DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

EXISTENCE AND TESTABLE IMPLICATIONS OF EXTREME STABLE MATCHINGS

Federico Echenique
California Institute of Technology

SangMok Lee
California Institute of Technology

M. Bumin Yenmez
Microsoft Research and Carnegie Mellon University

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Abstract

We investigate the testable implications of the theory that markets produce matchings that are optimal for one side of the market; i.e. stable *extremal* matchings. A leading justification for the theory is that markets proceed as if the deferred acceptance algorithm were in place. We find that the theory of stable extremal matching is observationally equivalent to requiring that there be a unique matching, or that the matching be consistent with unrestricted monetary transfers. We also present results on rationalizing a matching as the median stable matching.

We work with a general model of matching, which encompasses aggregate and random matchings as special cases. As a consequence, we need to work with a notion of strong stability, and extend the standard theory on the existence and structure of extremal matchings.

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Existence and Testable Implications of Extreme Stable Matchings*

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1 Introduction

The celebrated *deferred acceptance algorithm* was first introduced by Gale and Shapley (1962). It has been widely adapted to clearinghouses for centralized two-sided matching markets. The algorithm produces the outcome which is the best outcome for one side of the market and the worst for the other side (Roth and Sotomayor, 1990); dubbed as the *extremal* outcomes. For instance, in the market for medical residents, the clearinghouse selects the optimal matching for doctors, while in the application to school choice the student-optimal matching is chosen (Roth, 2008). In fact, the deferred acceptance algorithm is not only viewed as a recipe of how we should clear centralized markets, but also as a template for how actual decentralized markets behave: Roth (2008, p. 550) notes that the algorithm corresponds to a "folk model" of how markets proceed when they work in an orderly fashion. Indeed, if offers in a labor market are driven by the firms then we may expect a firm optimal matching. If men make most marriage proposals, then perhaps marriages are optimal for men.

To empirically verify this intuition for decentralized markets, we need to understand when there exist underlying preferences for agents that rationalize a matching as extremal. We suppose that we have data on who matches with whom, but that we do not observe agents' preferences. We want to know which observed matchings are compatible with the outcome of the deferred acceptance algorithm: this exercise is the main focus of our paper. In other words, we study the testable implications of extremal selections in cases where agents' preferences are not known.

*We thank Lars Ehlers for questions that motivated the current research.

Our assumption on data is entirely realistic; as we shall see, a special case of our model is the class of data empirical researchers use for studying marriage matching. We want to know when we can say that agents have matched as in the optimal matching for one side. For example, if we have information on the decentralized matching of college freshmen to different dorms, can we determine if the matching in this decentralized market is also optimal for the freshmen? In a dating or marriage market, can we say if the observed relationships correspond to a matching that is optimal for either men or women? It turns out that our model can also accommodate random matching, so that given a table of probabilistic assignments (for example one matching children to schools, see Abdulkadirolu, Pathak, and Roth (2005)), we can tell whether the random matching is compatible with stability and extremal stability.

To lay some necessary groundwork, we first study stability in a general model of population matching; including, as special cases, *aggregate* and *random* matching. In doing so, we extend the basic theory of stable matching to our general model; including proofs of the existence and polarity properties of *strongly stable* matchings. These results are important because, in order to study extremal matchings empirically, we must first establish that they exist and behave in the usual ways.

Our model encompasses aggregate matchings (Choo and Siow, 2006; Dagsvik, 2000; Echenique, Lee, and Shum, 2010), and random matching (Hylland and Zeckhauser, 1979; Roth, Rothblum, and Vate, 1993; Kesten and Ünver, 2009; Alkan and Gale, 2003) as special cases. We look at the notion of strong stability, which is the natural notion of stability for aggregate matching. For random matching, strong stability captures the possibility of ex-ante trades among agents, or, alternatively, it captures basic fairness properties. In contrast with standard (or weak) stability, strong stability results in non-linear constraints on matchings. As a result, the geometry of stable matchings is complicated, and we need to use fixed-point methods instead of the linear programming approach.

Our main results are as follows. We characterize the matchings that are rationalizable as the optimal matching for one side of the market.[1] The main implication of the characterization is that a matching is rationalizable as optimal if and only if it is rationalizable as the unique stable matching in the market; in turn this happens if and only if the matching is rationalizable if the market allowed for transfers. Hence, the empirical

[1]We do not study the deferred acceptance procedure *per se*, which is shown to produce extremal outcomes for less general matching markets (Roth and Sotomayor, 1990), but the extremal outcomes themselves. It is clear that in aggregate markets the Gale-Shapley result still holds. Kesten and Ünver (2009) adopt the deferred acceptance algorithm to environments when matchings can be non-integer.

content of the hypothesis that marriage matching is man-optimal, or woman-optimal, is the same as the hypothesis that there are unlimited quasilinear monetary transfers in the market. In other words, the theory of one-sided optimal stable matching is observationally equivalent to the theory of unique stable matching; and observationally equivalent to the theory of transferable-utility matching.

The theory of stable matching without transfers was first developed by Gale and Shapley (1962). Gale and Shapley also propose the algorithm that finds an extremal stable matching. On the other hand, the theory of matching with transfers was developed by Shapley and Shubik (1971), and (with great impact) applied to marriage matchings by Becker (1973). *Our results imply that from the purely empirical viewpoint, these theories are equivalent. They have the same empirical content.*

Our results are directly applicable to actual data on matchings. For examples, there are "marriage tables" readily available: these show the numbers of agents of a particular age, level of education and income (for example) that are married in a particular year. By simply inspecting these tables one can test for extremal stability (or equivalently for unique stable matching). Similarly, one could analyze a random matching, such as the one proposed by a particular randomized mechanism for assigning students to schools. For example, if we guarantee a set of students a strictly positive probability of matching with a set of schools, then our results imply the resulting random matching cannot be student (or school) optimal.

1.1 Related literature

A number of papers study the empirical content of stability for aggregate matching (Choo and Siow, 2006; Dagsvik, 2000; Echenique, Lee, and Shum, 2010). There are not results, however, on the joint hypothesis that matchings are stable and optimal for one side of the market. Echenique, Lee, and Shum (2010) characterize the aggregate matchings that are rationalizable as stable, but their results are silent on which stable matching is selected when a matching is rationalized.

Random and fractional matchings are studied by Vande Vate (1989); Rothblum (1992); Roth, Rothblum, and Vate (1993); Kesten and Ünver (2009), but not from the revealed preference perspective of understanding which random matchings are rationalizable when preferences are unobserved. With the exception of Kesten and Ünver (2009), these papers focus on the standard notion of stability for fractional matching. Roth, Rothblum, and Vate (1993) introduce the idea of strong stability, as we use it, but obtain only very weak results: as we explain in Section 5, the problem is that strong stability gives rise to a system of quadratic equations (Vande Vate (1989); Rothblum (1992); Roth, Rothblum, and Vate (1993) use linear programming techniques). Our existence results are closer to Alkan and Gale (2003), who present a general approach to stable matching. Alkan and Gale work with choice functions instead of the incomplete first-order stochastic dominance preference relation that we use, but it is likely that their methods can be adapted to show existence of extremal matchings in our model. Of course, Alkan and Gale do not treat the issue of rationalizing a matching as stable when preferences are unknown.

We use Tarski's fixed point theorem to show the existence of stable matchings that are optimal for one side of the market. This approach has been used in the matching literature before. For example, see Roth and Sotomayor (1988); Adachi (2000); Fleiner (2003); Echenique and Oviedo (2004, 2006); Echenique and Yenmez (2007); Ostrovsky (2008); Hatfield and Milgrom (2005); Komornik, Komornik, and Viauroux (2010).

2 Model

2.1 Preliminary definitions

If S is a set, and $\leq$ is a partial order on S we say that the pair $(S, \leq)$ is a partially ordered set. We say that $x, y \in S$ are *comparable* if $x \leq y$ or $y \leq x$.

A partially ordered set $(S, \leq)$ is a *lattice* if, for every $x, y \in S$, the least upper bound, and the greatest lower bound of $\{x, y\}$ exist in S for the partial order $\leq$. We denote the least upper bound of $\{x, y\}$ by $x \vee y$; and the greatest lower bound of $\{x, y\}$ by $x \wedge y$. A lattice $(S, \leq)$ is *distributive* if the following holds: for all $x, y, z \in S$:

- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

An (undirected) *graph* is a pair $G = (V, L)$, where V is a set and L is a subset of $V \times V$. A *path* in G is a sequence $p = \langle v_0, \ldots, v_N \rangle$ such that for $n \in \{0, \ldots, N-1\}$, $(v_n, v_{n+1}) \in L$. We write $v \in p$ to denote that v is a vertex in p. A path $\langle v_0, \ldots, v_N \rangle$ *connects* the vertices v_0 and v_N. A path $\langle v_0, \ldots, v_N \rangle$ is *minimal* if there is no proper subsequence of $\langle v_0, \ldots, v_N \rangle$ which is also a path connecting the vertices v_0 and v_N. The *length* of a path $\langle v_0, \ldots, v_N \rangle$ is N.

A *cycle* in G is a path $c = \langle v_0, \ldots, v_N \rangle$ with $v_0 = v_N$. A cycle is *minimal* if for any two vertices v_n and $v_{n'}$ in c, the paths in c from v_n to $v_{n'}$ and from $v_{n'}$ to v_n are minimal. We call v and w *adjacent in* c if there is n such that $v_n = v$ and $v_{n+1} = w$ or $v_n = w$ and $v_{n+1} = v$. If c and c' are two cycles, and there is a path from a vertex of c to a vertex of c', then we say that c and c' are *connected*.

