Noise-induced subdiffusion in strongly localized quantum systems

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We consider the dynamics of strongly localized systems subject to dephasing noise with arbitrary correlation time. Although noise inevitably induces delocalization, transport in the noise-induced delocalized phase is subdiffusive in a parametrically large intermediate-time window. We argue for this intermediate-time subdiffusive regime both analytically and using numerical simulations on single-particle localized systems. Furthermore, we show that normal diffusion is restored in the long-time limit, through processes analogous to variable-range hopping. Our qualitative conclusions are also valid for interacting systems in the many-body localized phase.

The effects of disorder on quantum transport and dynamics have been a topic of longstanding interest [1, 2]. Both noninteracting [3] and interacting [4–13] systems of electrons in a random potential can get "localized" by the disorder, causing their d.c. conductivity to vanish in the limit of a fully isolated system. Isolated localized systems not only have vanishing transport coefficients, but also fail to reach thermal equilibrium starting from generic initial conditions [7]. Yet, in any practical situation the system of interest is coupled to a thermalizing environment, which restores equilibrium and transport. The nature of equilibration in the presence of a bath has been a topic of recent interest [14–25]; however the implications for *transport* have not yet been investigated in general (but see Refs. [14, 20]).

One reason to expect unusual transport properties in *imperfectly* localized states is that *isolated* localized states exhibit a broad distribution of timescales. This feature was recently noticed as a property of the *nearly localized* regime in the vicinity of the many-body localization (MBL) transition [26–31]. However, properties such as overlap integrals between localized orbitals also exhibit broad distributions deep in the localized phase, both in the single-particle case and in the many-body case [32–34]. Consequently, the inter-orbital hopping rates induced by the bath are broadly distributed [21–23]. One might expect such broad distributions to have anomalous transport signatures, particularly in one-dimensional systems, where single weak links can blockade transport.

In the present work, we explore this question, for localized systems coupled to generic non-Markovian dephasing noise. The Markovian limit was previously considered in Refs. [21–23]; these works noted a broad distribution of relaxation times, leading to stretched-exponential decay of the "contrast" (as measured in Ref. [9]). We find that *slowly* fluctuating noise can have even more dramatic effects: for strong disorder and slowly fluctuating noise, we find a large intermediate time window in which the system exhibits anomalous diffusion (Fig. 1). This anomalous subdiffusive regime vanishes in the limit of fast Markovian noise, and also crosses over to diffusion in the long-time limit. The existence of

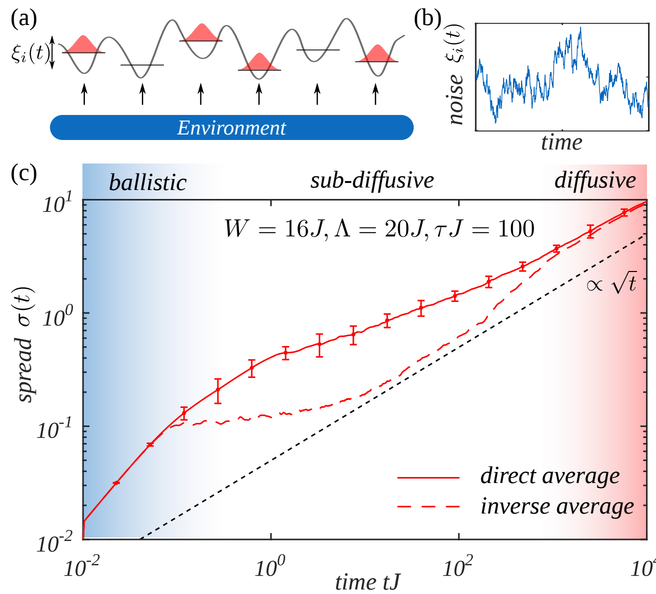

FIG. 1. **Noise-induced delocalization.** (a) We consider strongly-localized fermions in a random potential that are weakly coupled to an environment. (b) In that limit, the environment can be modeled by classical noise $\xi_i(t)$ that couples locally to the density. (c) We study the noise-induced transport, by preparing the system in a wavepacket and computing its spread $\sigma(t)$ in time. For finite coupling to the environment three regimes can be distinguished: (i) a short-time ballistic expansion where the spread is proportional to time $\sigma(t) \sim t$, (ii) a parametrically large regime of subdiffusive transport $\sigma(t) \sim t^\beta$ with a continuously increasing power β that approaches (iii) a diffusive regime $\sigma(t) \sim \sqrt{t}$ at late times. Numerical data taken as direct (solid) and inverse (dashed) average of the spread over individual realizations are shown for disorder strength $W = 16J$, noise strength $\Lambda = 20J$, and noise correlation time $\tau J = 100$.

a subdiffusive regime is notable because we do not explicitly *introduce* any broad distributions, as opposed to the cases in Refs. [35]. Rather, a broad distribution of hopping rates *emerges* as a result of the interplay between the disorder and noise; we present an analytic understanding of this effect. Furthermore, unlike the subdiffusive regime prefiguring the many-body localization transition [27], the phenomenon we discuss here persists in the noninteracting limit.

