Further work may be concerned with the estimation of the derivative of $R(x)$ by

$$\hat{R}_n^{(1)}(x)=\sum_{k=0}^{N(n)} a_{kn}\frac{d}{dx}g_k(x) \tag{16}$$

where $\hat{a}_{kn}$ is given by (7). One may apply the algorithm (16) for identifying the weighting function $w(t)$ in the system (13).

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Nonlinear Model Identification by Analysis of Feedback-Stimulated Bifurcation

BOGUSLAW KUSZTA AND JAMES E. BAILEY

Abstract—**Local topological equivalence between two rival models breaks down when one of the models is at a state of bifurcation. Creation of bifurcation conditions by feedback consequently may be a useful and sensitive approach for model identification.**

INTRODUCTION

Many dynamical systems are described mathematically by nonlinear ordinary differential equations. Every model is an approximation, the complexity of which depends upon the model's intended use and environmental conditions such as the spectra and amplitudes of input signals. The model's acceptability rests on a compromise between its incorporation of known phenomenological relationships for the process and its suitability for engineering analyses and control. In some situations the goal of modeling is to interpret experimental data in an effort to deduce the underlying physical attributes of the system.

An important aspect of any dynamic mathematical modeling study is determination of sensitive tests for evaluating and comparing models, both among themselves and with the observed behavior of the physical process. Such methods depend upon means of revealing significant differences between rival models. The theory of smooth dynamical systems on manifolds, specifically the central manifold theorem [1] or the reduction theorem [2], indicates that deeper insight into model structure can be obtained when the system is at the state of bifurcation. Since many nonlinear systems do not exhibit bifurcation for any time-invariant input, it is desirable for modeling to modify such systems so that bifurcation occurs. This can be done using feedback.

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NONLINEAR MODEL EQUIVALENCE AND BIFURCATION

Consider two nonlinear systems described by

$$\dot{x}=N(x,u,\beta), \qquad x\in X\subset R^n; \quad x(0)=x^0 \tag{1}$$

$$\dot{y}=M(y,u,\beta), \qquad y\in Y\subset R^m; \quad y(0)=y^0. \tag{2}$$

In both models u denotes the input and β is a vector of manipulated parameters. Here manipulated parameters denote process operating conditions, such as the temperature or mean residence time of a chemical reactor, which can be varied from experiment to experiment. Both models will usually also contain unknown parameters, the presence of which will remain implicit here.

Considering first the case of time-invariant input $u(t)=u^*$, both models are autonomous. Then the two models are said to be *topologically equivalent* if there exists a continuous one-to-one mapping h: $X\to Y$ such that trajectories of the first system are transformed to the trajectories of the second [3], [4]. Thus, in a certain sense, there is no basis for discriminating between topologically equivalent models.

A necessary condition for topological equivalence is that the Jacobian matrices of the two systems, evaluated at the stationary point corresponding to the applied u^* and β values, must have the same spectrum on the imaginary axis. This follows from the reduction theorem for nonlinear differential equations [2]. If the spectrum on the imaginary axis is nonempty for either model, that model is at a state of bifurcation [4]. Therefore, if only one of the systems is at the bifurcation state, the models are not equivalent. Comparing experimentally observed bifurcations with bifurcations of models is consequently a potentially sensitive discriminator and test for model validity. Bifurcation is identified experimentally as a qualitative change in the local phase space behavior as u^* and/or β values are smoothly changed. Appearance or disappearance of limit cycles is one common kind of bifurcation; the most familiar type of bifurcation to a limit cycle is the Hopf bifurcation, which corresponds to an imaginary-axis spectrum containing a single pair of conjugate imaginary values.

Unfortunate from the viewpoint of using bifurcation in modeling is the absence of any bifurcation in many physical systems and their corresponding models. This situation can often be modified by introduction of feedback

$$u=u^*-K(x-x^*) \tag{3}$$

where attention for the moment will be restricted to system (1); analogous comments apply to (2). Consider, for example, the following second-order nonlinear system:

$$\dot{x}_1=-x_2-bx_1^2-(2c+\delta)x_1x_2+du_1x_2 \tag{4.1}$$

$$\dot{x}_2=x_1+ax_1^2+(2b+\epsilon)x_1x_2-cu_2x_2. \tag{4.2}$$

Taking

$$u_i=-k_ix_2; \qquad i=1,2 \tag{5}$$

it may be shown that the resulting autonomous system exhibits a Hopf bifurcation if one of the following conditions is satisfied [5].

i) $a+ck_2=b+dk_1=0.$ (6.1)

ii) $(a+ck_2)=\delta(b+dk_1)$
and (6.2)
$a\epsilon^3-(3b+\epsilon)\epsilon^2\delta+(3ck_2+\delta)\epsilon\delta^2-dk_1=0.$

iii) $\epsilon+5(b+dk_1)=\delta+5(a+ck_2)$
$=ack_2+bdk_1+2(a^2+d^2k_1^2)=0.$ (6.3)

While other values may satisfy (6.2) or (6.3), use of the gains

$$k_1=-b/d, \quad k_2=-a/c \tag{6.4}$$

will satisfy condition (6.1) and will therefore produce a Hopf bifurcation.

