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REDUCED FORM AUCTIONS REVISITED

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Abstract

This note uses Farkas's Lemma to prove new results on the implementability of general, asymmetric auctions, and to provide simpler proofs of known results for symmetric auctions. The tradeoff is that type spaces are taken to be finite.

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1 Auctions

Following Maskin and Riley [3], Matthews [4], and Border [1], I analyze the following question. A seller wishes to dispose of a single indivisible object. There are N potential buyers, or *bidders*, indexed by $i = 1, \dots, N$. Each bidder i has a *type* belonging to a set T_i. For this note, assume each T_i is finite. A *profile* is an element $\boldsymbol{t}$ of

$$\boldsymbol{T} = T_1 \times \cdots \times T_N.$$

An *auction* is an ordered list of functions $\boldsymbol{p} = (p_1, \dots, p_N)$, $p_i \colon \boldsymbol{T} \to [0,1]$, $i = 1, \dots, N$, satisfying the feasibility condition

$$\sum_{i=1}^{N} p_i(\boldsymbol{t}) \leqslant 1 \tag{\textbf{F}}$$

for each $\boldsymbol{t} \in \boldsymbol{T}$. Here $p_i(\boldsymbol{t})$ is the probability that bidder i wins the auction in profile $\boldsymbol{t}$. The feasibility condition $\mathbf{F}$ is just that the probability that some bidder wins cannot exceed unity. It may be less than unity, if there are circumstances under which the seller keeps the object. The justification for writing the probability of winning directly as a function of types as opposed to behaviors, is that each type determines the bidder's behavior, so p represents the composed mapping from types to behavior to outcome (the "revelation principle").

The seller and the bidders view the bidders' types as being selected by nature at random, according to the probability measure μ on $\boldsymbol{T}$. From bidder i's point of view, what is important to him is the probability that he wins given his type. To facilitate the discussion of these probabilities, we write $\mu(\boldsymbol{t})$ instead of $\mu(\{\boldsymbol{t}\})$, and define as usual $\boldsymbol{T}^{-i} = \prod_{j:j \neq i} T_j$, and write $\boldsymbol{t}^{-i}$ for a typical element of $\boldsymbol{T}^{-i}$. We also write $\boldsymbol{t} \in \boldsymbol{T}$ as $(\boldsymbol{t}_i, \boldsymbol{t}^{-i}) \in T_i \times \boldsymbol{T}^{-i}$, and more generally $(\tau, \boldsymbol{t}^{-i})$ is the tuple $\boldsymbol{t} \in \boldsymbol{T}$ with $\boldsymbol{t}_i = \tau$ and $\boldsymbol{t}_j = \boldsymbol{t}_j^{-i}$ for $j \neq i$. Let $\mu_i^{\bullet}$ denote the marginal probability on T_i and $\mu_i(\boldsymbol{t}^{-i}|\tau)$ denote the conditional probability of $\boldsymbol{t}^{-i}$ given that bidder i has type τ. That is,

$$\mu_i^{\bullet}(\tau) = \sum_{\boldsymbol{t}^{-i} \in \boldsymbol{T}^{-i}} \mu(\tau, \boldsymbol{t}^{-i}), \qquad \mu_i(\boldsymbol{t}^{-i}|\tau) = \frac{\mu(\tau, \boldsymbol{t}^{-i})}{\mu_i^{\bullet}(\tau)} \quad \text{if } \mu_i^{\bullet}(\tau) > 0.$$

Notice that I have not defined probability conditional on types of probability zero. If every type has positive marginal probability, this is not an issue, but if we are careful, the existence of zero probability types is still not an issue.

An ordered list of functions $\boldsymbol{P} = (P_1, \dots, P_N)$, where each $P_i \colon T_i \to [0,1]$ is the *reduced form* of the auction $\boldsymbol{p} = (p_1, \dots, p_N)$, if for each bidder i and each type $\tau \in T_i$,

$$P_i(\tau) = \sum_{\boldsymbol{t}^{-i} \in \boldsymbol{T}^{-i}} p_i(\tau, \boldsymbol{t}^{-i}) \mu_i(\boldsymbol{t}^{-i} | \tau) \quad \text{if } \mu_i^{\bullet}(\tau) > 0. \tag{\textbf{R}}$$

That is, $P_i(\tau)$ is a bidder's expected probability of winning given his own type is τ for types with positive probability. If the probability is zero, no restriction is placed on $P_i(\tau)$ (other than $0 \leqslant P_i(\tau) \leqslant 1$, which is implied by $P_i \colon T_i \to [0,1]$). If $\boldsymbol{P}$ is the reduced form of some auction $\boldsymbol{p}$, we may also say that $\boldsymbol{P}$ is *implementable*.

From the seller's point of view the reduced form of an auction is usually easier to work with, and summarizes (along with self-selection constraints) all the incentives faced by the buyers. It is therefore highly desirable to find a simple criterion for whether or not $\boldsymbol{P}$ is a reduced form. The literature starting with Maskin and Riley [3] dealt with the case where types are independently and identically distributed. They argued that in that case, for most reasonable seller's objective functions it is enough to consider only symmetric auctions. (A formal definition is given below.) That is, P_i could be taken to be independent of i. But in the general case, symmetric auctions are not general enough, and even in the i.i.d. case, the seller may wish to discriminate on something other than the bidder's type. For instance, the seller may prefer to sell to someone of his own ethnic group, or more virtuously, as in the case of the FCC recently, the seller may wish to advantage businesses owned by underrepresented minorities. One could attempt to finesse this problem by making these nonbehavioral attributes part of the type, indeed one could incorporate the bidder's name into his type. Doing so makes the i.i.d. assumption invalid, so the results for the symmetric case do not apply.

In the symmetric case, Matthews [4] conjectured that the only restriction on $\boldsymbol{P}$ for implementability was the necessary condition that the probability that the "winner" had a type in the a subset A could not exceed the probability that there was a bidder with type in A (the **MRM** condition). Maskin and Riley [3, Theorem 7] proved something like this result for increasing step functions P on the unit interval. Their proof is long, tedious, and unintuitive. Matthews extended their result to general increasing functions on the unit interval, and conjectured this form of the theorem. Border [1] proved the conjecture for general abstract measure spaces of types, which need not have an order, so the notion of increasing need not be defined. All these papers rely heavily on symmetry and the latter papers use topological and/or functional analytic techniques and are mildly opaque.

However when T is a finite set (ordered or not), P is a reduced form if the finite system (**F**)-(**R**) of linear inequalities in p has a nonnegative solution. By Farkas's Lemma, if this system has no nonnegative solution, then its dual system possesses a solution. The proof of sufficiency thus reduces to showing that the existence of a solution to the dual implies that the **MRM** condition must be violated. Given this insight, the proof practically writes itself. In this framework, the general result, Theorem 7 below, is easier to prove than the symmetric case, which is presented in Theorem 1. The main problem with carrying out this program of proof in the symmetric case is notational. The natural system of inequalities together with the symmetry conditions on p are unwieldy. It is actually simpler to recognize that for a symmetric auction, only a bidder's own type and the distribution of the other bidders' types matters, and to rewrite the problem in these terms.

The next section deals with the symmetric case and is followed by an example, which is redundant, but convinces me that the subscripts are correct. The following section deals with the general case. An appendix derives the particular variant of Farkas's Lemma that is used.