If $X = (x_{i,j})$ is a matrix then $x_{i,\cdot}$ is the ith row and $x_{\cdot,j}$ the jth column. When it is not ambiguous, we write x_i for $x_{i,\cdot}$ and x_j for $x_{\cdot,j}$.

2.2 Model

The primitives of the model are represented by a four-tuple $\langle M, W, P, K \rangle$, where

- M and W are finite and disjoints sets of, respectively *types of men*, and *types of women*.
- P is a *preference profile*: a list of preferences P_m for every type of man m and P_w for every type of woman w. Each P_w is a linear order over $W \cup \{\emptyset\}$, and each P_m is a linear order over $M \cup \{\emptyset\}$. Here, $\emptyset$ represents the alternative of being unmatched. The weak order associated with P_a is denoted by R_a for any $a \in M \cup W$.
- K is a list of non-negative real numbers K_m for each $m \in M$ and K_w for each $w \in W$. There are K_m men of type m and K_w women of type w.

Suppose that there are l types of women and n types of men. Therefore, we can enumerate W as $\{w_1, \ldots, w_l\}$ and M as $\{m_1, \ldots, m_n\}$.

Let $\mathcal{X}_m = \left\{x \in \mathbf{R}_+^{l+1} : \sum_i x_i = K_m\right\}$.[2] Define a partial order $\leq_m$ on $\mathcal{X}_m$ as follows $y \leq_m x$ iff

$$\forall w \in W \quad \sum_{i: w_i R_m w} y_i \leq \sum_{i: w_i R_m w} x_i;$$

interpret w_{l+1} as $\emptyset$, the option of remaining single. Note that $\leq_m$ is defined by analogy to first order stochastic dominance. Letting $\mathcal{X}_w = \left\{x \in \mathbf{R}_+^{n+1} : \sum_i x_i = K_w\right\}$, we define $\leq_w$ in an analogous way.

[2] $\mathbf{R}_+$ denotes the set of non-negative real numbers.

A *matching* is a matrix $X = (x_{m,w})_{(m,w)\in M\times W}$ such that $x_{m,w} \in \mathbf{R}_+$, $\sum_w x_{m,w} \leq K_m$ and $\sum_m x_{m,w} \leq K_w$.

We introduce two partial orders on matchings. Suppose X and Y are matchings, then:

- $X \leq_M Y$ if, for all m, $x'_m \leq_m y'_m$
- $X \leq_W Y$ if, for all w, $x'_w \leq_w y'_w$.

where x'_m is the vector in $\mathbf{R}^{l+1}_+$ obtained from $x_{\cdot,w} \in \mathbf{R}^l_+$ by assigning $K_m - \sum_w x_{m,w}$ to the entry corresponding to $l+1$ ($K_m - \sum_w x_{m,w}$ being the mass of single m-agents in X). Similarly for y'_m, x'_w and y'_w.

Definition 1. *A matching X is individually rational if $x_{m,w} > 0$ implies that $w \; R_m \; \emptyset$ and $m \; R_w \; \emptyset$.*

A pair (m, w) is a blocking pair for A if there is m' and w' such that $m \, P_w \, m'$, $w \, P_m \, w'$, $x_{m,w'} > 0$, and $x_{m',w} > 0$.

A matching X is stable if it is individually rational and there are no blocking pairs for X.

Denote by $S(M, W, P, K)$ the set of all stable matchings in $\langle M, W, P, K\rangle$.

Two special cases of our model are worth emphasizing: The model of *random matching* is obtained when $K_m = 1$ for all m, and K_w is a positive integer, for all w. The interpretation of random matching is that men are "students" and women are "schools." Students are assigned a school at random, and each school w has K_w seats available for students. In real-life school choice, the randomization often results from indifferences in schools' preferences over students (Abdulkadirolu, Pathak, and Roth, 2005); Matching theory requires strict preferences, so a random "priority order" is produced for the schools in order to break indifferences. Random matchings arise in many other situations as well because random assignment is often a basic consequence of fairness considerations. Here we are mainly interested in situations where a random assignment is given in unambiguous terms, but preferences are unobserved (it is also possible that they are observed but we suspect that they have been misrepresented, or observed with error).

The second model is that of *aggregate matching*, where all numbers K_m and K_w are natural numbers, and all entries of matchings X are natural numbers. The interpretation is that there are K_m men of type m, and K_w women of type w, and that a matching X exhibits in $x_{m,w}$ how many men of type m matched to women of type w. The model of aggregate matching captures actual observations in marriage models. We observe that men and women are partitioned into types according to their observable characteristics (age, income, education, etc.); and we are given a table showing how many men of type m married women of type w. These observations are essentially "flow" observations (marriages in a given year), so the aggregate matchings do not have any single agents.

Finally, *canonical* matchings are those matchings X for which the entries $x_{m,w}$ are either 0 or 1. Some of the results depend only on whether entries are zero or positive, so canonical matchings play a key role in our analysis.

3 Testable Implications

We study the testable implications of extremal matchings. First, we establish that they exist for our general model of matchings. In fact, the standard theory of the structure of stable matching extends.[3]

Theorem 2. *$(S(M, W, P, K), \leq_M)$ and $(S(M, W, P, K), \leq_W)$ are nonempty, complete, and distributive lattices; in addition, for $X, Y \in S(M, W, P, K)$*

1. *$X \leq_M Y$ iff $Y \leq_W X$;*
2. *for all types $a \in M \cup W$, either $x_a \leq_a y_a$ or $y_a \leq_a x_a$;*
3. *for all m and w, $\sum_{w \in W} x_{m,w} = \sum_{w \in W} y_{m,w}$ and $\sum_{m \in M} x_{m,w} = \sum_{m \in M} y_{m,w}$.*

Theorem 2 implies that there are two stable matchings, X^M and X^W, such that for all stable matchings X,

$$X^W \leq_M X \leq_M X^M$$
$$X^M \leq_W X \leq_W X^W,$$

we refer to X^M as the *man-optimal* (M-optimal) stable matching, and to X^W as the *woman-optimal* (W-optimal) stable matching. A matching X is the unique stable matching if $S(M, W, P, K) = \{X\}$; in this case X coincides with the M- and the W-optimal matchings.

[3] As we explain in Section 5, since we work with strong stability, the existing results on stable random matchings do not apply to our case. For the model of aggregate matching, there are no previous results on the structure of stable matchings.

Theorem 2 also presents versions of other classical results on matching for our model. Statement (1) is a "polarity of interests" results, saying that a stable matching X is better for men if and only if it is worse for women. Statement (2) says that the outcomes in two stable matchings are always comparable for an agent: note that $\leq_a$ is an incomplete preference relation. Statement (3) is the "rural hospitals theorem," which says that single agents in any two stable matchings are the same.

Having established that extremal matchings always exist, we turn to the revealed preference question of when we can rationalize a given matching as extremal stable. Formally, a matching X is *M-optimal rationalizable* if there is a preference profile $P = ((P_m)_{m\in M}, (P_w)_{w\in W})$ such that X is the M-optimal stable matching in $\langle M, W, P, K\rangle$. Analogously, it is *W-optimal rationalizable* if there is a preference profile such that X is the W-optimal matching in the corresponding market. We assume that there are no singles in our data: we observe only formed couples.[4]

It is worth remarking that rationalizing preferences must be strict. If we assume weak preferences, there is a trivial rationalization by making each agent be indifferent between all potential partners.

Given a matching X, we use a graph defined by the strictly positive entries in X, where there is an edge between two entries in the same row, and an edge between two entries in the same column. Formally, consider the graph (V, L) for which the set of vertices is $V := \{(m, w)|m \in M, w \in W \text{ such that } x_{m,w} > 0\}$, and $((m_i, w_j), (m_k, w_l)) \in L$ if (m_i, w_j), $(m_k, w_l) \in V$ and $m_i = m_k$ or $w_j = w_l$.

Theorem 3. *Let X be a matching. The following statements are equivalent:*

1. *X is rationalizable as a M-optimal stable matching;*
2. *X is rationalizable as a W-optimal stable matching;*
3. *X is rationalizable as the unique stable matching;*
4. *the graph (V, L) associated to X has no cycles.*

The theorem is easily applicable to aggregate matchings or random matchings. For example consider a random matching where there is a set S of students and C of schools. Suppose that there is a set of two students S_0 which have a positive chance of being admitted at every school. Then the resulting matching cannot be student optimal, no

[4]This is the case in actual data: see Echenique, Lee, and Shum (2010).

matter what the students' preferences are, or what schools preferences (priorities) are chosen. To see this simple point, imagine that students are men and schools are women in the notation above. Then if we consider the rows corresponding to the students in S_0 we will have only positive entries:

$$\begin{pmatrix} & \vdots & \vdots & \\ \cdots & p_{s,c} & p_{s,c'} & \cdots \\ \cdots & p_{s',c} & p_{s',c'} & \cdots \\ & \vdots & \vdots & \end{pmatrix}$$

The graph (V, L) would then contain as a subgraph the cycle

$$\begin{tikzcd} p_{s,c} \arrow[r, dash] \arrow[d, dash] & p_{s,c'} \arrow[d, dash] \\ p_{s',c} \arrow[r, dash] & p_{s',c'} \end{tikzcd}$$

By Theorem 3, this random matching is not rationalizable as either student optimal or school optimal, regardless of what preferences-priorities one may assume for the two sides.

