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Choosing with the Worst in Mind: A Reference-Dependent Model

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Abstract

We develop an axiomatic model of reference-dependent preferences in which reference points are menu-dependent. In particular, we focus on choices from menus of two-attribute alternatives, and the reference point for the given menu is a vector that consists of the minimums of each dimension of the menu. We characterize this model by two weakenings of the Weak Axiom of Revealed Preference (WARP) in addition to standard axioms. Our model is consistent with the *attraction effect* and the *compromise effect*, well-known deviations from rational choice theory, and it provides a connection between the two effects and *diminishing sensitivity*, a widely used behavioral property in economics. We apply the model to two different contexts, intertemporal choice and risky choice, and diminishing sensitivity has interesting implications. In intertemporal choice, the main implication of the model is that borrowing constraints produce a psychological pressure to move away from the constraints even if they are not binding. This implication provides a rationale for two empirical puzzles of life-cycle consumption profiles. In risky choice, the model allows different degrees of risk-aversion.

Keywords: Reference-Dependence; Menu-Dependent Reference; WARP; Attraction Effect; Compromise Effect; Diminishing Sensitivity; Intertemporal Choice; Risky Choice.

JEL Classification Numbers: D01, D81, D91.

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1 Introduction

Rational choice theory does not allow a *preference reversal* which is a situation that an agent prefers alternative a over alternative b in some cases, but she prefers b over a in other cases. However, many experimental, marketing, and field studies suggest that the presence of an irrelevant third alternative could cause a preference reversal. Preference reversals are typically explained by models of reference-dependent behavior in which the agent’s basis for decision making, i.e., a *reference point*, changes depending on the situations that she faces. In this paper, we focus on two well-known preference reversals, the *compromise effect* and the *attraction effect* (defined below), and develop an axiomatic model of reference-dependent preferences that is consistent with them.

Reference-dependent behavior is first described by Kahneman and Tversky (1979) in the context of risky choice. Later, Tversky and Kahneman (1991) provided the first explicit model of reference-dependent preferences on riskless choice by extending Kahneman and Tversky (1979). In their model, they treated reference points as exogenous¹ and explained many experimental findings that the standard model cannot accommodate using the following two properties on preferences: i) people are more sensitive to losses than gains (*loss aversion*) and ii) the marginal values of both gains and losses decrease with their distance from the reference point (*diminishing sensitivity*). Despite its explanatory power, models of reference-dependent preferences with exogenous reference points can be consistent with essentially any choice behavior by freely choosing reference points. In order to have a model of reference-dependent preferences that makes testable predictions for observed behavior, we need to endogenize (explicitly model) reference points in a way that they are determined by observable factors. We propose a model in which reference points are determined by the menu that the agent faces in a very specific manner and explain the aforementioned preference reversals using the diminishing sensitivity property alone.

Before introducing our modelling approach, let us discuss the *compromise effect* and the *attraction effect*, two well-documented deviations of the standard choice theory.² We focus on alternatives with two attributes. Take two alternatives (x, p) and (y, q) such that $x > y$ and $p < q$ and suppose (x, p) is preferred over (y, q) in the binary comparison.³ The effects

¹Exogenous in sense that reference points are not explicitly modeled and can be chosen freely without any restrictions in order to explain experimental findings.

²The compromise effect and the attraction effect are first documented in the experimental study of Simonson (1989) and Huber et al. (1982), respectively and confirmed by many studies in consumer choice (e.g., Simonson and Tversky (1992), Tversky and Simonson (1993), Doyle et al. (1999), Chernev (2004), Sharpe et al. (2008)). These effects are also demonstrated in the contexts of choice over risky alternatives (Herne (1999)), choice over policy issues Herne (1997), choice over political candidates (Sue O’Curry and Pitts (1995)), among others.

³For example, the first dimension represents quality of good and the second dimension represents price

Figure 1: Compromise and Attraction Effects

relate to the consequences of adding a third alternative which has z in its the first dimension where $x > y > z$ (Figure 1).

In the compromise effect, the introduction of a third alternative (z, r) with $r > q$ into the menu $\{(x, p), (y, q)\}$ reverses the original binary comparison; that is, (y, q) is preferred over (x, p) , because the third alternative makes (x, p) an extreme alternative and people have a tendency to avoid extremes (the left hand side of Figure 1).⁴

In the attraction effect⁵, the introduction of a third alternative (z, t) with $q > t > p$ into the binary menu reverses binary comparison because the third alternative makes (y, q) more attractive since (z, t) is dominated by (y, q) , but not by (x, p) (the right hand side of Figure 1).

Do these two effects have similarities? Yes. If we look at the ranges of the two attributes, in the compromise effect, $[y, x] \times [p, q]$ changes to $[z, x] \times [p, r]$, but in the attraction effect, $[y, x] \times [p, q]$ changes to $[z, x] \times [p, q]$. In both effects, the range of the first attribute $[y, x]$ changes to $[z, x]$. Then note that both effects can be rationalized by the following rationale: *adding z hurts x over y* . In our model, this rationale will be an implication of the aforementioned diminishing sensitivity property of Tversky and Kahneman (1991). More specifically, we develop a model in which the reference point for the given menu is a vector that consists of the minimum of each dimension of the menu. Therefore, since adding z decreases the minimum of the first dimension of the binary menu $\{(x, p), (y, q)\}$, the marginal value of x over y decreases by diminishing sensitivity. In other words, z hurts x over y . We call the

of good. In that case, (x, p) is an expensive high quality good, while (y, q) is a cheap low quality good.

⁴Note that x is the largest among x, y , and z , while p is the smallest among p, q , and r .

⁵It is sometimes called the asymmetric dominance effect

above rationale as *weak diminishing sensitivity* and it not only rationalizes the two effects, but also has two additional implications consistent with results of lab experiments.

We focus on a choice theoretic environment in which an agent makes choices from menus of two-attribute alternatives. To introduce our model, we use the following usual used formulation of Tversky and Kahneman (1991): the utility of a two-attribute alternative (x, p) for given reference point $\mathbf{r} = (r_1, r_2)$ is

$$(1) \quad U((x, p)|\mathbf{r}) = f(u(x) - u(r_1)) + g(w(p) - w(r_2)).$$

where f , g , u , and w are strictly increasing functions. Now we turn to our model which is a special case of (1). We specify (1) in the following two ways. First, we explicitly model reference points in order to obtain predictive power. In particular, the agent uses a vector that consists of the minimum of each dimension of the menu as a reference point; that is, (x^A, p^A) is the reference point of menu A where x^A is the minimum of the first dimension of A and p^A is the minimum of the second dimension of A . Second, we also require $f = g$ in order to rule out violations of transitivity on binary comparisons.⁶ Then in our model the utility of (x, p) in menu A is

$$U_A(x, p) = f(u(x) - u(x^A)) + f(w(p) - w(p^A)).$$

We take this particular approach using the minimums as references for several reasons. First, as we discussed earlier, in order to be consistent with the two effects, the reference point should decrease as the range of the first dimension $[y, x]$ changes to $[z, x]$. Taking the minimums as references is consistent with that property since the minimum of the first dimension decreases y to z as the range $[y, x]$ changes to $[z, x]$. Second, if we consider more general models in which reference points are determined by several factors in the menu, then the predictive power of the model will be much weaker. In particular, if we allow any kind of menu-dependence for reference points, then the model will not have any testable restriction on observed behavior (Section 5). So we need to restrict our attention to a particular kind of menu-dependence such as the minimums of menu.⁷ Lastly, our model is one of the most restrictive specifications of (1) because it has only two additional parameters f and (x^A, p^A) compared to the standard additive utility model. Therefore, our model is restrictive enough

⁶In the Appendix C, we weaken transitivity and obtain a general representation with $f \neq g$.

⁷Moreover, it is likely that references are increasing in the minimums in order to rationalize the two effects (e.g., a model in which references are determined by only maximums cannot be consistent with the attraction effect.) The behavior of such general models could be captured by the model that has only the minimums. For example, in Appendix B, we discuss a model with reference points which depend on both minimums and maximums. Although some predictions become weaker than using only minimums, we still obtain very similar results.

to make strong predictions (Section 4) but flexible enough to capture deviations of the standard choice theory (Section 2).

One of the main contributions of the paper is that we axiomatically characterize the model by two novel axioms called *Independence from Non-Extreme Alternatives* (INEA) and *Reference Translation Invariance* (RTI) in addition to standard axioms. Both INEA and RTI are weakenings of the Weak Axiom of Revealed Preference (WARP). INEA simply requires that if a non-extreme alternative (i.e., alternative which does not include minimums) is not chosen from a menu, then eliminating that alternative from the menu does not change the choice. RTI requires that if we can obtain a triple x', y' , and z' from a triple x, y , and z by some distance preserving shift (to be define later), then the relative advantage of x over y in the presence of z is equal to that of x' over y' in the presence of z' .⁸

We also behaviorally characterize both diminishing sensitivity and weak diminishing sensitivity. In particular, we show that weak diminishing sensitivity is equivalent to observing the attraction effect in some neighborhood while diminishing sensitivity is equivalent to observing the attraction effect in smaller neighborhood (Section 3.2). This provides a further connection between diminishing sensitivity and the two effects.

We apply the model to two different contexts in Section 4: intertemporal choice and risky choice. In intertemporal choice, the main implication of the model is that borrowing constraints produce a psychological pressure to move away from the constraints even if they are not binding. To illustrate the intuition, consider a model with two periods, and suppose there is a constraint in today. First, the constraint decreases the maximum level for today's consumption. Assuming that a consumer consumes her total discounted income, the minimum level for tomorrow's consumption will increase. As the reference for tomorrow increases, by diminishing sensitivity, the consumer increases tomorrow's consumption and decreases today's consumption. This implication helps us to explain two empirical observations of life-cycle consumption profiles: *excess sensitivity of consumption to income* and a *hump-shaped consumption profile* (see Attanasio and Weber (2010)). It is usual to explain deviations of the standard model of consumer choice (both consumption and financial) by liquidity constraints. But empirically it is inconclusive whether constraints were binding at the time of decision. Our model gives a justification for those explanations with constraints since liquidity constraints can have an effect on choices even if they are not binding. In risky choice, our model allows different degrees of risk-aversion (in a sense of EUT) without relying on loss aversion. In both applications, the driving force of the results is diminishing sensitivity.

There are many models of reference-dependent preferences and we will discuss them in

⁸It will be clear that RTI is an implication of WARP after we formally define relative distances.

the related literature. However, to the best of our knowledge, Ok et al. (2014) is the only paper that axiomatically developed a model with purely endogenous reference points. They also weakened WARP and can accommodate the attraction effect. The main difference is that we explicitly modelled reference points while they identify reference points from observed choices in the spirit of classical revealed preference analysis. The explicit modelling gives us more flexibility to apply our model in many different contexts and to use diminishing sensitivity to draw some testable predictions. Moreover, they cannot capture the compromise effect because in their model an alternative that causes a preference reversal (e.g., (z, r) of $\{(x, p), (y, q), (z, r)\}$ in the compromise effect) should be unambiguously worse than the chosen alternative $((y, q)$ of $\{(x, p), (y, q), (z, r)\}$).⁹

The paper is organized as follows. In Section 2, we formally define the model and diminishing sensitivity. We demonstrate that our model is not only consistent with the compromise and attraction effects, but it also connects the two effects by weak diminishing sensitivity. Section 3 discusses behavioral foundation: axioms and the main representation theorem. Section 4 discusses the applications to intertemporal and risky choice. Section 5 shows that models with a general menu-dependent reference points have no testable implications on observed behavior. Section 6 discusses the related literature. Section 7 concludes the paper by discussing the main contributions and proofs and two extensions of the model are collected in the Appendixes.

2 Model

Let $\mathcal{X} \equiv \mathbb{R}_+^2$ be the set of all alternatives and $\mathcal{A} \subseteq 2^{\mathcal{X}} \setminus \{\emptyset\}$ be a collection of compact subsets (menus) of $\mathcal{X}$. Suppose $\mathcal{A}$ includes all doubleton and tripleton sets; that is, for any $a, b, c \in \mathcal{X}$, we have $\{a, b\}, \{a, b, c\} \in \mathcal{A}$. We denote generic members of $\mathcal{X}$ by $(x, p), (y, q)$ and (z, r) where x, y, z are for the first dimension and p, q, r are for the second dimension. We write $(x_1, x_2) > (y_1, y_2)$ if $(x_1, x_2) \neq (y_1, y_2)$, $x_1 \geq y_1$, and $x_2 \geq y_2$, and $(x_1, x_2) \gg (y_1, y_2)$ if $x_1 > y_1$ and $x_2 > y_2$.

Let $(x^A, p^A) \equiv (\min_{(x,p) \in A} x, \min_{(x,p) \in A} p)$ for each $A \in \mathcal{A}$ (i.e., the meet of A). The primitive is a choice correspondence $C: \mathcal{A} \rightrightarrows \mathcal{X}$ where $C(A)$ is non-empty and $C(A) \subseteq A$ for each $A \in \mathcal{A}$.

Definition 1. A choice correspondence C is an *additive reference dependent choice* (ARDC)

⁹See Proposition 2 of Ok et al. (2014).

if there are strictly increasing functions f, u, w such that $f(0) = 0$ and for any menu $A \in \mathcal{A}$,

$$(2) \quad C(A) = \arg \max_{(x,p) \in A} \{f(u(x) - u(x^A)) + f(w(p) - w(p^A))\}.$$

Denote an ARDC with functions f, u, w by $C_{(f,u,w)}$. Note that when f is linear, ARDC reduces to the standard additive utility model.

Now we define two preference relations for given C . The first preference $\succeq$ represents binary comparisons. For example, if a is chosen over b from $\{a, b\}$, then we have $a \succ b$. The second preference defined from choices on triplexton menus. For example, for given c , if a is chosen over b in the presence of d , then we have $a \succ_d b$. More formally,

Definition 2 (Induced Preferences). For any $a, b, d \in \mathcal{X}$,

- i) $a \succeq b$ if and only if $\{a\} \in C(\{a, b\})$
- ii) $a \succeq_d b$ if and only if $\{a\} \in C(\{a, b, d\})$

Note that for any ARDC $C_{(f,u,w)}$, $\succeq$ satisfies transitivity because for any $(x, p), (y, q) \in \mathcal{X}$ with $x > y$ and $p < q$, since the reference point of $\{(x, p), (y, q)\}$ is (y, p) , we have

$$\begin{aligned} (x, p) \succeq (y, q) &\Leftrightarrow f(u(x) - u(y)) + \cancel{f(w(p) - w(p))} \geq \cancel{f(u(y) - u(y))} + f(w(q) - w(p)) \\ &\Leftrightarrow u(x) + w(p) \geq u(y) + w(q). \end{aligned}$$

2.1 Diminishing Sensitivity

Diminishing sensitivity is a widely used behavioral property in economics and will be a driving force of our results. We define it in terms of functional properties on f as in Tversky and Kahneman (1991). It requires that the marginal increase of utility is decreasing in the distance from reference point. Formally,

Definition 3 (Diminishing Sensitivity). An ARDC $C_{(f,u,w)}$ satisfies *diminishing sensitivity* if for any x, y, r_1, r'_1 such that $x > y$ and $r_1 > r'_1$,

$$(3) \quad f(u(x) - u(r_1)) - f(u(y) - u(r_1)) > f(u(x) - u(r'_1)) - f(u(y) - u(r'_1)).$$

Diminishing sensitivity is equivalent to the strict concavity of f and it is illustrated in Figure 2.¹⁰

¹⁰Diminishing sensitivity may be an implication of a general law of human perception in psychology, called the Weber-Fechner law. The law states that perceived intensity is proportional to the logarithm of the stimulus (or to more general concave functions). In other words, the law states that people exhibits diminishing sensitivity to stimuli in general. This connection is suggested by Bruni and Sugden (2007) and also discussed in Bordalo et al. (2012) and Bordalo et al. (2013)).