2 The symmetric i.i.d. case

An environment is *symmetric* if $T_i = T$ for each $i = 1, \dots, N$ (so $\boldsymbol{T} = T^N$), and μ is a product measure λ^N on T^N, where λ is a probability on T. (Types are independently and identically distributed.) Let T^* denote the support of the probability measure λ, that is, $T^* = \{\tau \in T : \lambda(\tau) > 0\}$. An auction for a symmetric environment is *symmetric* if for each permutation π on $\{1, \dots, N\}$, each $\boldsymbol{t} \in \boldsymbol{T}$, and each i,

$$p_i(\boldsymbol{t}_1, \dots, \boldsymbol{t}_N) = p_{\pi^{-1}(i)}(\boldsymbol{t}_{\pi(1)}, \dots, \boldsymbol{t}_{\pi(N)}). \tag{\textbf{S}}$$

A symmetric auction is completely determined by p_1, which we may refer to simply as p. Likewise the reduced form can be summarized by P_1, which we shall refer to as simply P. Thus we may say that a function $P\colon T \to [0,1]$ is the reduced form of the symmetric auction $\boldsymbol{p} = (p_1, \dots, p_N)$ if

$$P(\tau) = \sum_{\boldsymbol{t}^{-1} \in \boldsymbol{T}^{-1}} p_1(\tau, \boldsymbol{t}^{-1}) \lambda^{N-1}(\boldsymbol{t}^{-1}) \tag{\textbf{R}$'$}$$

for each $\tau \in T^*$. Clearly not every $P\colon T \to [0,1]$ is a reduced form. For instance, let $T = \{\tau\}$. Then $P(\tau) = 1$ cannot be a reduced form (unless there is only one bidder), for every bidder would have to win with probability one.

1 Theorem (Maskin–Riley–Matthews–Border) *A function $P\colon T \to [0,1]$ is the reduced form of a symmetric auction if and only if for every subset A of T, it satisfies the Maskin–Riley–Matthews (MRM) condition*

$$N \sum_{\tau \in A} P(\tau)\lambda(\tau) \leqslant 1 - \lambda(A^c)^N. \tag{\textbf{MRM}}$$

For the remainder of this section, I shall also abuse notation and identify the set of types with the set integers $\{1, \ldots, T\}$. That is, T denotes both the number of types and the set of types,

$$T = \{1, \ldots, T\}.$$

You should not get confused.

Reformulation in terms of censuses

In the symmetric case, all that matters to bidder i about a profile is his own type and the number of other bidders of each type. Let us call the information about the number of bidders of each type a *census*. Formally, a census is a nonnegative integer-valued measure on T, which we can think of as an element of $\mathcal{D} = \mathbb{N}^T$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers including 0. Given a census d, write d_τ instead of $d(\{\tau\})$, and define the size of the census by

$$|d| = d(T)$$

and

$$\mathcal{D}_n = \{d \in \mathcal{D} : |d| = n\}.$$

The set $\mathcal{D}_n$ is the set of censuses that can arise from n draws (with replacement) from T. We shall be mainly interested in the two cases $n = N - 1$ and $n = N$.

Now instead of profiles being drawn at random from λ^N, we may think of censuses as being drawn at random from a multinomial distribution. The chance that the census d results from $|d|$ i.i.d. draws (with replacement) from T is

$$c(d) = \frac{|d|!}{d_1! \cdots d_T!} \lambda(1)^{d_1} \cdots \lambda(T)^{d_T}. \tag{1}$$

Thus

$$\sum_{d \in \mathcal{D}_n} c(d) = 1 \qquad \text{for each } n.$$

Let $\kappa \colon \bigcup_{n=1}^{\infty} T^n \to \mathcal{D}$ assign to each profile $\boldsymbol{t} \in T^n$ its census. That is,

$$\kappa_\tau(\boldsymbol{t}) = |\{j : \boldsymbol{t}_j = \tau\}|,$$

where $|\cdot|$ denotes the cardinality of a set.

Given a type $\tau \in T$ and census $d \in \mathcal{D}_n$, the census $d \oplus \tau$ in $\mathcal{D}_{n+1}$ that results by adding an individual of type τ is given by

$$(d \oplus \tau)_\sigma = \begin{cases} d_\sigma & \sigma \neq \tau \\ d_\tau + 1 & \sigma = \tau. \end{cases}$$

Likewise, given a census $m \in \mathcal{D}_{n+1}$ and a type $\tau \in T$, if $m_\tau > 0$ define the census $m \ominus \tau$ in $\mathcal{D}_n$ that results by removing an individual of type τ by

$$(m \ominus \tau)_\sigma = \begin{cases} m_\sigma & \sigma \neq \tau \\ m_\tau - 1 & \sigma = \tau. \end{cases}$$

Clearly

$$(m \ominus \tau) \oplus \tau = m, \qquad (d \oplus \tau) \ominus \tau = d,$$

and there is a one-to-one correspondence between $\{(\tau, m) \in T \times \mathcal{D}_n : m_\tau > 0\}$ and $T \times \mathcal{D}_{n-1}$ via $(\tau, m) \leftrightarrow (\tau, m \ominus \tau)$.

Direct computation yields the following useful results.

$$c(d \oplus \tau) = \frac{(|d| + 1)c(d)\lambda(\tau)}{d_\tau + 1}, \tag{2}$$

and if $m_\tau > 0$, then

$$c(m \ominus \tau) = \frac{m_\tau c(m)}{\lambda(\tau)|m|} \qquad \text{provided } \lambda(\tau) > 0. \tag{3}$$

We may now recast the the discussion of symmetric auctions in terms of censuses rather than profiles.

Start with the **MRM** condition. The term $\lambda(A^c)^N$ is the probability that in N i.i.d. draws from T no element of A appears, in other words, the census m of types has $m_\tau = 0$ for all $\tau \in A$. Thus $1 - \lambda(A^c)^N$ is just the probability that $m(A) > 0$ (that is, there is at least one type in the set A), so the **MRM** condition can be rewritten as

$$N \sum_{\tau \in A} P(\tau)\lambda(\tau) \leqslant \sum_{m \in \mathcal{D}_N : m(A) > 0} c(m). \tag{$\mathbf{MRM'}$}$$

We now describe auctions in terms of censuses rather than profiles. Define the function $r: T \times \mathcal{D}_{N-1} \to [0, 1]$ by

$$r(\tau; d) = p(\tau, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_N) \qquad \text{where } \kappa(\boldsymbol{t}_2, \ldots, \boldsymbol{t}_N) = d.$$

Symmetry guarantees that this is well defined. We can recover $p = p_1$ from r by

$$p_1(\boldsymbol{t}) = r(\boldsymbol{t}_1; \kappa(\boldsymbol{t}_2, \ldots, \boldsymbol{t}_N)).$$

We can express the feasibility condition **F** on p in terms of r as follows. Let $\boldsymbol{t}'$ be derived from $\boldsymbol{t}$ by interchanging $\boldsymbol{t}_1$ and $\boldsymbol{t}_i$. Then $\kappa(\boldsymbol{t}) = \kappa(\boldsymbol{t}')$ and

$$p_i(\boldsymbol{t}) = p_1(\boldsymbol{t}') = r(\boldsymbol{t}'_1; \kappa(\boldsymbol{t}') \ominus \boldsymbol{t}'_1) = r(\boldsymbol{t}_i; \kappa(\boldsymbol{t}) \ominus \boldsymbol{t}_i).$$

