- "ARPANET routing algorithm improvements, second semiannual Rep.," Bolt, Beranek, and Newman, Inc., BBN Rep. prepared for ARPA and DCA, Oct. 1978.
- [5] J. McQuillan, G. Falk, and I. Richer, "A review of the development and performance of the ARPANET routing algorithm," *IEEE Trans. Commun.*, vol. COM-26, Dec. 1978.
 - [6] M. Schwartz, *Computer-Communication Network Design and Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1977.
 - [7] L. Kleinrock, *Queueing Systems*, Vol. I and II. New York: Wiley, 1976.
 - [8] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*. New York: Academic, 1970.
 - [9] R. Gallager, "A minimum delay routing algorithm using distributed computation," *IEEE Trans. Commun.*, vol. COM-25, pp. 73-85, 1977.
 - [10] E. Jury, *Theory and Application of the z-Transform Method*. New York: Wiley, 1964.
 - [11] J. McQuillan, I. Richer, and E. Rosen, "The new routing algorithm for the ARPANET," *IEEE Trans. Commun.*, vol. COM-28, pp. 711-719, 1980.
 - [12] D. P. Bertsekas, "A class of optimal routing algorithms for communication networks," in *Proc. 5th Int. Conf. Computer Communication (ICCC'80)*, Atlanta, GA, pp. 71-76, Oct. 1980.
 - [13] D. P. Bertsekas, E. Gafni, and R. G. Gallager, "Second derivative algorithms for minimum delay distributed routing in networks," L.I.D.S. working paper, Massachusetts Inst. Technol., Cambridge, Jan. 1981 (submitted for publication).

Dimitri P. Bertsekas (S'70-SM'77) was born in Athens, Greece, in 1942. He received the Mechanical and Electrical Engineering Diploma from the National Technical University of Athens, Greece, in 1965, the M.S.E.E. degree from George Washington University, Washington, DC, in 1969, and the Ph.D. degree in system science from the Massachusetts Institute of Technology, Cambridge, in 1971.

He has held faculty positions with the Engineering-Economic Systems Department, Stanford University, Stanford, CA, and the Electrical Engineering Department of the University of Illinois, Urbana. He is currently in the faculty of the Department of Electrical Engineering and Computer Science of the Massachusetts Institute of Technology. He consults regularly with private industry and he is an Associate Editor for the *SIAM Journal on Control and Optimization*. He has done research in the areas of estimation and control of stochastic systems, nonlinear and dynamic programming, and data communication networks, and has written numerous papers in each of these areas. He is the author of *Dynamic Programming and Stochastic Control* (Academic, 1976), *Constrained Optimization and Lagrange Multiplier Methods* (Academic, 1981), and the coauthor of *Stochastic Optimal Control: The Discrete-Time Case* (Academic, 1978).

Short Papers

Reconstruction of Atmospheric Pollutant Concentrations from Remote Sensing Data—An Application of Distributed Parameter Observer Theory

MASATO KODA AND JOHN H. SEINFELD

Abstract—The reconstruction of a concentration distribution from spatially averaged and noise-corrupted data is a central problem in processing atmospheric remote sensing data. Distributed parameter observer theory is used to develop reconstructibility conditions for distributed parameter systems having measurements typical of those in remote sensing. The relation of the reconstructibility condition to the stability of the distributed parameter observer is demonstrated. The theory is applied to a variety of remote sensing situations, and it is found that those in which concentrations are measured as a function of altitude satisfy the conditions of distributed state reconstructibility.

I. INTRODUCTION

In the remote sensing of tropospheric species, a ground-, aircraft-, or satellite-based platform performs an instantaneous scan of a region of the atmosphere and measures the species burden within the field of view. With aircraft or satellite remote sensing the platform is in motion and the field of view is constantly changing. An object of remote sensing of the atmosphere is to enable reconstruction of the concentration distribution of trace species over an entire region based on the data available from the instrument.

The reconstruction of a concentration distribution from spatially averaged and possibly noise-corrupted data is a central problem in processing remote sensing data. In the absence of a mathematical model describing the spatial and temporal concentration distributions, the reconstruction can be carried out by standard data interpolation methods. However, when a mathematical model exists, the problem becomes one of matching the remote sensing data to the model solution in such a way that the incomplete data can be used in conjunction with the model to produce an estimate of the region-wide concentration distribution. This problem of the matching or assimilation of remote sensing data into mathematical models for atmospheric constituents is the subject of this paper.

There exist a few recent studies that assess the capabilities of remote sensing for monitoring regional air pollution episodes. For example, Barnes *et al.* [1] conducted a comparative analysis of satellite visible channel imagery in ground-based aerosol measurements. For three cases, each of which represented a significant pollution episode based on low surface visibility and high sulfate levels, the results show that the extent and transport of the haze pattern can be monitored from satellite data. The study demonstrated the potential of the satellite to monitor both magnitude and aerial extent of pollution episodes. In a related study, Lyons *et al.* [2] reported on a demonstration project showing that currently available synchronous satellite data can detect the aerial extent of large-scale hazy air masses associated with sulfate and ozone episodes.

A study related to that of the present work was reported by Diamante *et al.* [3] in which they considered the comparison of remote and in situ data on pollutant concentrations from point sources. They considered typical remote sensing geometries to provide insight on estimation of plume properties from these measurements. In a study also related to the present, Kibbler and Suttles [4] considered the estimation of unknown parameters in a pollutant dispersion model by comparing model predictions with remotely sensed air-quality data. A ground-based sensor provided relative pollutant concentration measurements as a function of space and time. The measured data were compared with the dispersion model output through a numerical estimation procedure to yield parameter estimates that best fit the data.

Manuscript received September 9, 1980; revised September 22, 1980 and March 9, 1981. Paper recommended by A. Ephremides, Past Chairman of the Estimation Committee. This work was supported by NASA under Research Grant NAG-1-71.