3.1 Transferable Utility

We now turn to the model of matching with transfers. The focus of our paper is on the models without transfers, but it is instructive to compare extremal rationalizability with rationalizability under transfers.

Fix sets M, W and list K. For a matching X, suppose that a type m man who matches with a type w woman can generate a surplus $\alpha_{m,w} \in \mathbf{R}$. So the total surplus generated by the matchings of types m and w in X is $x_{m,w}\alpha_{m,w}$. The surpluses are contained in a matrix

$$\alpha = (\alpha_{m,w})_{|M|\times|W|}.$$

We want to know when a matching X is rationalizable using the standard model of matching with transfers.

Let X be a matching. Say that X is *TU-rationalizable* by a matrix of surplus α if X is the unique solution to the following problem.

$$\begin{array}{ll} \max_{\tilde{X}} \sum_{m,w} \alpha_{m,w}\tilde{x}_{m,w} & \\ \text{s.t.} \begin{cases} \forall w \ \sum_m \tilde{x}_{m,w} = K_m \\ \forall m \ \sum_w \tilde{x}_{m,w} = K_w \end{cases} & . \end{array} \tag{1}$$

Remark 4. Note that we require X to be the unique maximizer in (1). If we instead require X to be only one of the maximizers of (1), then any matching can be rationalized with a constant surplus. ($\alpha_{i,j} = c$ for all i, j). Uniqueness in the TU model holds for almost all real matrices α, and it seems to be the natural way to phrase the revealed preference question.

Corollary 5. *A matching X is TU-rationalizable if and only if it is M-optimal rationalizable.*

The corollary follows from Theorem 3 above and Theorem 3.6 in Echenique, Lee, and Shum (2010) that a matching X is TU rationalizable if and only if the associated graph (V, L) has no cycles.

4 Median Stable Matching: Existence and Testable Implications

Extremal stable matchings may be unreasonable because they favor one side over the other. One may instead be interested in matchings that present a compromise: median stable matchings are such a compromise. Informally, the median stable matching gives each agent the partner that is in the median of his/her preference list, once the preference list is restricted to his/her partners in some stable matching. To calculate the median we weight each partner by the number of stable matching in which the two are matched.

It is not obvious that median stable matchings exist: for the standard models of (deterministic) matching, existence was proven by Teo and Sethuraman (1998); see also Klaus and Klijn (2010), and Schwarz and Yenmez (2007) for other matching markets. For our model, we shall first show the existence of median stable matching, and then present a sufficient condition for rationalizability as median stable matchings.

In this section, we only consider aggregate matchings. So we assume a market $\langle M, W, P, K\rangle$, where all numbers K_m, K_w and entries of matchings X are non-negative integers. As a result, the number of stable matchings is finite, say k. Let $S(M, W, P, K) =$

$\{X^1, \ldots, X^k\}$ be the set of stable matchings. For each agent a we consider all the stable outcomes and rank them according to $\geq_a$. We can rank all the outcomes by (2) of Theorem 2. More formally, let $\{x_a^{(1)}, \ldots, x_a^{(k)}\} = \{x_a^1, \ldots, x_a^k\}$ and $x_a^{(1)} \geq_a \ldots \geq_a x_a^{(k)}$. Using these ranked outcomes for all the agents we construct the following matrices: $y_m^{(i)} = x_m^{(i)}$ and $y_w^{(i)} = x_w^{(k+1-i)}$ for all $i = 1, \ldots, k$. The matrices $Y^{(i)}$ give each type of men the i-th best outcome, and each type of women the i-th worst outcome out of the outcomes in a stable matching.

Proposition 6. *$Y^{(i)}$ is a stable aggregate matching.*

If k is odd we term $Y^{(k+1/2)}$ the *median stable matching*. If k is even we refer to $Y^{(k/2)}$ as the median stable matching (but of course this choice is arbitrary). In our proofs we construct preferences such that the number of stable matchings is odd, so our results do not depend on this choice.

Corollary 7. *The median stable matching exists.*

Having established that median matching always exists in the model of aggregate matching, we proceed to understanding their testable implications. We want to know when a matching can be rationalized as the median stable matching. Formally, an aggregate matching X is *median rationalizable* if there is a preference profile P such that X is a median matching in $\langle M, W, P, K\rangle$.

Recall that if $\langle v_0, \ldots v_N\rangle$ is a cycle, then N is an even number. Say that a cycle c is balanced if

$$\min\{v_0, v_2, \ldots, v_{N-2}\} = \min\{v_1, v_3, \ldots, v_{N-1}\}.$$

Theorem 8. *An aggregate matching X is median rationalizable if it is rationalizable and if all cycles of the associated graph (V, L) are balanced.*

Corollary 9. *A canonical matching X is either not rationalizable or it is median rationalizable.*

Unfortunately, the result in Theorem 8 is only a sufficient condition for median rationalizability. Corollary 9 gives a characterization, but for the limited case of canonical matching. To sketch the boundaries of these results, we present two examples. Example 10 shows that there are indeed matchings that are not rationalizable as median matchings: so Corollary 9 does not extend to all aggregate matchings. Example 11 shows that the sufficient condition in Theorem 8 is not necessary.

Example 10. A rationalizable aggregate matching that cannot be rationalized as the median:

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[d, dash] & 3 \arrow[d, dash] \\ 3 \arrow[r, dash] & 1 \end{tikzcd}$$

The only way to rationalize this aggregate matching is to direct the edges so that the cycle is a flow, i.e., if $w_1 P_{m_1} w_2$ then $m_2 P_{w_1} m_1$, $w_2 P_{m_2} w_1$, and $m_1 P_{w_2} m_2$. Similarly, if $w_2 P_{m_1} w_1$ then the preferences of agents are such that there is a flow. Regardless of how the flow is directed, all the feasible aggregate matchings are stable, so there are 5 stable aggregate matchings in total. Therefore, the median aggregate stable matching is the one where each agent is matched to both possible partners twice.

Example 11. The following example shows that the sufficient condition in Theorem 8 is not necessary. Suppose that there are four types of men, three types of women, and consider the aggregate matching

$$X = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \\ 3 & 1 & 0 \end{pmatrix}.$$

We choose the preferences such that the graph corresponding to X is as follows. Note that we indicate preference with an arrow, so that for example $x_{i,j} \to x_{i,h}$ means that $w_h \, P_{m_i} \, w_j$.

$$\begin{tikzcd}
& 0 \arrow[d, dotted] \arrow[dd, dotted, bend right] & 2 \arrow[l, dotted] \\
& 2 \arrow[r] & 1 \arrow[u] \\
1 \arrow[r] & 3 \arrow[u] \arrow[d] & \\
3 \arrow[u] & 1. \arrow[l] &
\end{tikzcd}$$

Note that there is a cycle $\langle x_{3,1}, x_{3,2}, x_{4,2}, x_{4,1} \rangle$ and that ...

By "rotating" the cycle $\langle x_{3,1}, x_{3,2}, x_{4,2}, x_{4,1} \rangle$ in a clockwise direction we obtain the aggregate matching

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{pmatrix},$$

which is better for the men. However, by counterclockwise rotations we obtain the matchings

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \\ 1 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix},$$

which are all better for women than X.

Now, *with the rationalization in the arrows above,* the following "joint" rotation of the cycle and the upper entries of the matchings is stable as well:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

These two matchings are better for the men. In all, then, under the rationalizing preferences in the arrows, there are 7 stable matchings: and X is the median matching.

There is a crucial aspect of the example that makes this possible. Note that, if we are to rotate the upper part of the graph, we need preferences to be as indicated by the arrows. In particular, we must have $x_{2,3} \to x_{1,3}$ for the graph to be rationalizable; then, to accommodate a > 0 entry in $x_{1,2}$ after the rotation, we must have $x_{1,3} \to x_{1,2}$ or we would get a blocking pair (m_1, w_3). But since we have positive entries in $x_{1,3}$ and in $x_{3,2}$, $x_{1,3} \to x_{1,2}$ implies that we need $x_{1,2} \to x_{3,2}$ (the long dotted arrow in the graph). Now, a modification of X that has a positive entry in $x_{1,2}$ is only possible if we simultaneously set $x_{3,1} = 0$, as $x_{3,1} \to x_{3,2}$ and $x_{1,2} \to x_{3,2}$. Hence the rotation of the upper side of the graph *is not feasible under any of the modification of X that improve the matches of the women.*

There is, therefore, an asymmetry in the graph that allows us to offset the unbalancedness of the number of men and women in the cycle.

5 Strong Stability

In this section we argue that (a) strong stability is the natural notion of stability for aggregate matchings, and of *ex-ante* stability for random matchings; and (b) that one cannot analyze strong stability using the standard linear programming tools.

First, it is rather obvious that strong stability is the natural notion of stability for aggregate matchings. Concretely, if there is a pair (m, w) that can block an aggregate matching X, then there are two agents: one man of type m and one woman of type w, such that the two agents can block in the usual sense. However, the case of random matching requires more explaining.

If X is a random matching, we can view the row x_m as a probability distribution over the set of women who may be partners of m; similarly for x_w. Then (m, w) is a blocking pair (in the strong sense) if and only if there are distributions x'_m and x'_w such that

- x'_m first order stochastically dominates x_m and x'_w first order stochastically dominates x_w;
- (m, w) can achieve x'_m and x'_w by mutual agreement, without the consent of any other agents, because $x'_{m,\tilde{w}} \leq x_{m,\tilde{w}}$ and $x'_{\tilde{m},w} \leq x_{\tilde{m},w}$ for any $\tilde{m}$ and $\tilde{w}$.

