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## NOMINATION PROCESSES AND POLICY OUTCOMES

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# Nomination Processes and Policy Outcomes\*

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## Abstract

We model and compare three different processes by which political parties nominate candidates for a general election: nominations by party leaders, nominations by a vote of party members, and nominations by a spending competition among potential candidates. We show that in equilibrium, non-median outcomes can result when two parties compete using nominations via any of these processes. We also show that more extreme outcomes can emerge from spending competition than from nominations by votes or by party leaders. When voters (and potential nominees) are free to switch political parties, then median outcomes ensue when nominations are decided by a vote but not when nominations are decided by spending competition.

Keywords: Nominations, elections, primaries, party competition, spending competition

JEL Classification numbers: P16, D72, C72

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# 1 Introduction

A general perception regarding elections is that the median voter's preferences determine the outcome whenever the candidates can be ordered on a one-dimensional left-right spectrum, and the voters have single-peaked preferences. This follows from fundamental models dating back to Hotelling (1929) and Black (1958), and from the fact that the median outcome is the Condorcet winner. While the logic that predicts median outcomes is simple, observed political outcomes often deviate significantly from the median (as discussed in more detail below). We show that by modeling the nomination processes by which parties choose their candidates, we can account for outcomes that differ significantly from the policy preferred by the median voter.

As different processes by which political parties nominate their candidates have not been modeled before, our contribution is not only in understanding how nomination procedures affect outcomes, but also in providing simple models of nomination processes. In particular, we present and contrast three different processes by which two political parties nominate candidates for a general election. Each party selects one of its members to serve as its candidate, and if elected that candidate then chooses his or her most preferred policy from a one-dimensional set of potential policies. All voters (and hence potential candidates) have single-peaked preferences. The three different nomination processes are as follows:

1. A party leader, who is a member of the party (and thus one of the potential candidates), unilaterally chooses the party's nominee.
2. Party members vote over who should be the party's nominee.
3. Party members compete for the nomination by spending. The party member who spends (or is willing to spend) the most money wins the nomination.

Our models of these processes should prove to be useful beyond the current paper, especially when models of nomination processes become part of more general election models. In each case we define an equilibrium to be a pair of nominees, one for each party, such that the following is true.

- **Nomination by party leaders:** neither party leader would want to change his or her nominee, given the nominee put forth by the other party and anticipating the eventual election against the other party's nominee.
- **Nomination by a vote of party members:** there is no other party member who would defeat the party's nominee in a majority vote of the party's members, anticipating the eventual election against the other party's nominee.

- **Nomination by spending competition:** no other party member would be willing to spend more than the party's nominee in order to secure the party nomination, anticipating the eventual election against the other party's nominee.

These three nomination processes can lead to very different outcomes. We first analyze the nomination by party leaders. There we show that the winner can come from either party, but lies between the overall median and the leader of the party that contains the median. The outcome can range anywhere between these points. We then show that nominations by party vote are equivalent to situations where nominations are made by party leaders, but where the party leaders are the medians of the parties. Elections by spending competition differ more dramatically from the other nomination processes and depend on the preference intensities of various party members in complex and subtle ways. Most importantly, nominations by spending competition can lead to extremist nominees from either or both parties, and can lead to extreme policy outcomes.

Furthermore, we show that endogenizing party membership has some effect, but only leads to a convergence to the median in the case of nomination by votes. If nominations are by spending competitions, then extremist outcomes can ensue even with endogenous parties.

Before presenting the formal model, let us discuss some of the related literature. Our results that non-median outcomes can emerge from the nomination processes are consistent with several empirical studies. For example, Stone and Rapoport (1994) show that the candidates competing for and winning U.S. Presidential nominations cover a wide range of political ideologies. In terms of the prediction of our model, this suggests that party leadership and/or spending competition play roles in party nomination processes.<sup>1</sup> Morton and Gerber (1998) show that differences in the laws governing electoral primaries can have an effect on the outcome. They examine the consequences of different primary laws across states in the U.S. and show that closed primaries can lead to more extreme nominations, while semi-closed primaries (allowing voters to declare a party on election day and for independents to vote in a primary) lead to even more moderate nominees than completely open primaries (where strategic voting across parties can occur). Our model is one where party members are the only ones who vote, and so it is a closed system. However, the differences between nomination by party leadership and nomination by party members' vote can be seen as reflecting different degrees of closure. Moreover, once we endogenize party membership, we move closest to a semi-closed system. There, we find that the outcome converges to the overall median, which is consistent with their finding that semi-closed systems are the most moderate. Our analysis of nomination by spending competition is harder to connect to their classification. However, as Morton and Gerber (1998) point out, while there is a literature

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<sup>1</sup>There are also other studies focusing on the U.S. presidential nomination process, e.g., Arterton (1977), Aldrich (1980), and Gurian (1993); but these are unrelated to our model here.

that has examined primaries and nomination processes,<sup>2</sup> there is no systematic analysis of nomination procedures.

