Supplemental Material:
Noise-induced subdiffusion in strongly localized quantum systems

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CONTENTS

S1. PERTURBATIVE TREATMENT

We discuss how to establish analytical insights in the noise-induced dynamics by a perturbative treatment in small hopping $J \ll \Lambda, W$. The equations of motion for the annihilation operator c_j set by Hamiltonian Eq. (1) read

$$i\frac{dc_j}{dt} = -J(c_{j-1} + c_{j+1}) + [\epsilon_j + \xi_j(t)]c_j. \tag{S1}$$

We solve these equations order by order in the hopping J [1]. In the absence of interactions we can represent the quantum operator c_j by a complex amplitude A_j. The dynamics of the wave function amplitude to leading order A_j^0 is determined by

$$i\frac{dA_j^0}{dt} = [\epsilon_j + \xi_j(t)]A_j^0, \tag{S2}$$

which describes the accumulation of phase

$$A_j^0(t) = A_j^0 e^{-i\epsilon_j t - i\int_0^t \xi_j(t')dt'} = A_j^0 e^{-i\epsilon_j t} e^{-i\phi_j(t)}. \tag{S3}$$

To leading order transport is absent. However, it is restored by evaluating the next-to-leading order correction

$$i\frac{dA_i^1}{dt} - [\epsilon_i + \xi_i(t)]A_i^1 = -J(A_{j+1}^0 + A_{j-1}^0). \tag{S4}$$

Introducing $\mu_j = e^{i\phi_j(t)}$, we rewrite the equation as $\frac{i}{\mu_j}\frac{d(A_i^1\mu_j)}{dt} = -J(A_{j+1}^0 + A_{j-1}^0)$, which has the solution

$$A_j^1(t) = A_j^0(t) + \frac{iJ}{\mu_j(t)}\int_0^t dt' \mu_j(t')[A_{j+1}^0(t') + A_{j-1}^0(t')]. \tag{S5}$$

Next, we express the Heisenberg equations of motion in terms of the probability distribution $p_j = |A_j|^2$

$$\frac{dp_j}{dt} = -2J\,\mathrm{Im}[A_j^* A_{j+1} + A_j^* A_{j-1}]. \tag{S6}$$

Plugging in the next-to-leading order result for the amplitudes A_j^1 and taking the average over the noise, we obtain the rate equation (3) for the probability distribution with the rates

$$\Gamma(\epsilon_i-\epsilon_j)=2J^2\,\mathrm{Re}\langle\int_0^t dt' e^{-i[\phi_j(t)-i\phi_j(t')]}e^{i(\phi_i(t)-i\phi_i(t'))}\rangle=2J^2\int_0^t dt'\cos[(\epsilon_j-\epsilon_i)t']\,|C^{\phi}(t')|^2. \tag{S7}$$

Hence, in the asymptotic limit, $t\to\infty$, the rate is determined by the Fourier transform of the kernel $|C^{\phi}(t)|^2=\exp\left[-2\int_0^t(t-x)C(x)dx\right]$ evaluated at the energy difference of the neighboring sites. We evaluate the rate $\Gamma(\omega)$ for our noise model, Eq. (2), which in the strong noise limit $\Lambda\tau\gtrsim 1$ yields Eq. (5). The rate thus exhibits an intermediate Gaussian regime that exists for large noise correlation times τ. This strong decay of the rate with frequency ω leads to bottlenecks and is the origin of the subdiffusive transport.

S2. SUBDIFFUSIVE TRANSPORT

The strong decay of the rate $\Gamma(\omega)$ in the intermediate Gaussian regime leads to bottlenecks. We introduce a cutoff Γ_0 and define that rates that are smaller than Γ_0 realize bottlenecks and block transport

$$\Gamma(\omega)=\frac{2J^2}{\Lambda}e^{-\omega^2/4\Lambda^2}<\Gamma_0. \tag{S8}$$

