results in other settings, e.g., fermionic systems in higher dimensions. Assume without loss of generality that $\mathrm{tr} H_i = 0$ (traceless) and $\|H_i\| \le 1$ (bounded operator norm).

Let A, B, C, D be local (not necessarily unitary) operators with unit operator norm. The residual value of late-time OTOC is

$$\mathrm{OTOC}_\infty^{A,B,C,D} := \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau dt \langle AB(t)CD(t)\rangle, \tag{4}$$

where $\langle X\rangle := (1/d)\mathrm{tr}X$ denotes the expectation value of an operator at infinite temperature.

Let $\{|1\rangle, |2\rangle, \ldots, |d\rangle\}$ be a complete set of eigenstates of H with corresponding energies $E_1 \le E_2 \le \cdots \le E_d$ in nondescending order. Let $X_{jk} = \langle j|X|k\rangle$ be the matrix element of an operator in the energy eigenbasis. Define

$$\langle A, B, C, D\rangle_j = (AC)_{jj} B_{jj} D_{jj} + A_{jj} C_{jj} (BD)_{jj} - A_{jj} B_{jj} C_{jj} D_{jj}. \tag{5}$$

In strongly chaotic systems, we propose the following formula for late-time OTOC:

$$\mathrm{OTOC}_\infty^{A,B,C,D} \approx \frac{1}{d} \sum_j \langle A, B, C, D\rangle_j. \tag{6}$$

Based on this formula, we argue for (i) $\mathrm{OTOC}_\infty^{A,B,A^\dagger,B^\dagger}$ for traceless local operators A, B vanishes in the thermodynamic limit $n \to \infty$. (ii) In finite-size systems, OTOC $\langle AB(t)A^\dagger B^\dagger(t)\rangle$ saturates to $\Theta(1/n)$ if either A or B (or both) has a finite overlap with the Hamiltonian H. We not only derive the prefactor hidden in the big-Theta notation, but also provide a (not necessarily tight) upper bound on the remainder,

$$\mathrm{OTOC}_\infty^{A,B,A^\dagger,B^\dagger} = \frac{\langle AA^\dagger\rangle |\langle HB\rangle|^2 + \langle BB^\dagger\rangle |\langle HA\rangle|^2}{\langle HH_i\rangle n} + \tilde{O}(n^{-1.5}). \tag{7}$$

This is our main result. It is an example where certain properties of quantum chaotic systems can be calculated analytically. For comparison, Table I summarizes the finite-size scaling of late-time OTOC of generic traceless local operators for various types of quantum dynamics.

In the remainder of this Letter, by assuming a "generic" energy spectrum we first present a simple derivation of (7) for the special case where the local operators in OTOC are terms in the Hamiltonian. Then, we extend this approach to the general case using the eigenstate thermalization hypothesis (ETH) [26–28]. Thus, we give a rigorous proof of Eqs. (6) and (7) based on two very mild assumptions for chaotic systems: a generic spectrum and ETH. Next, we propose a heuristic physical picture for our results from the perspective of interpreting chaotic dynamics with random unitaries. We introduce a previous approach, which takes into account the unitarity of the dynamics by approximating the time-evolution operator e^{-iHt} with a random unitary. Unfortunately, this approximation remains too crude, for it still suggests that the residual value of late-time OTOC is exponentially small in the system size. We show that once energy conservation is also taken into account by requiring the random unitary to act within small energy windows, the finite-size scaling of late-time OTOC becomes inverse polynomial. Finally, we support our analytical arguments with numerical simulations of a nonintegrable spin chain. The numerical results suggest that the remainder in (7) can be improved to $O(n^{-2})$.

Special case.—In the case where the local operators in OTOC are terms in the Hamiltonian, we give a simple rigorous proof of (7) assuming only a generic spectrum.

TABLE I. Finite-size scaling of generic late-time OTOC for various types of quantum dynamics.

Types of dynamics	Late-time OTOC
Haar random unitary	$e^{-\Theta(n)}$ [11,19]
Chaotic Hamiltonian dynamics	$1/\mathrm{poly}\, n$ [this work]
Many-body localization	$\Theta(1)$ [20–25]

In strongly chaotic systems, one might expect that the energy spectrum satisfies the generic condition.

Assumption 1: (generic spectrum; see, e.g., Ref. [29]). $E_p + E_r = E_q + E_s$ implies $((p = q)$ and $(r = s))$ or $((p = s)$ and $(r = q))$.

This assumption is necessary in the sense that it rules out certain integrable (e.g., free-fermion) systems.