Figure 2: Diminishing Sensitivity when $u(x) = x$

We can phrase diminishing sensitivity in the following way: the relative value of i -th dimension increases when its reference increases. This will be useful when we discuss the compromise and attraction effects. Figure 3 demonstrates how indifference curves change as reference points change under diminishing sensitivity. Here curves I_r represent indifference curves for the reference point r . For example, $I_{(1,1)}$ (solid thick curve) and $I_{(3,1)}$ (dashed curve) are indifference curves such that $\sqrt{x-1} + \sqrt{p-1} = 2$ and $\sqrt{x-3} + \sqrt{p-1} = 2$, respectively. As the reference for the first dimension 1 increases to 3 (the reference point $(1,1)$ shifts to $(3,1)$), the indifference curves $I_{(1,1)}$ (solid) become steeper since the first dimension is more valuable now and shift to $I_{(3,1)}$ (dashed curve). On the other hand, as the reference for the second dimension 1 increases to 3 (the reference point $(1,1)$ shifts to $(1,3)$), the indifference curves $I_{(1,1)}$ become flatter since the second dimension is more valuable now and shift to $I_{(1,3)}$ (dotted curve). This intuition will be the driving force of our results in Section 2.2 and Section 4.

Finally, note that the relative value of x over y in the presence of r_1 decreases when r'_1 is added since $f(u(x) - u(r_1)) - f(u(y) - u(r_1)) > f(u(x) - u(r'_1)) - f(u(y) - u(r'_1))$. Therefore, we can rephrase diminishing sensitivity in the following way: *adding r'_1 hurts x over y in the presence of $r_1 \in (z, y]$* . Now remember that in the introduction we suggested a rationale that can rationalize both compromise and attraction effects: *adding z hurts x over y* . This rationale can be obtained from diminishing sensitivity by setting $r_1 = y$ and $r'_1 = z$. We call this rationale *weak diminishing sensitivity* and define it formally in the following way:

Definition 4 (Weak Diminishing Sensitivity). An ARDC $C_{(f,u,w)}$ satisfies *weak diminishing*

Figure 3: The effect of a shift in reference points when $f(t) = \sqrt{t}$.

sensitivity if for any x , y , and z with $x > y > z$,

$$(4) \quad f(u(x) - u(y)) + f(u(y) - u(z)) > f(u(x) - u(z)).$$

Now we will discuss the compromise and attraction effects and implications of diminishing sensitivity. Section 3.2 provides two behavioral postulates (on choices) equivalent to two diminishing sensitivities.

2.2 Implications of Diminishing Sensitivity

In this subsection, weak diminishing sensitivity plays an important role. In particular, first, we show the equivalence between the attraction and compromise effects and weak diminishing sensitivity (Proposition 1). Second, we show that more concavity means more likely to observe the compromise and attraction effects (Proposition 2). Third, we discuss additional implications of weak diminishing sensitivity that are testable in lab experiments (Observation 1 and Proposition 3). Although weak diminishing sensitivity is enough to rationalize the compromise and attraction effects, the intuition of diminishing sensitivity will be more useful to interpret two effects (e.g., Figure 4). Moreover, in section 4, we will apply our model to two different contexts and diminishing sensitivity will be necessary to obtain our results. Now we turn to the first implication of weak diminishing sensitivity.

Figure 4: Compromise Effect, Attraction Effect, and Diminishing Sensitivity

Proposition 1. Suppose $C = C_{(f,u,w)}$. For any x, y, z, p, t, q , and r such that $x > y > z$ and $p < t < q < r$, the following statements are equivalent.

- i) (compromise effect) The alternative (x, p) is chosen over (y, q) from menu $\{(x, p), (y, q)\}$, but (y, q) is chosen from $\{(x, p), (y, q), (z, r)\}$ when (z, r) is added to the binary menu.
- ii) (attraction effect) The alternative (x, p) is chosen over (y, q) from the menu $\{(x, p), (y, q)\}$, but (y, q) is chosen from $\{(x, p), (y, q), (z, t)\}$ when (z, t) is added to the binary menu.
- iii) (weak diminishing sensitivity) C exhibits weak diminishing sensitivity at x, y, z and satisfies

$$f(u(x) - u(y)) > f(w(q) - w(p)) > f(u(x) - u(z)) - f(u(y) - u(z)).$$

The first two statements i) and ii), are the formal definitions of the compromise and attraction effects, respectively. Note that in the first case, adding (z, r) to $\{(x, p), (y, q)\}$ makes (x, p) an extreme option and (y, q) a compromise option. In the second case, (z, t) is dominated by (y, q) since $y > z$ and $q > t$.

The intuitive argument of Proposition 1 is given in Figure 4. Here $I_{(y,p)}$ (solid curve) is the indifference curve for reference point (y, p) of $\{(x, p), (y, q)\}$ and $I_{(z,p)}$ (dashed curve) is the indifference curve for reference point (z, p) of $\{(x, p), (y, q), (z, r)\}$ (also $\{(x, p), (y, q), (z, t)\}$). In the both compromise and attraction effects, adding a third alternative changes the reference point (y, p) to (z, p) . Then by diminishing sensitivity, the second dimension becomes more valuable since the reference of the first dimension decreased. Therefore, the indifference

Figure 5: Two Decoy Effect and Diminishing Sensitivity

curve $I_{(y,p)}$ (solid curve) becomes flatter and should shift left to $I_{(z,p)}$ (dashed curve) which helps (y, q) to be chosen over (x, p) .

Note that our explanations are consistent with common rationales for two effects. A common rationale for the compromise effect is that people avoid extreme options. Similarly, in our model, the agent also avoids extremes because of the strict concavity of f . A rationale for the attraction effect is that a decoy (z, t) makes (y, q) more attractive, i.e., the *decoy effect*. However, it must be that (y, q) becomes attractive because of the second dimension rather than the first dimension since $y < x$. Similarly, in our model, the second dimension becomes more valuable compared to the first dimension because of diminishing sensitivity. Next, we will show that if f is more concave, then it is more likely to observe the compromise and attraction effects.

Proposition 2 (More Concave, More Preference Reversals). *Take any two additive reference dependent choices $C_{(f,u,w)}$ and $C_{(f',u,w)}$. If f' is more concave than f ; that is, there is some strictly concave and increasing function h such that $f' = h(f)$, then $(x, p) \prec_{(z,t)} (y, q)$ implies $(x, p) \prec'_{(z,t)} (y, q)$ and $(x, p) \prec_{(z,r)} (y, q)$ implies $(x, p) \prec'_{(z,r)} (y, q)$.¹¹*

Now we will discuss three additional implications of diminishing sensitivity.

Two Decoy Effect: First, we will discuss what happens if we add the fourth alternative (k, s) to $\{(x, p), (y, q), (z, s)\}$ where (k, s) is dominated by (x, p) (i.e., $y < k < x$ and $s < p$). The model predicts that the decoy effect of (z, t) to (y, q) should be cancelled out with the

¹¹Here the preference relations $\succeq_d$ and $\succeq'_d$ are defined from $C_{(f,u,w)}$ and $C_{(f',u,w)}$, respectively.

decoy effect of (k, s) to (x, p) . We call it the *two decoy effect* and the intuition is given in Figure 5.

Note that when (k, s) is added, the reference shift down to (z, s) since $s < p$. The indifference curve $I_{(z, p)}$ (dashed curve) becomes steeper and shifts to $I_{(z, s)}$ (dotted curve) as the reference point (z, p) shifts to (z, s) .¹² In other words, although z hurts x over y , it should be cancelled out by s since s also hurts q over p . Therefore, it is more likely that (x, p) is chosen from $\{(x, p), (y, q), (z, s), (k, s)\}$. This prediction is consistent with the experimental result of Teppan and Felfernig (2009).

Symmetric Dominance: Now note that the above argument also suggests that if we add (z, s) (symmetrically dominated) to $\{(x, p), (y, q)\}$, it is less likely to observe a preference reversal compared to the compromise and attraction effects (See Figure 5). In fact, this prediction is also consistent with the experimental result by Masatlioglu and Uler (2013). Observation 1 shows this argument formally.

Observation 1: Suppose $C_{(f, u, w)}$ satisfies weak diminishing sensitivity. For any x, y, z, q, t, p , and s with $x > y > z$ and $q > t > p > s$, if $(x, p) \prec_{(z, s)} (y, q)$, then $(x, p) \prec_{(z, t)} (y, q)$.

Intuitively, Observation 1 states that it is easier to observe the attraction effect than to observe a preference reversal when the third alternative is symmetrically dominated. To see Observation 1, note that $(x, p) \prec_{(z, s)} (y, q)$ implies

$$f(u(x) - u(z)) - f(u(y) - u(z)) < f(w(q) - w(s)) - f(w(p) - w(s)).$$

Then, we will have $(x, p) \prec_{(z, t)} (y, q)$ because weak diminishing sensitivity at q, p , and s implies that

$$f(u(x) - u(z)) - f(u(y) - u(z)) < f(w(q) - w(s)) - f(w(p) - w(s)) < f(w(q) - w(p)).$$

Now let us summarize Proposition 1, the two decoy effect, and Observation 1 in the following way ($\geq$ represents the difficulty of observing a preference reversal):

Two Decoy $\doteq$ Symmetric Dominance $\succ$ Attraction (Decoy) $\doteq$ Compromise.

Finally, we show that weak diminishing sensitivity provides an additional behavioral prediction that can be easily tested in experimental settings. Roughly speaking, Proposition 3 provides an observable restriction on pairs of alternatives that can be involved with preference reversals. Formally,

¹²Note that $I_{(z, s)}$ (dotted curve) is much similar to $I_{(y, q)}$ (solid curve) compared to $I_{(z, p)}$ (dashed curve).

Proposition 3 (Preference Reversal). *Suppose $C_{(f,u,w)}$ satisfies weak diminishing sensitivity. For any alternatives $a, b \in \mathcal{X}$ and menus $A, B \in \mathcal{A}$ such that $a, b \in A \cap B$, if $a \in C(A)$ and $b \notin C(A)$, but $b \in C(B)$ and $a \notin C(B)$, then either a is non-extreme option of A or b is non-extreme option of B .¹³*

We can check whether the attraction and compromise effects are consistent Proposition 3. In both effects, we observed a preference reversal for a pair (x, p) and (y, q) . The two effects are consistent with Proposition 3 since (y, q) is non-extreme option of $\{(x, p), (y, q), (z, r)\}$ and $\{(x, p), (y, q), (z, t)\}$.

3 Behavioral Foundation

In this section, we axiomatically characterize our model by two novel axioms called *Independence from Non-Extreme Alternatives* (INEA) and *Reference Translation Invariance* (RTI) in addition to three standard axioms (Theorem 1). The role of axioms and the sketch of the proof of the main theorem will be discussed in Section 3.3. We also characterize diminishing sensitivity by another novel axiom called *Advantage of Extreme Alternatives* (AE) (Proposition 4) in Section 3.2.

3.1 Axioms and Representation Theorem

We impose five axioms. The first axiom Regularity is a collection of standard properties that guarantee to have a well-behaved representation.

AXIOM 1. (Regularity)

- i) (Monotonicity) For any $A \in \mathcal{A}$ and $a, a' \in A$, if $a' > a$, then $a \notin C(A)$.
- ii) (Continuity) For any $A \in \mathcal{A}$ and a sequence $\{a_n\}_{n=1}^{\infty}$ of $\mathcal{X}$ such that $a_n \notin A$ for each n and $\lim_{n \rightarrow \infty} a_n = a^*$ for some $a^* \notin A$,
 - if $a_n \in C(A \cup \{a_n\})$ for each n , then $a^* \in C(A \cup \{a^*\})$,
 - if $b \in C(A \cup \{a_n\}) \setminus \{a_n\}$ for each n , then $b \in C(A \cup \{a^*\}) \setminus \{a^*\}$,
 - if $a_n \notin C(A \cup \{a_n\})$ for each n and $a^* \notin C(A \cup \{a^*\})$, then the limit $\lim_{n \rightarrow \infty} C(A \cup \{a_n\})$ exists and is equal to $C(A \cup \{a^*\})$,

¹³We say an alternative $a = (x, p) \in A$ is an extreme option of A if either $x = x^A$ or $p = p^A$. So $a \in A$ is non-extreme option of A if $a \gg (x^A, p^A)$.

- iii) (Solvability) For any $p, q, y \in \mathbb{R}_+$ with $p < q$ and $d \in \mathcal{X}$, there exists $x \in \mathbb{R}_+$ such that $(x, p) \succ (y, q)$ and $(x, p) \succeq_d (y, q)$.

Monotonicity and Continuity are standard properties. Solvability requires that (x, p) can be preferred over (y, q) as long as x is large enough. It is rather technical, but frequently used in the literature.

With the next two axioms, we can use standard methods to obtain additive representations. It is well-known that two axioms are necessary and sufficient to have an additive representation: Transitivity and Cancellation (see Krantz et al. (1971), Fishburn and Rubinstein (1982), Wakker (1988), and Tversky and Kahneman (1991)).¹⁴ In particular, we use the pair, Transitivity and Cancellation, for both $\succeq$ and $\{\succeq_d\}_{d \in \mathcal{X}}$.

AXIOM 2 (Transitivity). For any $a, b, c, d \in \mathcal{X}$ with $a, b, c > d$,

- i) if $a \succeq b$ and $b \succeq c$, then $a \succeq c$.
- ii) if $a \succeq_d b$ and $b \succeq_d c$, then $a \succeq_d c$.

Transitivity (ii) requires that if d is dominated by all a, b , and c , then transitivity should be satisfied at a, b, c in a sense of standard binary comparisons; i.e., Transitivity (i). Lastly, we define Cancellation.

AXIOM 3 (Cancellation). For any $(x_1, x_2), (y_1, y_2), (z_1, z_2)$, and $d \in \mathcal{X}$ such that $(x_1, x_2), (y_1, y_2), (z_1, z_2) > d$,

- i) if $(x_1, x_2) \succeq (y_1, z_2)$ and $(y_1, y_2) \succeq (z_1, x_2)$, then $(x_1, y_2) \succeq (z_1, z_2)$.
- ii) if $(x_1, x_2) \succeq_d (y_1, z_2)$ and $(y_1, y_2) \succeq_d (z_1, x_2)$, then $(x_1, y_2) \succeq_d (z_1, z_2)$.