Inverting this equation, we obtain a bound on the energy $|\omega|>2\Lambda\sqrt{-\log\frac{\Lambda\Gamma_0}{2J^2}}\equiv 2\Lambda\sqrt{-\log\tilde{\Gamma}_0}$. We first consider that diffusion is initiated by resonant processes between nearest neighbor sites. Thus the frequency ω needs to be resonant with a random variable x drawn from the distribution of the nearest neighbor energy differences, which is a Gaussian of width $\sqrt{2}W$, where W is the local disorder strength: $N(x,\sqrt{2}W)=\frac{1}{\sqrt{\pi}2W}e^{-x^2/4W^2}$. The cumulative probability distribution of finding rates that are smaller than the cutoff is thus

$$P(\Gamma<\Gamma_0)=P(x>2\Lambda\sqrt{-\log\tilde{\Gamma}_0})=\int_{2\Lambda\sqrt{-\log\tilde{\Gamma}_0}}^{\infty}N(x,\sqrt{2}W)dx=\frac{1}{2}\,\mathrm{erfc}[\frac{\Lambda}{W}\sqrt{-\log\tilde{\Gamma}_0}]. \tag{S9}$$

In the asymptotic limit of small $\tilde{\Gamma}_0$ we approximate $\mathrm{erfc}[z]\sim\frac{\exp[-z^2]}{z\sqrt{\pi}}$ and hence find that the cumulative distribution function obeys (up to logarithmic corrections) a powerlaw

$$P(\Gamma<\Gamma_0)\sim e^{\frac{\Lambda^2}{W^2}\log\tilde{\Gamma}_0}\sim\tilde{\Gamma}_0^{\frac{\Lambda^2}{W^2}}. \tag{S10}$$

Interpreting the local transition rates as inverse resistors, we make an analogy with a random resistor network model and find subdiffusive transport when the exponent of $P(\Gamma<\Gamma_0)$ is less than one [2, 3]

$$\Lambda<W. \tag{S11}$$

In summary, we expect subdiffusion for $\Lambda<W<2\Lambda\sqrt{\log\Lambda\tau}$. Thus, τ has to be large enough to enable this anomalous transport regime.

S3. CROSSOVER TO DIFFUSION

Thus far we only considered hopping processes between nearest neighbors. However, once we find a small nearest-neighbor rate, it does not automatically mean that we do have global subdiffusion. Analogously to variable range hopping, we consider higher-order hopping processes to more distant neighbors which scale as $J(J/W)^{(n-1)}$. Only if none of these transition rates is large, the site can act as a bottleneck. Using the renormalized hopping, the transition rate at order n is given by $\Gamma_i^{(n)}\simeq\frac{2J^2}{\Lambda}\left(\frac{J}{W}\right)^{2(n-1)}\exp\left[-\frac{\omega^2}{4\Lambda^2}\right]$. The corresponding cumulative distribution function reads

$$P(\Gamma_i^{(n)}<\Gamma_0)=\left[\tilde{\Gamma}_0(W/J)^{2(n-1)}\right]^{\frac{\Lambda^2}{W^2}}. \tag{S12}$$

The probability of finding a series of such slow sites (taking them as independent processes) is

$$\tilde{P}(\Gamma_0|n^*)=\prod_{n=1}^{n^*}\left[\frac{\Gamma_0\Lambda}{2J^2}(W/J)^{2(n-1)}\right]^{\frac{\Lambda^2}{W^2}}, \tag{S13}$$

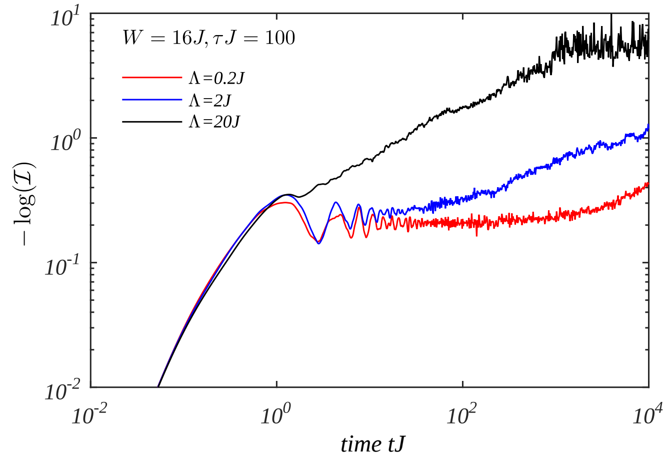

FIG. S1. **Stretched-exponential decay of the imbalance in a noisy environment.** The contrast of an initial density-wave pattern of occupied even and unoccupied odd lattice sites, denoted as imbalance $\mathcal{I}$, is shown for strong disorder $W = 16J$, large noise correlation times $\tau J = 100$ and three different values of the noise strength Λ. The asymptotic stretched exponential decay of the imbalance, Eq. (S20), can be inferred from plotting $-\log \mathcal{I}$ on a double logarithmic plot, in which the stretching exponent α can be directly read off from the slope of the linear growth at late times.