The ex-ante perspective makes sense if we think that agents can trade probabilities or time shares — see Hylland and Zeckhauser (1979). After agents trade, any random matching can be implemented physically by a decomposition into deterministic, simultaneous, matchings (using the Birkhoff von-Neumann theorem). If trades in probabilities or time-shares are somehow ruled out, then one can still justify strong stability on fairness grounds (Kesten and Ünver, 2009).

The standard notion of stability can be analyzed by linear programming methods (Vande Vate, 1989; Rothblum, 1992; Roth, Rothblum, and Vate, 1993). The reason is that stability results in a collection of linear constraints on matchings: a fractional matching X is stable if and only if it satisfies a set of linear inequalities. Strong stability, on the other hand, does not result in linear constraints.

Specifically, the property that an individually rational X is strongly stable is equivalent to the following statement. For all pairs (m, w),

$$\left(\sum_{w' : w P_m w'} x_{m,w'} \right) \left(\sum_{m' : m P_w m'} x_{m',w} \right) = 0.$$

Hence X is a strongly stable matching if and only if it is feasible, individually rational, and satisfies a set of quadratic constraints. The resulting set of matrices does not have the geometry that one can exploit for fractional stable matching. On the other hand, observe that the weak notion of stability is captured by the constraints: for all (m, w)

$$\sum_{w':wP_m w'} x_{m,w'} + \sum_{m':mP_w m'} x_{m',w} \leq 1;$$

which conforms a linear system of inequalities.

6 Conclusion

The deferred acceptance algorithm was first introduced in Gale and Shapley (1962). However, it was already being used in several markets even before then (see Roth (2008)). Indeed, it is viewed as not only a recipe of how we should organize clearinghouses for two-sided matching markets, but also a folk model of how decentralized markets behave. Although economists have built clearinghouses based on the deferred acceptance algorithm (NY and Boston school mechanisms, the National Residence Matching Program, etc.), there has been no analysis of whether the decentralized market outcomes can be the deferred acceptance algorithm outcomes. We have filled in this void.

We show for a general two-sided matching market that a matching can be rationalized as the outcome of the deferred acceptance algorithm, which produces extremal outcomes in the lattice of stable matchings, if and only if there are no cycles in the graph associated with the matching which in turn is equivalent to the rationalizability of the matching as a stable matching if transfers were allowed between agents. Therefore, the empirical content of rationalizability as an extremal outcome without transfers is the same as rationalizability in the transferable utility setup.

Thus, our analysis provides the tools necessary for the empirical analysis of matching data. To be specific, the question of whether a market behaves as in the deferred acceptance algorithm can now be studied.

7 Proofs

7.1 Proof of Theorem 2

We split up the proof in short propositions.

Lemma 12. $(\mathcal{X}_m, \leq_m)$ *is a complete lattice, with a largest and smallest element.*

Proof. That $(\mathcal{X}_m, \leq_m)$ is a partially ordered set follows from the definition of $\leq_m$. Take a subset S_m of X_m. We need to show that S_m has a least upper bound and a greatest lower bound in $(\mathcal{X}_m, \leq_m)$ to complete the proof.

Reorder $W \cup \{w_{l+1}\}$ according to m's preference such that $w_1 P_m w_2 P_m \ldots P_m w_{l+1}$.

Let $z_1 = \sup\{x_1 | x \in S_m\}$. Define z_i inductively as follows: $z_i = \sup\{x_1 + \ldots + x_i | x \in S_m\} - (z_1 + \ldots + z_{i-1})$. Note that by definition $z_i \geq 0$ and $\sum\limits_i z_i = K_m$, so $z \in \mathcal{X}_m$.

Claim: z is a least upper bound of S_m.

Proof: By definition, $\sum\limits_{j=1}^{j=i} z_j = \sup\{x_1 + \ldots + x_i | x \in S_m\}$ which is greater than $x_1 + \ldots + x_i$ for all $x \in S_m$ and i. Therefore, $z \geq_m x$ for all $x \in S_m$, which means that z is an upper bound. Suppose that z' is another upper bound of S_m. Therefore, $z'_1 + \ldots + z'_i \geq x_1 + \ldots + x_i$ for all $x \in S_m$ and i. If we take the supremum of the right hand side, then we get $z'_1 + \ldots + z'_i \geq \sup\{x_1 + \ldots + x_i | x \in S_m\}$ for all i. On the other hand, $z_1 + \ldots + z_i = \sup\{x_1 + \ldots + x_i | x \in S_m\}$ for all i by definition of z. The last two impressions imply $z'_1 + \ldots + z'_i \geq z_1 + \ldots + z_i$ for all i, so $z' \geq_m z$. Thus, z is a least upper bound.

Similarly we can construct a greatest lower bound as follows: $u_1 = \inf\{x_1 | x \in S_m\}$. Define u_i inductively: $u_i = \inf\{x_1 + \ldots + x_i | x \in S_m\} - (u_1 + \ldots + u_{i-1})$. The proof that u is a greatest lower bound is similar to the proof that z is a least upper bound, so it is omitted. □

For each m, let the choice C_m be defined as follows. For a vector $x \in \mathbf{R}_+^{l+1}$, let $C_m(x)$ be the vector in

$$\{y \in X_m : y_j \leq x_j, j = 1, \ldots, l\}$$

that is maximal for $\leq_m$. In other words, if x represents the quantities of women available for m, C_m chooses according to P_m from best choice downwards until filling quota K_m. Note that if $\emptyset P_m w_j$ then $y_j = 0$, and that $y_{l+1} = K_m - \sum_{j: w_j P_m \emptyset} y_j$. Define C_w analogously.

Proposition 13. *$(S(M,W,P,K), \leq_M)$ is a nonempty, complete lattice.*

Proof. A *man pre-matching* is a matrix $A = (a_{m,w})$ such that $a_{m,w} \in \mathbf{R}_+$ and $\sum_w a_{m,w} = K_m$. A *woman pre-matching* is a matrix $B = (b_{m,w})$ such that $b_{m,w} \in \mathbf{R}_+$ and $\sum_m b_{m,w} = K_w$.

We consider pairs (A, B), where A is a man prematching, and B is a woman prematching, ordered by a partial order $\leq$. The order $\leq$ is defined as $(A, B) \leq (A', B')$ if

$$\forall m \forall w \left(a_{m,\cdot} \leq_m a'_{m,\cdot} \wedge b'_{\cdot,w} \leq_w b_{\cdot,w} \right).$$

The order $\leq$ is a product order of complete lattices by Lemma 12, so that the set of all pairs (A, B) ordered by $\leq$ is a complete lattice.

We define a function C, mapping pairs (A, B) of prematchings into pairs of prematchings. Fix (A, B): For a man m, the number of women of type w who are willing to match with m at B is $\theta_{m,w} \equiv \sum_{i: m R_w m_i} b_{m_i,w}$. Let $\theta_m \equiv (\theta_{m,w_1}, \ldots, \theta_{m,w_l})$, i.e., the l-vector such that entry w is the number of women of type w who are willing to match with m at B. Similarly, for $w \in W$, let η_w be the vector for which entry m is the number of men of type m who are willing to match with w at A. Now let $C(A, B) = (A', B')$ where $a'_m = C_m(\theta_m)$ and $b'_w = C_w(\eta_w)$, with a'_m being m's row in A' and b'_w w's column in B'.

We now prove that C is isotone. Suppose that $(A, B) \leq (A', B')$. We prove that $C(A, B) \leq C(A', B')$. Fix m. For any w_j, note that if m is not the top man in R_{w_j} we obtain

$$\sum_{i: m R_{w_j} m_i} b_{i,j} = K_{w_j} - \sum_{i: m_i P_{w_j} m} b_{i,j} \leq K_{w_j} - \sum_{i: m_i P_{w_j} m} b'_{i,j} = \sum_{i: m R_{w_j} m_i} b'_{i,j},$$

as $b'_{w_j} \leq_{w_j} b_{w_j}$. If m is the top man in R_{w_j} we obtain $\sum_{i: m R_{w_j} m_i} b'_{i,j} = K_{w_j} = \sum_{i: m R_{w_j} m_i} b_{i,j}$. As a consequence, in B' man m has weakly more women of each type willing to match with him than in B: $\theta_m \leq \theta'_m$. Thus $C_m(\theta_m) \leq_m C_m(\theta'_m)$. Similarly, for women w, $C_w(\eta'_w) \leq_w C_w(\eta_w)$. It follows that $C(A, B) \leq C(A', B')$

By Tarski's fixed point theorem, there is a fixed point of C, the set of fixed points of C is a complete lattice when ordered by $\leq$.

Let $(A, B) = C(A, B)$ be a fixed point of C. Assume that $a_{mw} > b_{mw}$ for some m and w. $C_w(\eta_w) = b_w$ and $b_{mw} < a_{mw}$ implies that although a_{mw} number of type m men were available, only b_{mw} of them are chosen by C_w. Therefore, all nonnegative entries in b_w are at least as good as m with respect to R_w. This implies that $\theta_{mw} = b_{mw}$. Since $b_{mw} < a_{mw}$, we get $\theta_{mw} < a_{mw}$ which contradicts $a_m = C_m(\theta_m)$. Therefore, $a_{mw} = b_{mw}$ for all m and w. Hence a fixed point has the property that $A = B$ is an aggregate matching, not only prematchings.