M. Koda is with the Department of Chemical Engineering, California Institute of Technology, Pasadena, CA, on leave from the Department of Aeronautics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113, Japan.

J. H. Seinfeld is with the Department of Chemical Engineering, California Institute of Technology, Pasadena, CA 91125.

The object of this paper is to define the problem of the assimilation of atmospheric remote sensing data into mathematical models of pollutant behavior. Since the atmosphere is a three-dimensional system, models of pollutant behavior are of the distributed parameter type [5]. Remote sensing data represent spatial averages of concentrations, so that the assimilation problem is, in essence, one of distributed parameter state estimation.

First, the concept of distributed state reconstructibility is developed for the class of problems of interest. That is, the first question to be faced is— can the desired spatial-temporal concentration distribution information be recovered from the measurements in the absence of noise. The derivation of general conditions that allow one to answer this question is the subject of Section II. In Section III a variety of common remote sensing measurement configurations and atmospheric models are tested for reconstructibility. We conclude in Section IV with general observations concerning the inherent potential of remote sensing data in analyzing regional air pollution.

II. RECONSTRUCTIBILITY AND OBSERVERS FOR DISTRIBUTED PARAMETER SYSTEMS

Atmospheric pollutant models consist of partial differential equations, linear in the case in which the species does not react chemically or in which it is produced or destroyed by a first-order reaction of the form $A \rightarrow$. This case represents a wide class of important situations and is the one to which we direct our attention here. Nonlinear distributed models must be handled by linearization and therefore also fall within the present framework.

Our interest in this section is to derive distributed parameter observers for systems described by linear partial differential equations with inhomogeneous boundary conditions characteristic of atmospheric models. An observer is an algorithm that processes measurements of the state of a system to yield an estimate of the entire system state. An observer is most frequently employed when not all of the states of a system are accessible for measurement. In the present application, we will be generally interested in only a single state variable, the measurements of which have limited spatial resolution. The observer is stable if its estimated state converges to the true state after a sufficiently long time. The concept of state reconstructibility is useful as a condition for the stability of the observer. Thus, if a measurement strategy satisfies the condition of state reconstructibility, then the corresponding observer is stable, and, the state (i.e., the concentrations) can, in principle, be estimated from the measurements. The condition that allows the reconstruction of the system state on the entire field is called distributed state reconstructibility. Associated with distributed state reconstructibility, the concept of uniform n -mode reconstructibility can be developed. Both conditions, n -mode and distributed state reconstructibility, will be applied, in Section III, to typical remote sensing measurement configurations.

There exists some previous work on observer theory for distributed parameter systems [6]–[8]. Kitamura *et al.* [6] formally extended the lumped parameter observer to the distributed parameter case. Gressang and Lamont [7] developed a more complete theory of the distributed parameter observer, including reduced order observers. An application of distributed parameter observer theory has been presented by Köhne [9]. The most complete treatment of observer theory is that of Dolecki and Russell [8]. In the current work, distributed parameter observers are derived in a form appropriate for application to the class of systems representing atmospheric species behavior. In addition, a result of the present work is an explicit relation between distributed parameter reconstructibility and the stability of the observer. Observer stability is demonstrated using a technique of Hale [10] in which Lyapunov stability theory is extended to function spaces.

We consider the linear distributed parameter system,

$$\frac{\partial u(x, t)}{\partial t} = L_x u(x, t) + B(x, t)f(x, t) \quad (1)$$

defined for $t > 0$, $x \in D$. The domain D is a connected subset of a d -dimensional Euclidean space E^d with boundary surface ∂D . The d -dimensional spatial coordinate vector is denoted by x . The state $u(x, t)$ is

a scalar function and L_x is a linear partial differential operator with respect to x . It is assumed that the operator L_x is well-posed. The input $f(x, t)$ is a known scalar function and $B(x, t)$ is a known coefficient.

The boundary condition on (1) is

$$\beta_x u(x, t) = h(x, t) \quad x \in \partial D \quad (2)$$

where β_x is a linear, spatial differential operator of a suitable order over ∂D and $h(x, t)$ is a known function. The initial condition is assumed to be unknown or incompletely known.

We are interested in considering three types of measurements.

Case 1—Spatially Independent Integral Measurements: The measurement takes the form

$$w(t) = \int_D H(x, t)u(x, t) dx \quad (3)$$

where $H(x, t)$ is a spatial weighting function.

Case 2—Spatially Continuous Measurements:

$$w(x, t) = C(x, t)u(x, t) \quad (4)$$

where $C(x, t)$ is a square-integrable function, i.e., $C \in L_2$.

Case 3—Spatially Discrete Measurements:

$$w_i(t) = H_i(t)u(x_i, t), \quad i = 1, 2, \dots, l \quad (5)$$

where $w_i(t)$ denotes a measurement at the i th measurement location x_i . By taking the limit to small volumes of integration in (3), we can represent a system such as (5) by choosing $H(x, t) = H_i(t)\delta(x - x_i)$, $i = 1, 2, \dots, l$.

For the moment let us restrict the problem to one spatial dimension, i.e., $0 \leq x \leq 1$. Accordingly, boundary condition (2) can be expressed as

$$\begin{aligned} \beta_0 u(x, t) &= h_0(t), & x &= 0 \\ \beta_1 u(x, t) &= h_1(t), & x &= 1. \end{aligned} \quad (6)$$