The paper proceeds as follows. In Section 2, we introduce the elements of the model that are common to each of our three nomination processes. In Section 3, we delineate each of the three nomination processes and demonstrate equilibria for each process. In Section 4, we introduce endogenous parties, demonstrating both equilibria and non-existence for the nomination processes when the candidates are allowed to switch parties. Section 5 offers possible extensions and concluding remarks.

Throughout the paper we concentrate on two party systems.

## 2 The General Model

Our model is related to a citizen-candidate framework,<sup>3</sup> but one where the citizens cannot simply decide to run but must be nominated through their parties.

There are  $n$  voters, and voter  $i$ 's preferences are represented by a utility function  $u_i : [0, 1] \rightarrow \mathbb{R}$ . Voters have single-peaked preferences over the interval  $[0, 1]$ , and the peak of voter  $i$  is denoted  $x_i$ .

Voters are divided into two parties,  $P_1$  and  $P_2$ , that partition  $\{0, 1, \dots, n\}$ . In the first part of the paper, we analyze what happens when the two parties are fixed; later we return to study party formation. We use a notation of  $P_\ell$  and  $P_{-\ell}$  to indicate a generic party  $\ell$  and its competitor. In general, we allow for arbitrary party structures, so that it could be that the parties are not simply left and right parties. For instance, it could be that one party has some left and right-minded voters, and the other party has some centrists. We say that there is *no overlap* in parties if for each  $\ell \in \{1, 2\}$  and any  $i$  and  $j \in P_\ell$ , there does not exist any  $k \in P_{-\ell}$  such that  $x_i \leq x_k \leq x_j$ .

The political process is as follows.

- (1) Each party (simultaneously) nominates one of its members to serve as its candidate.
- (2) Voters vote for one of the two candidates, and a candidate is elected by majority rule with ties broken by a fair coin toss.
- (3) The policy outcome is the elected candidate's most preferred policy.

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<sup>2</sup>For instance, there is research relating the nomination process to party structure (e.g., Ranney (1975), Jewell (1984), Epstein (1986)), or modeling information dispersion and acquisition through primaries (e.g., Callander (2002), Meirowitz (2005), Bartels (1988)).

<sup>3</sup>See Osborne and Slivinski (1996) and Besley and Coate (1997).

We carefully model the nomination processes in (1) through equilibrium definitions, where everyone anticipates the election and outcome in (2) and (3). Given just two parties, it is a dominant strategy for each voter to vote for his or her preferred candidate in (2). (3) is motivated by a standard argument that candidates cannot credibly commit to follow any policy other than their most preferred policies.<sup>4</sup>

Let  $M$  be the overall median voter out of  $P_1 \cup P_2$ . To keep things simple, assume that  $n$  is odd. This implies that one of the parties has two medians. Also, assume that no voter is indifferent between any distinct candidates  $i$  and  $j$ .

Let  $W[i, j]$  denote the majority winner among any two candidates  $i$  and  $j$ .

Given that a candidate is identified with her ideal point, we abuse notation and let  $u_i(j)$  denote  $u_i(x_j)$ , or the utility that  $i$  gets if  $j$  wins the overall election.

Finally, let

$$d_i(j, k) = u_i(j) - u_i(k). \tag{1}$$

This is the difference in utility between what  $i$  gets if  $j$  is the overall winner versus what  $i$  gets if  $k$  is the overall winner.

### 3 Nominations with a Fixed Party Structure

Here, we analyze what happens when the distribution of voters across the two parties is fixed. As discussed above, we model three different processes for the ways that parties nominate a candidate.

- A party leader (one of the party members) unilaterally chooses the candidate,
- party members vote over who should be their candidate, and
- party members compete for the nomination by spending, with the nominated candidate being the party member who spent the most.

Each of these requires a corresponding definition of equilibrium.

#### 3.1 Equilibrium Definitions for the Three Nomination Procedures

The definitions of equilibrium for each of the nomination procedures are as follows.