Finally we prove that the set of fixed points of C is the set of stable aggregate matchings. More precisely, (A, A) is a fixed point of C if and only if A is a stable aggregate matching.

Suppose that a fixed point (A, A) is not stable. Then there is a blocking pair (m, w). That is, there is m' and w' such that $m\ P_w\ m'$, $w\ P_m\ w'$, $a_{m,w'} > 0$, and $a_{m',w} > 0$. Now, the number of women of type w who are willing to match with m is

$$\theta_{m,w} = \sum_{i:mR_w m_i} a_{i,w} \geq a_{m,w} + a_{m',w} > a_{m,w},$$

as $a_{m',w} > 0$. But $\theta_{m,w} > a_{m,w}$ and $a_m = C_m(\theta_m)$ contradicts that there is w' with $w P_m w'$ and $a_{m,w'} > 0$.

Suppose that A is a stable aggregate matching. Fix m and we show that $a_m = C_m(\theta_m)$ where $\theta_{m,w} = \sum_{i:mR_w m_i} a_{i,w}$. Denote w_j as the most preferred type of women such that $a_{m,w_j} \neq (C_m(\theta_m))_j$. By definition of C_m, $a_{m,w_j} > (C_m(\theta_m))_j$ is not feasible. For all $w_{j'}$ preferred to w_j, $a_{m,w_{j'}} = (C_m(\theta_m))_{j'}$. Thus, $a_{m,w_j} > (C_m(\theta_m))_j$ implies either $a_{m,w_j} > \theta_{m,w_j}$ or $\sum_{j':w_{j'} R_m w_j} a_{m,w_{j'}} > K_m$. On the other hand, $a_{m,w_j} < (C_m(\theta_m))_j$ contradicts that A is stable. Although there are type w_j women available more than a_{m,w_j}, some type m men are matched to less preferred women; That is there is j' such that $w_j P_m w_{j'}$ and $a_{m,w_{j'}} > 0$. (m, w_j) is a blocking pair. Similarly, we can show that $a_w = C_w(\theta_w)$, and therefore $(A, A) = C(A, A)$.

□

Proposition 14. *Suppose that X and Y are two stable matchings. Then for any men or women type a, either $x_a \leq_a y_a$ or $y_a \leq_a x_a$. Consequently, $x_a \vee_a y_a = \max_{\leq a}\{x_a, y_a\}$ and $x_a \wedge_a y_a = \min_{\leq a}\{x_a, y_a\}$.*

Proof. We only prove the first part: that either $x_a \leq_a y_a$ or $y_a \leq_a x_a$, in three steps depending on whether X and Y have integer, rational, and real entries. The second part is implied immediately: $x_a \vee_a y_a = \max_{\leq a}\{x_a, y_a\}$ and $x_a \wedge_a y_a = \min_{\leq a}\{x_a, y_a\}$.

Case 1 (Integer Entries) We first start with the case when X and Y have integer entries. Therefore, K_w and K_m are also integers. From $\langle M, W, P, K\rangle$, we create a many-to-one matching market (of colleges and students) as follows.

The set of men remains the same; interpreted as the set of colleges. A college m has a capacity of K_m. Whereas, each woman w is split into K_w copies, all of which have the same preferences P_w over men; women are interpreted as students. On the other hand, man m's preferences P'_m replaces woman w in P_m with her copies enumerated from 1 to K_w, in increasing order. Denote w_i's j-th copy by w_i^j. So $w_i^j \; P'_m \; w_i^k$ if and only if $k > j$. In addition, each man has responsive preferences over groups of women. The new matching market with $|M|$ men and $\sum_w K_w$ women is a many-to-one matching market where an outcome for a man is a group of women and an outcome for a woman is either a man or being single.

Now, we construct a new matching, X', in the new market from X. It is enough to describe the matches of women in X'. Rank woman w's outcomes in X in decreasing order according to her preference P_w. Let the jth copy of w_i, w_i^j, match to the jth highest outcome of w in X. Similarly construct Y' from Y.

We claim that X' and Y' are stable matchings in the new market. Suppose for contradiction that X' is not a stable matching. Since X' is individually rational by construction, there exists a blocking pair (m, w_i^j). This means that w_i^j's match is worse than m. Similarly, m's match includes an agent worse than w_i^j: this agent cannot be w_i^k where $k < j$ by definition of P', and it cannot be w_i^k where $k > j$ because by construction w_i^k's match is worse than w_i^j's match with respect to P_{w_i}. Hence, one of m's matches is worse than w_i^j, and not a copy of w_i. This means that (m, w_i) forms a blocking pair in X: A contradiction to the stability of X. Therefore, X' must be stable. Similarly, Y' is also stable.

Now, by Theorem 5.26 of Roth and Sotomayor (1990), for any man m the outcomes in X' and Y' are comparable. This means that the responsive preferences over groups of women inherited from P'_m, which is equivalent to the first order stochastic dominance, can compare the outcomes of m in these two stable matchings. Therefore, $\leq_m$ can compare the outcomes in X and Y since P_m is a coarser order than P'_m.

An analogous argument shows that $\leq_w$ can compare X_w and Y_w.

Case 2 (Rational Entries) Suppose for now that all entries of X and Y are rational numbers. Therefore, K_w and K_m are also rational numbers. Define a new matching market from $\langle M, W, P, K\rangle$ as follows.

M, W, and P are the same. We change the capacities as follows. Find a common factor of denominators in all entries in X and Y, say r, and multiply all capacities by r. Therefore, the new market is $\langle M, W, P, rK\rangle$. Define $X' \equiv rX$ and $Y' \equiv rY$ with non-negative integer entries. By the argument above, for any agent a, rX_a and rY_a can be compared with respect to $\leq_a$ which implies that X_a and Y_a can also be compared.

Case 3 (Real Entries) Suppose now that entries of X and Y are real numbers. We construct two sequences of matrices, $X^{(n)}$ and $Y^{(n)}$, as follows:

1. $X^{(n)}$ and $Y^{(n)}$ have rational entries,
2. $x_{ij}^{(n)} = 0 \iff x_{ij} = 0$ and $y_{ij}^{(n)} = 0 \iff y_{ij} = 0$ for all i, j,
3. the sum of entries in row i and column j is the same for $X^{(n)}$ and $Y^{(n)}$, and
4. $X^{(n)} \Rightarrow X$ and $Y^{(n)} \Rightarrow Y$ as $n \Rightarrow \infty$.

By construction the stability of X and Y imply stability of $X^{(n)}$ and $Y^{(n)}$ where the capacities of types are adjusted. By the argument above, for each type a, $x_a^{(n)}$ and $y_a^{(n)}$ can be compared with respect to $\leq_a$. Take a subsequence such that the ordering is the same for all entries. Therefore, $x_a^{(n)} \leq_a y_a^{(n)}$ for all n or $y_a^{(n)} \leq_a x_a^{(n)}$ for all n. By taking n to ∞ we get that either $x_a \leq_a y_a$ in the former case, or $y_a \leq_a x_a$ in the latter. $\square$

Using the proposition above, we show that $(S(M, W, P, K), \leq_M)$ is distributive.

Proposition 15. *$(S(M, W, P, K), \leq_M)$ is a distributive lattice.*

Proof. We have shown that $(S(M, W, P, K), \leq_M)$ is a lattice in Proposition 2. We show that the lattice is distributive.

Suppose that X, Y, and Z are stable matchings. We are going to prove that type a have the same matching in $X \wedge (Y \vee Z)$ and $(X \wedge Y) \vee (X \wedge Z)$ for all agents a:

$$
\begin{aligned}
(X \wedge (Y \vee Z))_a &= \min\{x_a, \max\{y_a, z_a\}\} \\
&= \max\{\min\{x_a, y_a\}, \min\{x_a, z_a\}\} \\
&= \max\{(X \vee Y)_a, (X \vee Z)_a\} \\
&= ((X \wedge Y) \vee (X \wedge Z))_a,
\end{aligned}
$$

where min and max operators are defined with respect to $\leq_a$ and where we repeatedly use Proposition 14.

The proof that $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$ is analogous. $\square$

Proposition 16. *Suppose that X and Y are stable matchings. Then $\sum_{j=1}^{l} x_{ij} = \sum_{j=1}^{l} y_{ij}$ for all i and similarly $\sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} y_{ij}$ for all j.*

Proof. The proof has the same structure as in the proof of Proposition 14: it has three steps depending on whether X and Y have integer, rational, and real entries.

Case 1 (Integer Entries) We first start with the case when X and Y have integer entries. Therefore, K_a is also an integer for all a. From $\langle M, W, P, K \rangle$, we create a many-to-one matching market and also stable matchings X' and Y' in this new market as in the proof of Proposition 14.

Now, by Theorem 5.12 of (Roth and Sotomayor, 1990), the set of positions filled for any man m in X' and Y' are the same. Therefore, $\sum_{j=1}^{l} x_{ij} = \sum_{j=1}^{l} y_{ij}$ for all i. Similarly, $\sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} y_{ij}$ for all j.

Case 2 (Rational Entries) Suppose for now that all entries of X and Y are rational numbers. Therefore, K_a is also a rational number for all a. Define a new matching market from $\langle M, W, P, K \rangle$ as follows.

M, W, and P are the same. We change the capacities as follows. Find a common factor of denominators in all entries in X and Y, say r, and multiply all capacities by r. Therefore, the new market is $\langle M, W, P, rK \rangle$. Define $X' \equiv rX$ and $Y' \equiv rY$ with non-negative integer entries.