where n^* characterizes the distance beyond which all rates are small compared to Γ_0 by definition. We estimate this maximum distance by

$$\Gamma_0 = \frac{2J^2}{\Lambda}(J/W)^{2(n^*-1)}. \tag{S14}$$

Solving for n^* we obtain $n^* = \log(\Gamma_0\Lambda/2W^2)/2\log(J/W)$. Taking this maximal distance, the probability of finding a series of slow sites is

$$\tilde{P}(\Gamma_0|n^*) \simeq \left(\frac{\Lambda\Gamma_0}{2W^2}\right)^{-\Lambda^2/(2W^2)} \exp\left[-\frac{\Lambda^2}{4W^2\log W/J}\log^2\frac{\Gamma_0\Lambda}{2W^2}\right], \tag{S15}$$

which is decaying slightly faster than a powerlaw with $1/\Gamma_0$. Therefore, bottlenecks become ineffective at asymptotically late times and subdiffusive transport crosses over to diffusion.

We now estimate the diffusion constant, by computing the mean resistance and inverting it: via the Einstein relation, we can identify the dc conductance with the diffusion constant. Using the cumulative distribution function (S15) for sites with decay rates smaller than Γ_0, we proceed as follows. First, we note that the "resistance" R is identified with the inverse rate. Second, from Eq. (S15), we compute the probability density by computing the derivative of $\tilde{P}(\Gamma_0|n^*)$

$$p(R) = \frac{1}{R}\frac{\Lambda^2}{2W^2\log(W/J)}\left[\log\left(\frac{2W^2R}{\Lambda}\right) - \log\left(\frac{W}{J}\right)\right]\tilde{P}(1/R|n^*). \tag{S16}$$

Using this distribution, we can estimate the mean resistance, which is given by

$$\langle R\rangle \simeq \frac{\sqrt{\pi\log[W/J]}}{W}\left(\frac{W}{J}\right)^{(W/\Lambda+\Lambda/2W)^2} \tag{S17}$$

from which it follows that the asymptotic diffusion coefficient is given (for large W/Λ) by

$$D_{\text{VRH}} \sim \frac{W}{\sqrt{\pi\log[W/J]}}\left(\frac{J}{W}\right)^{W^2/\Lambda^2}. \tag{S18}$$

This expression only applies when $W > \Lambda$, and is only *controlled* when $W \gg \Lambda$. In the opposite limit, $W \ll \Lambda$, one gets diffusion even from incoherent single-site hopping. The diffusion constant in that regime can be found by computing the average resistance due to lowest-order hops, Eq. (S10), which leads to the result

$$D_{\text{single-hop}} \sim \frac{2J^2}{\Lambda}(1 - W^2/\Lambda^2), \tag{S19}$$

i.e., it vanishes as $\Lambda \to W$, and then crosses over to the VRH form above.