Writing out the matrix elements,

$$\langle AB(t)CD(t)\rangle = \frac{1}{d}\sum_{p,q,r,s} A_{pq}B_{qr}C_{rs}D_{sp}e^{i(E_q-E_r+E_s-E_p)t}. \tag{8}$$

Substituting into (4), we obtain

$$\mathrm{OTOC}_\infty^{A,B,C,D} = \frac{1}{d}\sum_{p,q,r,s} A_{pq}B_{qr}C_{rs}D_{sp}\delta_{E_p+E_r,E_q+E_s}, \tag{9}$$

where δ is the Kronecker delta. Assumption 1 implies

$$\mathrm{OTOC}_\infty^{A,B,C,D} = \frac{1}{d}\sum_{j,k} A_{jj}B_{jk}C_{kk}D_{kj} + \frac{1}{d}\sum_{j,k} A_{jk}B_{kk}C_{kj}D_{jj} - \frac{1}{d}\sum_{j} A_{jj}B_{jj}C_{jj}D_{jj}. \tag{10}$$

Given a Hamiltonian H, there are multiple ways to write it as a sum of local terms, $H = \sum_i H_i$. Without loss of generality, we fix this ambiguity by expanding H in the Pauli basis and assigning all Pauli string operators starting at site i to H_i [see (26) for an example]. This convention implies $\mathrm{tr}(H_jH_k) = 0$ for $j \neq k$. Hence, $\langle H_i^2\rangle = \langle HH_i\rangle = \langle H^2\rangle/n$ for any i due to translational invariance. Using this convention,

Theorem 1: Assumption 1 implies

$$\mathrm{OTOC}_\infty^{H_1,H_i,H_1,H_i} = 2\langle H_i^2\rangle^2/n + O(n^{-2}). \tag{11}$$

Proof.—We use the observation that $(H_i)_{jj} = E_j/n$ for any i due to translational invariance. For the present choice of local operators in OTOC, the first term on the right-hand side of (10) reads

$$\begin{aligned}\frac{1}{d}\sum_{j,k=1}^{d}&(H_1)_{jj}(H_i)_{jk}(H_1)_{kk}(H_i)_{kj}\\ &= \frac{1}{dn^2}\sum_{j,k=1}^{d} E_j\langle j|H_i|k\rangle E_k\langle k|H_i|j\rangle\\ &= \frac{1}{dn^2}\mathrm{tr}\left(\sum_{j=1}^{d}|j\rangle E_j\langle j|H_i\sum_{k=1}^{d}|k\rangle E_k\langle k|H_i\right)\\ &= \frac{\mathrm{tr}(HH_iHH_i)}{dn^2} = \frac{1}{n^2}\sum_{j,k=1}^{n}\langle H_jH_iH_kH_i\rangle.\end{aligned} \tag{12}$$

In the last sum, there are n^2 terms, most of which are 0 because $\mathrm{tr}H_j = \mathrm{tr}H_k = 0$. Furthermore, the convention stated above implies $\mathrm{tr}(H_jH_k) = 0$ for $j \neq k$. Hence, the number of nonvanishing terms in the last sum of Eq. (12) is $n + O(1)$ [n comes from the terms with $j = k$ and $O(1)$ accounts for the remainder]. Equation (12) equals

$$\begin{aligned}\frac{1}{n^2}\sum_{j=1}^{n}&\langle H_jH_iH_jH_i\rangle + O(n^{-2})\\ &= \langle H_i^2\rangle^2/n + O(n^{-2}) + O(n^{-2})\\ &= \langle H_i^2\rangle^2/n + O(n^{-2}).\end{aligned} \tag{13}$$

The second term on the right-hand side of (10) gives the same result. The last term on the right-hand side of (10) equals $(1/d)\sum_j E_j^4/n^4 = \langle H^4\rangle/n^4 = \Theta(n^{-2})$ [30]. This completes the proof. ■

General case.—We sketch an argument for (6) and (7). The argument is rigorous assuming a generic spectrum and ETH.

Technically it suffices to assume ETH for most eigenstates in the middle of the spectrum [30]. For simplicity, here we assume it for all eigenstates in the full spectrum.

Assumption 2: (eigenstate thermalization hypothesis). For any local operator X with $\|X\| \leq 1$, there is a function $f_X : [-1, 1] \to [-1, 1]$ such that

$$|X_{jj} - f_X(E_j/n)| \leq 1/\mathrm{poly}n \tag{14}$$

for all j. We assume that f_X is smooth in the sense of having a Taylor expansion to some low order.