Thus

$$\sum_{i=1}^{N} p_i(\boldsymbol{t}) = \sum_{\tau : m_\tau > 0} m_\tau r(\tau; m \ominus \tau) \leqslant 1 \qquad \text{for all } m \in \mathcal{D}_N. \tag{$\mathbf{F'}$}$$

The reduced form condition **R** can be rewritten in terms of r by means of a standard multinomial probability calculation.

$$\begin{aligned} P(\tau) &= \sum_{\boldsymbol{t}^{-1} \in \boldsymbol{T}^{-1}} p_1(\tau, \boldsymbol{t}^{-1}) \mu_1(\boldsymbol{t}^{-1} | \tau) \\ &= \sum_{\boldsymbol{t}^{-1} \in \boldsymbol{T}^{-1}} p_1(\tau, \boldsymbol{t}^{-1}) \lambda(\boldsymbol{t}_2) \cdots \lambda(\boldsymbol{t}_N) \\ &= \sum_{\boldsymbol{t}^{-1} \in \boldsymbol{T}^{-1}} r(\tau; \kappa(\boldsymbol{t}^{-1})) \lambda(\boldsymbol{t}_2) \cdots \lambda(\boldsymbol{t}_N) \\ &= \sum_{d \in \mathcal{D}_{N-1}} r(\tau; d) c(d). \end{aligned} \tag{$\mathbf{R''}$}$$

Restatement

In light of the discussion above we have shown that an equivalent definition of reduced form auctions is:

2 Proposition *A function $P\colon T \to [0,1]$ is a reduced form symmetric auction if there exists a function $r\colon T \times \mathcal{D}_{N-1} \to [0,1]$ satisfying*

$$P(\tau) = \sum_{d \in \mathcal{D}_{N-1}} r(\tau; d) c(d) \qquad (\text{for } \tau \in T^*), \tag{$\mathbf{R''}$}$$

and

$$\sum_{d \in \mathcal{D}_{N-1}} (d_\tau + 1) r(\tau; d) = \sum_{\tau : m_\tau > 0} m_\tau r(\tau; m \ominus \tau) \leqslant 1 \qquad \text{for all } m \in \mathcal{D}_N. \tag{$\mathbf{F'}$}$$

The theorem can be now be written as follows.

3 MRMB Theorem Recast *A function $P\colon T \to [0,1]$ satisfies conditions* $\mathbf{R''}$ *and* $\mathbf{F'}$*, that is, is the reduced form of a symmetric auction, if and only if for every subset A of T, it satisfies*

$$N \sum_{\tau \in A} P(\tau) \lambda(\tau) \leqslant \sum_{\substack{m \in \mathcal{D}_N : \\ m(A) > 0}} c(m). \tag{$\mathbf{MRM'}$}$$

The proof is presented in two parts.

4 Proposition (Necessity) *If P is a reduced form, then it satisfies the* **MRM′** *condition.*

Proof: If P is a reduced form, then

$$P(\tau) = \sum_{d \in \mathcal{D}_{N-1}} r(\tau; d) c(d)$$

if $\lambda(\tau) > 0$, so

$$\begin{aligned} P(\tau)\lambda(\tau) &= \sum_{d \in \mathcal{D}_{N-1}} r(\tau; d) c(d) \lambda(\tau) \\ &= \sum_{d \in \mathcal{D}_{N-1}} r(\tau; d) \frac{(d_\tau + 1) c(d \oplus \tau)}{N} \\ &= \sum_{\substack{m \in \mathcal{D}_N: \\ m_\tau > 0}} \frac{m_\tau r(\tau; m \ominus \tau) c(m)}{N} \end{aligned}$$

where the second equality follows from equation (2) and the fact that $|d| = N - 1$. Let us agree to interpret $m_\tau r(\tau; m \ominus \tau) = 0$ when $m_\tau = 0$, even though $r(\tau; m \ominus \tau)$ is not defined. Then we may write

$$\begin{aligned} N \sum_{\tau \in A} P(\tau)\lambda(\tau) &= \sum_{\tau \in A} \sum_{\substack{m \in \mathcal{D}_N: \\ m_\tau > 0}} m_\tau r(\tau; m \ominus \tau) c(m) \\ &\leqslant \sum_{\tau \in A} \sum_{\substack{m \in \mathcal{D}_N: \\ m(A) > 0}} m_\tau r(\tau; m \ominus \tau) c(m) \\ &= \sum_{\substack{m \in \mathcal{D}_N: \\ m(A) > 0}} c(m) \sum_{\tau \in A} m_\tau r(\tau; m \ominus \tau) \\ &\leqslant \sum_{\substack{m \in \mathcal{D}_N: \\ m(A) > 0}} c(m), \end{aligned}$$

where the second inequality follows from (**F′**). But this is just the **MRM′** condition. ∎

5 Proposition (Sufficiency) *If P satisfies the* **MRM′** *condition, then it is a reduced form.*

Proof: We shall prove the contrapositive, namely, if P is not a reduced form, then the **MRM′** condition is violated.

The function P is a reduced form if and only the system of linear inequalities (**F′**)–(**R″**) has a nonnegative solution r. We can express this system in matrix terms as follows. Columns are indexed by $(\tau; d) \in T \times \mathcal{D}_{N-1}$. There are rows indexed by $\sigma \in T^*$ that express condition $\mathbf{R''}$ and rows indexed by $m \in \mathcal{D}_N$ that express condition $\mathbf{F'}$.

$$
\begin{array}{ccccc}
\text{indices} & \begin{smallmatrix}(\tau;d)\\ \in T\times\mathcal{D}_{N-1}\end{smallmatrix} & & & \\
\begin{array}{c}
\vdots\\ \sigma\in T^*\\ \vdots\\ \hline \vdots\\ m\in\mathcal{D}_N\\ \vdots
\end{array}
&
\left[\begin{array}{ccc}
 & \vdots & \\
\cdots & \delta_{\sigma,\tau}c(d) & \cdots\\
 & \vdots & \\ \hline
 & \vdots & \\
\cdots & \delta_{m,d\oplus\tau}m_\tau & \cdots\\
 & \vdots &
\end{array}\right]
&
\begin{bmatrix}\vdots\\ r(\tau;d)\\ \vdots\end{bmatrix}
&
\begin{array}{c}\vdots\\ =\\ \vdots\\ \hline \vdots\\ \leqslant\\ \vdots\end{array}
&
\left[\begin{array}{c}\vdots\\ P(\sigma)\\ \vdots\\ \hline \vdots\\ 1\\ \vdots\end{array}\right]
\end{array}
\tag{4}
$$

where δ is the Kronecker symbol, $\delta_{a,b} = 1$ if $a = b$ and is zero otherwise.