Cancellation (i) is the standard version of Cancellation. Figure 6 illustrates Cancellation (i) when all pairs are indifferent (i.e., $(x_1, x_2) \sim (y_1, z_2)$, $(y_1, y_2) \sim (z_1, x_2)$, and $(x_1, y_2) \sim (z_1, z_2)$.) Solid and dashed curves represent indifferences. Intuitively, it requires that if the relative advantage of x_1 over y_1 is equivalent to that of z_2 over x_2 (dashed intervals) and the relative advantage of y_1 over z_1 is equivalent to that of x_2 over z_2 (dotted intervals), then the relative advantage of x_1 over z_1 is equivalent to that of z_2 over y_2 (combined intervals). Cancellation (ii) requires that if d is dominated by all three alternatives, then Cancellation should be satisfied in a sense of binary comparisons.

The last two axioms, INEA and RTI, are our main axioms and weakenings of the Weak Axiom of Revealed Preference (WARP). It is well-known that WARP is necessary and

¹⁴Cancellation is sometimes called Thomsen Condition or Double Cancellation.

Figure 6: Cancellation (i)

sufficient for choice correspondence to be consistent with utility maximization.¹⁵ We can phrase WARP in the following way: for any A and $a \notin A$, if $\{a\} \neq C(A \cup \{a\})$, then $C(A) = C(A \cup \{a\}) \setminus \{a\}$. INEA is a postulate that is very similar to WARP and can be stated in the following way: for any non-extreme a 's, WARP is satisfied, but for extreme a 's, WARP can be violated. More formally,

AXIOM 4 (Independence from Non-Extreme Alternatives (INEA)). For any $A \in \mathcal{A}$ and $a \notin A$,

$$\text{if } \{a\} \neq C(A \cup \{a\}) \text{ and } a \gg (x^A, p^A), \text{ then } C(A) = C(A \cup \{a\}) \setminus \{a\}.$$

Lastly, we define RTI, which is a modification of the standard *translation invariance*. Translation invariance requires that (x, p) is indifferent with (y, q) if and only if (x', p) is indifferent with (y', q) whenever $x - y = x' - y'$. In order to define RTI, we define relative distances.

Distance: For any x, y, x' , and y' , we say a *relative distance between x and y is equivalent to that of x' and y'* , denoted by $[x, y]D_1[x', y']$, if for any p and q , $(x, p) \sim (y, q)$ if and only if $(x', p) \sim (y', q)$. Similarly, for any p, q, p' , and q' , we say a *relative distance between p and q is equivalent to that of p' and q'* , denoted by $[p, q]D_2[p', q']$, if for any x and y , $(x, p) \sim (y, q)$ if and only if $(x, p') \sim (y, q')$.

RTI is illustrated in Figure 7. Suppose the relative distance between x and y is equivalent

¹⁵Note that under WARP, Transitivity (ii) is equivalent to Transitivity (i) and Cancellation (ii) is equivalent to Cancellation (i).

Figure 7: Reference Translation Invariance (RTI)

to that of x' and y' (dashed intervals) and the relative distance between y and z is equivalent to that of y' and z' (dotted intervals). Then RTI requires that (x, p) is indifferent with (y, q) in the presence of z (solid curve $I_{(z,p)}$) if and only if (x', p) is indifferent with (y', q) in the presence of z' (dashed curve $I_{(z',p)}$). More formally,

AXIOM 5 (Reference Translation Invariance (RTI)). For any $(x, p), (y, q), (z, r) \in \mathcal{X}$,

- i) for any $x', y', z' \in \mathbb{R}_+$ such that $[x, y]D_1[x', y']$ and $[y, z]D_1[y', z']$, $(x, p) \sim_{(z,p)} (y, q)$ if and only if $(x', p) \sim_{(z',p)} (y', q)$;
- ii) for any $p', q', r' \in \mathbb{R}_+$ such that $[p, q]D_2[p', q']$ and $[q, r]D_2[q', r']$, then $(z, r) \sim_{(z,p)} (x, q)$ if and only if $(z, r') \sim_{(z,p')} (x, q')$.

RTI is much weaker than the standard translation invariance. In fact, it is a weakening of WARP. Let us show that RTI (i) is a weakening WARP. Remember that WARP requires that the irrelevant third alternatives does not effect a comparison between (x, p) and (y, q) . Therefore, under WARP, the statement $(x, p) \sim_{(z,p)} (y, q)$ if and only if $(x', p) \sim_{(z',p)} (y', q)$ is equivalent to the statement $(x, p) \sim (y, q)$ if and only if $(x', p) \sim (y', q)$. However, the latter statement is equivalent to the definition of D_1 .

Finally, we can state our main theorem. Theorem 1 characterizes (2) and also provides a uniqueness result.

Theorem 1. *C satisfies Regularity, Transitivity, Cancellation, INEA, and RTI if and only if there exist strictly increasing and continuous functions f, u, w such that $f(0) = 0$, $f(+\infty) = u(+\infty) = w(+\infty) = +\infty$, and for any menu $A \in \mathcal{A}$,*

$$C(A) = \arg \max_{(x,p) \in A} \{f(u(x) - u(x^A)) + f(w(p) - w(p^A))\}.$$

Moreover, u and w are unique up to a linear transformation and for any two vectors of functions (f, u, w) and (f', u', w') such that $u(0) = u'(0) = w(0) = w'(0) = 0$, $f(1) = f'(1)$, and $u(1) = u'(1)$, if $C = C_{(f, u, w)} = C_{(f', u', w')}$, then $(f, u, w) = (f', u', w')$.

3.2 Characterizing Diminishing Sensitivity

In Section 2.2, we showed that diminishing sensitivity is sufficient to rationalize the compromise and attraction effects. In this subsection, we show that diminishing sensitivity is in fact necessary for the attraction effect by behaviorally characterizing two versions of diminishing sensitivity. Since Proposition 1 shows the equivalence between the attraction and compromise effects, we can say that diminishing sensitivity is also necessary for the compromise effect in our model. Now we will impose two novel axioms. The first axiom characterizes weak diminishing sensitivity while the second axiom characterizes diminishing sensitivity.

AXIOM 6 (Weak Advantage of Extreme Alternatives (WAE)). For any $x, y, z, p, q \in \mathbb{R}_+$ such that $x > y > z$ and $p < q$, if $(x, p) \sim (y, q)$, then $(x, p) \prec_{(z, p)} (y, q)$.

Roughly speaking, WAE requires that (z, p) gives an advantage to (y, q) over (x, p) . Evidently, WAE is very similar to the attraction effect. In fact, we now show that WAE is equivalent to observing the attraction effect in some neighborhood without assuming our model. Suppose $(x, p) \sim (y, q)$ and WAE is satisfied. Then with continuity and monotonicity, there are positive real numbers δ and ϵ with $\delta < q - p$ such that $(x + \epsilon, p) \succ (y, q)$ and $(x + \epsilon, p) \prec_{(z, p+\delta)} (y, q)$. Note that we obtained the attraction effect at alternatives $(x + \epsilon, p)$, (y, q) , and $(z, p + \delta)$. Therefore, WAE is equivalent to observing the attraction effect in neighborhood of a triple (x, p) , (y, q) , and (z, p) . In our model, since WAE is equivalent to weak diminishing sensitivity, observing the attraction effect at some neighborhood is equivalent to weak diminishing sensitivity. Now we define the second axiom.

AXIOM 7 (Advantage of Extreme Alternatives (AE)). For any $x, y, z, l, p, q \in \mathbb{R}_+$ such that $x > y > z > l$ and $p < q$, if $(x, p) \sim_{(z, p)} (y, q)$, then $(x, p) \prec_{(l, p)} (y, q)$ and $(x, p) \succ (y, q)$.

Roughly speaking, AE requires that (z, p) helps (y, q) over (x, p) , but (l, p) helps more when $l < z$. Now we illustrate a connection between AE and the attraction effect without assuming our model. Suppose $(x, p) \sim_{(z, p)} (y, q)$ and AE is satisfied. With AE, we have $(x, p) \prec_{(l, p)} (y, q)$ and $(x, p) \succ (y, q)$. Similar to the discussion of WAE, by continuity and monotonicity, there are positive real numbers δ and ϵ with $\delta < q - p$ such that $(x + \epsilon, p) \succ_{(z, p+\delta)} (y, q)$, $(x + \epsilon, p) \prec_{(l, p+\delta)} (y, q)$, and $(x + \epsilon, p) \succ (y, q)$. Now we argue that AE is summarized by two properties. First, notice that we obtained the attraction effect at $(x + \epsilon, p)$, (y, q) , and $(l, p + \delta)$. Second, $(x + \epsilon, p) \succ_{(z, p+\delta)} (y, q)$ implies that $(l, p + \delta)$ is more

likely to cause a preference reversal compared to $(z, p + \delta)$ since $z > l$. Therefore, AE is equivalent to observing the attraction effect in neighborhood of a triple (x, p) , (y, q) , and (l, p) while not in neighborhood of a triple (x, p) , (y, q) , and (z, p) . In other words, AE dictates the degree of observing the attraction effect in some neighborhood (recall Proposition 2). In our model, since AE is equivalent to diminishing sensitivity, observing the attraction effect in neighborhood of (x, p) , (y, q) , and (l, p) but not in that of (x, p) , (y, q) , and (z, p) where $z > l$ is equivalent to diminishing sensitivity.

Finally, we show that (W)AE characterizes (weak) diminishing sensitivity.

Proposition 4. *Suppose $C = C_{(f, u, w)}$ for strictly increasing continuous functions f , u , and w such that $f(+\infty) = u(+\infty) = w(+\infty) = +\infty$. Then*

i) C satisfies WAE if and only if it satisfies weak diminishing sensitivity.

ii) C satisfies AE if and only if it satisfies diminishing sensitivity.

3.3 Sketch of the Proof of Theorem 1

Lastly, we briefly discuss the roles of INEA and RTI in the representation theorem. To do so, we need to know the role of other three axioms, regularity, transitivity, and cancellation. First of all, with regularity, there are functions $\{V_A\}_{A \in \mathcal{A}}$, such that for any $A \in \mathcal{A}$,

$$C(A) = \arg \max_{(x, p) \in A} V_A(x, p).$$

Second, transitivity (i) and cancellation (i) on $\succeq$ are necessary and sufficient to have functions u and w such that $V_{\{(x, p), (y, q)\}}(\{(x, p)\}) = u(x) + w(p)$; that is,

$$(x, p) \succeq (y, q) \text{ if and only if } u(x) + w(p) \geq u(y) + w(q).$$

Similarly, transitivity (ii) and cancellation (ii) on $\succeq_d$ (for given d) are necessary and sufficient to have functions u_d and w_d such that for any $A = \{(x, p), (y, q), d\}$ with $(x, p), (y, q) > d$, $V_A(x, p) = u_d(x) + w_d(p)$; that is,

$$(x, p) \succeq_d (y, q) \text{ if and only if } u_d(x) + w_d(p) \geq u_d(y) + w_d(q).$$

Now note that there is no connection between utilities for different menus since above conditions are separately imposed on $\succeq$ and $\succeq_d$. In particular, there is no connection between the utilities for doubleton menus ($u + w$), tripleton menus with dominated alternative d ($u_d + w_d$), and other menus with at least three alternatives A (V_A).

In other words, INEA and RTI connect $u + w$, $u_d + w_d$, and V_A . We use two steps. In the first step, INEA connects $u_d + w_d$ and V_A . In particular, we show that the comparison between (x, p) and (y, q) in A is only affected by (x^A, p^A) ; that is, $V_A(x, p) = V_{\{(x, p), (y, q), (x^A, p^A)\}}(x, p)$ and $V_A(y, q) = V_{\{(x, p), (y, q), (x^A, p^A)\}}(y, q)$. In the second step, RTI connects $u + w$ and $u_d + w_d$. In particular, we show that we can obtain u_d and w_d from u and w by some common distortion function f ; that is, $u_d(x) = f(u(x) - u(d_1))$ and $w_d(p) = f(w(p) - w(d_2))$ for any $(x, p) > d = (d_1, d_2)$. While the first step is naturally expected from INEA, the second step is a non obvious implication of RTI because we need to prove that $u_d(x)$ is independent of d_2 and $u_d(x)$ is a function of the utility difference $u(x) - u(d_1)$.

Sketch of the proof of step 2: Most of the proof of Theorem 1 is devoted to prove the last step. Let us briefly explain how to prove it. Essentially we need to prove that $V_{\{(x, p), (y, q), (x^A, p^A)\}}(x, p) = f(u(x) - u(x^A)) + f(w(p) - w(p^A))$ for some f . With Transitivity (ii) and Cancellation (ii), we know that $V_{\{(x, p), (y, q), (x^A, p^A)\}}(x, p) = u_{(x^A, p^A)}(x) + w_{(x^A, p^A)}(p)$. Therefore, we can prove that $V_{\{(x, p), (y, q), (x^A, p^A)\}}(x, p) = W(u(x) - u(x^A), w(p) - w(p^A))$ for some W (Lemma 5), then we know that $u_{(x^A, p^A)}(x)$ is a function of $u(x) - u(x^A)$.

In order to prove Lemma 5, we will construct $W(u(x) - u(x^A), w(p) - w(p^A))$. In particular, for any (x, p) and (x^A, p^A) , we find x^* such that $(x^*, p^A) \sim_{(x^A, p^A)} (x, p)$ and set $W(u(x) - u(x^A), w(p) - w(p^A)) = u(x^*) - u(x^A)$. Finally, we need to make sure that that W is well-defined.

For example, for any x' and $x^{A'}$ such that $u(x') - u(x^{A'}) = u(x) - u(x^A)$, we shall prove that $W(u(x) - u(x^A), w(p) - w(p^A)) = W(u(x') - u(x^{A'}), w(p) - w(p^A))$. In other words, we need to have that $u(x^*) - u(x^A) = u(x^{**}) - u(x^{A'})$ for x^{**} such that $(x^{**}, p^A) \sim_{(x^{A'}, p^A)} (x', p)$. Now recall the definition of D_1 . If we note that $u(x') - u(x^{A'}) = u(x) - u(x^A)$ is equivalent to $[x', x^{A'}]D_1[x, x^A]$ and $u(x^{**}) - u(x^{A'}) = u(x^*) - u(x^A)$ is equivalent to $[x^{**}, x^{A'}]D_1[x^*, x^A]$, then RTI(i) concludes the proof.

4 Applications

Explicit modelling of reference points allows us to apply our model to many different contexts. In order to demonstrate that, we apply our model into two different contexts and diminishing sensitivity plays an important role in both cases. First, we discuss the standard intertemporal consumption choice. The main implication of the model is that if there is a borrowing constraint, then the consumer prefers to consume away from the constraint even if the constraint is not binding. We show that this implication will lead to two empirical observations of life-cycle consumption profiles: *excess sensitivity of consumption to income* and a *hump-shaped consumption profile*.

Second, we discuss risky choice. Our model can explain contradicting risk behavior (in sense of the expected utility). More precisely, the model can allow behaviors that the expected utility model implies an agent has both high and low degrees of risk-aversion, i.e., the agent prefers riskier options in some cases and the safer options in other cases.