##### Equilibrium with Nominations by Party Leaders

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<sup>4</sup>This assumption is not critical to our results. What is needed is that voters have some expectation regarding what policy would be enacted given each candidate and that the voters would not be indifferent across the candidates.

An *equilibrium* in the case of nominations by party leaders is a pair of nominations, denoted  $Nom(P_1) \in P_1$  and  $Nom(P_2) \in P_2$ , such that for each party  $\ell$ ,  $W[Nom(P_\ell), Nom(P_{-\ell})]$  is preferred by the leader of party  $\ell$  to  $W[x, Nom(P_{-\ell})]$ , for any  $x \in P_\ell$ .

This definition requires that neither party leader can benefit by changing her nomination.

### Equilibrium with Nominations by a Vote of Party Members

An *equilibrium* in the case of nominations by a vote of party members is a pair of nominations  $Nom(P_1) \in P_1$  and  $Nom(P_2) \in P_2$  such that there does not exist any  $x \in P_\ell$  such that  $W[x, Nom(P_{-\ell})]$  is preferred by a strict majority of voters in  $P_\ell$  to  $W[Nom(P_\ell), Nom(P_{-\ell})]$ .<sup>5</sup>

This definition requires that a party's nominee not be beaten in a head-to-head vote with some other potential nominee, given the other party's nomination. Thus, the nominee of a party must be a sort of internal Condorcet winner, given that voters anticipate the eventual election and overall outcome. This yields some intuitive interactions between the parties' nominees, as candidates who appeal to the party in the abstract might still be defeated for the nomination if they lack a chance of winning the subsequent election. Even though most of the interesting interaction under nomination by voting is between candidates that are viable given anticipations of what the other party will do, we still find that parties' nominees can drift away from the party and overall median voters.

### Equilibrium with Nominations by Spending Competitions

An *equilibrium* in the case of spending competition by party members is a pair of nominations  $i = Nom(P_1) \in P_1$  and  $k = Nom(P_2) \in P_2$  such that

$$u_i(W[i, k]) - u_i(W[j, k]) \geq u_j(W[j, k]) - u_j(W[i, k]) \quad (2)$$

for all  $j \in P_1$  and

$$u_k(W[k, i]) - u_k(W[h, i]) \geq u_h(W[h, i]) - u_h(W[k, i]) \quad (3)$$

for all  $h \in P_2$ .

This definition captures competition by candidates through spending. It requires that a party's nominee would not be beaten by some other nominee from the same party in a head-to-head spending competition, given the other party's nomination. That is, the party's nominee would be willing to outspend any challenger in order to keep the nomination. Here, for instance,  $u_i(W[i, k]) - u_i(W[j, k])$  represents the maximum that  $i$  is willing to spend in order to win the nomination instead of having  $j$  win it, given that  $k$  is the nominee of party 2.

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<sup>5</sup>We note that this definition is related to Duggan's (2001) definition of "group stable" equilibrium, which he defines for abstract games played between groups of players.

This definition captures the essential aspect of competition by spending, namely how much different candidates would be willing to pay in order to gain a nomination, without getting caught up in a detailed model of the process itself.<sup>6</sup> The definition is somewhat subtle since how much a candidate would be willing to spend can depend on whom they are bidding against. A candidate might be willing to spend more to defeat a candidate who differs more drastically from their own stance, than a candidate who is closer in stance.

The important difference between nomination by spending competition and the other nomination processes is that intensity of preferences matter under spending competition, while it is only ordinal and not cardinal preferences that matter in the party leadership and voting nomination settings. This is what allows for a wide variety of outcomes under this setting, depending on how much different candidates are willing to spend to win office. Also, there are some other effects that arise, as candidates might seek the nomination even though they would lose the subsequent election in cases where they wish to prevent another nominee from obtaining office.

### 3.2 Nomination by Party Leaders

We now characterize equilibrium under each of the nomination procedures, starting with the case of a nomination by party leaders.

**EXAMPLE 1** *Multiple Equilibria Under Party Leaders, No Overlap*

There are seven voters,  $N = \{1, \dots, 7\}$ , and two parties that partition  $N$  as follows:  $P_1 = \{1, 2\}$  and  $P_2 = \{3, 4, 5, 6, 7\}$ . The voters' ideal points are ordered by their labels.