Our focus here is on free-fermion systems coupled to classical colored noise, as relatively large systems are accessible in numerical simulations for this case. As we discuss, however, our qualitative conclusions can also be adapted to *interacting* systems in the many-body localized phase. Moreover, we expect that our model can be directly extended from classical noise to quantum dephasing, through the arguments of Ref. [36].

Model.—We consider non-interacting electrons in one dimension, subject to a random disorder potential and time dependent noise, as described by the Hamiltonian

$$H = -J\sum_{\langle ij\rangle} c_i^\dagger c_j + \sum_i [\epsilon_i + \xi_i(t)] c_i^\dagger c_i, \tag{1}$$

where J represents the tunneling matrix element and $c_i^\dagger$ (c_i) creates (destroys) an electron on lattice site i. The on-site energies ϵ_i are uncorrelated, and are drawn from a Gaussian distribution of width W and zero mean. The noise $\xi_i(t)$ is characterized by its strength Λ and correlation time τ. We consider spatially uncorrelated noise generated by an Ornstein-Uhlenbeck process [37] and having the correlation function

$$C(t) = \langle \xi_i(t)\xi_i(0)\rangle = \Lambda^2 \exp[-t/\tau]. \tag{2}$$

We will be particularly interested in how transport changes as a function of the disorder strength W and the noise strength Λ as its correlation time is tuned from the Markovian, white noise limit, $\tau \to 0$, to the limit of quasistatic noise, $\tau \to \infty$.

Perturbative treatment.—Analytical insight into the noise-induced dynamics can be obtained by working in the deeply localized limit where the single-particle hopping is the smallest energy scale. In this limit, the system dephases completely between successive hops, so transport is purely incoherent. To compute the incoherent transport rate it suffices to consider a two-site problem [38], and to solve the equations of motion generated by the Hamiltonian (1) perturbatively in the hopping $J \ll \Lambda, W$. In the absence of the hopping, each site simply accumulates phase, and its wavefunction amplitude at time t, denoted A_j^0, is given by $A_j^0(t) = A_j^0 e^{-i\epsilon_i t - i\phi_j(t)}$, where $\phi_j(t) = \int_0^t \xi_j(t')dt'$. To describe transport, we expand the equations of motion to the lowest nontrivial order in the hopping, resulting in the following rate equation for the probability distribution $p_j \equiv |A_j|^2$ for the particle position at time t, see Supplemental Material [39]:

$$\frac{dp_j}{dt} = \Gamma_{j,j+1}p_{j+1} + \Gamma_{j,j-1}p_{j-1} - (\Gamma_{j+1,j} + \Gamma_{j-1,j})p_j \tag{3}$$

with a locally varying rate $\Gamma_{i,j} = \Gamma(\epsilon_i - \epsilon_j)$ that depends on the *energy difference* between neighboring sites i and j:

$$\Gamma(\omega) = 2J^2 \int_0^\infty dt \cos(\omega t)\, |C^\phi(t)|^2, \tag{4}$$

where $C^\phi(t)$ is the phase correlation function $C^\phi(t) = \langle e^{-i\phi_j(t)} e^{i\phi_j(0)}\rangle = e^{-\int_0^t (t-t')C(t')dt'}$ with the noise correlation function $C(t)$, Eq. (2), and we have performed a Gaussian average over noise trajectories. For our specific noise model and $\Lambda\tau \gtrsim 1$, the rate $\Gamma(\omega)$ has the form

$$\frac{\Gamma(\omega)}{2J^2} = \begin{cases} \frac{\Lambda}{\omega^2+\Lambda^2} & \omega < \tau^{-1} \\ \Lambda^{-1} e^{-\omega^2/(4\Lambda^2)} & \tau^{-1} < \omega < 2\Lambda\sqrt{\log(\Lambda\tau)} \\ \frac{\Lambda^2}{2\tau\omega^4} & \omega > 2\Lambda\sqrt{\log(\Lambda\tau)} \end{cases} \tag{5}$$

Note that Eq. (3) has the form of a random walk with locally varying transition rates. In the disorder-free limit [38], Γ has no spatial dependence, and Eq. (4) reduces to a discretized diffusion equation with a diffusion constant given by $\Gamma(0)$.

Subdiffusive regime.—In the disordered system, the transition rate Γ_{ij} between a particular pair of neighboring sites depends on their energy difference ω through Eq. (5). For very small or very large ω, the rate decreases polynomially with ω. However, in the intermediate regime, which exists only for sufficiently large τ, $\Gamma(\omega)$ decreases very rapidly as ω increases. This rapid decrease, as we now discuss, is the origin of anomalous diffusion.

To this end, we estimate the density of very weak links in this regime. Recall that the on-site energies are Gaussian distributed with a characteristic width W. Then the cumulative distribution function of finding a bottleneck, defined by the transition rate being smaller than a certain cut-off Γ_0, follows a power-law relation [39]

$$P(\Gamma < \Gamma_0) \sim \left(\frac{\Lambda\Gamma_0}{2J^2}\right)^{\frac{\Lambda^2}{W^2}}. \tag{6}$$

In order to elucidate the consequences of the power law in $P(\Gamma < \Gamma_0)$, we make an analogy with a random resistor network model by interpreting the local transition rates as conductances. For resistors that are power-law distributed $P(R) = (R_0/R)^{\mu+1}$, the mean resistance is finite for $\mu > 1$ (leading to regular diffusion) but ill-defined for $\mu < 1$ (leading to subdiffusion [27, 35]). Our rate distribution corresponds to a heavy-tailed ($\mu < 1$) resistance distribution and thus to subdiffusion when $\Lambda < W$.