Then the solution of (1) and (6) with initial condition $u(x, 0) = u_0(x)$ can be expressed in the form¹

$$\begin{aligned} u(x, t) &= \int_0^1 \Phi^*(r, 0; x, t)u_0(r) dr \\ &+ \int_0^t \int_0^1 \Phi^*(r, \tau; x, t)B(r, \tau)f(r, \tau) dr d\tau \\ &+ \int_0^t \int_0^1 \Phi^*(r, \tau; x, t)g(r, \tau) dr d\tau \end{aligned} \quad (7)$$

where

$$g(x, t) = 2h_1(t)\delta(x - 1) - 2h_0(t)\delta(x). \quad (8)$$

The adjoint Green's function $\Phi^*(x, t; y, \tau)$ is governed by

$$\frac{\partial \Phi^*(x, t; y, \tau)}{\partial t} + L_x^* \Phi^*(x, t; y, \tau) = 0 \quad (9)$$

with the terminal condition

$$\Phi^*(x, t; y, t) = \delta(x - y) \quad (10)$$

and boundary conditions

$$\begin{aligned} \beta_0^* \Phi^* &= 0 \\ \beta_1^* \Phi^* &= 0. \end{aligned} \quad (11)$$

¹The explicit form of operators L_x , β_0 , and β_1 are assumed as follows:

$$L_x(\cdot) = \alpha_2(x, t) \frac{\partial^2(\cdot)}{\partial x^2} - \alpha_1(x, t) \frac{\partial(\cdot)}{\partial x} - \alpha_0(x, t)(\cdot)$$

$$\beta_0(\cdot) = \alpha_2(0, t) \frac{\partial(\cdot)}{\partial x} + \theta_0(t)(\cdot)$$

$$\beta_1(\cdot) = \alpha_2(1, t) \frac{\partial(\cdot)}{\partial x} + \theta_1(t)(\cdot).$$

The operators L_x^* , β_0^* , and β_1^* are the adjoints of the operators L_x , β_0 , and β_1 , respectively.

The extension of the adjoint Green's functions to higher spatial dimensions is straightforward. In higher dimensions, (9) and (10) remain the same with the general boundary conditions

$$\beta_x^* \Phi^*(x, t; y, \tau) = 0, \quad x \in \partial D \quad (12)$$

where β_x^* is the adjoint of the operator β_x . In general, we note that Φ^* is related to the Green's function Φ associated with the system (1) with homogeneous boundary conditions by the relationship $\Phi(x, t; y, \tau) = \Phi^*(y, \tau; x, t)$.

The adjoint Green's function for well-posed distributed parameter systems can be constructed in a variety of ways. Expansion in spatial eigenfunctions and construction of the adjoint Green's function from eigenvalues and eigenfunctions is a powerful method for linear systems. Let us assume that L_x^* has an infinite series of discrete eigenvalues $\{\lambda_i\}$, $i = 1, 2, \dots$. Using standard methods, the adjoint Green's function that satisfies (9)–(12) is found to be [11]

$$\Phi^*(x, t; y, \tau) = \sum_{n=1}^{\infty} \phi_n(x) \phi_n(y) e^{-\lambda_n(t-\tau)} \quad (13)$$

where the eigenfunctions $\{\phi_i\}$, $i = 1, 2, \dots$, are the solution of the equation. $L_x^* \phi_i = \lambda_i \phi_i$, satisfying the boundary conditions (11) or (12).

A. Reconstructibility Conditions

The objective of an observer is to reconstruct the system state when the measurements are incomplete. To be able to reconstruct the state the observer must be asymptotically stable.

An identity or nonreduced observer for the system (1) with measurements (4) takes the form

$$\frac{\partial \hat{u}(x, t)}{\partial t} = L_x \hat{u}(x, t) + B(x, t)f(x, t) + G[w(x, t) - C(x, t)\hat{u}(x, t)] \quad (14)$$

where $\hat{u}(x, t)$ is the observer output and G is a suitably chosen integral operator with the kernel $G(x, y, t)$.

Before presenting a derivation of the observer, we will establish the conditions under which the system (1) and (4) is reconstructible. We define the reconstructibility kernel function by

$$Q(x, y, t) = \int_0^t \int_D \Phi^*(x, t; r, \tau) C^2(r, \tau) \Phi^*(y, t; r, \tau) dr d\tau. \quad (15)$$

It will be shown later that the observer (14) is stable if $Q(x, y, t)$ has a so-called generalized inverse, i.e., if there exists $P(x, y, t)$ such that

$$\int_D P(x, r, t) Q(r, y, t) dr = \delta(x - y). \quad (16)$$

By formal differentiation of (15) with respect to time and use of the properties of the adjoint Green's function (9)–(12), it is found that $Q(x, y, t)$ satisfies the following Lyapunov equation,

$$\frac{\partial Q(x, y, t)}{\partial t} = -L_x^* Q(x, y, t) - Q(x, y, t) L_y^* + C^2(x, t) \delta(x - y) \quad (17)$$

with the initial condition

$$Q(x, y, 0) = 0 \quad (18)$$

and boundary conditions

$$\beta_x^* Q = 0, \quad Q \beta_y^* = 0 \quad (19)$$

where $L_x^* Q = Q L_y^*$. Although $Q(x, y, t)$ is formally defined by (15), it is

important to note that $Q(x, y, t)$ may be computed from (17)–(19) without using the adjoint Green's function.

By using the identity

$$\frac{\partial P(x, y, t)}{\partial t} = - \int_D \int_D P(x, r, t) \frac{\partial Q(r, s, t)}{\partial t} P(s, y, t) ds dr \quad (20)$$

$P(x, y, t)$ can be shown to obey the following Riccati equation,²

$$\frac{\partial P(x, y, t)}{\partial t} = L_x P(x, y, t) + P(x, y, t) L_y - \int_D P(x, r, t) C^2(r, t) P(r, y, t) dr \quad (21)$$

with boundary conditions

$$\beta_x P = 0 \quad P \beta_y = 0 \quad (22)$$

$P(x, y, t)$ may be considered as the kernel of the integral operator P defined as

$$Pf(x) = \int_D P(x, y, t) f(y) dy \quad (23)$$

for $f \in L_2$.