First, note that in this example, the winner will come from  $P_2$  regardless of who the leaders are. This follows since if 3 is nominated, 3 will win against any nominee from  $P_1$ , and all members of  $P_2$  prefer 3 to either nominee of  $P_1$ .

In this example, there are multiple equilibria, but all equilibria have the same outcome: the winner is the member of  $P_2$  who is most preferred by the leader of  $P_2$  out of those who beat 2. The winner must always lie in the interval between 4 (the median) and the leader of  $P_2$ . For example, if the leader of  $P_2$  is 3, then 3 is the outcome. Note that here we already

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<sup>6</sup>One could model this via a sort of auction process. One natural process would be an “all-pay” auction, where each candidate spends as they wish and the winner is the candidate that spends the most. An equilibrium of that auction where candidates are aware of each other’s willingness to pay corresponds to the equilibrium we define here. That is, a candidate that is willing to spend more than each other candidate would win the auction by spending a minimal amount as no other candidate would want to spend given that they anticipate eventually being outspent. Our setting is slightly more complicated, as a candidate’s willingness to spend depends on whom they are bidding against, but the equilibrium can be constructed as an easy extension of that where there are private values (e.g., Dekel, Jackson, and Wolinsky (2005)).

see the multiplicity of equilibria;  $P_1$  is willing to nominate either 1 or 2, as it is irrelevant. Either nomination leads to the same outcome. If the leader is 4, then 4 is the equilibrium outcome. If the leader is 5, then the outcome is either 4 if 2 beats 5, but is 5 if 5 beats 2. If the leader is 6, then the outcome is in  $\{4, 5, 6\}$ , and is the highest indexed member of this set that beats 2.

Some features of this example generalize. We find that there may be a multiplicity of equilibria, but that they always lie in a well-defined interval between the overall median and the party leader of the party containing the overall median.

**PROPOSITION 1** *There always exists an equilibrium under a nomination by party leaders. The winning candidate in any equilibrium lies in the interval between (and including) the overall median voter and the leader of the party which contains the overall median voter.*

The proof appears in the appendix.

The fact that the winner always comes from the interval between the overall median,  $M$ , and the leader  $k$  of the party that contains  $M$  is relatively straightforward. If the winner came from the other side of the median from  $k$ , then  $k$  could improve by nominating  $M$ . If the winner came from the other side of  $k$ , then  $k$  could improve by nominating him or herself. The more specific details of the equilibrium structure get more complicated and there is no simple formula.

When there is no overlap in parties, then the winner is the same in all equilibria.

**PROPOSITION 2** *If there is no overlap in parties, then there is a unique equilibrium winner. The winning candidate comes from the party that contains the overall median, and the outcome is that party's leader's most preferred member from the set of those who beat all members of the other party.*

The proof is straightforward, following the logic of Example 1, and is left to the reader.

While the case with no overlap produces a unique winner, things are more complicated when there is overlap in parties. In that case there can exist multiple equilibrium outcomes, and depending on the configuration of parties, the winning nominee can come from either party. To get some feeling for this, consider the following example.

**EXAMPLE 2** *Multiple Equilibria Under Party Leaders*

There are seven voters,  $N = \{1, \dots, 7\}$ , and two parties that partition  $N$  as follows:  $P_1 = \{2, 3, 6\}$  and  $P_2 = \{1, 4, 5, 7\}$ . The voters' ideal points are ordered by their labels. The party leaders are 6 and 7. Let preferences be such that  $W[i, 5] = i$  unless  $i = 6$  or  $i = 7$ .

There is an equilibrium where the nominees are 6 and 7. There is also an equilibrium where the nominees are 3 and 4. This is an equilibrium even though both leaders would

prefer the other equilibrium.<sup>7</sup> Note that these two equilibria have different parties winning. Note also that the set of equilibria is not connected in the sense that there is no equilibrium where 5 is the winner. The only equilibrium outcomes are 4 or 6.

We can refine the set of equilibria using strong equilibrium. Then, we end up with the selection of equilibria where the winner lies between the peaks of the party leaders. We provide the details of this refinement in the appendix.

### 3.3 Nomination by a Vote of Party Members

We now turn to nomination processes by a vote of party members. As we show below, nominations by a vote of party members are equivalent to having nominations by party leaders where the party leaders are the medians of the parties.

**EXAMPLE 3** *Nomination by Voting*

Reconsider Example 1 where there are seven voters,  $N = \{1, \dots, 7\}$ , and two parties,  $P_1 = \{1, 2\}$  and  $P_2 = \{3, 4, 5, 6, 7\}$ . The voters' ideal points are ordered by their labels.