Crossover to diffusion.—Within our noise model, there are two mechanisms that result in a crossover to diffusion in the long-time limit. We call these respectively the "variable-range hopping" (VRH) and "ultraviolet" (UV) mechanisms. We begin by discussing the VRH mechanism, which is more generally applicable. This mechanism involves processes whereby the system avoids a bottleneck by tunneling virtually through it. Crucially, for a site to act as a bottleneck, *all* transitions out of it, not just nearest-neighbor hops, must be blocked. The matrix element for an n-site virtual process is $J(J/W)^{n-1}$, and the corresponding incoherent rate is given by

$$\Gamma_i^{(n)} \simeq \frac{2J^2}{\Lambda}\left(\frac{J}{W}\right)^{2(n-1)} \exp\left[-\frac{\omega_n^2}{4\Lambda^2}\right], \tag{7}$$

where ω_n is the energy difference between sites i and $i \pm n$. For a site to act as a bottleneck we require that

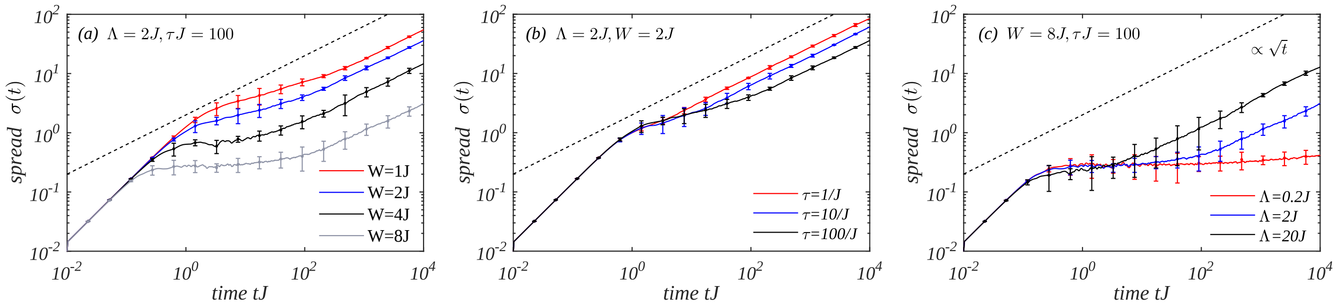

FIG. 2. **Time evolution of a localized wavepacket.** The numerically evaluated, inverse-averaged spread $\sigma(t)$ of an initially localized wavepacket for systems of size $L = 400$ is shown for (a) fixed $\Lambda = 2J$, $\tau J = 100$, (b) $\Lambda = 2J$, $W = 2J$, and (c) $W = 8J$, $\tau J = 100$. Dotted lines indicate diffusive expansion $\sigma(t) \sim \sqrt{t}$. Errorbars are obtained from the sample average of 250 noise and disorder realizations.

$\prod_n \Gamma_i^{(n)} \lesssim \Gamma_0$, i.e., each link must independently act as a bottleneck. In effect, this product only runs over $n \leq n^* = \log(\Gamma_0 \Lambda / 2W^2)/2\log(J/W)$, as more distant links are slower than Γ_0 regardless of the energy difference [39]. The probability of finding a series of such sites can be estimated (Supplemental Material [39]) as

$$\tilde{P}(\Gamma_0|n^*) \sim \exp\left[-c \log^2 \frac{\Gamma_0 \Lambda}{2W^2}\right], \tag{8}$$

with a constant $c \simeq \Lambda^2/(4W^2 \log[W/J])$. This probability decays slightly faster than a powerlaw in $1/\Gamma_0$ and, hence, bottlenecks are asymptotically always sufficiently rare such that diffusion is recovered. Specifically, as the mean inverse transition rate (i.e., "resistance") is well-defined, we can compute the asymptotic diffusion constant by taking the inverse of this mean resistance. For $W \gg \Lambda$ we find that [39]

$$D_{\mathrm{VRH}} \simeq \frac{W}{\sqrt{\pi \log[W/J]}} \left(\frac{J}{W}\right)^{W^2/\Lambda^2}. \tag{9}$$

We now turn to the "ultraviolet" mechanism. At very large ω the noise ceases to fall off as a Gaussian with ω. Instead it falls off as $1/\omega^4$, Eq. (5). The origin of this power-law is the "cuspy" short-time behavior of the noise correlation function Eq. (2). Thus, unlike the VRH mechanism, the ultraviolet mechanism is quite model-dependent. In this large-ω limit, $P(\Gamma < \Gamma_0) \sim \exp(-\mathrm{const.}\Gamma_0^{-1/2})$, so that weak links are sparse and irrelevant for transport. Thus, the overall diffusion constant is set by the weakest *common* links, for which $\omega \simeq 2\Lambda\sqrt{\log \Lambda\tau}$. The density of these links is $n_{\mathrm{UV}} \sim \exp[-\Lambda^2 \log(\Lambda\tau)/2W^2]$, and the rate across each such link is $2J^2/(\Lambda^2\tau)$. Thus the effective UV diffusion constant is

$$D_{\mathrm{UV}} \simeq \frac{2J^2}{\Lambda} \frac{1}{(\Lambda\tau)^{1-(\Lambda/W)^2}}. \tag{10}$$

Within our noise model, the asymptotic diffusion coefficient is set by $\max(D_{\mathrm{VRH}}, D_{\mathrm{UV}})$.