A linear distributed parameter system (1) and (2) with measurement (4) is said to be *distributed state reconstructible* if and only if $Q(x, y, t)$ defined by (15) has a bounded generalized inverse $P(x, y, t)$ for $t > 0$. It may be shown that $Q(x, y, t)$ has a bounded generalized inverse when $Q(x, y, t)$ is bounded and positive definite for $t > 0$ [11].³ The system (1), (2), and (4) will be defined to be *uniformly n -mode reconstructible* if there exists positive constants M_1 , M_2 , and σ such that

$$M_1 \leq \int_D \int_D \phi_n(x) Q^o(x, y, t) \phi_n(y) dx dy \leq M_2 \quad (24)$$

for all $t > 0$, where $\phi_n(x)$ is the eigenfunction of L_x^* and the modified reconstructibility kernel $Q^o(x, y, t)$ is defined by

$$Q^o(x, y, t) = \int_{t-\sigma}^t \int_D \Phi^*(x, t; r, \tau) C^2(r, \tau) \Phi^*(y, t; r, \tau) dr d\tau. \quad (25)$$

The system is distributed state reconstructible if (24) is satisfied for each of the eigenfunctions. The uniform n -mode reconstructibility test (24) is useful when $P(x, y, t)$ cannot be found directly from $Q(x, y, t)$. Since it is straightforward to extend the concept of distributed state reconstructibility to measurement Cases 1 and 3, detailed discussion is omitted here.

B. Minimum Variance Observers

1) *Observer for Case 1:* For the system described by (1) and (3), we define the reconstructibility kernel function by

$$Q(x, y, t) = \int_0^t \int_D \Phi^*(x, t; r, \tau) H(r, \tau) dr \int_D H(s, \tau) \Phi^*(y, t; s, \tau) ds d\tau \quad (26)$$

where $Q(x, y, t)$ obeys

$$\frac{\partial Q(x, y, t)}{\partial t} = -L_x^* Q(x, y, t) - Q(x, y, t) L_y^* + H(x, t) H(y, t) \quad (27)$$

with initial and boundary conditions given by (18) and (19). Assuming that the system is distributed state reconstructible, the existence of the generalized inverse $P(x, y, t)$ of $Q(x, y, t)$, that satisfies

²The impact of observation error on the design of an observer can be assessed from (21) by comparing P to that from the corresponding distributed parameter filter.

³Positive-definiteness of the kernel Q implies that

$$\int_D \int_D f(x) Q(x, y, t) f(y) dx dy > 0, \quad \text{for all } t > 0 \text{ and } f \in L_2.$$

$$\frac{\partial P(x, y, t)}{\partial t} = L_x P(x, y, t) + P(x, y, t) L_y - \int_D P(x, r, t) H(r, t) dr \int_D H(s, t) P(s, y, t) ds \quad (28)$$

will establish the observer for the system (1)–(3).

Following Meditch [12], we define the cost functional associated with the observer as

$$J_0 = \frac{1}{2} \int_D [u(x, 0) - u_0(x)] A_x [u(y, 0) - u_0(y)] dx + \frac{1}{2} \int_0^{t_f} \left\{ w(t) - \int_D H(x, t) u(x, t) dx \right\}^2 dt \quad (29)$$

where t_f is an arbitrary final time, $u_0(x)$ is an initial estimate of $u(x, 0)$, and

$$A_x(\cdot) = \left[\int_D P_0(x, y) \{ \cdot \} dy \right]^{-1} \quad (30)$$

$P_0(x, y)$ is a bounded, symmetric, and positive-definite weighting function. The observer is found by selecting $u(x, t)$ so as to minimize (29) subject to (1) and (2). By minimizing the augmented functional,

$$J = J_0 + \int_0^{t_f} \int_D \lambda(x, t) \left[\frac{\partial u(x, t)}{\partial t} - L_x u(x, t) - B(x, t) f(x, t) \right] dx dt \quad (31)$$

the result is the Euler–Lagrange equation,

$$\frac{\partial \lambda(x, t)}{\partial t} = -L_x^* \lambda(x, t) - H(x, t) \left[w(t) - \int_D H(y, t) \hat{u}(y, t) dy \right] \quad (32)$$

with the transversality conditions,

$$\begin{aligned} \lambda(x, 0) &= A_x [\hat{u}(y, 0) - u_0(y)] \\ \lambda(x, t_f) &= 0. \end{aligned} \quad (33)$$

Equations (32) and (33) constitute a two-point boundary value problem that may be solved by the sweep method. We assume the following Riccati transformation for $\hat{u}(x, t)$,

$$\hat{u}(x, t) = \int_D P(x, y, t) \lambda(y, t) dy + p(x, t) \quad (34)$$

where the kernel $P(x, y, t)$ and $p(x, t)$ have to be determined.

Substitution of (34) into (1), (32), and (33) yields

$$\begin{aligned} \frac{\partial p(x, t)}{\partial t} &= L_x p(x, t) + B(x, t) f(x, t) \\ &+ \int_D P(x, r, t) H(r, t) dr \left[w(t) - \int_D H(s, t) p(s, t) ds \right] \end{aligned} \quad (35)$$

$$p(x, 0) = u_0(x) \quad (36)$$

$$\beta_x p(x, t) = h(x, t), \quad x \in \partial D \quad (37)$$

$$\begin{aligned} \frac{\partial P(x, y, t)}{\partial t} &= L_x P(x, y, t) + P(x, y, t) L_y \\ &- \int_D P(x, r, t) H(r, t) dr \int_D H(s, t) P(s, y, t) ds \end{aligned} \quad (38)$$

$$P(x, y, 0) = P_0(x, y) \quad (39)$$

$$\beta_x P(x, y, t) = 0, \quad P(x, y, t) \beta_y = 0, \quad x, y \in \partial D. \quad (40)$$

Equations (33) and (34) imply that $p(x, t_f) = \hat{u}(x, t_f)$ is the state estimate at an arbitrary final time t_f . It is important to note that (38) is identical to (28). Thus we may conclude that the symmetric positive-definite kernel $P(x, y, t)$ completely characterizes the minimum variance observer.