The dual system

Assume now that P is not a reduced form, that is, assume that the system (4) has no nonnegative solution. Then from Farkas's Lemma (see Lemma 11 in the appendix), the dual system has a solution. The dual system has variables $Z = (Z_\sigma)_{\sigma \in T^*}$ (unrestricted signs) and nonnegative variables $u = (u_m)_{m \in \mathcal{D}_N}$, and consists of:

$$\sum_{\sigma \in T^*} \delta_{\sigma,\tau} Z_\sigma c(d) - \sum_{m \in \mathcal{D}_N} \delta_{m,d\oplus\tau} m_\tau u_m \leqslant 0. \qquad \forall (\tau; d) \in T \times \mathcal{D}_{N-1} \tag{5}$$

and

$$\sum_{\sigma \in T^*} Z_\sigma P(\sigma) - \sum_{m \in \mathcal{D}_N} u_m > 0, \tag{6}$$

and the nonnegativity condition $u \geqq 0$.

Now equation (5) has an inequality for each $(\tau; d) \in T \times \mathcal{D}_{N-1}$. There are two cases. If $\lambda(\tau) > 0$, that is, $\tau \in T^*$, then the (τ, d) inequalities can be written

$$Z_\tau c(m \ominus \tau) \leqslant m_\tau u_m \qquad \forall (\tau, m) \in T^* \times \mathcal{D}_N : m_\tau > 0. \tag{5$'$}$$

But if $\lambda(\tau) = 0$, equation (5) reduces to $0 \leqslant (d_\tau + 1) u_m$, which is redundant.

Properties of the dual solution

The first thing to note is that if the dual system has a solution (Z, u), then by increasing u if needed, there is a solution with $Z_\sigma \geqslant 0$ for every $\sigma \in T^*$. To see this just note that if $Z_\sigma < 0$, then setting $Z_\sigma = 0$ only strengthens inequalities (6), and the nonnegativity of u makes (5') superfluous. Now observe that given a nonnegative solution, by increasing each u_m slightly, we can increase each Z_σ to get a solution with $Z_\sigma > 0$ for every $\sigma \in T^*$.

Finally, given a solution with each $Z_\sigma > 0$, fixing Z, we can look for a minimal u that solves the dual.

Let $\mathcal{D}^* = \{m \in \mathcal{D}_N : c(m) > 0\}$. Thus if $m \in \mathcal{D}^*$ and $m_\tau > 0$, then $\tau \in T^*$. Since each $P(\tau) \geqslant 0$ and each $u_m \geqslant 0$, equation (6) implies the stronger inequality

$$\sum_{\tau \in T^*} Z_\tau P(\tau) - \sum_{m \in \mathcal{D}^*} u_m > 0.$$

We now proceed to break up these sums into pieces. Renumbering the members of T^* if necessary, we may assume the types are numbered so that $\lambda(\tau) > 0$ and $Z_\tau > 0$ for $\tau = 1, \ldots, K = |T^*|$, and

$$\frac{Z_1}{\lambda(1)} \geqslant \cdots \geqslant \frac{Z_K}{\lambda(K)} > 0. \tag{7}$$

(Note that $K \geqslant 1$.) Then

$$Z_1 P(1) + \cdots + Z_K P(K) - \sum_{m \in \mathcal{D}^*} u_m > 0 \tag{8}$$

Now let's break up $\mathcal{D}^*$ into pieces. Let

$$E_1 = \{m \in \mathcal{D}^* : m_1 > 0\},$$

and recursively define $E_1, \ldots, E_K$ by

$$E_{\tau+1} = \{m \in \mathcal{D}^* \setminus (E_1 \cup \cdots \cup E_\tau) : m_{\tau+1} > 0\}.$$

That is, $m \in \mathcal{D}^*$ belongs to E_τ if and only if $m_\tau > 0$ and $m_\sigma = 0$ for $\sigma < \tau$ so the sets E_τ are disjoint. Two key properties are that

$$E_1 \cup \cdots \cup E_k = \{m \in \mathcal{D}^* : m_\tau > 0 \text{ for some } \tau \leqslant k\}, \tag{9}$$

and

$$E_1 \cup \cdots \cup E_K = \mathcal{D}^*.$$

Now from equation (5′),

$$u_m \geqslant \frac{Z_\tau c(m \ominus \tau)}{m_\tau} \qquad \forall\, \tau : m_\tau > 0,$$

so for $m \in \mathcal{D}^*$, by equation (3),

$$u_m \geqslant \frac{Z_\tau c(m)}{\lambda(\tau) N} \qquad \forall m \in \mathcal{D}^*,\ m_\tau > 0. \tag{10}$$

Therefore, to find a solution with minimal u_m, we may decrease u_m until we have equality in (10) for the type with the largest value of the ratio $\frac{Z_\tau c(m)}{\lambda(\tau) N}$. But this is how we constructed the E_τ sets, so

$$\text{if } m \in E_\tau, \text{ then } u_m = \frac{Z_\tau c(m)}{\lambda(\tau) N}, \qquad \tau = 1, \ldots, K.$$

Thus equation (8) becomes

$$Z_1P(1) + \cdots + Z_KP(K) - \sum_{m\in E_1} \frac{Z_1c(m)}{\lambda(1)N} - \cdots - \sum_{m\in E_K} \frac{Z_Kc(m)}{\lambda(K)N} > 0,$$

or

$$\sum_{\tau=1}^{K} \frac{Z_\tau}{\lambda(\tau)} \Big(NP(\tau)\lambda(\tau) - c(E_\tau) \Big) > 0, \tag{11}$$

where $c(E_\tau) = \sum_{m\in E_\tau} c(m)$.

Now here comes the crux of the argument—it corresponds to Lemma 5.3 in [1]. Observe that if for some k we have

$$\sum_{\tau=1}^{k} NP(t)\lambda(\tau) - c(E_\tau) > 0,$$

then by equation (9) we have a violation of condition $\mathbf{MRM'}$, for $A = \{1, \ldots, k\}$. In particular, if $K = 1$, then equation (11) implies that we have a violation of the $\mathbf{MRM'}$ condition for $A = \{1\}$.

So assume that for $K > 1$ and for $k = 1, \ldots, K-1$,

$$\sum_{\tau=1}^{k} NP(\tau)\lambda(\tau) - c(E_\tau) \leqslant 0. \tag{12}$$

Now take the $\tau = 1$ term in equation (11) to the right hand side and divide by $Z_1/\lambda(1)$ to get

$$\sum_{\tau=2}^{K} \frac{\lambda(1)Z_\tau}{\lambda(\tau)Z_1} \Big(NP(\tau)\lambda(\tau) - c(E_\tau) \Big) > c(E_1) - NP(1)\lambda(1) \geqslant 0. \tag{13}$$

where the second inequality follow from the hypothesis (12). Now multiply the left-hand side of equation (13) by $Z_1\lambda(2)/Z_2\lambda(1) \geqslant 1$ (by equation (7)) to get a stronger inequality. Take the $\tau = 2$ term to the right to get

$$\sum_{\tau=3}^{K} \frac{\lambda(2)Z_\tau}{\lambda(t)Z_2} \Big(NP(\tau)\lambda(\tau) - c(E_t) \Big) > c(E_1) - NP(1)\lambda(1) + c(E_2) - NP(2)\lambda(2) \geqslant 0,$$

Continue in this fashion until reaching

$$0 > c(E_1) - NP(1)\lambda(1) + \cdots + c(E_K) - NP(K)\lambda(K).$$

This means that for some $k = 1, \ldots, K$, condition (12) is false, and thus the $\mathbf{MRM'}$ condition is violated for some $A = E_1 \cup \cdots \cup E_k$.