By the argument above, $\sum_{j=1}^{l} rx_{ij} = \sum_{j=1}^{l} ry_{ij}$ for all i and $\sum_{i=1}^{n} rx_{ij} = \sum_{i=1}^{n} ry_{ij}$ for all j. The conclusion follows.

Case 3 (Real Entries)
Suppose now that entries of X and Y are real numbers. We construct two sequences of matrices, $X^{(n)}$ and $Y^{(n)}$, as follows:

1. $X^{(n)}$ and $Y^{(n)}$ have rational entries,
2. $x_{ij}^{(n)} = 0 \iff x_{ij} = 0$ and $y_{ij}^{(n)} = 0 \iff y_{ij} = 0$ for all i, j,
3. the sum of entries in row i and column j is the same for $X^{(n)}$ and $Y^{(n)}$, and
4. $X^{(n)} \Rightarrow X$ and $Y^{(n)} \Rightarrow Y$ as $n \Rightarrow \infty$.

By construction, stability of X and Y imply stability of $X^{(n)}$ and $Y^{(n)}$ where the capacities of agents are adjusted. By the argument above, $\sum_{j=1}^{l} x_{ij}^{(n)} = \sum_{j=1}^{l} y_{ij}^{(n)}$ for all i and $\sum_{i=1}^{n} x_{ij}^{(n)} = \sum_{i=1}^{n} y_{ij}^{(n)}$. If we take the limit of these equalities as $n \Rightarrow \infty$, we get the desired equalities. □

7.2 Proof of Theorem 3

We proceed by proving first that rationalizability as either M- or W-optimal (i.e. extremal rationalizability) stable matching implies the absence of cycles. Second, we prove that the absence of cycles implies rationalizability as a unique stable matching. Since a unique stable matching is trivially both M- and W-optimal, the result follows.

7.2.1 Proof that extremal rationalizability implies the absence of cycles

Let X be an aggregate matching that is extremal rationalizable. There exists a preference profile P such that X is M-optimal or W-optimal stable matching in $\langle M, W, P, K \rangle$.

By means of contradiction, suppose that the graph (V, L) associated to X has a minimal cycle $c = \langle v_0, \ldots, v_N \rangle$. We denote m_n for the type of the men in v_n, and w_n for the type of the women in v_n, respectively. We call an edge $((m, w), (m', w')) \in L$ *vertical* if $w = w'$ and *horizontal* if $m = m'$. The following result is Lemma 14 and 15 of Echenique, Lee, and Shum (2010).

Lemma 17. *If $c = \langle v_0, \ldots, n_N \rangle$ is a minimal cycle, then no vertex appears twice in c, and*

$$(v_n, v_{n+1}) \in L \text{ is vertical } \Rightarrow (v_{n+1}, v_{n+2}) \in L \text{ is horizontal}$$
$$(v_n, v_{n+1}) \in L \text{ is horizontal } \Rightarrow (v_{n+1}, v_{n+2}) \in L \text{ is vertical}$$

An *orientation* of (V, L) is a mapping $d : L \to \{0, 1\}$. We shall often write $d((m_i, w_j), (m_i, w_k))$ as d_{m_i,w_j,w_k} and $d((m_i, w_j), (m_l, w_j))$ as d_{w_j,m_i,m_l}. Fix an orientation d of (V, L). A path $\langle v_n \rangle_{n=0}^{N}$ is a *flow* for d if $d(v_n, v_{n+1}) = 1$ for all $n = 0, \ldots, N-1$.

A preference profile (P_{m_i}, P_{w_j}) defines an orientation d by setting $d_{w_j,m_i,m_l} = 1 \iff m_i \; P_{w_j} \; m_l$ and $d_{m_i,w_j,w_k} = 1 \iff w_j \; P_{m_i} \; w_k$. Let d be the orientation defined by the preference profile P that rationalizes X as an extremal matching.

Lemma 18. *The index of the cycle c can be chosen such that the path $\langle v_n \rangle_{n=0}^{N-1}$ is a flow for d.*

A result similar to Lemma 18 is shown by Echenique, Lee, and Shum (2010). By Lemma 18, for all $n = 0, 1, \ldots, N-1$, if edge (v_n, v_{n+1}) is *vertical* (i.e. $w_n = w_{n+1}$), we have $d_{w_n,m_n,m_{n+1}} = 0$, and when the edge is *horizontal*, we have $d_{m_n,w_n,w_{n+1}} = 0$.

In the following proof, we show that we can make the men (women) weakly better (worse) off by "rematching" men and women whose matches are involved in the cycle c while preserving stability. We can also make women (men) weakly better (worse) off with a similar rematching. Therefore, X is neither M-optimal nor W-optimal stable matching.

We capture "rematching," using a matrix of differences in matches: let $\mathcal{A}$ be the set of all $|M| \times |W|$ matrices A such that for all i and j:

1. $a_{i,j} = 0$ if (i, j) is not in the cycle c;
2. $\sum_h a_{i,h} = 0$, $\sum_l a_{l,j} = 0$; and
3. $\forall (i', j') \quad |a_{i,j}| \leq x_{i',j'}$.

Claim 19. *For all $A \in \mathcal{A}$, the matrices $X + A$ and $X - A$ are stable in $\langle M, W, P, K \rangle$; and either*

- *$X - A \leq_M X \leq_M X + A$ and $X - A \geq_W X \geq_W X + A$, or*
- *$X + A \leq_M X \leq_M X - A$ and $X + A \geq_W X \geq_W X - A$.*

Proof. For any $A \in \mathcal{A}$, $X + A$ is a well defined matching: by Property (2) the row and column sum of $X + A$ respect the feasibility constrains; by Property (1) and (3), the entries of $X + A$ are non-negative. The matrix $X + A$ is also a stable matching, as

$$(X + A)_{i,j} > 0 \Rightarrow X_{i,j} > 0.$$

Indeed, if there were a blocking pair of type m_i and type w_j under $X + A$, it would also be a blocking pair under X. Since $A \in \mathcal{A} \Rightarrow -A \in \mathcal{A}$, $X - A$ is also well defined and stable.

Observe that as a consequence of Properties (2)-(1), a_{m_n,w_n} alternate in sign. So that if $a_{m_n,w_n} \geq 0$ then $a_{m_{n+1},w_{n+1}} \leq 0$; and $a_{m_n,w_n} > 0$ then $a_{m_{n+1},w_{n+1}} < 0$. This implies, first, that if $a_{m_n,w_n} = 0$ for some n then $A = 0$. And, second, that one of the following two cases has to hold. (a) For all n, if $m_n = m_{n+1} = m$ then $a_{m,w_n} > 0$ and $a_{m,w_{n+1}} < 0$, and if $w_n = w_{n+1} = w$ then $a_{m_n,w} < 0$ and $a_{m_{n+1},w} > 0$. (b) For all n, if $m_n = m_{n+1} = m$ then $a_{m,w_n} < 0$ and $a_{m,w_{n+1}} > 0$, and if $w_n = w_{n+1} = w$ then $a_{m_n,w} > 0$ and $a_{m_{n+1},w} < 0$.

Clearly, if $A = 0$ then there is nothing to prove. We shall proceed by assuming that we are in case (a), and we shall prove that $X - A \leq_M X \leq_M X + A$. It will become clear that if we instead assume that we are in case (b), we would establish that $X + A \leq_M X \leq_M X - A$.

Fix $m \in M$. By definition of minimal cycle, there is at most one n such that $v_n, v_{n+1} \in c$ and $m_n = m_{n+1} = m$. If no such n exists, by Property (1) of $\mathcal{A}$, $(X - A)_{m,\cdot} = (X + A)_{m,\cdot} = x_{m,\cdot}$. Thus $(X - A)_{m,\cdot} \leq_m x_{m,\cdot} \leq_m (X + A)_{m,\cdot}$.

If, on the other hand, there is $v_n, v_{n+1} \in c$ such that $m_n = m_{n+1} = m$, then (v_n, v_{n+1}) is horizontal and (v_{n+1}, v_{n+2}) is vertical. From the orientation d, we have $d_{m,w_n,w_{n+1}} = 1$, which implies $w_n \, P_m \, w_{n+1}$.

In $A_{m,\cdot}$, only a_{m_n,w_n} and $a_{m_{n+1},w_{n+1}}$ are non-zero, and $0 < a_{m_n,w_n} = -a_{m_{n+1},w_{n+1}}$, as we have assumed that we are in case (a). By definition of $\leq_m$, $w_n \, P_m \, w_{n+1}$ implies that $(X - A)_{m,\cdot} \leq_m X_{m,\cdot} \leq_m (X + A)_{m,\cdot}$.

Since the type m was arbitrary, we obtain $X - A \leq_M X \leq_M X + A$. By Theorem 2, this also implies that $X - A \geq_W X \geq_W X + A$.

$\square$

7.2.2 Proof of that the absence of cycles implies unique rationalizability.

We prove that if the graph (V, L) associated to X has no cycles, then there are preferences P such that (M, W, P, K) has X as its unique stable matching. The matching X is therefore both M- and W-optimal.