In the case where 5 beats 2 in an election, then the unique equilibrium outcome and nominee from  $P_2$  is 5, while there are two equilibria in that  $P_1$  can nominate either 1 or 2. To verify this, it is enough to check that 5 would be the nominee of party 2 regardless of party 1's nomination. Voters 5, 6, and 7 prefer to have 5 nominated than either 3 or 4 (either of whom would win in the subsequent election against either candidate from party 1), and so it is clear that 5 would defeat 3 and 4 for the nomination, regardless of party 1's nomination. So consider, a nominee of 6 or 7. If that nominee would win against the nominee of party 1, then 3, 4 and 5 would all rather have 5 nominated. If that nominee would lose against the nominee of party 1, then 5, 6, and 7 would all prefer to have 5 nominated. This leaves 5 as the equilibrium nomination from party 2 in all equilibria.

If 2 beats 5, then one can verify that all equilibria have  $P_2$  nominate 4, who wins the subsequent election.

We now show that at least one equilibrium always exists and relate the equilibrium structure under voting to the nominations by party leaders.

**PROPOSITION 3** *There always exists an equilibrium under a vote by party members. The set of equilibria coincides with that where the median voter in a party is a "party leader".<sup>8</sup> The*

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<sup>7</sup>Note that this is an equilibrium in undominated strategies given that 1 beats 6 (as 1 beats 5).

<sup>8</sup>Given that one party will have two medians, this refers to a union of the sets of equilibria where each one of the two medians is party leader.

*winning candidate lies between the overall median and the median<sup>9</sup> of the party containing the overall median.*

The proof appears in the appendix.

The intuition for a party acting as if the median were a party leader is much more subtle than it would seem. For example, note that it is *not* always true that given a comparison between two arbitrary candidates, if the median prefers one to the other then so does a majority. It is possible that when comparing candidates from opposite sides of the median, the median's preferences are in the minority.

Nonetheless, the claim is true because the set of viable candidates has structure to it. To understand this, consider the nomination of one party taking the nomination of the other party as given.<sup>10</sup> It is relatively straightforward to show that the set of possible nominees who could defeat the nominee of the other party is either (i) an interval including the median of the party, or (ii) an interval lying entirely to one side of the median and to the side of the other party's nominee. In case (i) where the set of viable nominees includes the median, then the median would be preferred to the nominee from the party by a majority of the voters of the party, as the comparison would always boil down to a comparison of the median of the party and some other outcome. In that case, the median is the only possible nominee in response to the other party's nominee. If instead, case (ii) applies and the interval is entirely on one side of the median, then the critical observation is that the interval must lie on the same side of the party median as the other party's nominee (since the set of candidates that beat the nominee of the other party is a connected set around the overall median that extends out to the other party's nominee). If we consider two viable nominees from that interval, then they both lie on the same side of the party median and so a majority of the party will have preferences that agree with the party median's preferences.

While the nomination by party voting allows for non-median outcomes overall, the chosen candidate still comes from a well-defined interval between the overall median and the median of the party containing the overall median.

As we shall now see, the equilibrium looks very different when we consider nominations by party spending.

### 3.4 Nomination by Spending Competition

We begin the analysis of nomination by spending competition with some examples. First, we show an equilibrium where there is an extreme outcome in terms of each party's nominee and the overall winner.

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<sup>9</sup>This is the furthest median voter of the party, if there are an even number of voters.

<sup>10</sup>Consider the case where there the first party has a single median and see the appendix for the case with two medians.

EXAMPLE 4 *Nomination by Spending Competition*

Again, reconsider Example 1 where there are seven voters,  $N = \{1, \dots, 7\}$ , and two parties,  $P_1 = \{1, 2\}$  and  $P_2 = \{3, 4, 5, 6, 7\}$ . The voters' ideal points are ordered by their labels.

Note first, that there are preference configurations where the nominee of  $P_2$  is 3, even though all other members of party 2 would prefer to nominate 4, and even though that nominee does not lie between the overall median and the median of  $P_2$  (in contrast to the case of nomination by voting). For example, if  $d_3(3, i) > d_i(i, 3)$  for all  $i > 3$ , then 3 wins the nomination of  $P_2$  and the overall election.

