Supplemental material for "Entanglement area laws for long-range interacting systems"

Zhe-Xuan Gong,[1,2] Michael Foss-Feig,[3] Fernando G. S. L. Brandão,[4] and Alexey V. Gorshkov[1,2]

[1]*Joint Quantum Institute, NIST/University of Maryland, College Park, Maryland 20742, USA*
[2]*Joint Center for Quantum Information and Computer Science, NIST/University of Maryland, College Park, Maryland 20742, USA*
[3]*United States Army Research Laboratory, Adelphi, MD 20783, USA*
[4]*IQIM, California Institute of Technology, Pasadena CA 91125, USA*

In this supplemental material, we prove Lemma 2 in the main text by generalizing the following result of Ref. [1]:

For any operators A and B acting on sites i and j respectively, there exists constants $v = \mathcal{O}(1)$ such that for $\alpha > 2D$, $\gamma = \frac{D+1}{\alpha-2D}$, and $0 < t < t_R \equiv (\frac{R}{6v})^{\frac{1}{1+\gamma}}$,

$$\|[A(t), B\| \leq \|A\| \, \|B\| \left[\mathcal{O}(e^{vt - r_{ij}/t^{\gamma}}) + \mathcal{O}(\frac{t^{\alpha(1+\gamma)}}{r_{ij}})\right]. \tag{1}$$

In order to prove Lemma 2 in the main text, the above Lieb-Robinson-type bound must be extended to allow for operators A and B that are supported on an arbitrary number of sites. To proceed we will need to recall how Eq. (1) is derived. Let us first quote Eq. (11-12) in Ref. [1],

$$\|[A(t), B]\| \leq \sum_{\ell} \left[\|[\mathcal{A}^{\ell}(t), B]\| + 4c \sum_{k=1}^{\infty} \frac{t^a}{a!} \mathcal{J}_a(i,j)\right], \tag{2}$$

$$\mathcal{J}_a(i,j) = 4^n \sum_{\ell, \xi_1, \ldots, \xi_a} e^{-\ell} D_{\mathrm{i}}(\xi_1) \|\mathcal{W}_{\xi_1}\| D(\xi_1, \xi_2) \|\mathcal{W}_{\xi_2}\| \times \ldots \cdots \times \|\mathcal{W}_{\xi_{a-1}}\| D(\xi_{a-1}, \xi_a) \|\mathcal{W}_{\xi_a}\| D_{\mathrm{f}}(\xi_a). \tag{3}$$

We will not introduce all of the notation used in the above two equations, as it can be found Ref. [1], but we will explain the aspects of the notation that are relevant for our purposes. The operator $\mathcal{A}^{\ell}(t)$ is exclusively supported on the set $\mathscr{B}_{\ell}(i)$, which is the set of sites with distance $\leq (vt + \ell)\chi$ from site i, with $v = \mathcal{O}(1)$ the so-called Lieb-Robinson velocity. Importantly, $\left\|\mathcal{A}^{l}(t)\right\| \leq \|A\| \, \mathcal{O}(e^{-l})$ (see Eq. (S8) in Ref. [1]). The collective index $\xi_k = (x_k, y_k, m_k, n_k)$ specifies the support of the operator $\mathcal{W}_{\xi_k}$, which is an interaction connecting sites in $\mathscr{B}_{m_k}(x_k)$ to sites in $\mathscr{B}_{n_k}(y_k)$ (and is supported on the unions of these two sets). The quantity $D_i(\xi_1) = 1$ when $\mathscr{B}_{\ell}(i) \cap \mathscr{B}_{m_1}(x_1) \neq \varnothing$ and vanishes otherwise, while the quantity $D_{\mathrm{f}}(\xi_a)$ is unity when $j \in \mathscr{B}_{n_a}(y_a)$ and vanishes otherwise. Similarly, $D(\xi_{k-1}, \xi_k) = 1$ when $\mathscr{B}_{n_{k-1}}(y_{k-1}) \cap \mathscr{B}_{m_k}(x_k) \neq \varnothing$, and vanishes otherwise. Intuitively, these quantities ensure that $\mathcal{J}_a(i,j)$ connects operators A to B and contributes to the commutator $[A(t), B]$.

Now we assume that A and B act on two arbitrary sets X and Y respectively, which results in a number of changes. First, $\mathcal{A}^{\ell}(t)$ will instead act on $\mathscr{B}_{\ell}(X)$, which is defined as the set of sites with distance $\leq (vt + \ell)\chi$ from any site in X. But $\left\|\mathcal{A}^{l}(t)\right\| \leq \|A\| \, \mathcal{O}(e^{-l})$ holds independent of the size of X, because it is obtained using a finite-range Lieb-Robinson bound [2]. Second, $D_i(\xi_1)$ in Eq. (3) should now be replaced by $D_X(\xi_1)$, which restricts the summation over ξ_1 to the cases where $\mathscr{B}_{\ell}(X) \cap \mathscr{B}_{m_1}(x_1) \neq \varnothing$. Finally, $D_j(\xi_a)$ should be replaced by $D_Y(\xi_a)$ such that the summation over ξ_a is restricted such that $\mathscr{B}_{\ell}(Y) \cap \mathscr{B}_{n_a}(y_a) \neq \varnothing$.