We introduce a particular set of preferences. Let $U = (u_{m,w}) \in \mathcal{R}_+^{|M|\times|W|}$ in which $u_{m,w} \neq u_{m',w'}$ for all $(m, w) \neq (m', w')$. For each m and w, a man of type m and a woman of type w both receive utility $u_{m,w}$ by being matched to each other. We denote by P_U the preferences induced by such utilities, called *perfectly correlated preferences.*

Lemma 20. *If preferences are perfectly correlated, there exists a unique aggregate stable matching.*

Proof. Suppose for contradiction that X and Y are two distinct stable matchings. Let $\mathcal{U}$ be the set of numbers $u_{m,w}$ for m and w such that $x_{m,w} \neq y_{m,w}$. Let (m^*, w^*) be such that $u_{m^*,w^*} \in \mathcal{U}$ and $u_{m^*,w^*} \geq u$ for all $u \in \mathcal{U}$. Suppose, without loss of generality, that $x_{m^*,w^*} < y_{m^*,w^*}$. Note that

$$\sum_{m:mP_{w^*}m^*} x_{m,w^*} = \sum_{m:mP_{w^*}m^*} y_{m,w^*}$$
$$\sum_{w:wP_{m^*}w^*} x_{m^*,w} = \sum_{w:wP_{m^*}w^*} y_{m^*,w},$$

because $m\ P_{w^*}\ m^* \Rightarrow u_{m,w^*} > u_{m^*,w^*} \Rightarrow x_{m,w^*} = y_{m,w^*}$ by construction of (m^*, w^*), and similarly for the second equality.

Then,

$$\begin{aligned}
\sum_{m:m^*P_{w^*}m} x_{m,w^*} &= K_{w^*} - \sum_{m:mR_{w^*}m^*} x_{m,w^*} \\
&= K_{w^*} - x_{m^*,w^*} - \sum_{m:mP_{w^*}m^*} x_{m,w^*} \\
&> K_{w^*} - y_{m^*,w^*} - \sum_{m:mP_{w^*}m^*} y_{m,w^*} \\
&\geq 0,
\end{aligned}$$

as $x_{m^*,w^*} < y_{m^*,w^*}$. Similarly, $\sum_{w:w^*P_{m^*}w} x_{m^*,w} > 0$ and (m^*, w^*) is a blocking pair of X; which contradicts the stability of X. $\square$

We prove the result by using the absence of cycles to assign cardinal utilities $U = (u_{m,w})$ so that agents' preferences are perfectly correlated. Then Lemma 20 guarantees that X is the unique stable matching.

We first prove the case when all nodes in V are connected, and later generalize to the case where there are multiple connected components of (V, L).

Suppose that the graph (V, L) associated to X has no minimal cycles; so it has no cycles. Choose a vertex v_0 in V. Since (V, L) contains no cycles, for each $v \in V$ there is a unique minimal path connecting v_0 to v in (V, L). Let $\eta(v)$ be the length of the minimal path connecting v_0 to v.

We construct correlated preferences that rationalize X by constructing numbers $U = (u_{m,w})$. For $v \in V$ (i.e. $x_{m_v,w_v} > 0$), $u_{m_v,w_v} = (1 + \eta(v)) + \varepsilon_{m_v,w_v}$, and for all other (m, w) with $x_{m,w} = 0$, $u_{m,w} = \varepsilon_{m,w}$. All ε_{m_v,w_v} and $\varepsilon_{m,w}$ are positive, and distinct real numbers; we assume all $\varepsilon_{m,w}$ are small enough that if $\eta(v) > \eta(v')$ then $u_{m_v,w_v} > u_{m_{v'},w_{v'}}$. Specifically, let $(\varepsilon_{m,w})$ be a collection of distinct real numbers such that $0 < \varepsilon_{m,w} < 1/3$ for all (m, w).[5]

Suppose that a man of type m and a woman of type w both receive the same utility $u_{m,w}$ by being matched to each other. We show that X is a stable matching in $\langle M, W, P_U, K\rangle$. It follows that X is the unique stable matching because preferences are correlated (Lemma 20).

Suppose for contradiction that a pair (m_i, w_j) blocks X. There exist $m_{i'}$ and $w_{j'}$ such that $x_{m_i,w_{j'}} > 0$, $x_{m_{i'},w_j} > 0$, and $u_{m_i,w_j} > u_{m_i,w_{j'}}$ and $u_{m_i,w_j} > u_{m_{i'},w_j}$. Since $x_{m_i,w_{j'}} > 0$ and $x_{m_{i'},w_j} > 0$, they are nodes in V, and $u_{m_i,w_{j'}} > 1$ and $u_{m_{i'},w_j} > 1$. Thus $u_{m_i,w_j} > \max\{u_{m_{i'},w_j}, u_{m_i,w_{j'}}\} > 1$, by definition of U, which implies $x_{m_i,w_j} > 0$. Therefore, $\langle(m_{i'}, w_j), (m_i, w_j), (m_i, w_{j'})\rangle$ is a path.

There are unique paths from v_0 to each $(m_{i'}, w_j)$, (m_i, w_j), and $(m_i, w_{j'})$.

Note that $u_{m_i,w_j} > u_{m_{i'},w_j}$ implies that $\eta((m_i, w_j)) \geq \eta((m_{i'}, w_j))$. Observe that if $(v, v') \in L$ then $\eta(v) \neq \eta(v')$ because if we had $\eta(v) = \eta(v')$ then v would not lie in the path $\langle v_0, \ldots, v'\rangle$ and v' would not lie in the path $\langle v_0, \ldots, v\rangle$, so $(v, v') \in L$ would imply the existence of a cycle. So we establish that $\eta(v) \neq \eta(v')$. So $\eta((m_i, w_j)) \geq \eta((m_{i'}, w_j))$ implies that

$$\eta((m_i, w_j)) > \eta((m_{i'}, w_j)). \tag{2}$$

[5] We use ε_{m_v,w_v} and $\varepsilon_{m,w}$ only to ensure strict preferences.

Now, (2) is only possible if the unique path from v_0 to (m_i, w_j) contains $(m_{i'}, w_j)$. Then $\eta(m_i, w_{j'}) > \eta(m_i, w_i)$; but this contradicts that $u_{m_i,w_j} > u_{m_i,w_{j'}}$. So (m_i, w_j) cannot be a blocking pair.

When (V, L) has multiple components, $\{(V_1, L_1), \ldots, (V_N, L_N)\}$, we can partition M and W as $(M_1, \ldots, M_N)$ and $(W_1, \ldots, W_N)$ such that for all $v \in V_n$, $m_v \in M_n$ and $w_n \in W_n$.

For each (V_n, L_n) with associated sets M_n and W_n, we assign utilities $(u_{m,w})_{(m,n)\in M_n\times W_n}$ similar to the single component case. For other m and w, we assign $u_{m,w} = \varepsilon_{m,w}$. For all (m, w), $\varepsilon_{m,w}$ are small and positive real number, and $\varepsilon_{m,w} \neq \varepsilon_{m',w'}$ when $(m, w) \neq (m', w')$.

Suppose a type m man and a type w woman are not matched under X. If there is n such that $(m, w) \in M_n \times W_n$, then (m, w) is not a blocking pair by the proof above for the case of a single connected component. If $(m, w) \in M_n \times W_l$ with $n < l$, then, by the construction of $u_{m,w}$, $w' \, P_m \, w$ for any w' with $x_{m,w'} > 0$. Thus (m, w) is again not a blocking pair; X is stable matching. By Lemma 20 it is the unique stable matching.

7.3 Proof of Theorem 8

Let X be a rationalizable aggregate matching such that all cycles of the associated graph (V, L) are balanced. Direct the edges of (V, L) such that each cycle is oriented as follows: if $\langle v_0, \ldots v_N\rangle$ is a cycle, then the edge $(v_n, v_{n+1}) \in L$ is oriented such that $d(v_{n+1}, v_n) = 1$, which we denote by $v_n \to v_{n+1}$. For each path $\langle v_0, \ldots v_N\rangle$, direct the edges in a similar way. If the matching X is rationalizable, then such an orientation of the edges exists and defines a rationalizing preferences profile P (Echenique, Lee, and Shum, 2010). The rationalizing preferences have the property that if $x_{i,j} = 0$ and $x_{i',j'} > 0$ then $w_{j'} \, P_{m_i} \, w_j$ if $i = i'$, and $m_{i'} \, P_{w_j} \, m_i$ if $j = j'$.

First, if X has no cycles, then it is rationalizable as the unique stable matching (Theorem 3), so there is nothing to prove, as a unique stable matching is also the median stable matching. Suppose then that X has at least one cycle $c = \langle v_0, \ldots v_N\rangle$. Enumerate the vertexes of the cycle such that $v_n \to v_{n+1}$ in the orientation (directed graph) of (V, L) above, and v_0 lies in the same row as v_1. Let

$$\Theta = \min\{v_0, v_2, \ldots, v_{N-2}\} = \min\{v_1, v_3, \ldots, v_{N-1}\}.$$

Let $\mathcal{A}$ be the set of all $|M| \times |W|$ matrices A of integer numbers such that

- $a_{i,j} = 0$ if (i, j) is not in the cycle c;
- for all i and j, $\sum_h a_{i,h} = 0$ $\sum_l a_{l,j} = 0$; and
- $|a_{i,j}| \leq \Theta$.

We want to make two observations about the matrices in $\mathcal{A}$. First, $(X + A)_{i,j} > 0 \Rightarrow x_{i,j} > 0$, so $X + A$ is a stable matching for all $A \in \mathcal{A}$. Second, $A \in \mathcal{A}$ if and only if $-A \in \mathcal{A}$; and $A \in \mathcal{A}$ is such that $x_{i,\cdot} \leq_{m_i} (X + A)_{i,\cdot}$ iff $(X - A)_{i,\cdot} \leq_{m_i} x_{i,\cdot}$. Similarly, $A \in \mathcal{A}$ is such that $x_{\cdot,j} \leq_{w_j} (X + A)_{\cdot,j}$ iff $(X - A)_{\cdot,j} \leq_{w_j} x_{\cdot,j}$.