It was proposed analytically [29] and supported by numerical simulations [31] that the right-hand side of (14) can be improved to $e^{-\Omega(n)}$. For our purposes, however, a (much weaker) inverse polynomial upper bound suffices.

Lemma 1: ([30]). For any traceless local operator A, assumption 2 implies

$$\begin{aligned}f_{AA^\dagger}(0) &= \frac{1}{d}\mathrm{tr}(AA^\dagger),\\ \frac{1}{d}\sum_j|A_{jj}|^2 &= \frac{|\mathrm{tr}(HA)|^2}{dn\mathrm{tr}(HH_i)} + O(n^{-2}).\end{aligned} \tag{15}$$

Let $J \subseteq \mathbb{R}$ be an energy interval. Define $P_J = \sum_{j:E_j\in J}|j\rangle\langle j|$ as the projector onto J.

Lemma 2. ([32]). Let $\epsilon < \epsilon'$. For any local operator X,

$$\|P_{(-\infty,\epsilon)}XP_{(\epsilon',\infty)}\| \leq \|X\|e^{-\Omega(\epsilon'-\epsilon)}. \tag{16}$$

This lemma states that local operators cannot (up to an exponentially small error) connect projectors that are far away from each other in the spectrum.

Proof.—(Justification of (6)). Consider the first term on the right-hand side of (10),

$$\frac{1}{d}\sum_{j,k} A_{jj}B_{jk}C_{kk}D_{kj} \approx \frac{1}{d}\sum_{j}\sum_{k:|E_k-E_j|\,\text{small}} A_{jj}B_{jk}C_{kk}D_{kj}$$
$$\approx \frac{1}{d}\sum_{j}\sum_{k:|E_k-E_j|\,\text{small}} A_{jj}B_{jk}C_{jj}D_{kj}$$
$$\approx \frac{1}{d}\sum_{j,k} A_{jj}C_{jj}B_{jk}D_{kj}$$
$$= \frac{1}{d}\sum_{j} A_{jj}C_{jj}(BD)_{jj}, \quad (17)$$

where we used lemma 2 in the first and third steps: The presence of off-diagonal matrix elements B_{jk}, D_{kj} allows us to upper bound the total contribution of all terms for which $|E_k - E_j|$ is not small. In the second step of (17), we replace C_{kk} by C_{jj} using ETH (assumption 2), which states that eigenstates with similar energies have similar local expectation values. A detailed and rigorous error analysis for (17) with a quantitative definition of smallness is given in the full version [30] of the present paper.

Equation (17) shows that the first term on the right-hand side of (10) corresponds to the second term on the right-hand side of (5). Similarly, the second term on the right-hand side of (10) corresponds to the first term on the right-hand side of (5). Obviously, the third terms on the right-hand sides of (5) and (10) are the same. Thus, we obtain (6). ■

Lemma 3. (concentration of eigenvalues [30]). Almost all eigenstates have zero energy density,

$$|\{j : |E_j| \geq n^{0.51}\}|/d \leq n^{-\omega(1)}. \quad (18)$$

This lemma is related to the fact that E_j's approach a normal distribution in the thermodynamic limit $n \to \infty$ [33,34]. Indeed, $|E_j| = \Theta(\sqrt{n})$ for almost all j.

Proof.—(Justification of (7)). Specializing to $\langle AB(t)A^\dagger B^\dagger(t)\rangle$, Eq. (6) reads

$$\mathrm{OTOC}_\infty^{A,B,A^\dagger,B^\dagger} \approx \frac{1}{d}\sum_j (AA^\dagger)_{jj}|B_{jj}|^2 + |A_{jj}|^2(BB^\dagger)_{jj} - |A_{jj}B_{jj}|^2. \quad (19)$$

Consider the first term on the right-hand side,

$$\frac{1}{d}\sum_j (AA^\dagger)_{jj}|B_{jj}|^2 \approx \frac{1}{d}\sum_{j:|E_j|<n^{0.51}} (AA^\dagger)_{jj}|B_{jj}|^2$$
$$\approx \frac{1}{d}\sum_{j:|E_j|<n^{0.51}} f_{AA^\dagger}(0)|B_{jj}|^2$$
$$\approx \frac{f_{AA^\dagger}(0)}{d}\sum_j |B_{jj}|^2$$
$$\approx \frac{\mathrm{tr}(AA^\dagger)|\mathrm{tr}(HB)|^2}{d^2 n\,\mathrm{tr}(HH_i)}, \quad (20)$$

where we used lemma 3 in the first and third steps, the continuity of $f_{AA^\dagger}(x)$ at $x = 0$ in the second step, and lemma 1 in the last step.