To summarize, we have shown that if P is not a reduced form, then the dual system has a solution, so the $\mathbf{MRM'}$ condition is violated. Thus by contraposition, if the $\mathbf{MRM'}$ condition is satisfied, then P is a reduced form. ∎

3 A symmetric example

Since I am easily confused by subscripts, here is an example of the primal in its “natural” form, and then reformulated.

6 Example Consider the case of $N = 3$ bidders, and 2 types, $T = \{1, 2\}$, with probabilities $\lambda(1) > 0$, $\lambda(2) > 0$. Given a potential reduced form $P = (P_1, P_2)$, $0 \leqslant P_i \leqslant 1$, $i = 1, 2$, we wish to find a symmetric auction function $p\colon T^3 \to [0, 1]$ satisfying the following (in)equalities:

$$\begin{aligned}
p(1;1,1)\lambda(1)^2 + p(1;1,2)\lambda(1)\lambda(2) + p(1;2,1)\lambda(1)\lambda(2) + p(1;2,2)\lambda(2)^2 &= P_1 \\
p(2;1,1)\lambda(1)^2 + p(2;1,2)\lambda(1)\lambda(2) + p(2;2,1)\lambda(1)\lambda(2) + p(2;2,2)\lambda(2)^2 &= P_2
\end{aligned}$$

$$\begin{array}{rcll}
p_1(1,1,1) + p_2(1,1,1) + p_3(1,1,1) & = & p(1;1,1) + p(1;1,1) + p(1;1,1) & \leqslant 1 \\
p_1(1,1,2) + p_2(1,1,2) + p_3(1,1,2) & = & p(1;1,2) + p(1;1,2) + p(2;1,1) & \leqslant 1 \\
p_1(1,2,1) + p_2(1,2,1) + p_3(1,2,1) & = & p(1;2,1) + p(2;1,1) + p(1;2,1) & \leqslant 1 \\
p_1(1,2,2) + p_2(1,2,2) + p_3(1,2,2) & = & p(1;2,2) + p(2;1,2) + p(2;2,1) & \leqslant 1 \\
p_1(2,1,1) + p_2(2,1,1) + p_3(2,1,1) & = & p(2;1,1) + p(1;2,1) + p(1;1,2) & \leqslant 1 \\
p_1(2,1,2) + p_2(2,1,2) + p_3(2,1,2) & = & p(2;1,2) + p(1;2,2) + p(2;1,2) & \leqslant 1 \\
p_1(2,2,1) + p_2(2,2,1) + p_3(2,2,1) & = & p(2;2,1) + p(2;2,1) + p(1;2,2) & \leqslant 1 \\
p_1(2,2,2) + p_2(2,2,2) + p_3(2,2,2) & = & p(2;2,2) + p(2;2,2) + p(2;2,2) & \leqslant 1
\end{array}$$

Because of symmetry, $p(1;1,2) = p(1;2,1)$ and $p(2;1,2) = p(2;2,1)$, so we can reduce the system to:

$$\begin{aligned}
p(1;1,1)\lambda(1)^2 + 2p(1;1,2)\lambda(1)\lambda(2) + p(1;2,2)\lambda(2)^2 &= P_1 \\
p(2;1,1)\lambda(1)^2 + 2p(2;1,2)\lambda(1)\lambda(2) + p(2;2,2)\lambda(2)^2 &= P_2 \\
3p(1;1,1) &\leqslant 1 \\
2p(1;1,2) + p(2;1,1) &\leqslant 1 \\
p(1;2,2) + 2p(2;1,2) &\leqslant 1 \\
3p(2;2,2) &\leqslant 1
\end{aligned}$$

In matrix form this becomes

$$\begin{array}{c|ccc|ccc|c|c|c|}
\text{indices} & (1{\cdot}11) & (1{\cdot}12) & (1{\cdot}22) & (2{\cdot}11) & (2{\cdot}12) & (2{\cdot}22) & & & \\
(1) & \lambda(1)^2 & 2\lambda(1)\lambda(2) & \lambda(2)^2 & 0 & 0 & 0 & p_{1\cdot11} & = & P_1 \\
(2) & 0 & 0 & 0 & \lambda(1)^2 & 2\lambda(1)\lambda(2) & \lambda(2)^2 & p_{1\cdot12} & = & P_2 \\
\hline
(111) & 3 & 0 & 0 & 0 & 0 & 0 & p_{1\cdot22} & \leqslant & 1 \\
(112) & 0 & 2 & 0 & 1 & 0 & 0 & p_{2\cdot11} & \leqslant & 1 \\
(122) & 0 & 0 & 1 & 0 & 2 & 0 & p_{2\cdot12} & \leqslant & 1 \\
(222) & 0 & 0 & 0 & 0 & 0 & 3 & p_{2\cdot22} & \leqslant & 1
\end{array} \tag{14}$$

Since we eliminated the redundant conditions resulting from symmetry, all we need to do to convert this system to the same form as equation (4), is to change the indices. To do this just note that for $T = \{1, 2\}$, there are three censuses in $\mathcal{D}_2$, namely $(2, 0)$, $(1, 1)$ and $(0, 2)$. The mapping κ from T^2 to $\mathcal{D}_2$ is

$$(1,1) \mapsto (2,0), \quad (1,2) \mapsto (1,1), \quad (2,1) \mapsto (1,1), \quad (2,2) \mapsto (0,2).$$

The mapping from T^3 to $\mathcal{D}_3$ is

$$\begin{array}{llll} (1,1,1) \mapsto (3,0), & (1,1,2) \mapsto (2,1), & (1,2,1) \mapsto (2,1), & (2,1,1) \mapsto (2,1), \\ (1,2,2) \mapsto (1,2), & (2,1,2) \mapsto (1,2), & (2,2,1) \mapsto (1,2), & (2,2,2) \mapsto (0,3). \end{array}$$

The new indices are

$$\begin{array}{c|ccc|ccc|c|c|c}
\text{indices} & \substack{\tau;d=\\1;(2,0)} & \substack{\tau;d=\\1;(1,1)} & \substack{\tau;d=\\1;(0,2)} & \substack{\tau;d=\\2;(2,0)} & \substack{\tau;d=\\2;(1,1)} & \substack{\tau;d=\\2;(0,2)} & & & \\
\sigma{=}1 & \lambda(1)^2 & 2\lambda(1)\lambda(2) & \lambda(2)^2 & 0 & 0 & 0 & r(1;(2,0)) & = & P_1 \\
\sigma{=}2 & 0 & 0 & 0 & \lambda(1)^2 & 2\lambda(1)\lambda(2) & \lambda(2)^2 & r(1;(1,1)) & = & P_2 \\
\hline
m{=}(3,0) & 3 & 0 & 0 & 0 & 0 & 0 & r(1;(0,2)) & \leqslant & 1 \\
\hline
m{=}(2,1) & 0 & 2 & 0 & 1 & 0 & 0 & r(2;(2,0)) & \leqslant & 1 \\
m{=}(1,2) & 0 & 0 & 1 & 0 & 2 & 0 & r(2;(1,1)) & \leqslant & 1 \\
m{=}(0,3) & 0 & 0 & 0 & 0 & 0 & 3 & r(2;(0,2)) & \leqslant & 1
\end{array}$$

Note that the coefficients are as advertised: above the line they are $c(d)$ or zero depending on whether $\sigma = \tau$, and below the line they are $d_\tau + 1$ when $m = d \oplus \tau$.