It is also possible to have extremists from both parties nominated. For instance, suppose that all members of  $P_2$  prefer any member of  $P_2$  to any member of  $P_1$ . In this case, the nominee of party 2 will win the election and so it is as if there were just one party and spending competition among its members. If  $d_7(7, i) > d_i(i, 7)$  for each  $i \in \{3, 4, 5, 6\}$ , then the unique equilibrium outcome would be that 7 wins the nomination and then the overall election. As the nominee from  $P_1$  is irrelevant, we could see extreme nominees from both parties.

This example shows the contrast between nomination by spending competition and nomination by voting. Under spending competition the outcome could be any member of  $P_2$ , while in the voting case it would have to be either 4 or 5.

While the possible outcomes under nominations by spending competition are more varied than under nominations by voting, we can still say something about the outcome, at least in the case where there is no overlap in the parties which is a very natural case to consider.

**PROPOSITION 4** *If there is no overlap in parties, then any equilibrium winner under nomination by spending competition is from the party containing the median, and is a candidate who defeats all candidates from the other party.*

The proof again appears in the appendix, but is easy to explain. In this case, all members of the party containing the median prefer the candidate  $k$  closest to the other party to any nominee of the other party. This means that any candidate willing to outspend  $k$  must also be able to win the election.

Proposition 4 does not mention the issue of existence. This is due to the fact that another contrast between nomination under spending competition and the other nomination procedures is that under spending competition an equilibrium need not always exist, as shown in the next example. In fact, the example shows nonexistence even in the no overlap case.

EXAMPLE 5 *Non-existence of Equilibrium Under Party Spending*

There are five voters  $N = \{1, \dots, 5\}$  and two parties,  $P_1 = \{1, 2\}$  and  $P_2 = \{3, 4, 5\}$ .

Suppose that every member of  $P_2$  prefers any member of  $P_2$  to any member of  $P_1$ . So, it is clear that the nominee of  $P_1$  is irrelevant. Let  $d_4(4, 3) > d_3(3, 4)$ . Then 3 cannot be the nominee as 3 would be outspent by 4. Also, let  $d_5(5, 4) > d_4(4, 5)$ . Then 4 cannot be the nominee as 4 would be outspent by 5. This leaves only 5 as the potential nominee. However, if  $d_3(3, 5) > d_5(5, 3)$ , then 5 cannot be the nominee either. Thus, there are situations where there is no equilibrium.

The nonexistence of equilibrium in the case of spending competition follows from the fact that intensities of preferences matter and might not be ordered across party members in any nice way.

### 3.4.1 Sufficient Conditions for Existence Under Party Spending with No-Overlap in Parties

We have seen that an equilibrium may not exist under nominations by spending competition, even in a five-voter<sup>11</sup> world with single-peaked preferences and no overlap in parties. We now look for sufficient conditions on preferences for an equilibrium to exist.

In the case of no overlap, an intuitive condition is sufficient to rule out the cycle exhibited in the above example and to restore existence. We abuse notation and let  $i < j$  denote that  $x_i$  is to the left of  $x_j$ .

Let us say that preferences satisfy the *extremist* condition if  $d_i(i, k) \geq d_j(j, k)$  whenever  $i \leq j \leq k$  or  $i \geq j \geq k$ .

This condition says that if one voter is willing to spend a given amount to move the outcome in a given direction (say to the left), then voters further to the left would be willing to spend at least as much for the same change. Under this condition, there is a consistent ordering to the intensity of voters preferences and this is enough to avoid the cycles from the example above and guarantee existence.

The extremist condition is clearly very strong, and one would expect to find many settings where it fails. However, as we see from Example 5, something on the order of this condition is really needed to establish equilibrium existence. There are cases where the extremist condition is satisfied. For instance, if preferences are Euclidean (so that utility is just the opposite of the distance between the outcome and the peak, as is often assumed in the literature), then the condition is clearly satisfied.

**PROPOSITION 5** *If there is no overlap in parties and the extremist condition is satisfied, then there exists an equilibrium under nomination by spending competition.*

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<sup>11</sup>One could even simplify the example further having only one party, and reduce it to a three voter world.

The proof of the proposition is constructive and appears in the appendix. The idea is that under the extremist condition, the relevant candidates are only extreme ones. We have to be a bit careful, as the relevant ones in some cases need to be defined relative to those who win against nominees of the other party.