The second term on the right-hand side of (19) can be estimated similarly. The third term on the right-hand side of (19) is of higher order in $1/n$ [30]. Thus, Eq. (7) is proved based on assumptions 1 and 2. ■

Physical picture.—We rederive (6) using techniques from the theory of random unitaries. The derivation is not rigorous, but provides a heuristic picture showing the extent to which chaotic dynamics can be approximated by a random unitary.

To improve the approximation described by (2), we first take into account the unitarity of the dynamics. In strongly chaotic systems, it is tempting to expect

Assumption 3: The time-evolution operator e^{-iHt} for large t behaves like a random unitary.

Based on this assumption, late-time OTOC can be estimated from

$$\mathrm{OTOC}_\infty^{A,B,C,D} = \int dU\langle A(U^\dagger BU)C(U^\dagger DU)\rangle, \quad (21)$$

where U is taken from the unitary group $\mathcal{U}(d)$ with respect to the Haar measure.

Lemma 4: ([11,19]).

$$\int dU\langle AU^\dagger BUCU^\dagger DU\rangle = \langle A,B,C,D\rangle - \frac{\langle AC\rangle_c\langle BD\rangle_c}{d^2-1}, \quad (22)$$

where $\langle XY\rangle_c \coloneqq \langle XY\rangle - \langle X\rangle\langle Y\rangle$ is the connected correlator and

$$\langle A,B,C,D\rangle \coloneqq \langle AC\rangle\langle B\rangle\langle D\rangle + \langle A\rangle\langle C\rangle\langle BD\rangle - \langle A\rangle\langle B\rangle\langle C\rangle\langle D\rangle. \quad (23)$$

Note that the right-hand side of (5) resembles that of (23) in the sense of replacing every $\langle\cdots\rangle$ (expectation value at infinite temperature) by $\langle j|\cdots|j\rangle$ (expectation value in an eigenstate).

Corollary 1: ([11,19]). Assumption 3 and lemma 4 imply

$$\mathrm{OTOC}_\infty^{A,B,C,D} = \langle A,B,C,D\rangle - \frac{\langle AC\rangle_c\langle BD\rangle_c}{d^2-1}. \quad (24)$$

Therefore, (i) $\mathrm{OTOC}_\infty^{A,B,A^\dagger,B^\dagger}$ for traceless operators A, B vanishes in the thermodynamic limit $n \to \infty$. (ii) In finite-size systems, the saturation value of OTOC $\langle AB(t)A^\dagger B^\dagger(t)\rangle$ is exponentially small in the system size (because $d = 2^n$).

The approximation stated in assumption 3 is still too crude. We propose a refined version of assumption 3 by incorporating energy conservation and argue (nonrigorously) that Eq. (6) follows from this refinement.

We observe that the time evolution conserves energy and that local operators can only additively change the energy of a state by $O(1)$ (lemma 2). Thus, the action of OTOC $AB(t)CD(t)$ is approximately restricted to each microcanonical ensemble. This observation motivates a refinement of assumption 3 in strongly chaotic systems.

Assumption 4: The time-evolution operator e^{-iHt} for large t behaves like a random unitary in each microcanonical ensemble.

Conceptually, this assumption is related to the so-called random diagonal unitaries [35,36].

Based on assumption 4, we argue for (6). Since the bandwidth of H is $\Theta(n)$, we decompose the energy spectrum into a disjoint union of $\Theta(n/\Delta)$ microcanonical ensembles with bandwidth Δ. Let $J_k := [k\Delta, (k+1)\Delta)$ and define $[A, B, C, D]_k$ as the right-hand side of (23) with every $\langle\cdots\rangle$ replaced by the expectation value $\mathrm{tr}(P_{J_k}\cdots)/\mathrm{tr}P_{J_k}$ in the microcanonical ensemble. We expect

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau dt\,\frac{\mathrm{tr}(P_{J_k}AB(t)CD(t))}{\mathrm{tr}P_{J_k}} \approx [A,B,C,D]_k \approx \frac{1}{\mathrm{tr}P_{J_k}}\sum_{j:E_j\in J_k}\langle A,B,C,D\rangle_j. \quad (25)$$

The first step is a consequence of lemma 4 and assumption 4. Indeed, it is just (24) restricted to the microcanonical ensemble P_{J_k}. The last step of (25) used ETH. Equation (6) follows immediately from (25).