Next, we observe that the summation over ξ_1 with the constraint $\mathscr{B}_{\ell}(X) \cap \mathscr{B}_{m_1}(x_1) \neq \varnothing$ is upper bounded by the summation over ξ_1 with the constraint $\mathscr{B}_{\ell}(i) \cap \mathscr{B}_{m_1}(x_1) \neq \varnothing$ *plus* an extra summation over all $i \in X$. This is true because for $\mathscr{B}_{m_1}(x_1)$ to overlap with $\mathscr{B}_{\ell}(X)$, it has to overlap with $\mathscr{B}_{\ell}(i)$ for some $i \in X$. In summing over all $i \in X$, we may count the same $\mathscr{B}_{m_1}(x_1)$ that overlaps multiple times, but since the summand in Eq. (3) is always non-negative we nevertheless obtain an upper bound. A similar treatment will be applied to the summation over ξ_a as well.

As a result, we will replace $\mathcal{J}_a(i,j)$ in Eq. (2) by $\sum_{i \in X, j \in Y} \mathcal{J}_a(i,j)$. The summation $\sum_{\ell} \|[\mathcal{A}^{\ell}(t), B]\|$ in Eq. (2) is now restricted only to ℓs satisfying $(vt + \ell)\chi \geq r_{XY}$, These two changes together lead to a modified version of Eq. (1), which reads

$$\|[A(t), B]\| \leq \|A\| \, \|B\| \left[\mathcal{O}(e^{vt - r_{XY}/t^{\gamma}}) + \sum_{i \in X, j \in Y} \mathcal{O}(\frac{t^{\alpha(1+\gamma)}}{r_{ij}^{\alpha}})\right]. \tag{4}$$

Finally, we set $B = U_R$, which is an arbitrary unitary acting only on sites with distance larger than or equal to R from all sites in X, and obtain (whenever $\alpha > D$)

$$\|[A(t), U_R]\| \leq \|A\| \, |X| [\mathcal{O}(e^{vt - R/t^{\gamma}}) + \mathcal{O}(\frac{t^{\alpha(1+\gamma)}}{R^{\alpha-D}})]. \tag{5}$$

Using $\|A(t)-A(t,R)\| \le \int d\mu(U_R)\, \|[A(t),U_R]\|$, we have proven Lemma 2 in the main text.

As mentioned towards the end of the main text, it can be shown that Eq. (4) is the optimal generalization of Eq. (1). To see this, consider a long-range Ising model $H=\sum_{ij} J_{ij}\sigma_i^z\sigma_j^z$ with $J_{ij}=r_{ij}^{-\alpha}$ if $i\in X$ and $j\in Y$, and $J_{ij}=0$ otherwise. We define

$$|\psi_0\rangle = \left[\frac{1}{\sqrt{2}}\left(\bigotimes_{i\in X\cup Y}|\sigma_i^z=1\rangle + \bigotimes_{i\in X\cup Y}|\sigma_i^z=-1\rangle\right)\right]\bigotimes_{i\notin X\cup Y}|\sigma_i^z=1\rangle, \tag{6}$$

which is a GHZ state for all sites in X and Y. Now let us choose $A=\prod_{i\in X}\sigma_i^+$ and $B=\sum_{j\in Y}\sigma_j^+$. It is not hard to find that

$$\langle\psi_0|\prod_{i\in X}\sigma_i^+(t)\prod_{j\in Y}\sigma_j^+|\psi_0\rangle = \frac{1}{2}e^{2i\sum_{i\in X,j\in Y}J_{ij}t}, \tag{7}$$

$$\langle\psi_0|\prod_{j\in Y}\sigma_j^+\prod_{i\in X}\sigma_i^+(t)|\psi_0\rangle = \frac{1}{2}e^{-2i\sum_{i\in X,j\in Y}J_{ij}t}. \tag{8}$$

As a result, for $t<\pi/(4\sum_{i\in X,j\in Y}J_{ij})$, we have

$$\|[A(t),B]\| \ge |\langle\psi_0|[A,B]|\psi_0\rangle| = \sin(2t\sum_{i\in X,j\in Y}J_{ij}) > \frac{4}{\pi}t\sum_{i\in X,j\in Y}r_{ij}^{-\alpha}. \tag{9}$$

Although the time dependence is different from Eq. (4), the distance dependence (for large r_{ij}) and the summations over $i\in X$ and $j\in Y$ match those in Eq. (4). Thus we claim that Eq. (4) is the optimal generalization of Eq. (1).

[1] M. Foss-Feig, Z.-X. Gong, C. W. Clark, and A. V. Gorshkov, Phys. Rev. Lett. **114**, 157201 (2015).
[2] Z.-X. Gong, M. Foss-Feig, Spyridon Michalakis, and A. V. Gorshkov, Phys. Rev. Lett. **113**, 030602 (2014).