Numerical results.—We quantitatively study the subdiffusive transport by performing exact numerical simulations of a particle localized in the center of our system. To this end, we first compute stochastic noise trajectories based on the Ornstein-Uhlenbeck process. Second, we numerically solve the equations of motion set by the Hamiltonian (1). We consider systems of size $L = 400$ and times to $tJ = 10^4$. A typical example for the spread $\sigma(t) = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$ is shown in Fig. 1 (c). Here, the expectation values $\langle \dots \rangle$ are taken with respect to the time evolved wave function $|\psi(t)\rangle$. At times $tJ \lesssim 1$, the expansion of the wavepacket is ballistic. At later times the spread crosses over to sub-diffusive behavior $\sigma(t) \sim t^\beta$. In that regime, the *direct* sample average of the spread over disorder and noise realizations $\langle \sigma(t) \rangle$, solid line, and the *inverse* average of the inverse spread $1/\langle \sigma^{-1}(t) \rangle$, dashed lines, strongly disagree. This is a manifestation of the probability distribution, Eq. (6), having ill defined moments. The apparent subdiffusion exponent β increases with time and slowly approaches the diffusive limit $\beta = 1/2$ at late times $tJ \sim 10^4$.

Simulations of the spread of an initially localized wavepacket are shown in Fig. 2 for a range of parameters. Generally, we observe (i) an initial regime of ballistic expansion, followed by (ii) an intermediate subdiffusive regime that gradually crosses over to (iii) diffusive transport. With increasing disorder, the wave-packet spread $\sigma(t)$ decreases and the crossover to diffusive transport is pushed to later times, Fig. 2 (a). Moreover, with increasing noise correlation time τ, the intermediate subdiffusive regime is extended, leading to a decrease of the asymptotic diffusion constant with τ, Fig. 2 (b). This suggests that for the relevant parameters subdiffusion is cut off by the "ultraviolet" mechanism (10). Finally, at strong disorder, transport is facilitated with increasing noise strength, Fig. 2 (c). At weak disorder, on the other hand, noise *impedes* transport (data not shown).

We evaluate the subdiffusion exponent $\sigma(t) \sim t^\beta$ by fitting the numerical data in the regime $1 < tJ < \tau J$, Fig. 3. We choose such a small range to capture the exponent at the onset of the subdiffusive regime. For weak noise strength and strong disorder, the subdiffusion exponent β is close to zero. When lowering the disorder strength, β strongly increases and

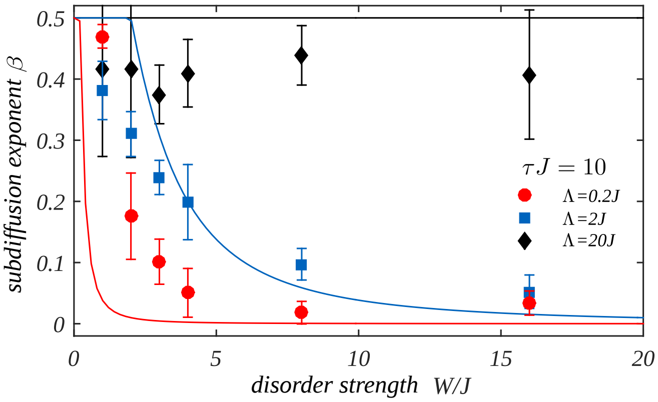

FIG. 3. **Short-time subdiffusion exponent.** The power law exponent β characterizing the initial subdiffusive transport $\sigma(t) \sim t^\beta$ is extracted from fitting the numerical data in the range $1 < tJ < \tau J$ for $\tau J = 10$. In the weak noise limit $\Lambda \lesssim J$, the exponent depends strongly on the disorder strength W approaching zero with increasing W whereas it is constant for strong-noise $\Lambda = 20J$. For weak disorder strength $W \to 0$, β universally approaches within the errorbars the limit of diffusive transport with $\beta = 1/2$. Solid lines are predictions for the subdiffusion exponent, Eq. (11).

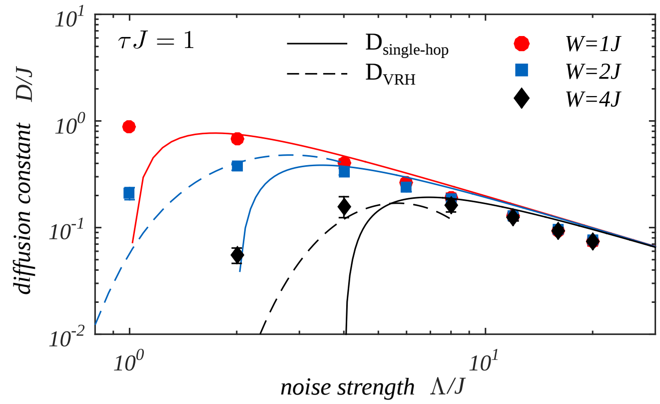

FIG. 4. **Asymptotic diffusion constant.** The diffusion constant D evaluated from the asymptotic spread of the wavepacket $\sigma(t \to \infty) = \sqrt{2Dt}$ is shown as a function of the the noise strength Λ for different values of the disorder strength W and fixed noise correlation time $\tau J = 1$, symbols. The numerical data is compared to the single-hop model valid for $\Lambda \gtrsim W$, solid line and the variable-range hopping model of Eq. (9) valid for $\Lambda \lesssim W$, dashed lines.