Equation (35) can be rewritten as

$$\begin{aligned} \frac{\partial \hat{u}(x, t)}{\partial t} &= L_x \hat{u}(x, t) + B(x, t) f(x, t) \\ &+ K(x, t) \left[w(t) - \int_D H(y, t) \hat{u}(y, t) dy \right] \end{aligned} \quad (41)$$

where a time-varying observer gain $K(x, t)$ is defined by

$$K(x, t) = \int_D P(x, y, t) H(y, t) dy. \quad (42)$$

The structure of the observer is identical to that of the distributed parameter filter [13].

We introduce the reconstruction or observer error $e(x, t) = \hat{u}(x, t) - u(x, t)$. Then we obtain the following equation for $e(x, t)$,

$$\frac{\partial e(x, t)}{\partial t} = L_x e(x, t) - K(x, t) \int_D H(y, t) e(y, t) dy \quad (43)$$

with initial and boundary conditions, $e(x, 0) = u_0(x) - u(x, 0)$, and $\beta_x e(x, t) = 0$. If the initial state is known exactly and the observer is initialized such that $\hat{u}(x, 0) = u(x, 0)$, then the observer will reconstruct the state exactly. It is not reasonable, however, to expect that the initial state will be known exactly. It is, therefore, important to insure that if errors are present in the initial conditions applied to the observer that the estimate will converge to the true value of the state, i.e., the reconstruction error $e(x, t)$ must have the property $\lim_{t \rightarrow \infty} \|e(x, t)\| = 0$, for all $e(x, 0)$.

Asymptotic stability of the observer can be demonstrated by using (16), (26), (27), and (43). We will consider a Lyapunov function defined by

$$V(e, t) = \int_D \int_D e(x, t) Q(x, y, t) e(y, t) dy dx. \quad (44)$$

It is first necessary to note that $Q(x, y, t)$ is positive-definite and bounded from below. Then the time derivative of the Lyapunov function is calculated using (27) and (43). The result is

$$\frac{d}{dt} V(e, t) = - \int_D \int_D e(x, t) H(x, t) H(y, t) e(y, t) dy dx \quad (45)$$

which is a negative-semidefinite quadratic form. This is sufficient to show that (43) is stable in the sense of Lyapunov [10].

2) *Observer for Case 2:* In a similar manner to that of Case 1, we can obtain the minimum variance observer for Case 2, i.e., for spatially continuous measurements (4). The observer dynamics are described by

$$\begin{aligned} \frac{\partial \hat{u}(x, t)}{\partial t} &= L_x \hat{u}(x, t) + B(x, t) f(x, t) \\ &+ \int_D G(x, y, t) [w(y, t) - C(y, t) \hat{u}(y, t)] dy \end{aligned} \quad (46)$$

with initial and boundary conditions

$$\hat{u}(x, t) = u_0(x) \quad (47)$$

$$\beta_x \hat{u}(x, t) = h(x, t), \quad x \in \partial D \quad (48)$$

where the optimal gain kernel $G(x, y, t)$ is defined by

$$G(x, y, t) = P(x, y, t) C(y, t). \quad (49)$$

The Riccati equation for $P(x, y, t)$ in (49) is identical to (21) with boundary conditions given by (22). The reconstruction error $e(x, t) = \hat{u}(x, t) - u(x, t)$ satisfies

$$\frac{\partial e(x, t)}{\partial t} = Fe(x, t) \quad (50)$$

where the integro-differential operator F is defined by

$$Fe(x, t) = L_x e(x, t) - \int_D G(x, y, t) C(y, t) e(y, t) dy. \quad (51)$$

We can demonstrate the stability of the observer by using the reconstructibility kernel $Q(x, y, t)$ defined by (15) and the Lyapunov function (44). Under the reconstructibility assumption, the derivative of the Lyapunov function becomes

$$\frac{d}{dt} V(e, t) = - \int_D e(x, t) C^2(x, t) e(x, t) dx \quad (52)$$

which is a negative-semidefinite quadratic form.

3) *Observer for Case 3:* For the spatially discrete measurements (5), i.e., Case 3, the observer is given by the following system:

$$\begin{aligned} \frac{\partial \hat{u}(x, t)}{\partial t} &= L_x \hat{u}(x, t) + B(x, t) f(x, t) \\ &+ \sum_{i=1}^I G_i(x, t) [w_i(t) - H_i(t) \hat{u}(x_i, t)] \end{aligned} \quad (53)$$

where

$$G_i(x, t) = P(x, x_i, t) H_i(t) \quad (54)$$

and

$$\begin{aligned} \frac{\partial P(x, y, t)}{\partial t} &= L_x P(x, y, t) + P(x, y, t) L_y \\ &- \sum_{i=1}^I P(x, x_i, t) H_i(t) H_i(t) P(x_i, y, t) \end{aligned} \quad (55)$$

with initial and boundary conditions given by (47), (48), (39), and (40). Under the distributed state reconstructibility assumption, the stability of the observer can be demonstrated.

C. Comments

The relationship has been established between distributed state reconstructibility and the existence of an observer. Distributed state reconstructibility is defined through the existence of the generalized inverse to the reconstructibility kernel. The kernel associated with the observer gain satisfies the same Riccati equation as does the generalized inverse of the reconstructibility kernel.

III. REMOTE SENSING MEASUREMENTS AND ATMOSPHERIC MODELS

In this section we will test both the n -mode and state reconstructibility of common remote sensing measurements with models of atmospheric pollutant behavior. By far the predominant mode of remote sensing is to measure the integrated quantity (burden) of material between the ground and some known altitude. Thus, both cases we consider here involve vertically integrated data. Various assumptions concerning the horizontal characteristics of the measurements will be tested. Three-dimensional models of pollutant behavior are generally based on the atmospheric diffusion equation [5] that describes the flow and diffusion of species. The object of this section is to ascertain if the customary remote sensing measurements allow one, in principle, to reconstruct the detailed concentration distribution. The distributed parameter reconstructibility condition derived in Section II will, therefore, be tested in each case.