4.1 Intertemporal Choice with Non Binding Borrowing Constraint

4.1.1 The Effect of Non Binding Constraint

Let us consider a two-period intertemporal choice model. A consumer lives two periods and her utility function is

$$U(c_1, c_2 | \mathbb{B}) = (c_1 - c_1^{\mathbb{B}})^{\gamma} + \beta(c_2 - c_2^{\mathbb{B}})^{\gamma}$$

where β is discount factor, c_i is consumption in period $i \in \{1, 2\}$, and $c_i^{\mathbb{B}}$ is the possible minimum consumption level in period i for given budget set $\mathbb{B}$. Suppose $\gamma < 1$, so we will have diminishing sensitivity.

The consumer earns income y_i in period i and she can borrow at most $\bar{b}$ in the first period at the interest rate r . We assume the consumer exhausts all her discounted total income $M \equiv y_1 + \frac{y_2}{1+r}$. Then the budget set is

$$\mathbb{B} = \{(c_1, c_2) \in \mathbb{R}_+^2 \mid c_1 + \frac{c_2}{1+r} = M \text{ and } c_1 \leq y_1 + \bar{b}\}.$$

Now we calculate the optimal consumption profile for two cases, without a borrowing constraint and with a borrowing constraint, and compare them.

Without a Borrowing Constraint: Suppose there is no borrowing constraint; that is, $\bar{b} \geq \frac{y_2}{1+r}$. Then the consumer's maximization problem is:

$$\max_{(c_1, c_2) \in \mathbb{B}} U(c_1, c_2 | \mathbb{B}) = c_1^{\gamma} + \beta c_2^{\gamma}$$

where $\mathbb{B} = \{(c_1, c_2) \in \mathbb{R}_+^2 \mid c_1 + \frac{c_2}{1+r} = M\}$ and $(0, 0)$ is the reference point.

By a direct calculation, the optimal consumption levels are

$$c_1^* = \frac{M}{1 + \beta^{\frac{1}{1-\gamma}}(1+r)^{\frac{\gamma}{1-\gamma}}} \text{ and } c_2^* = \frac{M \cdot (\beta(1+r))^{\frac{1}{1-\gamma}}}{1 + \beta^{\frac{1}{1-\gamma}}(1+r)^{\frac{\gamma}{1-\gamma}}}.$$

Note that similar to the standard lifetime consumption model, the optimal consumption levels are proportional to M .

With a Borrowing Constraint: Now suppose there is a borrowing constraint; that is, $\bar{b} < \frac{y_2}{1+r}$. Then the consumer's maximization problem is:

$$\max_{(c_1, c_2) \in \mathbb{B}} U(c_1, c_2 | \mathbb{B}) = c_1^\gamma + \beta (c_2 - [y_2 - (1+r)\bar{b}])^\gamma$$

where $\mathbb{B} = \{(c_1, c_2) \in \mathbb{R}_+^2 \mid c_1 + \frac{c_2}{1+r} = M \text{ and } c_1 \leq y_1 + \bar{b}\}$ and $(0, y_2 - (1+r)\bar{b})$ is the reference point.

By a direct calculation, the new optimal consumption levels are

$$c_1^{**} = \frac{M - (\frac{y_2}{1+r} - \bar{b})}{1 + \beta^{\frac{1}{1-\gamma}}(1+r)^{\frac{\gamma}{1-\gamma}}} = \frac{y_1 + \bar{b}}{1 + \beta^{\frac{1}{1-\gamma}}(1+r)^{\frac{\gamma}{1-\gamma}}} \text{ and } c_2^{**} = \frac{M \cdot (\beta(1+r))^{\frac{1}{1-\gamma}} + (y_2 - \bar{b}(1+r))}{1 + \beta^{\frac{1}{1-\gamma}}(1+r)^{\frac{\gamma}{1-\gamma}}}.$$

In the standard model, the optimal consumption profiles are the same when the constraint is not binding; i.e., $y_1 + \bar{b} > c_1^*$. However, in our model, two cases are different even if the constraint is not binding. The first implication of the model and diminishing sensitivity is on the effect of non binding constraint to the optimal consumption profile.

Observation 2: The consumer decreases her the first period consumption; that is, $c_1^* > c_1^{**}$, even if the borrowing constraint is not binding; that is, $c_1^* < y_1 + \bar{b}$. Moreover, as the constraint gets tighter (as $\bar{b}$ decreases), she consumes less in the first period. In other words, c_1^{**} is increasing in $\bar{b}$.

The intuition of Observation 2 as follows. The borrowing constraint decreases the consumer's the maximum consumption level for today. If the consumer has to consume all of her total discounted income, then the minimum consumption level for tomorrow's will increase as the maximum consumption level for today decreases. Then by diminishing sensitivity ($\gamma < 1$), the relative value of tomorrow's consumption with respect to today's consumption will increase. Therefore, the consumer will decrease her today's consumption and increase tomorrow's consumption.

It is usual to explain deviations from the standard model of consumer choice (both consumption and financial) by liquidity constraints (e.g., Zeldes (1989)). But there is no strong empirical evidence that constraints were actually binding at the time of decision. For example, Jappelli (1990) directly asked consumers whether they applied for and were denied credit and only 12.5 percent of 1982 respondents answered they were denied credit. Moreover, Deaton (1991) shows that liquidity constraints are rarely binding by simulation. Our model gives a justification for those explanations since liquidity constraints can have an effect on choices even if they are not binding.

Figure 8: Average Income and Consumption by Education, Attanasio and Weber (2010)

4.1.2 Excess Sensitivity of Consumption to Income and Hump Shaped Consumption Profiles

Two of the well-known deviations of standard life cycle model is called *excess sensitivity of consumption to income*, consumption path is strongly correlated to income path, and *hump shaped consumption profiles*. Figure 8 borrowed from Attanasio and Weber (2010) reports life cycle profiles for two education groups in UK. The left hand side of Figure 8 shows income and consumption paths for the group with compulsory education; the dotted curve is disposable income and the solid curve is nondurable consumption. Note that consumption path closely follows income path; i.e., excess sensitivity of consumption to income, and consumption profiles are hump-shaped.¹⁶

However, the standard model of life cycle profiles cannot explain these behaviors since the optimal consumption levels are proportional to the total discounted income and generate a linear consumption path (see the first case of previous subsection). Now note that if consumptions are positively correlated to incomes after fixing the total discounted income M constant, then we can obtain two deviations. In fact, the next observation shows that we will have positive correlation when there is a borrowing constraint.

Observation 3: c_i^{**} is positively correlated with y_i fixing M constant.

Observation 3 is true even if the constraint is not binding. We could apply this argument in a different context. For example, in a context of financial decision, it is known that younger

¹⁶Moreover, Campbell and Mankiw (1991) showed that in many different countries a large number of consumption who follow a “rule of thumb” and set their consumption proportional to their income.

people invest in risky asset less than older people even after controlling the income effect via risk preference. Observation 3 suggests an additional income effect to younger people's financial decision. Therefore, the income effect via reference-dependent preference might be an additional source for younger people to invest less.

We can also make another interesting observation: when $\bar{b}$ is small, c_1^{**} is more dependent on y_1 . In fact, Zeldes (1989) observed that consumers with a low level of assets (low $\bar{b}$) link consumption to their income more.

4.2 Observing Different Degrees of Risk-Aversion

Now we turn to risky choice. In particular, we focus on binary lotteries (x, p) where (x, p) denotes a binary lottery that gives x dollars with probability p and nothing with probability $1 - p$. Now recall the utility of (x, p) for given menu A :

$$U_A(x, p) = f(u(x) - u(x^A)) + f(w(p) - w(p^A)).$$

The objective of this subsection is obtain contradicting risk behavior. Note that by diminishing sensitivity f introduces an additional concavity to the total value of alternatives since utilities are distorted by f and f is strictly concave. Moreover, the additional concavity is menu-dependent. Therefore, an agent will act as if her the degree of risk-aversion (in the standard sense) is high in some menus and as if it is low in other menus. Now let us show it more formally.

Suppose an agent has the following additive reference-dependent preference: For any menu A of binary lotteries and binary lottery $(x, p) \in A$,

$$U((x, p)|A) = (\alpha \log(x) - \alpha \log(x^A))^\gamma + (\log(p) - \log(p^A))^\gamma$$

Here α is the degree of risk-aversion (in the standard sense) because when $\gamma = 1$, the model reduces to the expected utility theory: the agent maximizes expected utility $p \cdot x^\alpha$.¹⁷

Take four binary lotteries (x, p) , (y, q) , (x', p') , and (y', q') with $x > y$ and $p < q$ and $x' > y'$ and $p' < q'$. Suppose the agent prefers (x, p) over (y, q) in the menu $\{(x, p), (y, q)\}$ (choosing riskier one), but prefers (y', q') over (x', p') in some menu A (choosing safer one). In the standard model, two choices contradict when $\frac{\log(q/p)}{\log(x'/y')} < \frac{\log(q/p)}{\log(x/y)}$ because choosing (x, p) over (y, q) implies that $\alpha > \frac{\log(q/p)}{\log(x/y)}$, but choosing (y', q') over (x', p') implies that $\alpha < \frac{\log(q'/p')}{\log(x'/y')}$.

However, in our model the agent could choose (y', q') over (x', p') because $f(x) = x^\gamma$

¹⁷Moreover, we need to emphasize that γ is independent from the agent's risk preference since the concavity of f is motivated from riskless choices.

introduces an additional menu-dependent concavity to utilities of alternatives. To illustrate, note that choosing (x, p) over (y, q) implies $\alpha > \frac{\log(q/p)}{\log(x/y)}$ as in the standard model, but choosing (y', q') over (x', p') implies

$$\alpha < \left(\frac{(\log(q'/p^A))^\gamma - (\log(p'/p^A))^\gamma}{(\log(x'/x^A))^\gamma - (\log(y'/x^A))^\gamma} \right)^{\frac{1}{\gamma}} \equiv \alpha(\gamma).$$

If the reference point of A satisfies $\frac{\log(x'/x^A)}{\log(y'/x^A)} < \frac{\log(q'/p^A)}{\log(p'/p^A)}$, then $\alpha(\gamma)$ converges to infinity as γ converges to zero. Therefore, as long as $\frac{\log(x'/x^A)}{\log(y'/x^A)} < \frac{\log(q'/p^A)}{\log(p'/p^A)}$ and γ is small enough,¹⁸ we can have $\frac{\log(q/p)}{\log(x/y)} < \alpha < \alpha(\gamma)$. Therefore, when γ is small, the agent could act as if she is very risk-averse even if α is large.

Lastly, we briefly discuss the estimation of risk preference when the agent has an additive reference-dependent preference. Because of diminishing sensitivity or the strict concavity of f , measuring risk preferences from observed choices could be misleading. However, if we estimate risk preferences only using binary comparisons, then it gives unambiguous measure for the degree of risk because the effect of f cancels out. More precisely, because of the following inequality: for any (x, p) and (y, q) ,

$$\begin{aligned} (x, p) \succeq (y, q) &\Leftrightarrow f(u(x) - u(y)) + \cancel{f(w(p) - w(p))} \geq \cancel{f(u(y) - u(y))} + f(w(q) - w(p)) \\ &\Leftrightarrow u(x) + w(p) \geq u(y) + w(q). \end{aligned}$$

5 Model with General Menu-Dependent References

As we mentioned in the introduction, (1) is too general to have testable implications on observed choice behavior since reference points are exogenously given. The objective of the current paper is to endogenize reference points and so we focused on menu-dependent reference points. In this section, we show that in fact we do not lose any generality by focusing on menu-dependent reference points. In other words, we will show that any observed choice data can be rationalized by a model with menu-dependent reference points. This result also suggests that we should have more specific models for reference points in order to obtain testable predictions.

For the sake of simplicity, suppose we observe choices from finite number of finite menus $A_1, A_2, \dots, A_n$. Suppose $C(A_i) \in A_i$ for each i . The next result shows that any such choices can be rationalized by a reference-dependent model with menu-dependent reference points

¹⁸For example, let $A = \{(x', p'), (y', q'), (z', r')\}$ where $x' > y' > z'$ and $p' < q' < r'$. Since $p^A = p'$ and $x^A = z'$, $\frac{\log(x'/x^A)}{\log(y'/x^A)} = \frac{\log(x'/z')}{\log(y'/z')} < \frac{\log(q'/p^A)}{\log(p'/p^A)} = \frac{\log(q'/p')}{\log(p'/p')} = \infty$ always hold.

as long as Monotonicity is satisfied.

Proposition 5. *If C satisfies Monotonicity, then there exist $\lambda > 0$ and reference-points $\mathbf{r}^1, \mathbf{r}^2, \dots, \mathbf{r}^n$ such that for each i , $\mathbf{r}_i \in A_i$ and*

$$C(A_i) = \arg \max_{(x,p) \in A_i} f(x - \mathbf{r}_1^i) + f(p - \mathbf{r}_2^i) \text{ where } f(t) = \begin{cases} t & \text{if } t \geq 0 \\ -\lambda t & \text{if } t < 0. \end{cases}$$

In fact, we can rationalize observed choices by a model with a very specific functional form. Therefore, Proposition 5 also suggests that a specific functional form may not give us more predictive power without the further specification of menu-dependence.

In this paper, we took very the specific approach of taking minimums of menus as a reference point. Another factor to determine reference point might be the maximums of the menu. However, as mentioned in the introduction, models in which reference points only depend on the maximums cannot explain the attraction effect. Although there are many other possibilities, as long as reference points are increasing in the minimums, it is possible to rationalize the compromise and the attraction effects. But behavioral predictions of the general models will be weaker compared to just using the minimums. In the appendix B, we will consider a model in which reference points depend on both the maximums and the minimums. The main implications stay the same, but behavioral predictions are weaker.

6 Related Literature

Tversky and Kahneman (1991) provided the first explicit model of reference-dependent preferences. Their model is an extension of the Prospect Theory of Kahneman and Tversky (1979) to the context of riskless choice. The model of Tversky and Kahneman (1991) is further developed by Munro and Sugden (2003) (to the n-dimensional commodity space), Sugden (2003) (to subjective expected utility), and Koszegi and Rabin (2006, 2007) (expectation-based reference points, to be discussed later).

The main objective of the current paper is to develop an axiomatic model of reference-dependent preferences with endogenous reference points. However, in most of the previous work reference points are exogenously given. To the best of our knowledge, Ok et al. (2014) is the only paper that axiomatically developed a model with purely endogenous reference points. As we explained in the introduction, the authors weaken WARP to allow for choice behavior that exhibits the attraction effect, but not the compromise effect.

The literature related to reference-dependent behavior and prospect theory is too large to be discussed here (see Barberis (2013)). We will now narrow our focus and discuss related

literature in the following three ways. First, we discuss two main approaches to model reference points in the literature. Second, we discuss papers that accommodate behavioral phenomena we study in this paper. Finally, we attempt to place our model in the broad choice theory literature.

1. Two main approaches to model reference points:

In the first approach, an agent uses an exogenously given alternative as a reference point. For example, that alternative is the status quo, the default option (Masatlioglu and Ok (2005), Sagi (2006), and Apesteguia and Ballester (2009)), and the initial endowment (Masatlioglu and Ok (2013)). However, there are many choice situations where no alternative is exogenously given as the status quo or the initial endowment. For example, there is no sensible status quo or initial endowment in the compromise and the attraction effects. Since in our model the reference point can be defined for any given menu, our approach compliments their approach in situations where there is no sensible default option or initial endowment.