approaches the diffusive limit $\beta \to 1/2$. By contrast at large disorder strength, the subdiffusion exponent sets off at a larger value and quickly saturates. We obtain an estimate for the initial subdiffusion exponent by relating it to the exponent of the cumulative distribution function (6) as [27]

$$\beta = \frac{\Lambda^2}{\Lambda^2 + W^2}. \tag{11}$$

The predictions of this model are indicated by the solid lines in Fig. 3. The data qualitatively reproduces the trend of the subdiffusion exponent but slight quantitative differences are present. Such discrepancies are not unexpected as our theoretical analysis is valid to lowest order in $J/W, J/\Lambda$, and these parameters are not small in the numerically accessible regime.

From the long-time asymptotics of the spread $\sigma(t)$, we extract the diffusion constant $\sigma(t \to \infty) = \sqrt{2Dt}$ for different values of the noise and disorder strength at fixed noise correlation time $\tau J = 1$ (Fig. 4). When the noise is strong compared with the disorder, the diffusion constant is largely disorder-independent, and *decreases* with increasing noise as $\sim 1/\Lambda$, Eq. (5), consistent with Ref. [38]. In the strong noise limit, $\Lambda \gg W$, diffusion is induced already by nearest neighbor hops, leading to $D_{\text{single-hop}} \sim 2J^2(1 - W^2/\Lambda^2)/\Lambda$ (solid lines) that describes the *approach* to the subdiffusive regime [39]. In the subdiffusive regime, it is challenging to propagate to sufficiently long times to see the eventual crossover to diffusion. However, we were able to extract a few values for the diffusion constant for $\Lambda \ll W$, and observe a reversed dependence: noise assists diffusion rather than impeding it as predicted by the variable range processes, Eq. (9) (dashed lines).

Discussion.—How robust are our conclusions to adding interactions, and to more general forms of correlated noise? Adapting our results to the case of an interacting, many-body localized system coupled to noise is straightforward in principle. Qualitatively, the main difference is that there are many more ways for an interacting system to "escape" a bottleneck: in addition to longer-range hops, the system can undergo many-particle rearrangements, which have a larger phase space [32]. Thus the variable-range hopping mechanism will be more effective, giving rise to a smaller subdiffusive window and a larger asymptotic diffusion constant. Interacting localized systems coupled to Markovian baths have been shown to exhibit a broad distribution of relaxation rates, leading to a stretched-exponential decay of the contrast of an initial density wave pattern [21–23]. We find similar behavior for our non-interacting system with a stretching exponent that is independent of the noise correlation time τ for weak noise $\Lambda \lesssim J$ but depends strongly on τ for strong noise $\Lambda \gtrsim J$; see supplemental material [39]. Comparing with Refs. [21–23], this also demonstrates that qualitatively the dynamics remains the same irrespective of interactions.

Our perturbative analysis suggests that our numerical results should be sensitive to the short-time correlations of the noise, which are nonuniversal. In particular, noise emanating from a physical system with a finite bandwidth will decay as a Gaussian, rather than an exponential, on timescales that are short compared with the bandwidth. Thus, the "ultraviolet" mechanism should be absent in such systems. This interdependence of slow and fast processes has also been observed in a mean-field treatment of the many-body localization transition [16], and appears to be a generic phenomenon, reminiscent of the "UV-IR mixing" phenomenon in field theory [40]. Extending our numerical studies to more general forms of colored noise is an important direction, which we shall explore in future work.

Conclusions.—We have studied noise-induced transport in strongly disordered quantum systems. We have argued that for slowly fluctuating noise, transport is governed by an incoherent hopping model with an *emergent* broad distribution of hopping rates, giving rise to anomalous diffusion on intermediate timescales, and regular diffusion (albeit with a strongly suppressed diffusion constant) at late times.

Our approach paves the way for developing a self-consistent theory for the metallic phase in disordered and interacting quantum systems where interactions can be treated by a self-consistent Hartree-Fock decoupling. Furthermore, such a self-consistent theory can also be developed with the prospect of studying the response of a many-body localized system coupled to a bath. Having technical approaches at hand, which go beyond conventional exact diagonalization of small quantum systems, will help to provide further insight in the many-body localized phase and its breakdown.