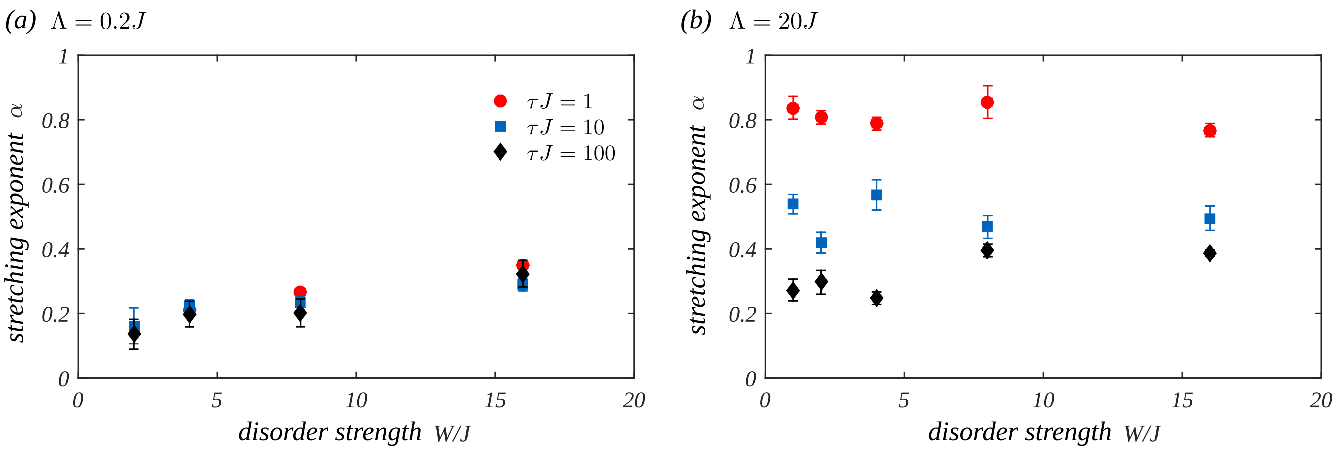

FIG. S2. **Stretching exponent of the imbalance.** The stretching exponent α is shown (a) in the weak noise limit $\Lambda = 0.2J$ and (b) in the strong noise limit $\Lambda = 20J$. For weak noise, the exponent does not depend on the noise correlation time τ but depends weakly on the disorder strength W. By contrast, for strong noise, the stretching exponent is very sensitive to the noise correlation time τ. For short correlation time $\tau J = 1$ the stretching exponent is close to one, indicating a nearly exponential decay of the imbalance $\mathcal{I}$.

S4. IMBALANCE

Many-body localized systems coupled to a Markovian bath have been shown to exhibit a large distribution of relaxation rates, which manifests itself in an asymptotic stretched exponential decay of the imbalance $\mathcal{I}$ of an initial charge density wave pattern of occupied even and unoccupied odd sites [4–6]

$$\mathcal{I}(t \to \infty) = \exp\left[-(t/\tau)^{\alpha}\right], \tag{S20}$$

where α is the stretching exponent. This quantity has been thoroughly investigated theoretically, since it has been used in experiments to establish the many-body localized phase [7, 8]. Here, we show that also for non-interacting systems in a noisy environment the imbalance decays as a stretched exponential, Fig. S1, which is best demonstrated by plotting $-\log \mathcal{I}$ on double logarithmic scales. In such a plot the stretching exponent α can directly be read off from the slope of the linear curve at late times. In the weak noise limit $\Lambda = 0.2J$ the imbalance remains constant up to late times $tJ \sim 10^3$ and then crosses over to a stretched-exponential decay. By contrast, in the strong noise limit $\Lambda = 20J$, the intermediate time plateau ceases to exist and after an initial decay on the single-particle timescale, the imbalance immediately turns to a stretched exponential. In the strong noise limit $\Lambda = 20\tau$, the curve saturates at late times which we attribute to the fact that the data hits the sample noise floor, as in this regime the imbalance is already $\mathcal{I} \lesssim 10^{-4}$.

We extract the stretching exponent α for a broad range of parameters, Fig. S2, and find that α is insensitive to the noise correlation time τ in the weak noise limit $\Lambda = 0.2J$ (a) but depends strongly on the noise correlation time for strong noise $\Lambda = 20J$ (b). In the latter regime the stretching exponent α approaches values near one for fast noise $\tau J = 1$, indicating an almost exponential decay, whereas for slow noise $\tau J = 100$, it remains appreciably smaller than one. Such a dependence of the stretching exponent on the noise correlation time cannot be studied in a Lindblad formalism [4–6], which assumes a Markovian bath with vanishing noise correlation times $\tau \to 0$.

S5. NOISE-INDUCED DYNAMICS IN THE MANY-BODY LOCALIZED PHASE

To study the noise-induced dynamics in the many-body localized phase, we consider disordered and interacting electrons

$$H = -\frac{J}{2}\sum_i (c_i^\dagger c_{i+1} + \text{h.c.}) + U\sum_i \hat{n}_i\hat{n}_{i+1} + \sum_i [\epsilon_i + \xi_i(t)]\hat{n}_i. \tag{S21}$$

Except for the second term, which describes the electron-electron interactions of strength U and a trivial rescaling by a factor $1/2$, the Hamiltonian is identical to Eq. (1) in the main text. We solve the quantum dynamics using Lanczos time evolution and update the Hamiltonian at each time step with a new spatial noise profile, sampled from an Ornstein-Uhlenbeck process. The initial state $|\tilde{\psi}\rangle$ is a random product state drawn the Haar measure. We polarize the initial state by applying the operator $\mathcal{P} = (|1\rangle\langle 1|)_{L/2}$ to the random state $|\psi\rangle = \mathcal{P}|\tilde{\psi}\rangle$. Starting with $|\psi\rangle$ we compute the time evolution of the system using