RESONANCE AND OTHER TYPES OF BIFURCATION

System (1) can be rewritten in the form

$$\dot{x} = Ax + \alpha^2(x) + \alpha^3(x) + \cdots \tag{7}$$

where α^r is a polynomial vector of degree r (i.e., vectors with components which are the sums of monomials of degree r). Let $\{\lambda_i\} = \{\lambda_1, \lambda_2, \cdots \lambda_n\}$ be a set of the eigenvalues of the matrix A. The set $\{\lambda_i\}$ is called resonant [4] if there exists a vector $m = (m_1, \cdots, m_n)$, the elements of which are nonnegative integers, which satisfies

$$\lambda_s = (m, \lambda) \triangleq \sum_j m_j \lambda_j \qquad \text{for some } s \in \{1, \cdots, n\} \tag{8}$$

and

$$\sum_{k=1}^{n} m_k = |m| \geq 2. \tag{9}$$

The integer $|m|$ is called the order of resonance.

By inspection it follows that resonance is a necessary condition for bifurcation of the kind discussed above. A zero eigenvalue is always resonant and Hopf bifurcation ($\lambda_1 = -\lambda_2 = j\omega$) is resonance of order three ($\lambda_1 + \lambda_2 = 0$ so that $\lambda_1 = 2\lambda_1 + \lambda_2$).

Resonance is connected with the qualitiative properties of nonlinear dynamic models in another important way. Other types of bifurcations, not involving zero real part Jacobian eigenvalues, may occur at resonant conditions. Examples of such bifurcations include a change from a T-periodic to a $2T$-periodic limit cycle [6]. On the other hand, when the vector field considered is real, i.e., $X \subset R^n$, resonance does not necessarily imply bifurcation. This is easily illustrated with the example of a linear system with $\lambda_1 = -1$ and $\lambda_2 = -2$, which, although resonance occurs, retains the same topological structure for small perturbations in the eigenvalues.

However, if the complex vector field is considered, resonance is sufficient for a change in local topological structure even for linear systems. This is significant because analysis of complex vector fields is often involved in bifurcation analysis, even when the original vector field is real. For example, for a system of dimension $n \geq 3$ with Hopf bifurcation and with the remaining n-2 eigenvalues in the left-hand plane, the nature of the bifurcated oscillation is studied on the central manifold, i.e., the system is reduced to two complex differential equations called normal Poincaré form [1].

CONCLUDING REMARKS

By introducing multivariable proportional feedback, a set of new and adjustable parameters, the feedback gains, influence system dynamics. The availability of a sufficient number of adjustable parameters is important for generation of complicated bifurcations in which three or more eigenvalues of the linear part of the model cross the imaginary axis simultaneously [4]. Here it is important not only to be able to locate the desired eigenvalues on the imaginary axis but also to preserve the structural stability of the arrangement in the presence of small changes in other system parameters [4]. Investigation of these situations should enhance the value of feedback-induced bifurcation in evaluating nonlinear process models.

Bifurcation analysis can also provide lower bounds on required model order and give information useful for parameter evaluation. For example, if the physical system exhibits Hopf bifurcation, a model of order at least two is needed; occurrence of additional bifurcation implies an order of three or more. Such information is very useful when the objective of dynamic experiments is elucidation of the structure of a sybsystem which can only be observed indirectly. For example, it is desirable to learn as much as possible about interactions on catalyst surface by means of transient manipulations and measurements of the adjacent fluid phase composition. In this context lower bounds on the required order of the surface reaction model can sometimes be obtained based on observations of oscillations in the fluid phase; simulation studies have shown that bifurcations can be generated in some chemical reaction systems of this type based only on feedback of available fluid phase composition measurements.

The value of feedback gain K at bifurcation can often be related to model parameters. Equation (6.4) above provides a simple illustration of this. Furthermore, the kind of oscillation observed after bifurcation (soft or hard, stable or unstable) gives further information on model form and parameter estimates. It is apparent that the study of bifurcation structure produced by feedback with adjustable gain K may be useful in investigating the local structure of nonlinear dynamic models for purposes of identification, validation, and discrimination.

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On-Line Identification of Time-Varying Systems by Nonparametric Techniques

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***Abstract*—A nonparametric algorithm, based on orthogonal series, is proposed for real-time identification of nonstationary systems. It is proved that the algorithm converges in the mean-square sense. The convergence conditions are closely related to those in Dupač [2].**

I. INTRODUCTION

An important problem in control engineering is the identification of time-varying systems. In this correspondence we focus our interest on systems described by

$$y_n = R_n(u_n) + z_n, \qquad n = 1, 2, \cdots \tag{1}$$

where $R_n(\cdot)$ are measurable functions, u_n and y_n are inputs and outputs of the system, z_n are measurement noises (independent of u_n) such that

$$Ez_n = 0, \quad Ez_n^2 = \sigma_n^2 < \sigma^2, \qquad n = 1, 2, \cdots. \tag{2}$$

The inputs u_n (a sequence of random s-vectors) have the same probability density $f(\cdot)$. Depending upon whether or not the functional form of $R_n(\cdot)$, $n = 1, 2, \cdots$, is known, two sequential approaches have been developed to identify the system (1).

1) If $R_n(\cdot)$, $n = 1, 2, \cdots$, are linear with unknown parameters $a_n^T = (a_n^1, \cdots, a_n^s)$, i.e.,

$$y_n = a_n^T u_n + z_n, \qquad n = 1, 2, \cdots, \tag{3}$$

several authors applied the stochastic approximation algorithms for the on-line estimation of a_n (see Kwatny [3] and references cited therein).

2) If $R_n(\cdot)$, $n = 1, 2, \cdots$, are completely unknown, the following model of $R_n(\cdot)$ may be proposed:

$$\tilde{R}_n(u) = c_n^T \Phi(u), \qquad n = 1, 2, \cdots \tag{4}$$

where $\Phi^T(u) = \{\Phi_1(u), \cdots, \Phi_k(u)\}$ are known basis functions and c_n is a

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