The dual system is:

$$Z_1\lambda(1)^2 - 3u_{3,0} \leqslant 0 \tag{15}$$

$$2Z_1\lambda(1)\lambda(2) - 2u_{2,1} \leqslant 0$$

$$Z_1\lambda(2)^2 - u_{1,2} \leqslant 0$$

$$Z_2\lambda(1)^2 - u_{2,1} \leqslant 0$$

$$2Z_2\lambda(1)\lambda(2) - 2u_{1,2} \leqslant 0$$

$$Z_2\lambda(2)^2 - 3u_{0,3} \leqslant 0 \tag{16}$$

$$Z_1P_1 + Z_2P_2 - u_{3,0} - u_{2,1} - u_{1,2} - u_{0,3} > 0 \tag{17}$$

It is apparent that if the dual system has a solution, then it has a solution with $Z_1, Z_2 > 0$. Renumbering types if necessary, assume

$$Z_1/\lambda(1) \geqslant Z_2/\lambda(2). \tag{18}$$

Fixing Z, we can choose u to make inequalities (15–16) bind. Simply set

$$\begin{aligned}
u_{3,0} &= Z_1\lambda(1)^2/3 \\
u_{2,1} &= \max\{Z_1\lambda(1)\lambda(2),\ Z_2\lambda(1)^2\} = Z_1\lambda(1)\lambda(2) \\
u_{1,2} &= \max\{Z_1\lambda(2)^2,\ Z_2\lambda(1)\lambda(2)\} = Z_1\lambda(2)^2 \\
u_{0,3} &= Z_2\lambda(2)^2/3,
\end{aligned}$$

where the maxima are given by equation (18). Then equation (17) becomes

$$Z_1 P_1 + Z_2 P_2 > Z_1 \lambda(1)^2/3 + Z_1 \lambda(1)\lambda(2) + Z_1 \lambda(2)^2 + Z_2 \lambda(2)^2/3. \tag{19}$$

Now this can be rewritten as

$$\begin{aligned} \frac{Z_1}{\lambda_1} P_1 \lambda(1) + \frac{Z_2}{\lambda_2} P_2 \lambda(2) \quad &> \quad \frac{Z_1}{3\lambda_1}\left(\lambda(1)^3 + \lambda(1)^2\lambda(2) + \lambda(1)\lambda(2)^2\right) + \frac{Z_2}{3\lambda(2)}\lambda(2)^3 \\ &= \quad \frac{Z_1}{3\lambda_1}\left(c((3,0)) + c((2,1)) + c((1,2))\right) + \frac{Z_2}{3\lambda(2)} c((0,3)), \end{aligned}$$

where the equality comes from equation (1). Multiply by $3\lambda(1)/Z_1$ to get

$$3\left(P_1\lambda(1) + \frac{Z_2\lambda(1)}{Z_1\lambda(2)} P_2 \lambda(2)\right) > \left(c((3,0)) + c((2,1)) + c((1,2))\right) + \frac{Z_2\lambda(1)}{Z_1\lambda(2)} c((0,3)) \tag{20}$$

Case 1. If

$$3P_1\lambda(1) > c((3,0)) + c((2,1)) + c((1,2)),$$

the $\mathbf{MRM'}$ condition is violated for $A = \{1\}$.

Case 2. Otherwise, rearrange equation (20) as

$$3\frac{Z_2\lambda(1)}{Z_1\lambda(2)}\left(P_2\lambda(2) - c((0,3))\right) > c((3,0)) + c((2,1)) + c((1,2)) - 3P_1\lambda(1).$$

By equation (18), we have $Z_2\lambda(1)/Z_1\lambda(2) \leqslant 1$, so we can strengthen the inequality by writing

$$3\left(P_2\lambda(2) - c((0,3))\right) > c((3,0)) + c((2,1)) + c((1,2)) - 3P_1\lambda(1)$$

which can be rewritten as

$$3(P_1\lambda(1) + P_2\lambda(2)) > c((3,0)) + c((2,1)) + c((1,2)) + c((0,3)) = 1.$$

This violates the $\mathbf{MRM'}$ condition for $A = \{1, 2\}$.

□

4 The general case

The statement of the general implementation condition is similar to the **MRM** condition, but the role of T is replaced by the collection of bidder-type pairs, $\mathcal{T}$, defined as

$$\mathcal{T} = \bigcup_{i=1}^{N} \{i\} \times T_i = \{(i,\tau) : 1 \leqslant i \leqslant N,\ \tau \in T_i\}.$$

Let

$$\mathcal{T}^* = \{(i,\tau) \in \mathcal{T} : \mu_i^\bullet(\tau) > 0\}.$$

Then the general implementation condition can be written as:

7 Theorem (General implementation condition) *The list $\boldsymbol{P} = (P_1, \ldots, P_N)$ of functions is the reduced form of a general auction $\boldsymbol{p} = (p_1, \ldots, p_N)$ if and only if for every subset $A \subset \mathcal{T}$ of individual-type pairs, we have*

$$\sum_{(i,\tau)\in A} P_i(\tau)\mu_i^{\bullet}(\tau) \leqslant \mu(\{\boldsymbol{t} \in \boldsymbol{T} : \exists (i,\tau) \in A,\ \boldsymbol{t}_i = \tau\}). \tag{\textbf{GI}}$$

The proof is divided into two parts.

8 Proposition (Necessity) *Let $\boldsymbol{P} = (P_1, \ldots, P_N)$ be the reduced form of a general auction. Then it satisfies condition* **GI**.

Proof: Let $\boldsymbol{P} = (P_1, \ldots, P_N)$ be the reduced form of the symmetric auction $\boldsymbol{p} = (p_1, \ldots, p_N)$. Let $A \subset \mathcal{T}$. Then

$$\begin{aligned}
\sum_{(i,\tau)\in A} P_i(\tau)\mu_i^{\bullet}(\tau) &= \sum_{(i,\tau)\in A} \sum_{\boldsymbol{t}^{-i}\in\boldsymbol{T}^{-i}} p_i(\tau, \boldsymbol{t}^{-i})\mu_i(\boldsymbol{t}^{-i}|\tau)\mu_i^{\bullet}(\tau) \\
&= \sum_{(i,\tau)\in A} \sum_{\boldsymbol{t}^{-i}\in\boldsymbol{T}^{-i}} p_i(\tau, \boldsymbol{t}^{-i})\mu(\tau, \boldsymbol{t}^{-i}) \\
&\leqslant \sum_{(i,\tau)\in A} \sum_{\boldsymbol{t}^{-i}\in\boldsymbol{T}^{-i}} \mu(\tau, \boldsymbol{t}^{-i}) \\
&\leqslant \mu(\{\boldsymbol{t} \in \boldsymbol{T} : \exists (i,\tau) \in A,\ \boldsymbol{t}_i = \tau\}).
\end{aligned}$$

∎

9 Proposition (Sufficiency) *If $\boldsymbol{P}$ satisfies condition* **GI**, *then it is the reduced form of some auction $\boldsymbol{p}$.*

Proof: The proof of sufficiency of condition **GI** proceeds by contraposition. That is, we shall prove that if $\boldsymbol{P}$ is not implementable, then condition **GI** is violated. Thus by contraposition, if condition **GI** is satisfied, then $\boldsymbol{P}$ is implementable.