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Supplemental Material: Noise-induced subdiffusion in strongly localized quantum systems

Perturbative treatment

We discuss how to establish analytical insights in the noise-induced dynamics by a perturbative treatment in small hopping $J \ll \Lambda, W$. The equations of motion for the annihilation operator c_j set by Hamiltonian Eq. (1) read

$$i\frac{dc_j}{dt} = -J(c_{j-1} + c_{j+1}) + [\epsilon_j + \xi_j(t)]c_j. \tag{S1}$$

We solve these equations order by order in the hopping J [38]. In the absence of interactions we can represent the quantum operator c_j by a complex amplitude A_j. The dynamics of the wave function amplitude to leading order A_j^0 is determined by

$$i\frac{dA_j^0}{dt} = [\epsilon_j + \xi_j(t)]A_j^0, \tag{S2}$$

which describes the accumulation of phase

$$A_j^0(t) = A_j^0 e^{-i\epsilon_j t - i\int_0^t \xi_j(t')dt'} = A_j^0 e^{-i\epsilon_j t} e^{-i\phi_j(t)}. \tag{S3}$$

To leading order transport is absent. However, it is restored by evaluating the next-to-leading order correction

$$i\frac{dA_i^1}{dt} - [\epsilon_i + \xi_i(t)]A_i^1 = -J(A_{j+1}^0 + A_{j-1}^0). \tag{S4}$$

Introducing $\mu_j = e^{i\phi_j(t)}$, we rewrite the equation as $\frac{i}{\mu_j}\frac{d(A_i^1\mu_j)}{dt} = -J(A_{j+1}^0 + A_{j-1}^0)$, which has the solution

$$A_j^1(t) = A_j^0(t) + \frac{iJ}{\mu_j(t)}\int_0^t dt' \mu_j(t')[A_{j+1}^0(t') + A_{j-1}^0(t')]. \tag{S5}$$

Next, we express the Heisenberg equations of motion in terms of the probability distribution $p_j = |A_j|^2$

$$\frac{dp_j}{dt} = -2J\,\mathrm{Im}[A_j^* A_{j+1} + A_j^* A_{j-1}]. \tag{S6}$$

Plugging in the next-to-leading order result for the amplitudes A_j^1 and taking the average over the noise, we obtain the rate equation (3) for the probability distribution with the rates

$$\Gamma(\epsilon_i - \epsilon_j) = 2J^2\,\mathrm{Re}\langle \int_0^t dt' e^{-i[\phi_j(t) - i\phi_j(t')]} e^{i(\phi_i(t) - i\phi_i(t'))}\rangle = 2J^2 \int_0^t dt' \cos[(\epsilon_j - \epsilon_i)t']\,|C^\phi(t')|^2. \tag{S7}$$

Hence, in the asymptotic limit, $t \to \infty$, the rate is determined by the Fourier transform of the kernel $|C^\phi(t)|^2 = \exp\left[-2\int_0^t (t-x)C(x)dx\right]$ evaluated at the energy difference of the neighboring sites. We evaluate the rate $\Gamma(\omega)$ for our noise model, Eq. (2), which in the strong noise limit $\Lambda\tau \gtrsim 1$ yields Eq. (5). The rate thus exhibits an intermediate Gaussian regime that exists for large noise correlation times τ. This strong decay of the rate with frequency ω leads to bottlenecks and is the origin of the subdiffusive transport.

Subdiffusive transport

The strong decay of the rate $\Gamma(\omega)$ in the intermediate Gaussian regime leads to bottlenecks. We introduce a cutoff Γ_0 and define that rates that are smaller than Γ_0 realize bottlenecks and block transport

$$\Gamma(\omega) = \frac{2J^2}{\Lambda} e^{-\omega^2/4\Lambda^2} < \Gamma_0. \tag{S8}$$

Inverting this equation, we obtain a bound on the energy $|\omega| > 2\Lambda\sqrt{-\log\frac{\Lambda\Gamma_0}{2J^2}} \equiv 2\Lambda\sqrt{-\log\tilde{\Gamma}_0}$. We first consider that diffusion is initiated by resonant processes between nearest neighbor sites. Thus the frequency ω needs to be resonant with a random variable x drawn from the distribution of the nearest neighbor energy differences, which is a Gaussian of width $\sqrt{2}W$, where W is the local disorder strength: $N(x,\sqrt{2}W) = \frac{1}{\sqrt{\pi}2W}e^{-x^2/4W^2}$. The cumulative probability distribution of finding rates that are smaller than the cutoff is thus

$$P(\Gamma < \Gamma_0) = P(x > 2\Lambda\sqrt{-\log\tilde{\Gamma}_0}) = \int_{2\Lambda\sqrt{-\log\tilde{\Gamma}_0}}^{\infty} N(x,\sqrt{2}W)dx = \frac{1}{2}\,\mathrm{erfc}[\frac{\Lambda}{W}\sqrt{-\log\tilde{\Gamma}_0}]. \tag{S9}$$

In the asymptotic limit of small $\tilde{\Gamma}_0$ we approximate $\mathrm{erfc}[z] \sim \frac{\exp[-z^2]}{z\sqrt{\pi}}$ and hence find that the cumulative distribution function obeys (up to logarithmic corrections) a powerlaw

$$P(\Gamma < \Gamma_0) \sim e^{\frac{\Lambda^2}{W^2}\log\tilde{\Gamma}_0} \sim \tilde{\Gamma}_0^{\frac{\Lambda^2}{W^2}}. \tag{S10}$$

Interpreting the local transition rates as inverse resistors, we make an analogy with a random resistor network model and find subdiffusive transport when the exponent of $P(\Gamma < \Gamma_0)$ is less than one [27, 35]

$$\Lambda < W. \tag{S11}$$

In summary, we expect subdiffusion for $\Lambda < W < 2\Lambda\sqrt{\log\Lambda\tau}$. Thus, τ has to be large enough to enable this anomalous transport regime.