A. Measurements in a Layer with Horizontal Homogeneity

The vertical concentration distribution of a pollutant in a layer with horizontal homogeneity can be described by the one-dimensional diffusion equation,

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial z^2} \quad (56)$$

subject to

$$K \frac{\partial u}{\partial z} = h_0(t), \quad z=0 \quad (57)$$

$$K \frac{\partial u}{\partial z} = 0, \quad z=1 \quad (58)$$

where h_0 is a given flux at the ground ($z=0$) and K is the turbulent diffusion coefficient.

The adjoint Green's function for the system (56)–(58) is

$$\Phi^*(z, t; z', \tau) = 1 + 2 \sum_{n=1}^{\infty} \cos(n\pi z) \cos(n\pi z') e^{-(n\pi)^2 K(t-\tau)}. \quad (59)$$

State reconstructibility is then to be assessed by condition (24) using the modified reconstructibility kernel (25).

We consider each of the measurement types (3)–(5). The condition for uniform n -mode reconstructibility is (24), which is written for $\phi_n = \cos(n\pi z)$, $n=0, 1, 2, \dots$, as

$$0 < M_1 \leq \int_0^1 \int_0^1 \cos(n\pi z) Q^a(z, z', t) \cos(n\pi z') dz dz' \leq M_2 < \infty. \quad (60)$$

For each of the three types of measurements, the integral in (60) is as follows.

Case 1—Spatially Independent Integral Measurements

$$\int_{t-\sigma}^t e^{2(n\pi)^2 K(t-\tau)} \left\{ \int_0^1 H(r, \tau) \cos(n\pi r) dr \right\}^2 d\tau. \quad (61)$$

Case 2—Spatially Continuous Measurements

$$\int_{t-\sigma}^t e^{2(n\pi)^2 K(t-\tau)} \int_0^1 \{C(r, \tau) \cos(n\pi r)\}^2 dr d\tau. \quad (62)$$

Case 3—Spatially Discrete Measurements

$$\int_{t-\sigma}^t e^{2(n\pi)^2 K(t-\tau)} \left\{ \sum_{i=1}^I H_i(\tau) \cos(n\pi z_i) \right\}^2 d\tau. \quad (63)$$

for $n=0, 1, 2, \dots$.

From (61)–(63), we see that uniform n -mode reconstructibility is completely dependent on the form of the measurement weighting functions $H(z, t)$, $C(z, t)$, and $H_i(t)$ and on the eigenfunction, $\cos(n\pi z)$. The condition (60) implies that

$$\int_0^1 H(z, t) \cos(n\pi z) dz \neq 0.$$

We may note that this inequality is essentially equivalent to the observability condition derived by McGlothin [14]. Similarly, (63) implies that the system state is reconstructible by point sensors if the sensors are not located at the zeros of any of the eigenfunctions.

In the remote sensing problem, the measurement weighting functions are often taken as $H(z, t)=1$ or $C(z, t)=1$. When $H(z, t)=1$, the condition (60) holds only for $n=0$ implying that the spatially independent integral measurements do not allow reconstruction of the system state on entire fields.⁴ This can be directly checked by computing the reconstructibility kernel $Q(z, z', t)$. The system with integral measurements cannot

⁴A mode associated with the eigenfunction $\phi_0 = 1$ ($n=0$) can be reconstructible and the appropriate observer is

$$\frac{\partial \hat{u}}{\partial t} = K \frac{\partial^2 \hat{u}}{\partial z^2} + \frac{1}{t} \left[u(t) - \int_0^1 \hat{u}(z', t) dz' \right].$$

Stability of the observer can be demonstrated by constructing the Lyapunov function

$$V(e, t) = \int_0^1 \int_0^1 e(z, t) e(z', t) dz dz'.$$

be distributed state reconstructible since the generalized inverse of $Q(z, z', t) = t$ does not exist, since Q is not an explicit function of z and z' .

When the measurements are spatially continuous and $C(z, t) = 1$, the system is distributed state reconstructible. From the definitions of $Q(x, y, t)$ and $P(x, y, t)$ in (15) and (16), we have

$$Q(z, z', t) = t + \frac{1}{\pi^2 K} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi z) \cos(n\pi z') \{e^{2(n\pi)^2 K t} - 1\} \quad (64)$$

and

$$P(z, z', t) = \frac{1}{t} + 4\pi^2 K \sum_{n=1}^{\infty} n^2 \cos(n\pi z) \cos(n\pi z') \{e^{2(n\pi)^2 K t} - 1\}^{-1}. \quad (65)$$

We may note that the integral equation (16) is satisfied when it is recognized that

$$\delta(z - z') = 1 + 2 \sum_{n=1}^{\infty} \cos(n\pi z) \cos(n\pi z') \quad (66)$$

$P(z, z', t)$ is bounded and positive-definite, and for $t > 0$, the series (65) is uniformly convergent.

B. Measurements of a Steady-State Point Source Plume

The concentration distribution in a plume from a continuously emitting, elevated point source can be described by

$$\frac{\partial u}{\partial t} = K_H \frac{\partial^2 u}{\partial y^2} + K_V \frac{\partial^2 u}{\partial z^2} \quad (67)$$

where t is the time an element of fluid spends in the plume from emission, equal to downwind distance x divided by the wind speed. The source is of strength q located at $t = 0$, $y = 1/2$, $z = z_H$ ($0 \leq z_H \leq 1$). The boundary conditions on (67) are

$$u(0, y, z) = q\delta(y - 1/2)\delta(z - z_H) \quad (68)$$

$$\frac{\partial u}{\partial y} = 0, \quad y = 0, 1 \quad (69)$$

$$K_V \frac{\partial u}{\partial z} = h_0, \quad z = 0 \quad (70)$$

$$\frac{\partial u}{\partial z} = 0, \quad z = 1. \quad (71)$$

The adjoint Green's function for this system is

$$\Phi^*(y, z, t; y', z', \tau) = \left\{ 1 + 2 \sum_{n=1}^{\infty} \cos(n\pi y) \cos(n\pi y') e^{(n\pi)^2 K_H(t-\tau)} \right\} \cdot \left\{ 1 + 2 \sum_{m=1}^{\infty} \cos(m\pi z) \cos(m\pi z') e^{(m\pi)^2 K_V(t-\tau)} \right\}. \quad (72)$$