In the second approach, an agent has probabilistic beliefs over possible outcomes and she uses her expectations of the outcome as a reference point (Koszegi and Rabin (2006, 2007)). This approach builds on Tversky and Kahneman (1991), as did we. Moreover, it is more natural in repeated or dynamic choice. However, there are two crucial differences. First, in their approach, the reference point is not purely endogenous because probabilistic beliefs are not directly observable, whereas in our model a reference point is completely determined by the menu. Second, their model reduces to the standard rational model when there is no underlying uncertainty. Therefore, their model cannot allow reference-dependent behavior such as the compromise and attraction effects since the effects are normally defined in riskless environment. On the other hand, our model can be applied to both risky and riskless environments (Section 4.2).

2. Papers that discuss behavioral phenomena we study:

In the literature, the attraction and compromise effects are usually used as motivations for menu-dependent or context-dependent preferences. Therefore, it is already well-known that these two effects can be explained by menu-dependent preferences or context-dependent preferences (e.g., Simonson and Tversky (1992), Wernerfelt (1995), Kamenica (2008), and Bordalo et al. (2013).)¹⁹ Although a connection between the compromise and attraction effects are pointed out in Simonson (1989) and Simonson and Tversky (1992), to the best of our knowledge, it is shown formally first in this paper.²⁰

¹⁹Moreover, there are number of recent works on stochastic choice that consistent with the two effects. For example, see Natenzon (2010), Echenique et al. (2013), and Fudenberg et al. (2013).

²⁰De Clippel and Eliaz (2012) rationalized the attraction effect and compromise effects as a result of single

Now, we narrow our focus to reference-dependent models which discuss either the attraction effect, the compromise effect, life-cycle consumption profile, or risky choice. Koszegi and Rabin (2009) and Pagel (2013) applied the expectation-based reference dependent model of Koszegi and Rabin (2006) to life-cycle consumption choice. They explained empirical observations about consumption profiles including excess sensitivity of consumption to income and a hump-shaped consumption profile. The main difference is that we rely on non binding borrowing constraint while their models rely on uncertainty since the expectation-based reference dependent model reduces to the standard model when there is no uncertainty.

Explaining two different risk attitudes and introducing reference-dependent behavior are two of the most important contributions of Kahneman and Tversky (1979). As mentioned in the introduction, loss aversion and diminishing sensitivity help to explain many different anomalies in risky choice (e.g., Benartzi and Thaler (1995), Koszegi and Rabin (2007) and Bordalo et al. (2012).) One contribution of our paper is to obtain two different risk behaviors relying on diminishing sensitivity, but not relying on loss aversion.

Lastly, the Saliency Theory of Bordalo et al. (2013) is important to mention in detail. The Saliency Theory focuses on alternatives with two attributes, quality and price. It generalizes standard models in a way that an agent uses menu-dependent weight functions, w_q and w_p , that distort evaluations of attributes of alternatives: if the quality dimension of alternative is more salient than the price dimension, then the agent overweights quality and underweights price ($w_q > 1 > w_p$) and vice versa ($w_q < 1 < w_p$).²¹ In their model, the agent uses the average quality ($\bar{q}$) and the average price ($\bar{p}$) of the menu as references in order to decide the saliency of dimensions. In particular, for given menu $A = \{(q_k, p_k)_{k=1}^n\}$,

$$C(A) = \arg \max_k \left\{ \underbrace{w_q(\sigma(q_k, \bar{q}), \sigma(p_k, \bar{p})) \cdot q_k - w_p(\sigma(q_k, \bar{q}), \sigma(p_k, \bar{p})) \cdot p_k}_{=I} \right\}$$

where the saliency of dimensions are determined by numbers $\sigma(q_k, \bar{q})$ and $\sigma(p_k, \bar{p})$ for some function σ .

Our model and the Saliency Theory have two important similarities. Let us consider the utilities of the first dimension in our model and the Saliency Theory: $f(u(x) - u(x^A))$ and I . First, both terms depend on some reference points which are purely determined by menu, x^A and $(\bar{q}, \bar{p})$. Second, diminishing sensitivity properties on two terms can explain the attraction and compromise effects.

However, since weight functions w_q and w_p depend on all possible factors in the menu

bargaining protocol (to be discussed in Part 3). However, behavioral postulates lead to the two effects are different.

²¹The authors discussed the Saliency Theory in the context of choice under risk in Bordalo et al. (2012).

through the reference point $(\bar{q}, \bar{p})$, the Saliency Theory can be seen as a model with a general menu-dependent reference points (see Section 5). Yet their model has testable predictions because of particular functional forms.

3. Choice theory literature:

In this paper, we weakened WARP in order to obtain a model that is consistent with observed behavioral phenomena. This approach is in the same spirit as a body of work that seeks to characterize models of non standard choice in terms of direct axioms on choice behavior. This body of work includes, for example, the rational shortlisting (Manzini and Mariotti (2007)), choosing the uncovered set of some preference relation (Ehlers and Sprumont (2008)), choosing the two finalists (Eliaz et al. (2011)), choosing an alternative that survived after a sequential elimination of alternatives by lexicographic semiorders (Manzini and Mariotti (2012b)), considering only alternatives that (belong to the best category Manzini and Mariotti (2012a); optimal according to some rationalizing criteria Cherepanov et al. (2013); pass menu-dependent threshold (Manzini et al. (2013)), multiself (e.g., De Clippel and Eliaz (2012)), choosing from a subset of menu because of limited attention (Masatlioglu et al. (2012)), comparing all pairs by certain orders (Apesteguia and Ballester (2013)).

Although of most of the above papers allow a preference reversal, only Masatlioglu et al. (2012) and De Clippel and Eliaz (2012) can be consistent with the attraction and two decoy effects (discussed in Section 2.2).²²

However, the main difference is that, except the fact that we study a reference-dependent behavior, in our paper attributes of alternatives are known as in consumer choice theory which allows us to obtain more stronger predictions while the above papers study alternatives with unknown attributes as in abstract choice theory.

The paper by De Clippel and Eliaz (2012) is important to mention; it axiomatically models choices as a solution to an intrapersonal bargaining problem among two selves of an individual. Each self is endowed with a preference relation, so having two preference relations for two selves is similar to having two-attribute alternatives. Their model can generate the compromise and attraction effects, but with two independent behavioral postulates called *Attraction* and *No Better Compromise*. Moreover, technically, they allow only a “weak” form the attraction and compromise effects: original two alternatives $((x, p)$ and $(y, q))$ are indifferent and the third alternative breaks the tie (in favor of (y, q)).²³

²²For example, the two decoy effect can be formulized by the following choices: $C(\{(x, p), (y, q)\}) = \{(x, p)\}$, $C(\{(x, p), (y, q), (z, t)\}) = \{(y, q)\}$, and $C(\{(x, p), (y, q), (z, t), (k, s)\}) = \{(x, p)\}$ where $x > k > y > z$ and $s < p < t < q$ (See Figure 5). It violates Weak WARP, an axiom that is necessary for the characterization of Manzini and Mariotti (2007), Manzini and Mariotti (2012a), and Cherepanov et al. (2013).

²³More precisely, they only allow $C(\{(x, p), (y, q)\}) = \{(x, p), (y, q)\}$ and $C(\{(x, p), (y, q), (z, r)\}) = \{(y, q)\}$, but not $C(\{(x, p), (y, q)\}) = \{(x, p)\}$ and $C(\{(x, p), (y, q), (z, r)\}) = \{(y, q)\}$.

7 Conclusion

We axiomatically developed a model of reference-dependent preferences in which reference points are determined by the minimums of menu. Our contribution is three-fold. First, we characterized our model by two weak versions of WARP, called INEA and RTI, in addition to standard axioms. Second, we showed the equivalence between the attraction and compromise effects, well documented preference reversals, and diminishing sensitivity, a widely used property in economics that is closely related to the Weber-Fechner law of human perception. Third, we applied our model to different environments such as riskless static choice, intertemporal choice, and risky choice, and explained empirical findings in those environments using diminishing sensitivity. In particular, we showed the effect of non binding borrowing constraints in intertemporal choice.

8 Appendix A: Proofs

8.1 Some Implications of Regularity

We use the following two lemmas frequently.

Lemma 1. *For any $p, q, y \in \mathbb{R}_+$ with $p < q$ and $c \in \mathcal{X}$, there exist x and x' such that i) $(x, p) \sim (y, q)$ and ii) $(x', p) \sim_c (y, q)$.*

Proof of Lemma 1. Since there is no essential difference between (i) and (ii), we only prove (i). Take any $p, q, y \in \mathbb{R}_+$ with $p < q$. We shall prove that there exists x such that $(x, p) \sim (y, q)$. By Solvability, there exists x^0 such that $(x^0, p) \succ (y, q)$. Now construct two sequences $\{x^n\}_{n=0}^\infty$ and $\{y^n\}_{n=0}^\infty$ by the induction. Let $y^0 \equiv y$. Note that $x^0 > y^0$. Suppose we have constructed $x^0, \dots, x^k$ and $y^0, \dots, y^k$ and we will define x^{k+1} and y^{k+1} in the following way: if $(\frac{x^k + y^k}{2}, p) \succeq (y^0, q)$, then let $x^{k+1} \equiv \frac{x^k + y^k}{2}$ and $y^{k+1} \equiv y^k$, and if $(\frac{x^k + y^k}{2}, p) \prec (y^0, q)$, then let $x^{k+1} \equiv x^k$ and $y^{k+1} \equiv \frac{x^k + y^k}{2}$. Since $\{x^n\}_{n=0}^\infty$ is a non-increasing sequence, $\{y^n\}_{n=0}^\infty$ is a non-decreasing sequence, and $\lim_{k \rightarrow \infty} x^k - y^k = \frac{x^0 - y^0}{2^k} = 0$, there exist x^* such that $\lim_{k \rightarrow \infty} x^k = \lim_{k \rightarrow \infty} y^k = x^*$. Moreover, by this construction, we have $(x^k, p) \succeq (y, q)$ and $(y, q) \succ (y^k, p)$ for each k . By continuity, $(x^*, p) \sim (y, q)$. $\square$

Lemma 2. *For any finite menu $A \in \mathcal{A}$ and $(x, p), (y, q) \notin A$, if $(x, p) = C(A \cup \{(x, p), (y, q)\})$, then there exist $\epsilon_1, \epsilon_2 > 0$ such that i) $(x - \epsilon_1, p) = C(A \cup \{(x - \epsilon_1, p), (y, q)\})$ and ii) $(x, p) = C(A \cup \{(x, p), (y + \epsilon_2, q)\})$.*

Proof of Lemma 2. We only prove i). Take any $A = \{a_1, \dots, a_n\} \in \mathcal{A}$ and $(x, p), (y, q) \notin A$ such that $(x, p) = C(A \cup \{(x, p), (y, q)\})$. We shall prove that there exists $\epsilon > 0$ such that

$(x - \epsilon, p) = C(A \cup \{(x - \epsilon, p), (y, q)\})$. By way of contradiction, suppose for any n with $x > \frac{1}{n}$, $(x - \frac{1}{n}, p) \neq C(A \cup \{(x - \frac{1}{n}, p), (y, q)\})$; that is, either $(y, q) \in C(A \cup \{(x - \frac{1}{n}, p), (y, q)\})$ for infinitely many n or for each n , there exists i_n such that $a_{i_n} \in C(A \cup \{(x - \frac{1}{n}, p), (y, q)\})$. If the former is true, then by continuity $(y, q) \in \lim_{n \rightarrow \infty} C(A \cup \{(x - \frac{1}{n}, p), (y, q)\}) = C(A \cup \{(x, p), (y, q)\})$. A contradiction. If the latter is true, then there exists i^* such that $a_{i^*} \in C(A \cup \{(x - \frac{1}{n}, p), (y, q)\})$ holds for infinitely many n . Then by continuity, $a_{i^*} \in C(A \cup \{(x, p), (y, q)\})$. A contradiction. □

8.2 Proof of Proposition 2

By $C_{(f, u, w)}, (x, p) \prec_{(z, t)} (y, q)$ is equivalent to

$$f(u(x) - u(z)) - f(u(y) - u(z)) < f(w(q) - w(p)).$$

Since h is strictly increasing, we have

$$h\left(f(u(x) - u(z)) - f(u(y) - u(z))\right) < h\left(f(w(q) - w(p))\right) = f'(w(q) - w(p)).$$

Then, in order to obtain $(x, p) \prec'_{(z, t)} (y, q)$, it is enough to prove that

$$h\left(f(u(x) - u(z))\right) - h\left(f(u(y) - u(z))\right) < h\left(f(u(x) - u(z)) - f(u(y) - u(z))\right).$$

It is enough to prove that $h(a) + h(b) > h(a + b)$ for any $a, b > 0$ when h is strictly concave.

Without loss of generality, suppose $a \geq b$. Figure 9 illustrates the intuitive proof. By the strict concavity, the angle $\angle \alpha$ is greater than the angle $\angle \beta$. Now since the tangent function is increasing function, we have $h(a + b) - h(a) = \tan(\beta)((a + b) - a) < \tan(\alpha)(b - 0) = h(b)$.

8.3 Proof of Proposition 3

Suppose $a = (x, p)$ and $b = (y, q)$ with $x > y$ and $p < q$. WLOG, let $C(\{a, b\}) \ni a$. Then we will prove that b is non-extreme option of B ; that is, $y > x^B$. By way of contradiction, suppose b is an extreme option of B ; that is, $y = x^B$. Note that $C(\{a, b\}) \ni a$ implies that $f(u(x) - u(y)) \geq f(w(q) - w(p))$ and $C(B) \ni b$ and $C(B) \not\ni a$ imply that $f(w(q) - w(p^B)) > f(u(x) - u(y)) + f(w(p) - w(p^B))$. Therefore, we obtain

$$f(w(q) - w(p^B)) > f(u(x) - u(y)) + f(w(p) - w(p^B)) \geq f(w(q) - w(p)) + f(w(p) - w(p^B))$$

Figure 9: Strictly Concave h

which contradicts with weak diminishing sensitivity.