### 3.4.2 Sufficient Conditions for Existence Under Party Spending: The General Case

When there is an overlap in parties, cycles turn out to be surprisingly robust to preference restrictions. Even the nice ordering of preferences under the extremist condition fails to be sufficient to guarantee existence. In fact, we show that equilibria fail to exist even under stronger preference restrictions. We examine two preference restrictions: First, an “strong extremist” property (that is a strengthening of the extremist condition), and second, an ordered preference intensities condition. The failures of these two conditions to guarantee existence helps illustrate another condition, which we call the “directional party” condition, which ensures existence.

Preferences satisfy the *strong extremist* condition if for all players  $i, j, k$  such that  $i \leq j \leq k$  and all alternatives  $h, t$  with  $i \leq h \leq t \leq k$ ,

1.  $d_i(h, t) > d_k(t, h)$  implies  $d_i(h', t') > d_j(t', h')$  for all  $i \leq h' \leq t' \leq j$  and,
2.  $d_k(t, h) > d_i(h, t)$  implies  $d_k(t', h') > d_j(h', t')$  for all  $j \leq h' \leq t' \leq k$ .

The strong extremist condition says that if one voter  $i$  has more intense preferences than another voter  $k$  regarding pairs of candidates in between those two ( $h$  and  $t$ ), then voter  $i$  has more intense preferences than some other voter  $j$  who lies in the same direction as  $k$ , over pairs of alternatives between  $i$  and  $j$ .

This, again, is a strong condition that imposes some consistency on preferences to rule out cycles. Similar to the extremist condition, while it is strong and only satisfied in special cases, it is satisfied by Euclidean preferences that are directly proportional to distance between an alternative and a voter’s peak.

Even with this strengthening of the extremist condition, there are situations where no equilibrium exists, provided there is overlap between the parties.

#### EXAMPLE 6 *Non-existence of Equilibrium Under the Strong Extremist Condition*

There are seven voters with ideal points at locations:  $x_1 = 0, x_2 = 1, x_3 = 3, x_4 = 6, x_5 = 7, x_6 = 9, x_7 = 10$ . Voters’ preferences are distance based, so they prefer candidates who

are closer to their ideal points to those farther away. Two parties partition  $N$  as follows:  $P_1 = \{1, 3\}$  and  $P_2 = \{2, 4, 5, 6, 7\}$ .

We suppose that the strong extremist condition is satisfied in terms of preference intensities and the following are true:<sup>12</sup>

$$d_7(7, 2) > d_2(2, 7)$$

$$d_1(2, 3) > d_3(3, 2)$$

$$d_2(3, 6) > d_6(6, 3).$$

Let us show that there is no equilibrium. We start by showing that there is no equilibrium with 1 as the nominee of  $P_1$ . Every candidate in  $P_2$  beats 1. Thus, by the strong extremist condition, the only candidates for nomination from  $P_2$  are 2 and 7. The nominee for  $P_2$  must then be 7, since  $d_7(7, 2) > d_2(2, 7)$ . However, if 7 is nominated by  $P_2$ , then both 1 and 3 in  $P_1$  would rather have 3 be nominated over 1. Thus, it is impossible to have an equilibrium with 1 as the nominee of  $P_1$ . So, let us consider 3 as the nominee of  $P_1$ . 2 cannot be the nominee of  $P_2$ , as then  $d_1(2, 3) > d_3(3, 2)$  implies that 1 would outbid 3 for the nomination of  $P_1$ . So, the nominee of  $P_2$  must come from  $\{4, 5, 6, 7\}$ . It cannot be 6, since 2 would outbid 6 given that  $d_2(3, 6) > d_6(6, 3)$ . By the strong extremist condition, this also means that it cannot be 5 or 4 for the same reason. So, we are left with 7. However if 7 is nominated, then 3 wins. 6 would then wish to outbid 7 (and 7 would be happy to be outbid). Thus, there is no equilibrium.

Suppose now that we can order the intensity of candidate preferences. Preferences satisfy the *ordered preference intensity* condition if every distinct pair of voters  $i$  and  $j$  can be ordered in terms of preference intensity such that either  $|d_i(h, k)| > |d_j(h, k)|$  (for all  $h \neq k$ )<sup>13</sup> or  $|d_j(h, k)| > |d_i(h, k)|$  (for all  $h \neq k$ ). Notice that having more intense preferences is a transitive relationship. Even this strong a condition is not enough to guarantee existence.