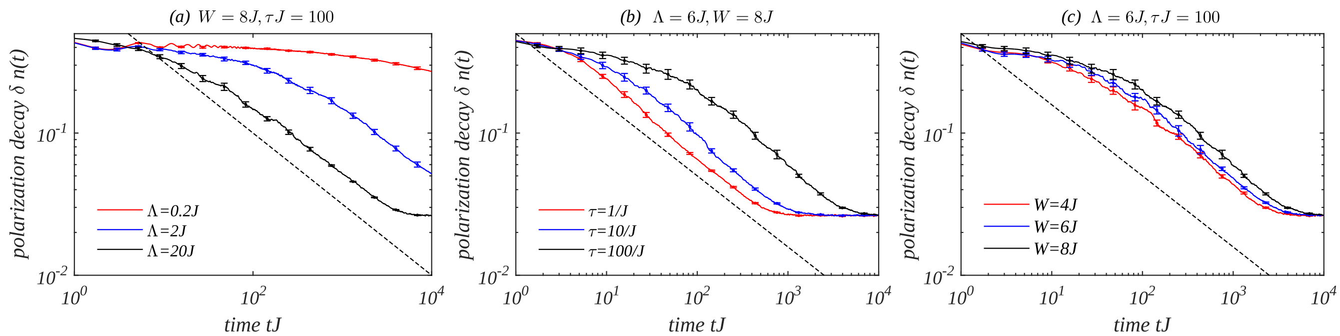

FIG. S3. **Noise-induced polarization decay for interacting and disordered fermions.** We numerically simulate the polarization decay $\delta n(t)$ for systems of size $L = 19$, $N = 10$ particles, interactions $U = -J$ for a broad range of disorder strength W, noise strength Λ, and noise correlation times τ, see legends. At late times and for strong enough noise the polarization decay crosses over to diffusion $\delta n(t) \sim 1/\sqrt{t}$ (dashed black line).

Lanczos algorithm $|\psi(t)\rangle = \mathcal{T} \exp[-i \int_0^t H(t')dt'] \, |\psi\rangle$ and measure the polarization decay in the center of the system: $\delta n(t) = \langle \psi(t)|\hat{n}_{L/2}|\psi(t)\rangle - n_0$, where n_0 is the static expectation value of the density that is determined by the total particle number which is conserved in our model.

Results of the polarization decay $\Delta n(t)$ for systems of size $L = 19$ with $N = 10$ particles to times $tJ = 10^4$ and interaction strength $U = -J$ are shown in Fig. S3 for a broad range of parameters. In order to minimize finite size effects, we consider systems with an odd number of sites. Otherwise, there would be a dangling particle in the surrounding of the initially polarized site, which leads to a late-time saturation plateau that deviates from the respective filling by a correction $\sim 1/L$.

In the absence of noise, the system is in the many-body localized phase for the chosen parameters. However, our data shows that noise inevitably induces delocalization. For strong noise $\Lambda = 20J$, Fig. S3 (a), we find that the system quickly approaches diffusive dynamics, as described by a $1/\sqrt{t}$ decay of $\delta n(t)$. At very late times, $tJ \sim 10^4$, the response saturates to a finite value which is a consequence of the finite system size. For weaker noise, $\Lambda = 2J$, it takes the system longer to approach the diffusive regime, and for extremely weak noise $\Lambda = 0.2J$ the polarization has almost not decayed on the simulated time scales. These numerical findings, are in agreement with our expectations discussed in the main text. In contrast to the non-interacting system, the crossover from subdiffusion to diffusion occurs more gradually, resulting from the many possible decay channels enabled by multi-particle rearrangements that are not allowed in the absence of interactions. Yet, at late times, diffusion sets the dynamics.

The polarization decay $\delta n(t)$, is shown in Fig. S3 (b) for fixed noise strength $\Lambda = 6J$ and disorder $W = 8J$, for different noise correlation time τ. After some initial dynamics, the polarization decay approaches the diffusive $1/\sqrt{t}$ limit. Finally, for fixed noise strength $\Lambda = 6J$, and noise correlation time $\tau J = 100$, (c), the system crosses over to diffusion irrespective of the disorder strength.

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