8.4 Proof of Proposition 4

First, take any $x, y, z \in \mathbb{R}_+$ such that $x > y > z$. There exist p and q such that $(x, p) \sim (y, q)$; equivalently, $u(x) - u(y) = w(q) - w(p)$. By WAE, we have $(y, q) \succ_{(z, p)} (x, p)$; equivalently,

$$f(u(x) - u(y)) + f(u(y) - u(z)) = f(w(q) - u(p)) + f(u(y) - u(z)) > f(u(x) - u(z)).$$

Second, take any x, y, z, l with $x > y > z > l$. There exist p and q such that $(x, p) \sim_{(l, p)} (y, q)$; equivalently, $f(u(x) - u(l)) - f(u(y) - u(l)) = f(w(q) - w(p))$. By AE, we have $(x, p) \succ_{(z, p)} (y, q)$; equivalently,

$$f(u(x) - u(z)) - f(u(y) - u(z)) > f(w(q) - u(p)) = f(u(x) - u(l)) - f(u(y) - u(l)).$$

8.5 Proof of Proposition 5

Let $\cup_{i=1}^n A_i \equiv A$ and $\lambda \equiv \max\{\max\{\frac{x-y}{q-p}, \frac{q-p}{x-y}\} | (x, p), (y, q) \in A \text{ with } x > y \text{ and } p < q\}$ and

$$f(t) = \begin{cases} t & \text{if } t \geq 0 \\ -(\lambda + \epsilon)t & \text{if } t < 0. \end{cases}$$

for some $\epsilon > 0$. Then if we set $\mathbf{r}^i \equiv C(A_i)$, then we will have

$$C(A_i) = \arg \max_{(x,p) \in A_i} f(x - r_1^i) + f(p - r_2^i).$$

To illustrate, take any i and let $C(A_i) = (x, p)$. Since $\mathbf{r}^i = (x, p)$, we have $f(x - r_1^i) + f(p - r_2^i) = 0$. Take any $(x', p') \in A_i \setminus (x, p)$. By Monotonicity, we cannot have $(x', p') \geq (x, p)$. When $x' \geq x$ and $p' < p$, $f(x' - r_1^i) + f(p' - r_2^i) = (x' - x) - (\lambda + \epsilon)(p - p') < 0$, by the definition of λ . Similarly, when $x' < x$ and $p' \geq p$, $f(x' - r_1^i) + f(p' - r_2^i) = -(\lambda + \epsilon)(x - x') + (p' - p) < 0$, by the definition of λ .

8.6 Two Implications of INEA

We discuss two implications of INEA.

Lemma 3. *Suppose C satisfies Regularity and INEA. For any $A \in \mathcal{A}$ and $a \notin A$,*

$$\text{if } a \neq C(A \cup \{a\}) \text{ and } a \succ (x^A, p^A), \text{ then } C(A) = C(A \cup \{a\}) \setminus \{a\}.$$

Proof of Lemma 3. Let $a = (x, p)$. If $a \gg (x^A, p^A)$, then by INEA, $C(A) = C(A \cup \{a\}) \setminus \{a\}$. Now, WLOG, suppose $x > x^A$ and $p = p^A$. We consider two cases.

Case 1: $a \notin C(A \cup \{a\})$.

Let $a_\epsilon = (x, p + \epsilon)$ for each $\epsilon \geq 0$. Note that $a_0 = a$. By Lemma 2, there is a sequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$ and $C(A \cup \{a_{\epsilon_n}\}) \not\supseteq a_{\epsilon_n}$. Since $a_\epsilon \gg (x^A, p^A)$, by INEA, we have $C(A \cup \{a_\epsilon\}) = C(A)$. Then, by continuity, $\lim_{\epsilon_n \rightarrow 0} C(A \cup \{a_{\epsilon_n}\}) = C(A) = C(A \cup \{a\})$.

Case 2: $a \in C(A \cup \{a\})$ and $a \neq C(A \cup \{a\})$.

Since $p = p^A$ and $a \notin A$, there exists $a' \in A$ such that $a' = (x', p^A) = (x', p)$ for some x' . Moreover, by monotonicity $a \in C(A \cup \{a\})$ implies $x' < x$. Let $a'_\epsilon = (x', p - \epsilon)$ for each $\epsilon \geq 0$. Note that $a'_0 = a'$. Let $A' = A \setminus \{a'\}$. By monotonicity, we have $a'_\epsilon \notin C(A' \cup \{a'_\epsilon\} \cup \{a\})$.

If there is a sequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$ and $C(A' \cup \{a'_{\epsilon_n}\} \cup \{a\}) = a$ for each n , then by continuity, $C(A' \cup \{a'\} \cup \{a\}) = C(A \cup \{a\}) = a$. A contradiction. Therefore, there exists $\epsilon^* > 0$ such that for any $\epsilon < \epsilon^*$, $C(A' \cup \{a'_\epsilon\} \cup \{a\}) \neq a$.

Since $a = (x, p) \gg (x^{A' \cup \{a'_\epsilon\}}, p^{A' \cup \{a'_\epsilon\}}) = (x^A, p - \epsilon)$ and $a \notin A' \cup \{a'_\epsilon\}$, by INEA, we have

$$C(A' \cup \{a'_\epsilon\} \cup \{a\}) \setminus a = C(A' \cup \{a'_\epsilon\}).$$

Therefore, by continuity,

$$C(A' \cup \{a'\} \cup \{a\}) \setminus a = \lim_{\epsilon \rightarrow 0} C(A' \cup \{a'_\epsilon\} \cup \{a\}) \setminus a = \lim_{\epsilon \rightarrow \infty} C(A' \cup \{a'_\epsilon\}) = C(A' \cup \{a'\}).$$

□

Now it is easy to see that, for any $A \in \mathcal{A}$ and $a \notin A$,

$$\text{if } a \neq C(A \cup \{a\}) \text{ and } a \geq (x^A, p^A), \text{ then } C(A) = C(A \cup \{a\}) \setminus \{a\}.$$

We also prove the following useful Corollary of Lemma 3.

Corollary 1. For any $x, y, z, p, q, r \in \mathbb{R}_+$ such that $x > y > z$ and $p < q < r$,

- i) $C(\{(x, p), (y, q), (z, r)\}) \neq (z, r)$ implies $C(\{(x, p), (y, q), (z, p)\}) = C(\{(x, p), (y, q), (z, r)\}) \setminus (z, r)$,
- ii) $C(\{(x, q), (z, p), (z, r)\}) = C(\{(x, q), (y, p), (z, r)\})$.

Proof of Corollary 1. First, we prove i). Let $A = \{(x, p), (y, q), (z, r)\}$ and $C(A) \neq (z, r)$. Consider a menu $A \cup \{(z, p)\}$. Since $(z, r) > (z, p)$, by monotonicity, $C(A \cup \{(z, p)\}) \not\supseteq (z, p)$. Therefore, since $(z, p) = (x^A, p^A)$ and $(z, p) \notin A$, by Lemma 3, $C(A \cup \{(z, p)\}) = C(A)$. Then by Lemma 3, $C(A \cup \{(z, p)\}) = C(\{(x, p), (y, q), (z, r)\}) \neq (z, r)$ implies that

$$C(A) \setminus (z, r) = C(A \cup \{(z, p)\}) \setminus (z, r) = C(\{(x, p), (y, q), (z, p)\}).$$

Second, we prove ii). Consider a menu $\{(x, q), (y, p), (z, p), (z, r)\}$. Since $(x, q) > (y, p) > (z, p)$, by monotonicity, $C(\{(x, q), (y, p), (z, p), (z, r)\}) \not\supseteq (z, p), (y, p)$. By Lemma 3,

$$C(\{(x, q), (y, p), (z, p), (z, r)\}) \not\supseteq (z, p) \text{ implies } C(\{(x, q), (y, p), (z, r)\}) = C(\{(x, q), (y, p), (z, p), (z, r)\})$$

and

$$C(\{(x, q), (y, p), (z, p), (z, r)\}) \not\supseteq (y, p) \text{ implies } C(\{(x, q), (z, p), (z, r)\}) = C(\{(x, q), (y, p), (z, p), (z, r)\}).$$

Therefore, we have

$$C(\{(x, q), (y, p), (z, r)\}) = C(\{(x, q), (y, p), (z, p), (z, r)\}) = C(\{(x, q), (z, p), (z, r)\}).$$

□

8.7 Useful Lemmas for the proof of Theorem 1

Now we turn to the proof of Theorem 1. First, we focus on binary choices.

Lemma 4. *If $\succeq$ satisfies Regularity, Transitivity (i), and Cancellation (i), then there exist strictly increasing and continuous functions u and w with $u(+\infty) = w(+\infty) = +\infty$ such that for any $(x, p), (y, q) \in \mathcal{X}$,*

$$(5) \quad (x, p) \succeq (y, q) \text{ if and only if } u(x) + w(p) \geq u(y) + w(q).$$

We omitted the proof of Lemma 4. See Krantz et al. (1971). Now we assume that we have functions u and w .

Lemma 5. *If C satisfies Regularity, Transitivity, Cancellation (i), and INEA, then there exists a strictly increasing and continuous function $W : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ such that $W(t, 0) = W(0, t) = t$ for each t and for any $A \in \mathcal{A}$,*

$$C(A) = \arg \max_{(x, p) \in A} \{W(u(x) - u(x^A), w(p) - w(p^A))\}.$$

Proof of Lemma 5. First, take a menu $B = \{(x, p), (y, q), (z, r)\}$ with $x > y > z$ and $p < q < r$. Let $W_B(x, p) \equiv u(x) - u(z)$ and $W_B(z, r) \equiv w(r) - w(p)$. Moreover, by Lemma 1, there exists x' such that $(x', p) \sim_{(z, p)} (y, q)$. Then let $W_B(y, q) \equiv u(x') - u(z)$.

Fact 1:

$$C(B) = \arg \max_{a \in B} \{W_B(a)\}.$$

To prove Fact 1, we consider two cases. First, when $C(B) \ni (x, p)$, we shall prove that

$$W_B(x, p) = u(x) - u(z) \geq \max\{W_B(z, r) = w(r) - w(p), W_B(y, q) = u(x') - u(z)\}.$$

By INEA, $(x, p) \succeq_{(y, q)} (z, r)$ implies $(x, p) \succeq (z, r)$; equivalently, $u(x) - u(z) \geq w(r) - w(p)$. Moreover, since $C(\{(x, p), (y, q), (z, r)\}) \ni (x, p)$ implies that $C(\{(x, p), (y, q), (z, r)\}) \neq (z, r)$, by Corollary 1 (i),

$$C(\{(x, p), (y, q), (z, r)\}) \setminus \{(z, r)\} = C(\{(x, p), (y, q), (z, p)\}) \ni (x, p).$$

Therefore, we obtain $(x, p) \succeq_{(z, p)} (y, q)$. Since $W_B(y, q) = u(x') - u(z)$ where $(x', p) \sim_{(z, p)} (y, q)$, by transitivity (ii) we have $u(x) - u(z) \geq u(x') - u(z)$.

Second, when $C(B) \ni (y, q)$, we shall prove that

$$W_B(y, q) = u(x') - u(z) \geq \max\{W_B(z, r) = w(r) - w(p), W_B(x, p) = u(x) - u(z)\}.$$

Since $C(B) \ni (y, q)$ implies that $C(\{(x, p), (y, q), (z, r)\}) \neq (z, r)$, by Corollary 1 (i),

$$C(\{(x, p), (y, q), (z, r)\}) \setminus \{(z, r)\} = C(\{(x, p), (y, q), (z, p)\}) \ni (y, q).$$

Therefore, we obtain $(y, q) \succeq_{(z, p)} (x, p)$. Since $W_B(y, q) = u(x') - u(z)$ where $(x', p) \sim_{(z, p)} (y, q)$, by transitivity (ii) we have $u(x') - u(z) \geq u(x) - u(z)$.

Now we construct W in the following way: for any menu $B = \{(x, p), (y, q), (z, r)\}$ with $x > y > z$ and $p < q < r$, let $W(u(x) - u(z), 0) = W_B(x, p) = u(x) - u(z)$, $W(0, w(r) - w(p)) = W_B(z, r) = w(r) - w(p)$, and $W(u(y) - u(z), w(q) - w(p)) = W_B(y, q)$. Finally, note that $W(u(y) - u(z), w(q) - w(p))$ is a function of y, z, p, q because $W_B(y, q)$ does not depend on x and r . Now we shall prove that W is well-defined.

Fact 2: W is well-defined; that is, for any y, z, p, q and y', z', p', q' , if $u(y) - u(z) = u(y') - u(z')$ and $w(q) - w(p) = w(q') - w(p')$, then $W(u(y) - u(z), w(q) - w(p)) = W(u(y') - u(z'), w(q') - w(p'))$.

First of all, note that $u(y) - u(z) = u(y') - u(z')$ is equivalent to $[y, z]D_1[y', z']$ and $w(q) - w(p) = w(q') - w(p')$ is equivalent to $[q, p]D_2[q', p']$. Now we prove Fact 2 in the following two steps.

Step 1: For any y, z, p, q and y', z' , if $u(y) - u(z) = u(y') - u(z')$, then $W(u(y) - u(z), w(q) - w(p)) = W(u(y') - u(z'), w(q) - w(p))$.

By the definition of W , $W(u(y) - u(z), w(q) - w(p)) = u(x) - u(z)$ where x satisfies $(x, p) \sim_{(z, p)} (y, q)$ and $W(u(y') - u(z'), w(q) - w(p)) = u(x') - u(z')$ where x' satisfies $(x', p) \sim_{(z', p)} (y', q)$. Since $[y, z]D_1[y', z']$, by ED (i), $(x, p) \sim_{(z, p)} (y, q)$ and $(x', p) \sim_{(z', p)} (y', q)$ imply $[x, y]D_1[x', y']$; that is, $u(x) - u(y) = u(x') - u(y')$. Therefore, we have

$$\begin{aligned} W(u(y) - u(z), w(q) - w(p)) &= u(x) - u(z) = u(x) - u(y) + u(y) - u(z) = u(x') - u(y') + u(y') - u(z') \\ &= u(x') - u(z') = W(u(y') - u(z'), w(q) - w(p)). \end{aligned}$$

Step 2: For any y, z, p, q and p', q' , if $w(q) - w(p) = w(q') - w(p')$, then $W(u(y) - u(z), w(q) - w(p)) = W(u(y) - u(z), w(q') - w(p'))$.

By the definition of W , $W(u(y) - u(z), w(q) - w(p)) = u(x) - u(z)$ where x satisfies $(x, p) \sim_{(z, p)} (y, q)$ and $W(u(y) - u(z), w(q') - w(p')) = u(x') - u(z)$ where x' sat-

isfies $(x', p') \sim_{(z, p')} (y, q')$. Let find r and r' such that $(x, p) \sim_{(z, p)} (y, q) \sim_{(z, p)} (z, r)$ and $(x', p') \sim_{(z, p')} (y, q') \sim_{(z, p')} (z, r')$. Since $[q, p]D_2[q', p']$, by ED (ii), $(y, q) \sim_{(z, p)} (z, r)$ and $(y, q') \sim_{(z, p')} (z, r')$ imply $[r, q]D_2[r', q']$; that is, $w(r) - w(q) = w(r') - w(q')$. Since $w(q) - w(p) = w(q') - w(p')$, we obtain $w(r) - w(p) = w(r') - w(p')$.

By Lemma 3, $(x, p) \sim_{(z, p)} (z, r)$ is equivalent to $(x, p) \sim (z, r)$; that is, $u(x) - u(z) = w(r) - w(p)$, and $(x', p') \sim_{(z, p')} (z, r')$ is equivalent to $(x', p') \sim (z, r')$; that is, $u(x') - u(z) = w(r') - w(p')$. Therefore, we have

$$\begin{aligned} W(u(y) - u(z), w(q) - w(p)) &= u(x) - u(z) = w(r) - w(p) = \\ &= w(r') - w(p') = u(x') - u(z) = W(u(y) - u(z), w(q') - w(p')). \end{aligned}$$

Evidently, Step 1-2 complete the proof of Fact 2. Now we will prove that we obtained desired W .