Crossover to Diffusion

Thus far we only considered hopping processes between nearest neighbors. However, once we find a small nearest-neighbor rate, it does not automatically mean that we do have global subdiffusion. Analogously to variable range hopping, we consider higher-order hopping processes to more distant neighbors which scale as $J(J/W)^{(n-1)}$. Only if none of these transition rates is large, the site can act as a bottleneck. Using the renormalized hopping, the transition rate at order n is given by $\Gamma_i^{(n)} \simeq \frac{2J^2}{\Lambda}\left(\frac{J}{W}\right)^{2(n-1)}\exp\left[-\frac{\omega^2}{4\Lambda^2}\right]$. The corresponding cumulative distribution function reads

$$P(\Gamma_i^{(n)} < \Gamma_0) = \left[\tilde{\Gamma}_0(W/J)^{2(n-1)}\right]^{\frac{\Lambda^2}{W^2}}. \tag{S12}$$

The probability of finding a series of such slow sites (taking them as independent processes) is

$$\tilde{P}(\Gamma_0|n^*) = \prod_{n=1}^{n^*}\left[\frac{\Gamma_0\Lambda}{2J^2}(W/J)^{2(n-1)}\right]^{\frac{\Lambda^2}{W^2}}, \tag{S13}$$

where n^* characterizes the distance beyond which all rates are small compared to Γ_0 by definition. We estimate this maximum distance by

$$\Gamma_0 = \frac{2J^2}{\Lambda}(J/W)^{2(n^*-1)}. \tag{S14}$$

Solving for n^* we obtain $n^* = \log(\Gamma_0\Lambda/2W^2)/2\log(J/W)$. Taking this maximal distance, the probability of finding a series of slow sites is

$$\tilde{P}(\Gamma_0|n^*) \simeq \left(\frac{\Lambda\Gamma_0}{2W^2}\right)^{-\Lambda^2/(2W^2)}\exp\left[-\frac{\Lambda^2}{4W^2\log W/J}\log^2\frac{\Gamma_0\Lambda}{2W^2}\right], \tag{S15}$$

which is decaying slightly faster than a powerlaw with $1/\Gamma_0$. Therefore, bottlenecks become ineffective at asymptotically late times and subdiffusive transport crosses over to diffusion.

We now estimate the diffusion constant, by computing the mean resistance and inverting it: via the Einstein relation, we can identify the dc conductance with the diffusion constant. Using the cumulative distribution function (S15) for sites with decay rates smaller than Γ_0, we proceed as follows. First, we note that the "resistance" R is identified with the inverse rate. Second, from Eq. (S15), we compute the probability density by computing the derivative of $\tilde{P}(\Gamma_0|n^*)$

$$p(R) = \frac{1}{R}\frac{\Lambda^2}{2W^2\log(W/J)}\left[\log\left(\frac{2W^2R}{\Lambda}\right) - \log\left(\frac{W}{J}\right)\right]\tilde{P}(1/R|n^*). \tag{S16}$$

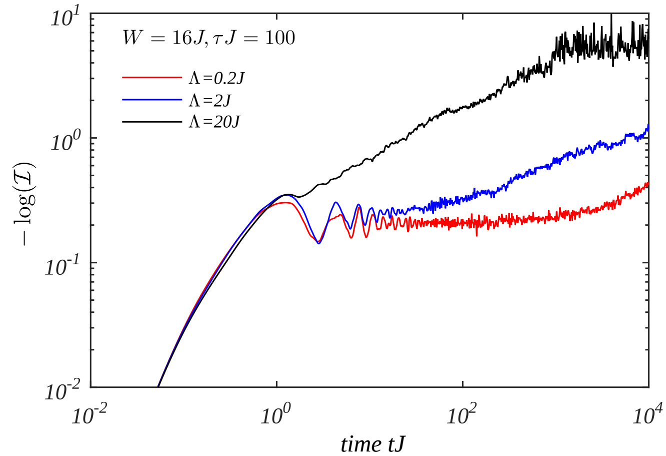

FIG. S1. **Stretched-exponential decay of the imbalance in a noisy environment.** The contrast of an initial density-wave pattern of occupied even and unoccupied odd lattice sites, denoted as imbalance $\mathcal{I}$, is shown for strong disorder $W = 16J$, large noise correlation times $\tau J = 100$ and three different values of the noise strength Λ. The asymptotic stretched exponential decay of the imbalance, Eq. (S20), can be inferred from plotting $-\log\mathcal{I}$ on a double logarithmic plot, in which the stretching exponent α can be directly read off from the slope of the linear growth at late times.