Consider first a scanning measurement performed at a horizontal position $y = y_*$.

$$w(t) = \int_0^1 J(z) u(t, y_*, z) dz \quad (73)$$

where the scanning data $w(t)$ are taken on a coordinate that moves along the t -axis. $J(z)$ in (73) is the altitude-dependent weighting function for the measurements. When $J = 1$, the reconstructibility kernel function becomes

$$Q(t, y, z; y', z') = t + \frac{2}{K_H} \sum_{n=1}^{\infty} \frac{\cos(n\pi y_*)}{(n\pi)^2} \cdot \{\cos(n\pi y) + \cos(n\pi y')\} \{e^{(n\pi)^2 K_H t} - 1\}$$

$$+ \frac{4}{K_H} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\cos(n\pi y_*) \cos(m\pi y_*)}{(n^2 + m^2)\pi^2} \cdot \cos(n\pi y) \cos(m\pi y') \{e^{(n^2 + m^2)\pi^2 K_H t} - 1\}. \quad (74)$$

The system is not distributed state reconstructible since the generalized inverse of $Q(t, y, z; y', z')$ does not exist. Therefore, we conclude that the scanning measurement (73) cannot, in principle, allow reconstruction of the system state.

The same results can be obtained for the following measurement systems:

$$w(t) = \int_0^1 \int_0^1 u(t, y, z) dy dz \quad (75)$$

$$w(t, y) = \int_0^1 u(t, y, z) dz \quad (76)$$

$$w(t, z) = \int_0^1 u(t, y, z) dy. \quad (77)$$

In these cases, the reconstructibility kernel function $Q(t, y, z; y', z')$ cannot be written explicitly in terms of all the spatial variables y, z , and y', z' . Thus, the generalized inverse $P(t, y, z; y', z')$ does not exist, which allows reconstruction of the system state on the whole domain. As a rule, if $Q(t, y, z; y', z')$ is expressible as an explicit function of all the spatial variables and if it satisfies the uniform n -mode reconstructibility test, then the system is distributed state reconstructible.

Indeed, we can show that the system is distributed state reconstructible for the measurement

$$w(t, y, z) = u(t, y, z). \quad (78)$$

In this case, we have

$$\begin{aligned} Q(t, y, z; y', z') &= t + \frac{1}{K_H} \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2} \cos(n\pi y) \cos(n\pi y') \{e^{2(n\pi)^2 K_H t} - 1\} \\ &+ \frac{1}{K_V} \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2} \cos(n\pi z) \cos(n\pi z') \{e^{2(n\pi)^2 K_V t} - 1\} \\ &+ 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\cos(n\pi y) \cos(m\pi z) \cos(n\pi y') \cos(m\pi z')}{(n\pi)^2 K_H + (m\pi)^2 K_V} \\ &\cdot \left[e^{2\{(n\pi)^2 K_H + (m\pi)^2 K_V\}t} - 1 \right]. \end{aligned} \quad (79)$$

The generalized inverse of (79) is given by

$$\begin{aligned} P(t, y, z; y', z') &= \frac{1}{t} + 4K_H \sum_{n=1}^{\infty} (n\pi)^2 \cos(n\pi y) \cos(n\pi y') \{e^{2(n\pi)^2 K_H t} - 1\}^{-1} \\ &+ 4K_V \sum_{n=1}^{\infty} (n\pi)^2 \cos(n\pi z) \cos(n\pi z') \{e^{2(n\pi)^2 K_V t} - 1\}^{-1} \\ &+ 8 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \{ (n\pi)^2 K_H + (m\pi)^2 K_V \} \\ &\cdot \cos(n\pi y) \cos(m\pi z) \cos(n\pi y') \cos(m\pi z') \\ &\cdot \left[e^{2\{(n\pi)^2 K_H + (m\pi)^2 K_V\}t} - 1 \right]^{-1} \end{aligned} \quad (80)$$

where (80) satisfies the Riccati equation associated with the measurement (78).

IV. CONCLUSION

This paper has examined the possibility of estimating atmospheric species concentration distributions from remote sensing data. Atmospheric concentrations can be modeled by partial differential equations of

the diffusion type. Remote sensing data generally represent spatial averages of the concentrations, frequently in the vertical direction. The essential problem, therefore, is to assess the possibility of estimating the state of a distributed parameter system on the basis of spatially averaged measurements. The theoretical basis of the assessment is a condition for state reconstructibility of distributed parameter systems. (The connection between state reconstructibility and the stability of the distributed parameter observer has also been developed.)

A variety of remote sensing measurement configurations were tested for reconstructibility. It was found, not unexpectedly, that those measurements based on integration of the vertical concentration distribution over the entire layer do not lead to distributed state reconstructibility, i.e., there does not exist a generalized inverse of the reconstructibility matrix kernel and therefore do not afford the possibility of estimating the concentration distribution over the entire field. Those measurement configurations that, on the other hand, enable sampling of the concentration at vertical positions lead to distributed state reconstructibility.