So assume that $\boldsymbol{P}$ is not implementable. Then the implementation and feasibility conditions **R** and **F** have no nonnegative solution $(p_j(\boldsymbol{t}))_{\substack{j\in N \\ \boldsymbol{t}\in\boldsymbol{T}}}$. The conditions can be written in matrix form, with columns indexed by $(j, \boldsymbol{t}) \in N \times \boldsymbol{T}$ and one set of rows indexed by $(i, \tau) \in \mathcal{T}^*$ expressing (**R**), and other rows indexed by $\boldsymbol{s} \in \boldsymbol{T}$ expressing (**F**):

$$
\begin{array}{cc}
\text{indices} & (j,\boldsymbol{t})\in N\times\boldsymbol{T} \\
\begin{array}{c} \vdots \\ (i,\tau)\in\mathcal{T}^* \\ \vdots \\ \hline \vdots \\ \boldsymbol{s}\in\boldsymbol{T} \\ \vdots \end{array} &
\left[\begin{array}{ccc} & \vdots & \\ \cdots & \delta_{i,j}\delta_{\tau,\boldsymbol{t}_j}\mu_i(\boldsymbol{t}^{-j}|\tau) & \cdots \\ & \vdots & \\ \hline & \vdots & \\ \cdots & \delta_{\boldsymbol{s},\boldsymbol{t}} & \cdots \\ & \vdots & \end{array}\right]
\end{array}
\left[\begin{array}{c} \vdots \\ p_j(\boldsymbol{t}) \\ \vdots \end{array}\right]
\begin{array}{c} \vdots \\ = \\ \vdots \\ \hline \vdots \\ \leqslant \\ \vdots \end{array}
\left[\begin{array}{c} \vdots \\ P_i(\tau) \\ \vdots \\ \hline \vdots \\ 1 \\ \vdots \end{array}\right]
\qquad
\begin{array}{c} (\mathbf{R}) \\ \hline (\mathbf{F}) \end{array}
$$

where again δ is the Kronecker symbol, $\delta_{a,b} = 1$ if $a = b$ and is zero otherwise. This system is illustrated for the case $N = 2$, $T_1 = \{a, b\}$, $T_2 = \{x, y\}$ in Figure 1.

Since this system has no solution, then by Farkas's Lemma (Lemma 11 in the appendix) the dual system must have a solution. The dual variables are $Z_{i,\tau}$, where $(i, \tau) \in \mathcal{T}^*$ and $u_{\boldsymbol{s}} \geqslant 0$, $\boldsymbol{s} \in \boldsymbol{T}$. The dual system is

$$Z_{j,\boldsymbol{t}_j} \mu_j(\boldsymbol{t}^{-j}|\boldsymbol{t}_j) - u_{\boldsymbol{t}} \leqslant 0 \qquad \forall\, (j, \boldsymbol{t}) \in N \times \boldsymbol{T} : (j, \boldsymbol{t}_j) \in \mathcal{T}^* \tag{21}$$

$$\sum_{(i,\tau)\in\mathcal{T}^*} Z_{i,\tau} P_i(\tau) - \sum_{\boldsymbol{t}\in\boldsymbol{T}} u_{\boldsymbol{t}} > 0. \tag{22}$$

Properties of the dual solution

If the dual system has a solution, it has a solution with each $u_{\boldsymbol{t}}$ as small as possible, namely

$$u_{\boldsymbol{t}} = \max_{(j,\boldsymbol{t}_j)\in\mathcal{T}^*} \{Z_{j,\boldsymbol{t}_j} \mu_j(\boldsymbol{t}^{-j}|\boldsymbol{t}_j)\} \vee 0. \tag{23}$$

We can use this now to partition $\boldsymbol{T}$. Start by enumerating $\mathcal{T}^*$ as $\{(i_1, \tau_1), \ldots, (i_M, \tau_M)\}$ so that

$$\frac{Z_{i_1,\tau_1}}{\mu^\bullet_{i_1}(\tau_1)} \geqslant \frac{Z_{i_2,\tau_2}}{\mu^\bullet_{i_2}(\tau_2)} \geqslant \cdots \geqslant \frac{Z_{i_K,\tau_K}}{\mu^\bullet_{i_K}(\tau_K)} > 0 \geqslant \frac{Z_{i_{K+1},\tau_{K+1}}}{\mu^\bullet_{i_{K+1}}(\tau_{K+1})} \geqslant \cdots \geqslant \frac{Z_{i_M,\tau_M}}{\mu^\bullet_{i_M}(\tau_M)}.$$

(By (22) for at at least one $(i, \tau) \in \mathcal{T}^*$ we must have $Z_{i,\tau} > 0$. In fact, if there are any solutions at all, there is at least one with $Z_{i,\tau} > 0$ for all $(i, \tau) \in \mathcal{T}^*$.) If $u_{\boldsymbol{t}} > 0$, then by equation (23), there is at least one (i_k, τ_k) for which $\boldsymbol{t}_{i_k} = \tau_k$ and $u_{\boldsymbol{t}} = Z_{i_k,\tau_k} \mu_{i_k}(\boldsymbol{t}^{-i_k}|\tau_k)$. Let $I(\boldsymbol{t})$ denote the least k for which this is true, and let $E_k = \{\boldsymbol{t} : I(\boldsymbol{t}) = k\}$, $k = 1, \ldots, K$. By construction these sets are disjoint. By equation (21), we have

$$u_{\boldsymbol{t}} \geqslant Z_{j,\boldsymbol{t}_j} \mu_{i_j}(\boldsymbol{t}^j|\boldsymbol{t}_j) = \frac{Z_{j,\boldsymbol{t}_j}}{\mu^\bullet_j(\boldsymbol{t}_j)} \mu(\boldsymbol{t}),$$

for each $(j, \boldsymbol{t}_j) \in \mathcal{T}^*$, so it follows that:

For all $\boldsymbol{t} \in \boldsymbol{T}$ and all $k = 1, \ldots, K$, $\quad \boldsymbol{t}_{i_k} = \tau_k \Longleftrightarrow \boldsymbol{t} \in E_1 \cup \cdots \cup E_k$.

That is, $E_1, \ldots, E_K$ is a partition of $\{\boldsymbol{t} \in \boldsymbol{T} : u_{\boldsymbol{t}} > 0\}$.