Using this distribution, we can estimate the mean resistance, which is given by

$$\langle R\rangle \simeq \frac{\sqrt{\pi \log[W/J]}}{W}\left(\frac{W}{J}\right)^{(W/\Lambda+\Lambda/2W)^2} \tag{S17}$$

from which it follows that the asymptotic diffusion coefficient is given (for large W/Λ) by

$$D_{\mathrm{VRH}} \sim \frac{W}{\sqrt{\pi \log[W/J]}}\left(\frac{J}{W}\right)^{W^2/\Lambda^2}. \tag{S18}$$

This expression only applies when $W > \Lambda$, and is only *controlled* when $W \gg \Lambda$. In the opposite limit, $W \ll \Lambda$, one gets diffusion even from incoherent single-site hopping. The diffusion constant in that regime can be found by computing the average resistance due to lowest-order hops, Eq. (S10), which leads to the result

$$D_{\text{single-hop}} \sim \frac{2J^2}{\Lambda}(1 - W^2/\Lambda^2), \tag{S19}$$

i.e., it vanishes as $\Lambda \to W$, and then crosses over to the VRH form above.

Imbalance

Many-body localized systems coupled to a Markovian bath have been shown to exhibit a large distribution of relaxation rates, which manifests itself in an asymptotic stretched exponential decay of the imbalance $\mathcal{I}$ of an initial charge density wave pattern of occupied even and unoccupied odd sites [21–23]

$$\mathcal{I}(t\to\infty) = \exp\left[-(t/\tau)^{\alpha}\right], \tag{S20}$$

where α is the stretching exponent. This quantity has been thoroughly investigated theoretically, since it has been used in experiments to establish the many-body localized phase [9, 13]. Here, we show that also for non-interacting systems in a noisy environment the imbalance decays as a stretched exponential, Fig. S1, which is best demonstrated by plotting $-\log\mathcal{I}$ on double logarithmic scales. In such a plot the stretching exponent α can directly be read off from the slope of the linear curve at late times. In the weak noise limit $\Lambda = 0.2J$ the imbalance remains constant up to late times $tJ \sim 10^3$ and then crosses over to a stretched-exponential decay. By contrast, in the strong noise limit $\Lambda = 20J$, the intermediate time plateau ceases to exist and

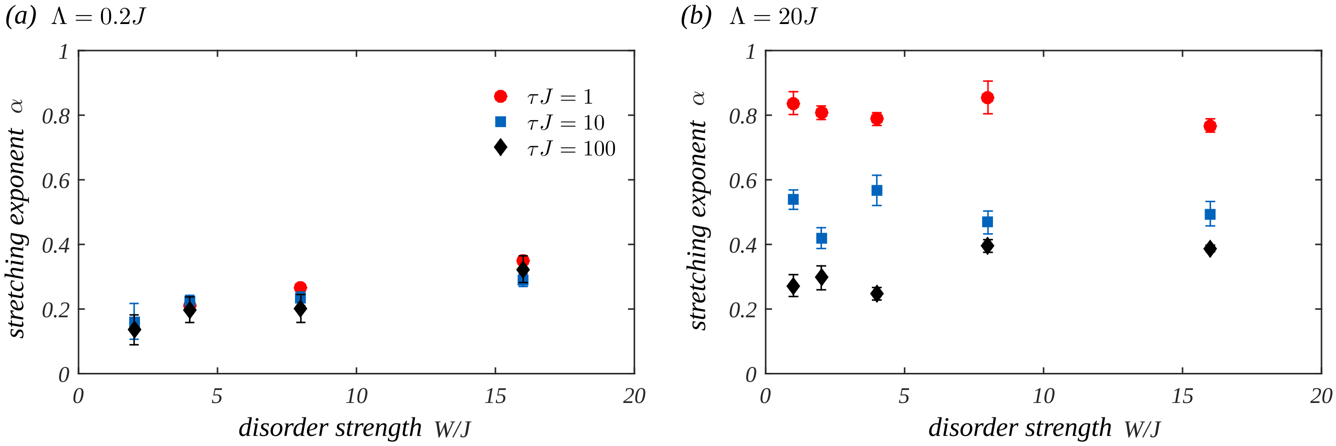

FIG. S2. **Stretching exponent of the imbalance.** The stretching exponent α is shown (a) in the weak noise limit $\Lambda = 0.2J$ and (b) in the strong noise limit $\Lambda = 20J$. For weak noise, the exponent does not depend on the noise correlation time τ but depends weakly on the disorder strength W. By contrast, for strong noise, the stretching exponent is very sensitive to the noise correlation time τ. For short correlation time $\tau J = 1$ the stretching exponent is close to one, indicating a nearly exponential decay of the imbalance $\mathcal{I}$.

after an initial decay on the single-particle timescale, the imbalance immediately turns to a stretched exponential. In the strong noise limit $\Lambda = 20\tau$, the curve saturates at late times which we attribute to the fact that the data hits the sample noise floor, as in this regime the imbalance is already $\mathcal{I} \lesssim 10^{-4}$.

We extract the stretching exponent α for a broad range of parameters, Fig. S2, and find that α is insensitive to the noise correlation time τ in the weak noise limit $\Lambda = 0.2J$ (a) but depends strongly on the noise correlation time for strong noise $\Lambda = 20J$ (b). In the latter regime the stretching exponent α approaches values near one for fast noise $\tau J = 1$, indicating an almost exponential decay, whereas for slow noise $\tau J = 100$, it remains appreciably smaller than one. Such a dependence of the stretching exponent on the noise correlation time cannot be studied in a Lindblad formalism [21–23], which assumes a Markovian bath with vanishing noise correlation times $\tau \to 0$.