Fact 3: For any menu $A \in \mathcal{A}$,

$$C(A) = \arg \max_{(x, p) \in A} \{W(u(x) - u(x^A), w(p) - w(p^A))\}.$$

Let $|A| = n$. It is obvious when $n = 2$. Now consider the cases when $n \geq 3$. We have already proved the case when $A = B = \{(x, p), (y, q), (z, r)\}$ with $x > y > z$ and $p < q < r$. We need to consider two other cases when $n = 3$.

Case 1: $A = \{(y, q), (y', q'), (z, p)\}$ with $y, y' > z$ and $q, q' > p$ and $C(A) \ni (y, q)$.

By the definition of W , we have $W(u(y) - u(z), w(q) - w(p)) = u(x') - u(z)$ where $(x', p) \sim_{(z, p)} (y, q)$ and $W(u(y') - u(z), w(q') - w(p)) = u(x'') - u(z)$ where $(x'', p) \sim_{(z, p)} (y', q')$. Then by transitivity (ii), $(x', p) \sim_{(z, p)} (y, q) \succeq_{(z, p)} (y', q') \sim_{(z, p)} (x'', p)$ implies $x' \geq x''$. Therefore, $W(u(y) - u(z), w(q) - w(p)) = u(x') - u(z) \geq W(u(y') - u(z), w(q') - w(p)) = u(x'') - u(z)$.

Case 2: $A = \{(x, q), (y, p), (z, r)\}$ with $x > y > z$ and $p < q < r$.

By the definition of W , we have $W(u(x) - u(z), w(q) - w(p)) = u(x') - u(z)$ where $(x', p) \sim_{(z, p)} (x, q)$. Moreover, by Corollary 1 (ii), we have

$$C(\{(x, q), (y, p), (z, r)\}) = C(\{(x, q), (z, p), (z, r)\}).$$

When $C(\{(x, q), (y, p), (z, r)\}) = C(\{(x, q), (z, p), (z, r)\}) \ni (x, q)$, we have $(x, q) \succeq_{(z, p)} (z, r)$. Therefore, by transitivity (ii), we have $(x', p) \sim_{(z, p)} (x, q) \succeq_{(z, p)} (z, r)$ which implies

$$u(x') - u(z) = W(u(x) - u(z), w(q) - w(p)) \geq W(u(z) - u(z), w(r) - w(p)) = w(r) - w(p).$$

When $C(\{(x, q), (y, p), (z, r)\}) = C(\{(x, q), (z, p), (z, r)\}) \ni (z, r)$, we have $(z, r) \succeq_{(z, p)} (x, q)$. Therefore, by transitivity (ii), we have $(z, r) \succeq_{(z, p)} (x', p) \sim_{(z, p)} (x, q)$ which implies

$$W(u(z) - u(z), w(r) - w(p)) = w(r) - w(p) \geq u(x') - u(z) = W(u(x) - u(z), w(q) - w(p)).$$

Now let us consider menus with more than three alternatives. Let $A = \{(x_1, p_1), \dots, (x_n, p_n)\}$ and $C(A) \ni (x_k, p_k)$. We shall prove that for any $i \neq k$,

$$W(u(x_k) - u(x^A), w(p_k) - w(p^A)) \geq W(u(x_i) - u(x^A), w(p_i) - w(p^A)).$$

WLOG, let $x_1 \geq x_2 \geq \dots \geq x_n$ and $p_s = p^A$ for some s . Then we shall prove that for any $i \neq k$,

$$W(u(x_k) - u(x_n), w(p_k) - w(p_s)) \geq W(u(x_i) - u(x_n), w(p_i) - w(p_s)).$$

We consider several cases.

Case 1: $s \neq i, n, k$ and $k = n$.

By INEA and Lemma 3, $C(\{(x_i, p_i), (x_s, p_s), (x_n, p_n)\}) \ni (x_n, p_n)$. Then by the previous argument for $n = 3$, $w(p_n) - w(p_s) \geq W(u(x_i) - u(x_n), w(p_i) - w(p_s))$.

Case 2: $s \neq i, n$ and $k = s$.

By INEA and Lemma 3, $C(\{(x_i, p_i), (x_s, p_s), (x_n, p_n)\}) \ni (x_s, p_s)$. Then by the previous argument for $n = 3$, $u(x_s) - u(x_n) \geq W(u(x_i) - u(x_n), w(p_i) - w(p_s))$.

Case 3: $s = n \neq k$.

By INEA and Lemma 3, $C(\{(x_i, p_i), (x_k, p_k), (x_n, p_n)\}) \ni (x_k, p_k)$. Then by the previous argument for $n = 3$, $W(u(x_k) - u(x_n), w(p_k) - w(p_n)) \geq W(u(x_i) - u(x_n), w(p_i) - w(p_n))$.

Case 4: $s = i$.

By INEA, $C(\{(x_i, p_i), (x_k, p_k), (x_n, p_n)\}) \ni (x_k, p_k)$. Then by the previous argument for $n = 3$, $W(u(x_k) - u(x_n), w(p_k) - w(p_i)) \geq u(x_i) - u(x_n)$.

Case 5: $s \neq i, n, k$ and $k \neq n$.

Let $A' = \{(x_i, p_i), (x_s, p_s), (x_k, p_k), (x_n, p_n)\}$. By INEA, $C(A') \ni (x_k, p_k)$. Let consider a menu $A' \cup \{(x_n, p_s)\} = \{(x_i, p_i), (x_s, p_s), (x_k, p_k), (x_n, p_n), (x_n, p_s)\}$. By monotonicity, $C(A' \cup \{(x_n, p_s)\}) \not\ni (x_n, p_s)$. Therefore, by Lemma 3, $C(A') = C(A' \cup \{(x_n, p_s)\})$. Since $C(A' \cup \{(x_n, p_s)\}) = C(A') \neq (x_n, p_n)$, by Lemma 3,

$$C(A' \cup \{(x_n, p_s)\} \setminus \{(x_n, p_n)\}) = C(A' \cup \{(x_n, p_s)\}) \setminus \{(x_n, p_n)\} = C(A') \setminus \{(x_n, p_n)\} \ni (x_k, p_k).$$

Moreover, $C(A' \cup \{(x_n, p_s)\} \setminus (x_n, p_n)) \neq (x_s, p_s)$, by Lemma 3, we have

$$\begin{aligned} C(A' \cup \{(x_n, p_s)\} \setminus \{(x_n, p_n)\} \setminus \{(x_s, p_s)\}) &= C(A' \cup \{(x_n, p_s)\} \setminus \{(x_n, p_n), (x_s, p_s)\}) \\ &= C(\{(x_i, p_i), (x_k, p_k), (x_n, p_s)\}). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} C(A') \setminus \{(x_n, p_n), (x_s, p_s)\} &= C(A' \cup \{(x_n, p_s)\} \setminus \{(x_n, p_n), (x_s, p_s)\}) \\ &= C(\{(x_i, p_i), (x_k, p_k), (x_n, p_s)\}) \ni (x_k, p_k). \end{aligned}$$

Then by the previous argument for $n = 3$, $W(u(x_k) - u(x_n), w(p_k) - w(p_s)) \geq W(u(x_i) - u(x_n), w(p_i) - w(p_s))$.

□

8.8 Proof of Theorem 1

By Lemma 5, there exist strictly increasing continuous functions u , w , and $W : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ such that $W(t, 0) = W(0, t) = t$ for each t and for any $A \in \mathcal{A}$,

$$C(A) = \arg \max_{(x, p) \in A} \{W(u(x) - u(x^A), w(p) - w(p^A))\}.$$

Fact: If C satisfies cancellation, then there exists a strictly increasing continuous function f such that $W(t_1, t_2) = f^{-1}(f(t_1) + f(t_2))$.

Let us define a binary relation $\succeq_W$ on $\mathbb{R}_+^2$ such that $(t_1, t_2) \succeq_W (t'_1, t'_2)$ if and only if $W(t_1, t_2) \geq W(t'_1, t'_2)$. Note that $\succeq_W$ is a continuous weak order. By Cancellation and Transitivity, there exist strictly increasing continuous functions f_1 and f_2 such that $(t_1, t_2) \succeq_W (t'_1, t'_2)$ if and only if $f_1(t_1) + f_2(t_2) \geq f_1(t'_1) + f_2(t'_2)$. See Krantz et al. (1971).

Take any t_1, t_2, t'_1, t'_2 such that $W(t_1, t_2) = W(t'_1, t'_2)$. Equivalently, we have $f_1(t_1) + f_2(t_2) = f_1(t'_1) + f_2(t'_2)$. Given that f_1 is continuous and strictly increasing, we obtain $t_1 = f_1^{-1}(f_1(t'_1) + f_2(t'_2) - f_2(t_2))$. Therefore, $W(t_1, t_2) = W(t'_1, t'_2)$ implies that

$$W(f_1^{-1}(f_1(t'_1) + f_2(t'_2) - f_2(t_2)), t_2) = W(t'_1, t'_2).$$

Since the right hand side does not depend on t_2 , the left hand side also should not depend on t_2 . Therefore, there exists a continuous and strictly increasing function F such that

$$W(f_1^{-1}(f_1(t'_1) + f_2(t'_2) - f_2(t_2)), t_2) = F(f_1(t'_1) + f_2(t'_2)) = W(t'_1, t'_2).$$

Since $W(t, 0) = W(0, t) = t$, we have $f_1 = f_2 = F^{-1} = f$ for some f . Therefore, $W(t_1, t_2) = f^{-1}(f(t_1) + f(t_2))$.

Now it is easy to see that for any $A \in \mathcal{A}$,

$$C(A) = \arg \max_{(x,p) \in A} \{f(u(x) - u(x^A)) + f(w(p) - w(p^A))\}.$$

Uniqueness: Take any two vectors of functions (f, u, w) and (f', u', w') such that $f(0) = f'(0) = u(0) = u'(0) = w(0) = w'(0) = 0$, $u(1) = u'(1)$, and $f(1) = f'(1)$. We prove the uniqueness in two steps.

Step 1. $u' = u$ and $w' = w$.

Take any $y, q \in \mathbb{R}_+$ with $0 < q$. By Lemma 1, there exists $x \in \mathbb{R}_+$ such that $(x, 0) \sim (y, q)$. Then we have

$$u(x) = u(y) + w(q) \Leftrightarrow u'(x) = u'(y) + w'(q); \text{ equivalently,}$$

$x = u^{-1}(u(y) + w(q)) = u'^{-1}(u'(y) + w'(q))$ since all functions are strictly increasing and continuous.

First, let $y = 0$. Then we have $u^{-1}(w(\cdot)) = u'^{-1}(w'(\cdot))$; that is, $u'(u^{-1}(\cdot)) = w'(w^{-1}(\cdot))$. Second, let $u(y) = t_1$, $w(q) = t_2$, and $h(t) = u'(u^{-1}(t))$. Since u and u' are strictly increasing and continuous and $u(\mathbb{R}_+) = u'(\mathbb{R}_+) = \mathbb{R}_+$, h is strictly increasing and continuous and $h(\mathbb{R}_+) = \mathbb{R}_+$. Then we have

$$h(t_1 + t_2) = u'(u^{-1}(t_1 + t_2)) = u'(u^{-1}(t_1)) + w'(w^{-1}(t_2)) = u'(u^{-1}(t_1)) + u'(u^{-1}(t_2)) = h(t_1) + h(t_2).$$

Since we obtained a typical Cauchy functional equation (See Kuczma (2008)), there exists $\alpha > 0$ such that $h(t) = \alpha t$; that is, $u' = \alpha u$ and $w' = \alpha w$. Moreover, since $u(1) = u'(1) > 0$, we have $u' = u$ and $w' = w$.

Step 2. $f' = f$.

Take any $y, q \in \mathbb{R}_+$ with $q > 0$. By Lemma 1, there exists x such that $(x, 0) \sim_{(0,0)} (y, q)$. Then we have

$$f(u(x)) = f(u(y)) + f(w(q)) \Leftrightarrow f'(u(x)) = f'(u(y)) + f'(w(q)).$$

Since u, w, f, f' are strictly increasing and continuous, similar to Step 1, we will obtain a Cauchy functional equation. Therefore, there exists $\beta > 0$ such that $f' \circ u = \beta \cdot f \circ u$ and $f' \circ w = \beta \cdot f \circ w$. Since $f'(1) = f(1)$, we have $f' = f$.

9 Appendix B: Using both Maximums and Minimums

Here we consider models in which reference points not only depend on the minimums of the menu, but also the maximums. Let $(x_A, p_A) \equiv (\max_{(x,p) \in A} x, \max_{(x,p) \in A} p)$ for each $A \in \mathcal{A}$ (i.e., the join of A). We focus on the following more general model: for any menu $A \in \mathcal{A}$,

$$(6) \quad C(A) = \arg \max_{(x,p) \in A} \{f(u(x) - u(r_1(x_A, x^A))) + f(w(p) - w(r_1(p_A, p^A)))\}$$

where r_1 and r_2 are strictly increasing reference functions. In Section 9.1-2, we discuss implications of (6) by reconsidering section 2.2.

9.1 Attraction Effect and Compromise Effect

First, we discuss a relation between the compromise and attraction effects. Previously, Proposition 1 showed that an agent exhibits the the compromise effect if and only if she exhibits the attraction effect. But in the model (6), we have a weaker prediction. In particular, we can show that if the agent exhibits the attraction effect, then she exhibits the compromise effect, which means that the compromise effect is easier to be exhibited.

To illustrate, suppose the agent exhibits the attraction effect at some (x, p) , (y, q) , and (z, t) with $x > y > z$ and $q > t > p$. That is, $(x, p) \succ (y, q)$ and $(x, p) \prec_{(z, t)} (y, q)$. By the representation (6), we have

$$f(u(x) - u(r_1(x, y))) - f(u(y) - u(r_1(x, y))) > f(w(q) - w(r_2(q, p))) - f(w(p) - w(r_2(q, p)))$$

and

$$f(u(x) - u(r_1(x, z))) - f(u(y) - u(r_1(x, z))) < f(w(q) - w(r_2(q, p))) - f(w(p) - w(r_2(q, p))).$$

Take any r such that $r > q$. Then since $r_2(r, p) > r_2(q, p)$, by diminishing sensitivity, we have

$$f(w(q) - w(r_2(q, p))) - f(w(p) - w(r_2(q, p))) < f(w(q) - w(r_2(r, p))) - f(w(p) - w(r_2(r, p))).$$

Therefore, we have

$$f(u(x) - u(r_1(x, z))) - f(u(y) - u(r_1(x, z))) < f(w(q) - w(r_2(r, p))) - f(w(p) - w(r_2(r, p)));$$

equivalently, $(x, p) \prec_{(z, r)} (y, q)$. In other words, the agent exhibits the compromise effect at

(x, p) , (y, q) , and (z, r) since $(x, p) \succ (y, q)$ and $(x, p) \prec_{(z, r)} (y, q)$.

9.2 Symmetric Dominance and Two Decoy Effect

Second, we discuss the effect of the symmetrically dominated third alternative (z, s) as in Observation 1. Consider again two alternatives (x, p) and (y, q) with $(x, p) \succ (y, q)$ and $x > y$ and $p < q$. Remember that Observation 1 showed that when $s < p < t < q$, observing $(x, p) \prec_{(z, s)} (y, q)$ is harder than $(x, p) \prec_{(z, t)} (y, q)$; that is, $(x, p) \prec_{(z, s)} (y, q)$ implies $(x, p) \prec_{(z, t)} (y, q)$. We obtain the same result as in Observation 1.