REFERENCES

- [1] J. C. Barnes, C. J. Bowley and H. K. Burke, "Evaluation of the capabilities of satellite imagery for monitoring regional air pollution episodes," NASA Contractor Rep. 159107, Langley Research Center, VA, Aug. 1979.
- [2] W. A. Lyons, J. C. Dooley, Jr., and K. T. Whitby, "Satellite detection of long-range pollution transport and sulfate aerosol hazes," *Atmos. Environ.*, vol. 12, pp. 621-631, 1978.
- [3] J. M. Diamante, T. S. Englar, Jr., and A. H. Jazwinski, "Final report phase I—Urban air quality estimation study," NASA Rep. CR-145097, Langley Research Center, VA, Jan. 1976.
- [4] J. F. Kibler and J. T. Suttles, "Air pollution model parameter estimation using simulated LIDAR data," *AIJA J.*, vol. 15, pp. 1381-1384, 1977.
- [5] J. H. Seinfeld, *Air Pollution: Physical and Chemical Fundamentals*. New York: McGraw-Hill, 1975.
- [6] S. Kitamura, S. Sakairi, and M. Nishimura, "Observers for distributed parameter systems of diffusion type," *Trans. IEE Jap.*, vol. 92-C, p. 407, 1972.
- [7] R. V. Greising and G. B. Lamont, "Observers for systems characterized by semi-groups," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 523-528, 1975.
- [8] S. Dolecki and D. L. Russell, "A general theory of observations and control," *SIAM J. Contr. Optimiz.*, vol. 15, pp. 185-220, 1977.
- [9] M. Köhne, "The control of vibrating elastic systems," in *Distributed Parameter Systems*, W. H. Ray and D. G. Lainiotis, Ed. New York: Dekker, 1977, pp. 387-456.
- [10] J. K. Hale, "Dynamical systems and stability," *J. Math. Anal. Appl.*, vol. 26, pp. 39-59, 1969.
- [11] K. Yosida, *Functional Analysis*. New York: Springer-Verlag, 1966.
- [12] J. S. Meditch, "Least squares filtering and smoothing for linear distributed-parameter systems," *Automatica*, vol. 7, pp. 315-322, 1971.
- [13] S. G. Tzafestas, "Distributed parameter state estimation," in *Distributed Parameter Systems*, W. H. Ray and D. G. Lainiotis, Ed. New York: Dekker, 1977, pp. 135-208.
- [14] G. E. McGlothlin, "Controllability, observability, and quality in a distributed parameter system with continuous and point spectrum," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 687-690, 1978.

"Optimal" Operating Rules for Automated Manufacturing Systems

J. A. BUZACOTT

Abstract—In modeling automated manufacturing systems such as transfer lines and flexible manufacturing systems it is necessary to make assumptions about how such systems will be operated. Hence, it is desirable to determine the "optimal" operating rules. In this paper a number of examples are given of how the "optimal" rules are determined so that consideration is given to the control options that can be used, the information available to the operator and the typical multilevel nature of the system's control.

Manuscript received September 29, 1980; revised February 5, 1981. Paper recommended by D. G. Fisher, Past Associate Editor, Applications, Systems Evaluation, and Components Committee.

The author is with the Department of Industrial Engineering, University of Toronto, Toronto, Ont., Canada.

INTRODUCTION

Automated manufacturing systems consist of a number of stations where production operations are carried out together with a material handling system linking the stations, with the whole system under some mixture of centralized and decentralized control and monitoring [1]. Typically, the operation of individual stations is controlled locally but the movement of parts between stations is controlled centrally. Furthermore, there will be central control of movement of parts into and out of the system with perhaps some linkage to even higher levels of control within the total manufacturing environment.

At present there are two general classes of multistation automated manufacturing systems.

One is automatic transfer lines such as are used in the automobile industry for machining cylinder blocks, transmission cases, and trans-axles. These have been in relatively widespread use for about 30 years. Their distinguishing feature is that all parts processed by the line follow the same sequence of operations and part movement is synchronized, that is at fixed intervals of time (the cycle time) every part in the system is transferred to the next station.

The other class is the flexible manufacturing system (FMS) where the basic concept has been around for about 10 years or so [1],[2]. The distinguishing feature of the FMS is that provision is made for a diversity of part routings so that all parts need not visit the same sequence of machines. The FMS can be regarded as equivalent to a conventional job shop with automated material handling [3].

More recent developments in transfer line design incorporate some aspects of the FMS, for example the introduction of parallel stations or segments and the avoidance of strict synchronism between all parts of the system [4].

Because of the way in which the stations of an automated manufacturing system are linked to form an integrated system it is necessary in designing them to give careful consideration to the effect of disturbances such as breakdowns of stations or variability in the processing times of the jobs. One way in which the effect of disturbances can be reduced is through providing in process storage space. While such space is expensive to provide because of the need to maintain workpiece orientation it has been shown that it can have a significant effect on system performance (for models of the role of in process storage, see [5] and [3]).

The focus of most of the literature on transfer lines has been on the way in which the performance is determined by the characteristics of the physical equipment, such as the processing rates of the stations, the speed and capacity for simultaneous job movement of the material handling system, and the location and capacity of the in-process inventory banks.

However, the performance of the system will also be influenced by the operating policies used in its control. The purpose of this paper is to discuss a number of issues relating to the operation of transfer lines and FMS and, where possible, identify "optimal" operating policies.

CONTROL OF AUTOMATED MANUFACTURING SYSTEMS

It is desirable to think of the control requirements from a hierarchical perspective, with three main levels (cf. [3]).

1) *Prerelease planning*: Deciding which jobs are to be manufactured by the system. Identifying constraints on operation sequence.

2) *Input or release control*: Determining the sequence and timing of the release of jobs to the system.

3) *Operational control*: Ensuring movement between machines and deciding which job is to be processed next by a machine.

A further aspect of system control is the monitoring of correct performance of the operations. This aspect will not be discussed in this paper.

At each level of control the physical configuration and the decisions made at higher levels set constraints on the alternative actions. In the manufacturing environment it has always proved to be difficult to collect and use reliable detailed information about the status of the machines and the jobs being processed by the system. Thus, in defining the control problem at each level is essential to specify clearly the available information on system status (cf. [6]).