Then (21)–(23) imply

$$\begin{aligned} Z_{i_1,\tau_1} P_{i_1}(\tau_1) + \cdots + & Z_{i_K,\tau_K} P_{i_K}(\tau_K) \\ & > \sum_{\boldsymbol{t}\in E_1} u_{\boldsymbol{t}} + \cdots + \sum_{\boldsymbol{t}\in E_K} u_{\boldsymbol{t}} \\ & = \sum_{\boldsymbol{t}\in E_1} Z_{i_1,\tau_1} \mu_{i_1}(\boldsymbol{t}^{-i_1}|\tau_1) + \cdots + \sum_{\boldsymbol{t}\in E_K} Z_{i_K,\tau_K} \mu_{i_K}(\boldsymbol{t}^{-i_K}|\tau_K) \end{aligned}$$

$$
\begin{array}{cc}
 & \begin{array}{cccc|cccc} (1,a,x) & (1,a,y) & (1,b,x) & (1,b,y) & (2,a,x) & (2,a,y) & (2,b,x) & (2,b,y) \end{array} \\
\begin{array}{c} (1,a) \\ (1,b) \\ (2,x) \\ (2,y) \\ \hline (a,x) \\ (a,y) \\ (b,x) \\ (b,y) \end{array}
&
\left[\begin{array}{cccc|cccc}
\mu_1(x|a) & \mu_1(y|a) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_1(x|b) & \mu_1(y|b) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_2(a|x) & 0 & \mu_2(b|x) & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_2(a|y) & 0 & \mu_2(b|y) \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c}
q_1(a,x) \\ q_1(a,y) \\ q_1(b,x) \\ q_1(b,y) \\ \hline q_2(a,x) \\ q_2(a,y) \\ q_2(b,x) \\ q_2(b,y)
\end{array}\right]
\begin{array}{c}
= \\ = \\ = \\ = \\ \hline \leqslant \\ \leqslant \\ \leqslant \\ \leqslant
\end{array}
\left[\begin{array}{c}
Q_1(a) \\ Q_1(b) \\ Q_2(x) \\ Q_2(y) \\ \hline 1 \\ 1 \\ 1 \\ 1
\end{array}\right]
\end{array}
$$

Figure 1: System (**R**)–(**F**) for the case $N = 2$, $T_1 = \{a, b\}$, $T_2 = \{x, y\}$.

$$
\begin{aligned}
&= \sum_{\boldsymbol{t}\in E_1} Z_{i_1,\tau_1}\frac{\mu(\tau_1,\boldsymbol{t}^{-i_1})}{\mu^{\bullet}_{i_1}(\tau_1)} + \cdots + \sum_{\boldsymbol{t}\in E_K} Z_{i_K,\tau_K}\frac{\mu(\tau_K,\boldsymbol{t}^{-i_K})}{\mu^{\bullet}_{i_K}(\tau_K)} \\
&= \frac{Z_{i_1,\tau_1}}{\mu^{\bullet}_{i_1}(\tau_1)}\mu(E_1) + \cdots + \frac{Z_{i_K,\tau_K}}{\mu^{\bullet}_{i_K}(\tau_K)}\mu(E_K).
\end{aligned}
$$

Thus

$$
\sum_{k=1}^{K} \frac{Z_{i_k,\tau_k}}{\mu^{\bullet}_{i_k}(\tau_k)}\big(P_{i_k}(\tau_k)\mu^{\bullet}_{i_k}(\tau_k) - \mu(E_k)\big) > 0. \tag{24}
$$

I now claim that for some $k \leqslant K$ we have

$$
\sum_{n=1}^{k} P_{i_n}(\tau_n)\mu^{\bullet}_{i_n}(\tau_n) - \mu(E_n) > 0.
$$

For suppose that

$$
\sum_{n=1}^{k} P_{i_n}(\tau_n)\mu^{\bullet}_{i_n}(\tau_n) - \mu(E_n) \leqslant 0 \tag{25}
$$

for all $k < K$. Then multiply equation (24) by $\mu^{\bullet}_{i_1}(\tau_{i_1})/Z_{i_1,\tau_1}$ and rearrange to get

$$
\sum_{n=2}^{K} \frac{Z_{i_n,\tau_n}\mu^{\bullet}_{i_1}(\tau_{i_1})}{Z_{i_1,\tau_1}\mu^{\bullet}_{i_n}(\tau_n)}\big(P_{i_n}(\tau_n)\mu^{\bullet}_{i_n}(\tau_n) - \mu(E_n)\big) > \mu(E_1) - P_{i_1}(\tau_{i_1})\mu^{\bullet}_{i_1}(\tau_{i_1}) \geqslant 0
$$

where the second inequality is just equation (25) for $k = 1$. Now multiply the left hand side by $\frac{Z_{i_1,\tau_1}\mu^{\bullet}_{i_2}(\tau_{i_2})}{Z_{i_2,\tau_2}\mu^{\bullet}_{i_1}(\tau_1)} \geqslant 1$ to strengthen the inequality and rearrange to get

$$
\begin{aligned}
&\sum_{n=3}^{K} \frac{Z_{i_n,\tau_n}\mu^{\bullet}_{i_2}(\tau_{i_2})}{Z_{i_2,\tau_2}\mu^{\bullet}_{i_n}(\tau_n)}\big(P_{i_n}(\tau_n)\mu^{\bullet}_{i_n}(\tau_n) - \mu(E_n)\big) \\
&\qquad > \quad \mu(E_1) - P_{i_1}(\tau_{i_1})\mu^{\bullet}_{i_1}(\tau_{i_1}) + \mu(E_2) - P_{i_2}(\tau_{i_2})\mu^{\bullet}_{i_2}(\tau_{i_2}) \quad \geqslant \quad 0,
\end{aligned}
$$

where the second inequality is just equation (25) for $k = 2$. Continue in this fashion until reaching the conclusion

$$
0 > \mu(E_1) - P_{i_1}(\tau_{i_1})\mu^{\bullet}_{i_1}(\tau_{i_1}) + \cdots + \mu(E_K) - P_{i_K}(\tau_{i_K})\mu^{\bullet}_{i_K}(\tau_{i_K}).
$$

That is, if equation (25) holds for $k = 1, \ldots, K-1$, it fails for $k = K$.

Thus for some k it must be that equation (25) fails. But this just says that condition **GI** is violated for $A = \{(i_1,\tau_1),\ldots,(i_k,\tau_k)\}$. This completes the proof of sufficiency. ∎

Appendix A Farkas's Lemma

The standard statement of Farkas's Lemma (see, e.g., Franklin [2, p. 56]) is this.

10 Lemma *Either the equation*

$$Ax = b$$

has a nonnegative solution $x \geqq 0$*, or else the inequalities*

$$A^* y \leqq 0, \quad y \cdot b > 0$$

have a solution y *(but not both).*

Here A^* is the transpose of A. The variant we need is this.

11 Lemma *Either the (in)equalities*

$$Ax = b, \quad Bx \leqq c$$

have a nonnegative solution $x \geqq 0$*, or else the inequalities*

$$A^* y - B^* u \leqq 0, \quad y \cdot b - u \cdot c > 0$$

have a solution (y, u) *with* $u \geqq 0$ *(but not both).*

Proof: Convert the first system of inequalities to a system of equations by introducing nonnegative slack variables z:

$$Ax = b, \quad Bx + z = c,$$

or

$$\begin{bmatrix} A & 0 \\ B & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}.$$

By the standard form of Farkas's Lemma, either this has a solution $(x, z) \geqq 0$ or else the alternative system

$$\begin{bmatrix} A^* & B^* \\ 0 & I \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} \leqq 0, \quad y \cdot b + v \cdot c > 0$$

has a solution (y, v). This system can be rewritten

$$A^* y + B^* v \leqq 0, \quad v \leqq 0, \quad y \cdot b + v \cdot c > 0,$$

and the conclusion follows by setting $u = -v$. ∎

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