We need to prove that there are no other stable matchings than the ones obtained through matrices in $\mathcal{A}$: Then X is a median stable aggregate matching.

Let $Y \neq X$ be another stable matching in the resulting market $\langle M, W, P, K \rangle$. We shall prove that $y_{i,j} \neq x_{i,j}$ only if $x_{i,j}$ is a vertex in a minimal cycle of X. Suppose then that $y_{i,j} \neq x_{i,j}$. The number of single agents of each type is the same in X as in Y (Proposition 16; in this case it is zero, as X has no single agents). So, if $x_{i,j} < y_{i,j}$ then there is $h \neq j$ and $l \neq i$ such that $y_{i,h} < x_{i,h}$ and $y_{l,j} < x_{l,j}$. Similarly, if $x_{i,j} > y_{i,j}$ then there is $h \neq j$ and $l \neq i$ such that $y_{i,h} < x_{i,h}$ and $y_{l,j} < x_{l,j}$.

We can apply the previous observation repeatedly to obtain a sequence $(i_1, j_1), \ldots (i_N, j_N)$ with $(i_1, j_1) = (i_N, j_N)$ such that for each $n \mod N$:

1. $(x_{i_n,j_n} - y_{i_n,j_n})(x_{i_{n+1},j_{n+1}} - y_{i_{n+1},j_{n+1}}) < 0$
2. $i_n \neq i_{n+1} \iff j_n = j_{n+1}$.

Claim 21. *For $n = 1, \ldots, N$, $x_{i_n,j_n} > 0$.*

Suppose, by way of contradiction, that $0 = x_{i_1,j_1} < y_{i_1,j_1}$. Without loss of generality, assume that $i_1 = i_2$. By definition of the rationalization P, we have that $w_{j_2} \, P_{m_{i_1}} \, w_{j_1}$, as $x_{i_1,j_1} = 0$ and $x_{i_1,j_2} > y_{i_1,j_2} \geq 0$. We can now show that if (i_n, j_n) and (i_{n+1}, j_{n+1}) differ in i, then j_n prefers $m_{i_{n+1}}$ to m_{i_n}; and that if they differ in j, then i_n prefers $w_{j_{n+1}}$ to w_{j_n}. This fact, which we prove in the next paragraph, establishes the contradiction: $i_N \neq i_{N-1}$, but $m_{i_{N-1}} \, P_{w_{i_N}} \, m_{i_N}$ by definition of P and because $0 = x_{i_N,j_N} = x_{i_1,j_1}$.

To prove the fact, we reason by induction. We have already established that $w_{j_2} \, P_{m_{i_1}} \, w_{j_1}$. Suppose that $w_{j_n} \, P_{m_{i_n}} \, w_{j_{n-1}}$. By Property 1 of the sequence $\langle (i_n, j_n) \rangle_{n=1}^{N}$, either

$x_{i_{n-1},j_{n-1}} > 0$ and $x_{i_{n+1},j_{n+1}} > 0$; or $y_{i_{n-1},j_{n-1}} > 0$ and $y_{i_{n+1},j_{n+1}} > 0$ (or both hold). Then the stability of X and Y implies that $m_{i_{n+1}} \, P_{w_{j_n}} \, m_{i_n}$. The proof for the case when $w_{i_n} \, P_{w_{i_n}} \, m_{i_{n-1}}$ is similar.

The claim implies that the sequence $\langle v_n \rangle = \langle x_{i_n,j_n} \rangle$ is a cycle in (V, L). Thus a stable Y can only differ from X in vertexes that are part of a cycle of (V, L). Let $A = Y - X$; we shall prove that $A \in \mathcal{A}$. We established above that $a_{i,j} \neq 0$ only if $x_{i,j}$ is a vertex in a cycle. We now prove that $|a_{i,j}| \leq \Theta$. Clearly, $a_{i,j} \geq -x_{i,j} \geq -\Theta$. We show that if $a_{i,j} > 0$ then there is h such that $a_{i,j} + a_{i,h} = 0$.

If $a_{i,j} > -a_{i,h}$ for all $h \neq i$ then there is h_1 and h_2 such that some men of type m_i who are married to women of type h_1 and h_2 in X are married to women of type w_j in Y. Then we can define two cycles, and $x_{i,j}$ would be a vertex in both of them. The first cycle has $(x_{i,j}, x_{i,h_1})$ as the first edge, and the remaining edges defined inductively, by the definition of $\langle (i_n, j_n) \rangle$ above. The second cycle has $(x_{i,j}, x_{i,h_2})$ as the first edge, and the remaining edges defined inductively. The resulting two cycles would be connected, which contradicts the hypothesis that X is rationalizable. So there must exist some h with $a_{i,j} \leq -a_{i,h}$. An analogous argument applied to $a_{i,h}$ implies that $a_{i,j} \geq -a_{i,h}$; so $a_{i,j} = -a_{i,h}$. Then, $a_{i,j} \leq \Theta$, as $a_{i,h} \geq -\Theta$.

7.4 Other proofs

Proof of Proposition 6. The proof that $Y^{(i)}$ is a stable matching follows from lattice structure with the operators $\vee$ and $\wedge$ and essentially the same as the proofs of median stable matchings presented in (Klaus and Klijn, 2006; Schwarz and Yenmez, 2007). □

References

Abdulkadirolu, A., P. A. Pathak, and A. E. Roth (2005): "The New York City High School Match," *The American Economic Review*, 95(2), pp. 364–367.

Adachi, H. (2000): "On a Characterization of Stable Matchings," *Economic Letters*, 68, 43–49.

Alkan, A., and D. Gale (2003): "Stable schedule matching under revealed preference," *Journal of Economic Theory*, 112(2), 289 – 306.

Becker, G. S. (1973): "A Theory of Marriage: Part I," *Journal of Political Economy*, 81(4), 813–846.

Choo, E., and A. Siow (2006): "Who Marries Whom and Why," *Journal of Political Economy*, 114(1), 175–201.

Dagsvik, J. K. (2000): "Aggregation in Matching Markets," *International Economic Review*, 41(1), 27–57.

Echenique, F., S. Lee, and M. Shum (2010): "Aggregate Matchings," Caltech SS Working Paper 1328.

Echenique, F., and J. Oviedo (2004): "Core Many-to-one Matchings by Fixed Point Methods," *Journal of Economic Theory*, 115(2), 358–376.

——— (2006): "A Theory of Stability in Many-to-Many Matching Markets," *Theoretical Economics*, 1(2), 233–273.

Echenique, F., and M. B. Yenmez (2007): "A Solution to Matching with Preferences over Colleagues," *Games and Economic Behavior*, 59(1), 46–71.

Fleiner, T. (2003): "A Fixed-Point Approach to Stable Matchings and Some Applications," *Mathematics of Operations Research*, 28(1), 103–126.

Gale, D., and L. S. Shapley (1962): "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, 69(1), 9–15.

Hatfield, J., and P. Milgrom (2005): "Auctions, Matching and the Law of Auctions Matching and the Law of Aggregate Demand," *American Economic Review*, 95(4), 913–935.

Hylland, A., and R. Zeckhauser (1979): "The efficient allocation of individuals to positions," *The Journal of Political Economy*, 87(2), 293–314.

Kesten, O., and U. Ünver (2009): "A Theory of School Choice Lotteries," mimeo, Boston College and Carnegie Mellon University.

Klaus, B., and F. Klijn (2006): "Median stable matching for college admissions," *International Journal of Game Theory*, 34(1), 1–11.

——— (2010): "Smith and Rawls share a room: stability and medians," *Social Choice and Welfare*, 35, 647–667, 10.1007/s00355-010-0455-8.

Komornik, V., Z. Komornik, and C. Viauroux (2010): "Stable schedule matchings by a fixed point method," *UMBC Economics Department Working Papers.*

Ostrovsky, M. (2008): "Stability in Supply Chain Networks," *American Economic Review*, 98(3), 897–923.

Roth, A., U. Rothblum, and J. Vate (1993): "Stable matchings, optimal assignments, and linear programming," *Mathematics of Operations Research*, 18(4), 803–828.

Roth, A., and M. Sotomayor (1988): "Interior Points in the Core of Two-Sided Matching Markets," *Journal of Economic Theory*, 45, 85–101.

——— (1990): *Two-sided Matching: A Study in Game-Theoretic Modelling and Analysis*, vol. 18 of *Econometric Society Monographs.* Cambridge University Press, Cambridge England.

Roth, A. E. (2008): "Deferred Acceptance Algorithms: History, Theory, Practice and Open Questions," *International Journal of Game Theory*, 36(3), 537–569.

Rothblum, U. (1992): "Characterization of stable matchings as extreme points of a polytope," *Mathematical Programming*, 54(1), 57–67.

Schwarz, M., and M. B. Yenmez (2007): "Median Stable Matching for Markets with Wages," Forthcoming, Journal of Economic Theory.

Shapley, L., and M. Shubik (1971): "The assignment game I: The core," *International Journal of Game Theory*, 1(1), 111–130.

Teo, C.-P., and J. Sethuraman (1998): "The geometry of fractional stable matchings and its applications," *Math. Oper. Res.*, 23(4), 874–891.

Vande Vate, J. H. (1989): "Linear programming brings marital bliss," *Oper. Res. Lett.*, 8(3), 147–153.