Numerics.—Finally, we support (7) with numerical simulations. Consider the spin-1/2 chain

$$H = \sum_{i=1}^{n} H_i,$$
$$H_i = \sigma_i^z\sigma_{i+1}^z - 1.05\sigma_i^x + 0.5\sigma_i^z + g\sigma_i^y\sigma_{i+1}^z \quad (26)$$

with periodic boundary conditions ($\sigma_{n+1}^z := \sigma_1^z$), where $\sigma_i^x, \sigma_i^y, \sigma_i^z$ are the Pauli matrices at site i. For $g = 0$, this model is nonintegrable in the sense of Wigner-Dyson level statistics [37,38]. Reference [9] calculated OTOC, focusing on the butterfly effect rather than the late-time behavior. Note that for $g = 0$, most energy levels are twofold degenerate so that assumption 1 does not hold.

We fix $g = 0.1$. Intuitively, the model is nonintegrable for any value of g. We have numerically confirmed the validity of assumption 1 for $n = 5, 6, \ldots, 12$. Presumably, assumption 1 holds for any integer $n \geq 5$. Let $F_n^x := \mathrm{OTOC}_\infty^{\sigma_1^x,\sigma_i^x,\sigma_1^x,\sigma_i^x}$ and $F_n^z := \mathrm{OTOC}_\infty^{\sigma_1^z,\sigma_i^z,\sigma_1^z,\sigma_i^z}$, whose values are independent of i. We compute F_n^x, F_n^z using exact diagonalization. The results are shown in the top panel of Fig. 1.

The leading terms in the finite-size scaling of F_n^x, F_n^z are calculated analytically from (7),

$$G_n^x := \frac{14}{15n} \approx \frac{0.933}{n}, \qquad G_n^z := \frac{40}{189n} \approx \frac{0.212}{n}. \quad (27)$$

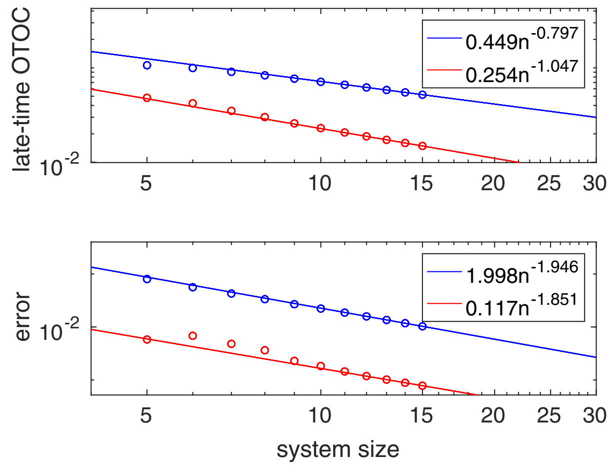

FIG. 1. Top panel: Finite-size scaling of late-time OTOC F_n^x (blue), F_n^z (red) for $n = 5, 6, \ldots, 15$. The lines are power-law fits $0.449n^{-0.797}$ (blue), $0.254n^{-1.047}$ (red) to the last few data points. Bottom panel: Finite-size scaling of the errors $|F_n^x - G_n^x|$ (blue), $|F_n^z - G_n^z|$ (red) for $n = 5, 6, \ldots, 15$. The lines are power-law fits $1.998n^{-1.946}$ (blue), $0.117n^{-1.851}$ (red) to the last few data points.

We expect that the noticeable differences between G_n^x, G_n^z and the power-law fits to F_n^x, F_n^z are due to finite-size effects. To justify this claim, we perform a scaling analysis of the errors $|F_n^x - G_n^x|$, $|F_n^z - G_n^z|$ in the bottom panel of Fig. 1. The numerics suggest that the errors should vanish as $\Theta(n^{-2})$ in the thermodynamic limit $n \to \infty$.

Conclusion.—We propose that in order to better approximate the late-time behavior of chaotic dynamics generated by a time-independent Hamiltonian, one needs to take into account energy conservation. In particular, we show that approximation schemes with and without energy conservation make different predictions about OTOC at late times: without energy conservation, late-time OTOC scales inverse exponentially with system size; with energy conservation, the scaling is inverse polynomial. The latter prediction has been rigorously confirmed based on two very mild assumptions and is consistent with numerical simulations of a nonintegrable spin chain.

An immediate open question is how good the energy-preserving approximation scheme proposed in this Letter is in predicting the late-time behavior of higher-order time-ordered or out-of-time-ordered correlators. A more general problem for future study is how to approximate the time-evolution process and capture other universal features of chaotic dynamics. See Refs. [39–42] for recent progress in this direction.

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