To illustrate, suppose $(x, p) \prec_{(z, s)} (y, q)$. By the representation (6), we have

$$f(u(x) - u(r_1(x, z))) - f(u(y) - u(r_1(x, z))) < f(w(q) - w(r_2(q, s))) - f(w(p) - w(r_2(q, s))).$$

By diminishing sensitivity and $r_2(q, p) > r_2(q, s)$, we have

$$f(w(q) - w(r_2(q, s))) - f(w(p) - w(r_2(q, s))) < f(w(q) - w(r_2(q, p))) - f(w(p) - w(r_2(q, p))).$$

Therefore, we have

$$f(u(x) - u(r_1(x, z))) - f(u(y) - u(r_1(x, z))) < f(w(q) - w(r_2(q, p))) - f(w(p) - w(r_2(q, p)));$$

equivalently, $(x, p) \prec_{(z, t)} (y, q)$.

Finally, note that the symmetric dominance and the two decoy effect are also equivalent as in Section 2.2 since the maximums and the minimums are identical. Therefore, we can summarize Section 9 in the following way ($\dot{\geq}$ represents the difficulty of observing a preference reversal):

Two Decoy $\dot{=}$ Symmetric Dominance $\dot{>}$ Attraction (Asymmetric) $\dot{>}$ Compromise.

10 Appendix C: Weakening Transitivity and Comparing Diminishing Sensitivities of Two Dimensions

In Section 2.2 and Section 3.2, we demonstrated that the equivalence between diminishing sensitivity and the compromise and attraction effects. In particular, Proposition 2 shows that if C is more diminishing sensitive (if f is more concave), then the compromise and attraction effects are more likely to be observed. So far we treated two dimensions in a symmetric way

in sense that two dimensions have a common distortion function f (recall and compare (1) and (2)). Now we consider a general representation in which two dimensions have different distortion functions, f and g as in (1). More formally,

Definition 5. A choice correspondence C is a *general additive reference dependent choice* (GARDC) if there are strictly increasing functions f, g, u, w such that $f(0) = g(0) = 0$ and for any menu $A \in \mathcal{A}$,

$$(7) \quad C(A) = \arg \max_{(x,p) \in A} \{f(u(x) - u(x^A)) + g(w(p) - w(p^A))\}.$$

Now we have two different diminishing sensitivities for two dimensions and they are defined as in (3). Indeed, diminishing sensitivities are equivalent to the strict concavity of f and g . The new representation allows us to compare two dimensions in terms of diminishing sensitivity; i.e., to know which of f and g is more concave. For example, if the first dimension is more diminishing sensitive than the second dimension; i.e., f is more concave than g , then the compromise and attraction effects are likely to be happen when the the third alternative (that is added to a two binary menu) has the minimum of the first dimension rather than the second dimension.

It turns out the comparison of two dimensions in terms of diminishing sensitivity is closely related to violations of transitivity of binary comparisons.²⁴ Violations of transitivity are consistently documented in experimental literature.²⁵

Now we discuss how to know which of two dimensions is more diminishing sensitivity; that is, which of f and g is more concave, from observed choices. Essentially, a direction of a violation of transitivity tells us which dimension is more diminishing sensitive. Indeed, f can be more concave than g for only a subset of $\mathbb{R}_+$, but in order to illustrate a connection between violations of transitivity and the relative diminishing sensitivity, we suppose $f(t) = t^\alpha$ and $g(t) = t^\beta$ for some α and β . Therefore, we have

$$C(A) = \arg \max_{(x,p) \in A} \{(u(x) - u(x^A))^\alpha + (w(p) - w(p^A))^\beta\}.$$

Our objective is to know either $\alpha = \beta$, $\alpha > \beta$, or $\alpha < \beta$. Take some alternatives (x, p) ,

²⁴In fact, we can obtain (7) by weakening transitivity of $\succeq$, but we will omit a behavioral foundation for it since it is very similar to Theorem 1.

²⁵For example, a violation of transitivity of pairwise comparisons of binary lotteries is documented in Tversky (1969), Lindman and Lyons (1978), Budescu and Weiss (1987), Loomes et al. (1991), and Day and Loomes (2010).

(y, q) , and (z, r) with $x > y > z$ and $r > q > p$ such that

$$(x, p) \sim (y, q) \text{ and } (y, q) \sim (z, r); \text{ equivalently,}$$

by our representation,

$$(u(x) - u(y))^{\frac{\alpha}{\beta}} = w(q) - w(p) \text{ and } (u(y) - u(z))^{\frac{\alpha}{\beta}} = w(r) - w(q).$$

There are three cases for the comparison between (x, p) and (z, r) : either i) $(x, p) \sim (z, r)$; ii) $(x, p) \prec (z, r)$; or iii) $(x, p) \succ (z, r)$.

In the first case, we have $\alpha = \beta$ because $(x, p) \sim (z, r)$ implies

$$(u(x) - u(y))^{\frac{\alpha}{\beta}} + (u(y) - u(z))^{\frac{\alpha}{\beta}} = w(q) - w(p) + w(r) - w(q) = w(r) - w(p) = (u(x) - u(z))^{\frac{\alpha}{\beta}}.$$

Therefore, if transitivity is satisfied, two dimensions are the same in terms of diminishing sensitivity ($\alpha = \beta$).

In the second case, we have $\alpha < \beta$ because $(x, p) \prec (z, r)$ implies

$$(u(x) - u(y))^{\frac{\alpha}{\beta}} + (u(y) - u(z))^{\frac{\alpha}{\beta}} = w(q) - w(p) + w(r) - w(q) = w(r) - w(p) > (u(x) - u(z))^{\frac{\alpha}{\beta}}.$$

Therefore, if transitivity is violated in a direction $(x, p) \sim (y, q) \sim (z, r) \succ (x, p)$, then the first dimension is more diminishing sensitive than the second dimension ($\alpha < \beta$).

Lastly, if transitivity is violated in a direction $(x, p) \sim (y, q) \sim (z, r) \prec (x, p)$, then the second dimension is more diminishing sensitive than the first dimension ($\alpha > \beta$).

References

- APESTEGUIA, J. AND M. A. BALLESTER (2009): “A theory of reference-dependent behavior,” *Economic Theory*, 40, 427–455.
- (2013): “Choice by sequential procedures,” *Games and Economic Behavior*, 77, 90–99.
- ATTANASIO, O. P. AND G. WEBER (2010): “Consumption and Saving: Models of Intertemporal Allocation and Their Implications for Public Policy,” *Journal of Economic Literature*, 48, 693–751.

- BARBERIS, N. C. (2013): “Thirty years of prospect theory in economics: A review and assessment,” *Journal of Economic Perspectives*, 27, 173–96.
- BENARTZI, S. AND R. H. THALER (1995): “Myopic loss aversion and the equity premium puzzle,” *The quarterly journal of Economics*, 110, 73–92.
- BORDALO, P., N. GENNAIOLI, AND A. SHLEIFER (2012): “Salience theory of choice under risk,” *The Quarterly Journal of Economics*, 127, 1243–1285.
- (2013): “Salience and Consumer choice,” *Journal of Political Economy*, 121, 803–843.
- BRUNI, L. AND R. SUGDEN (2007): “The road not taken: how psychology was removed from economics, and how it might be brought back*,” *The Economic Journal*, 117, 146–173.
- BUDESCU, D. AND W. WEISS (1987): “Reflection of transitive and intransitive preferences: A test of prospect theory,” *Organizational Behavior and Human Decision Processes*, 39, 184–202.
- CAMPBELL, J. Y. AND N. G. MANKIW (1991): “The response of consumption to income: a cross-country investigation,” *European economic review*, 35, 723–756.
- CHEREPANOV, V., T. FEDDERSEN, AND A. SANDRONI (2013): “Rationalization,” *Theoretical Economics*, 8, 775–800.
- CHERNEV, A. (2004): “Extremeness Aversion and Attribute-Balance Effects in Choice,” *Journal of Consumer Research*, 31, 249–263.
- DAY, B. AND G. LOOMES (2010): “Conflicting violations of transitivity and where they may lead us,” *Theory and decision*, 68, 233–242.
- DE CLIPPEL, G. AND K. ELIAZ (2012): “Reason-based choice: A bargaining rationale for the attraction and compromise effects,” *Theoretical Economics*, 7, 125–162.
- DEATON, A. (1991): “Saving and liquidity constraints,” *The Quarterly Journal of Economics*, 59, 1221–48.
- DOYLE, J. R., D. J. O’CONNOR, G. M. REYNOLDS, AND P. A. BOTTOMLEY (1999): “The robustness of the asymmetrically dominated effect: Buying frames, phantom alternatives, and in-store purchases,” *Psychology & Marketing*, 16, 225–243.
- ECHENIQUE, F., K. SAITO, AND G. TSERENJIGMID (2013): “The Perception-Adjusted Luce Model,” *Working Paper, Caltech*.

- EHLERS, L. AND Y. SPRUMONT (2008): “Weakened WARP and top-cycle choice rules,” *Journal of Mathematical Economics*, 44, 87–94.
- ELIAZ, K., M. RICHTER, AND A. RUBINSTEIN (2011): “Choosing the two finalists,” *Economic Theory*, 46, 211–219.
- FISHBURN, P. C. AND A. RUBINSTEIN (1982): “Time preference,” *International Economic Review*, 23, 677–694.
- FUDENBERG, D., R. IJIMA, AND T. STRZALECKI (2013): “Stochastic Choice and Revealed Perturbed Utility,” Tech. rep., Mimeo Harvard University.
- HERNE, K. (1997): “Decoy alternatives in policy choices: Asymmetric domination and compromise effects,” *European Journal of Political Economy*, 13, 575–589.
- (1999): “The effects of decoy gambles on individual choice,” *Experimental Economics*, 2, 31–40.
- HUBER, J., J. W. PAYNE, AND C. PUTO (1982): “Adding asymmetrically dominated alternatives: Violations of regularity and the similarity hypothesis,” *Journal of consumer research*, 90–98.
- JAPPELLI, T. (1990): “Who is credit constrained in the US economy?” *The Quarterly Journal of Economics*, 105, 219–234.
- KAHNEMAN, D. AND A. TVERSKY (1979): “Prospect theory: An analysis of decision under risk,” *Econometrica: Journal of the Econometric Society*, 263–291.
- KAMENICA, E. (2008): “Contextual inference in markets: On the informational content of product lines,” *The American Economic Review*, 98, 2127–2149.
- KOSZEGI, B. AND M. RABIN (2006): “A MODEL OF REFERENCE-DEPENDENT PREFERENCES.” *Quarterly journal of economics*, 121.
- (2007): “Reference-dependent risk attitudes,” *The American Economic Review*, 97, 1047–1073.
- (2009): “Reference-dependent consumption plans,” *The American Economic Review*, 99, 909–936.
- KRANTZ, D., D. LUCE, P. SUPPES, AND A. TVERSKY (1971): “Foundations of measurement, Vol. I: Additive and polynomial representations,” .

- KUCZMA, M. (2008): *An introduction to the theory of functional equations and inequalities: Cauchy's equation and Jensen's inequality*, Springer.
- LINDMAN, H. AND J. LYONS (1978): "Stimulus complexity and choice inconsistency among gambles," *Organizational Behavior and Human Performance*, 21, 146–159.
- LOOMES, G., C. STARMER, AND R. SUGDEN (1991): "Observing violations of transitivity by experimental methods," *Econometrica: Journal of the Econometric Society*, 425–439.
- MANZINI, P. AND M. MARIOTTI (2007): "Sequentially rationalizable choice," *The American Economic Review*, 1824–1839.
- (2012a): "Categorize then choose: Boundedly rational choice and welfare," *Journal of the European Economic Association*, 10, 1141–1165.
- (2012b): "Choice by lexicographic semiorders," *Theoretical Economics*, 7, 1–23.
- MANZINI, P., M. MARIOTTI, AND C. J. TYSON (2013): "Two-stage threshold representations," *Theoretical Economics*, 8, 875–882.
- MASATLIOGLU, Y., D. NAKAJIMA, AND E. Y. OZBAY (2012): "Revealed attention," *The American Economic Review*, 102, 2183–2205.
- MASATLIOGLU, Y. AND E. A. OK (2005): "Rational choice with status quo bias," *Journal of Economic Theory*, 121, 1–29.
- (2013): "A canonical model of choice with initial endowments," *The Review of Economic Studies*.
- MASATLIOGLU, Y. AND N. ULER (2013): "Understanding the reference effect," *Games and Economic Behavior*, 82, 403–423.
- MUNRO, A. AND R. SUGDEN (2003): "On the theory of reference-dependent preferences," *Journal of Economic Behavior & Organization*, 50, 407–428.
- NATENZON, P. (2010): "Random choice and learning," Tech. rep., Working paper, Princeton University.
- OK, E. A., P. ORTOLEVA, AND G. RIELLA (2014): "Revealed (p) reference theory," *Forthcoming, The American Economic Review*.
- PAGEL, M. (2013): "Expectations-Based Reference-Dependent Life-Cycle Consumption," .

- SAGI, J. S. (2006): “Anchored preference relations,” *Journal of Economic Theory*, 130, 283–295.
- SHARPE, K. M., R. STAELIN, AND J. HUBER (2008): “Using extremeness aversion to fight obesity: policy implications of context dependent demand,” *Journal of Consumer Research*, 35, 406–422.
- SIMONSON, I. (1989): “Choice based on reasons: The case of attraction and compromise effects,” *Journal of consumer research*, 158–174.
- SIMONSON, I. AND A. TVERSKY (1992): “Choice in context: tradeoff contrast and extremeness aversion.” *Journal of marketing research*.
- SUE O’CURRY, Y. P. AND R. PITTS (1995): “The attraction effect and political choice in two elections,” *Journal of Consumer Psychology*, 4, 85–101.
- SUGDEN, R. (2003): “Reference-dependent subjective expected utility,” *Journal of economic theory*, 111, 172–191.
- TEPPAN, E. C. AND A. FELFERNIG (2009): “Minimization of product utility estimation errors in recommender result set evaluations,” in *Proceedings of the 2009 IEEE/WIC/ACM International Joint Conference on Web Intelligence and Intelligent Agent Technology-Volume 01*, IEEE Computer Society, 20–27.
- TVERSKY, A. (1969): “Intransitivity of preferences.” *Psychological Review; Psychological Review*, 76, 31.
- TVERSKY, A. AND D. KAHNEMAN (1991): “Loss aversion in riskless choice: A reference-dependent model,” *The Quarterly Journal of Economics*, 106, 1039–1061.
- TVERSKY, A. AND I. SIMONSON (1993): “Context-dependent preferences,” *Management science*, 39, 1179–1189.
- WAKKER, P. (1988): “The algebraic versus the topological approach to additive representations,” *Journal of Mathematical Psychology*, 32, 421–435.
- WERNERFELT, B. (1995): “A rational reconstruction of the compromise effect: Using market data to infer utilities,” *Journal of Consumer Research*, 627–633.
- ZELDES, S. P. (1989): “Consumption and liquidity constraints: an empirical investigation,” *The Journal of Political Economy*, 97, 305.