**EXAMPLE 7** *Non-existence of Equilibrium when Preference Intensities are Ordered*

There are seven voters with ideal points  $x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 7, x_5 = 8, x_6 = 9, x_7 = 11$ , and who prefer outcomes closest to their own peaks. Two parties partition  $N$  as follows:  $P_1 = \{1, 4, 5, 6, 7\}$  and  $P_2 = \{2, 3\}$ . Preference intensities are ordered so that  $2 > 3 > 7 > 1 > 6 > 5 > 4$ , where ' $i > j$ ' means ' $i$  has more intense preferences than  $j$ '.

We now check that there is no equilibrium. No equilibrium can support the nomination of voter 2 in  $P_2$  without the nomination of 7 in  $P_1$  because 7 could win the final election and has the most intense preferences in  $P_1$ . But the pair (7, 2) is not an equilibrium either since voter 2 would be outspent by voter 3, as 3 is the best outcome that 2 can rationally

<sup>12</sup>These three relationships are consistent with the strong extremist condition.

<sup>13</sup>It would be more natural to require this only when  $h$  and  $k$  lie to one side of  $i$  and to one side of  $j$ , but even under this very strong condition equilibria fail to exist.

expect given the next round. Following the same logic, (7, 3) is not an equilibrium because 7 would be outspent by 4, 5 or 6. Furthermore, in each of (4, 3), (5, 3), (6, 3), voter 1 would outspend these other potential nominees from  $P_1$  as she has the most intense preferences in  $P_1$  after 7. Finally, voter 2 would not let voter 3 win the nomination under (1, 3), so that cannot be an equilibrium.

These last two examples suffer similar cycling issues: We first begin to move in one direction, but then someone on the opposite side breaks the directional trend by stealing the nomination, and starts a cycle. The following condition is sufficient to prevent cycling, thus implying equilibrium existence.

Preferences satisfy the *directional-party condition* if for each party  $\ell$ , either

1.  $d_i(h, t) \geq d_j(t, h)$  for all  $i \in P_\ell$  and  $j \in P_\ell$  and  $h, t \in N$  such that  $i \leq h < t \leq j$ , or
2.  $d_i(h, t) \leq d_j(t, h)$  for all  $i \in P_\ell$  and  $j \in P_\ell$  and  $h, t \in N$  such that  $i \leq h < t \leq j$ .

The directional party condition says that there is a consistent direction with respect to which a party's preferences can be ordered. Either it is always voters more to the left that care at least as much as voters to the right, or vice versa. Again, this condition is very strong, but satisfied when preferences are Euclidean (the opposite of the distance between an alternative and the voter's peak).

**PROPOSITION 6** *If preferences satisfy the directional-party condition, then an equilibrium under nomination by spending competition exists.*

The proof is in the appendix, and uses an algorithm that identifies an equilibrium under the directional party condition.

## 4 Endogenous Parties

We now turn to endogenizing the parties. This is important in order to understand how robust the equilibria identified in the earlier sections are to voters' incentives to switch parties. Interestingly, it turns out that with nominations by voting, endogenizing parties leads to median outcomes, while under nomination by spending competition, it is still possible to get extreme outcomes in both nominations and the overall winner.<sup>14</sup>

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<sup>14</sup>In this section we do not consider endogenous parties with party leaders, as it is not so clear how to properly define equilibrium in that case (e.g., who are the leaders if a leader switches parties?). Moreover, we already see an interesting contrast between the voting and spending competition cases, which is our more central focus.

## Equilibrium with Endogenous Parties

Consider a partition of the population into two parties,  $(P_1, P_2)$ , with the possibility that one of these is empty. We say that  $(P'_1, P'_2)$  is adjacent to  $(P_1, P_2)$  if there exists  $i$  such that  $(P'_1, P'_2) = (P_1 \setminus \{i\}, P_2 \cup \{i\})$  or  $(P'_1, P'_2) = (P_1 \cup \{i\}, P_2 \setminus \{i\})$ . Thus, adjacent pairs of parties are those where the only difference is that one voter has switched parties.

An equilibrium with endogenous parties is a pair of parties with the possibility that one is empty,  $(P_1, P_2)$  that partition the set of voters, and a pair of nominations that form an equilibrium  $(Nom(P_1), Nom(P_2))$ ,<sup>15</sup> as well as a specification of an equilibrium  $(Nom(P_1^*), Nom(P_2^*))$  for every adjacent partition into two parties  $(P_1^*, P_2^*)$ , such that:

$$u_i(W[Nom(P_\ell), Nom(P_{-\ell})]) \geq u_i(W[Nom(P_\ell \setminus \{i\}), Nom(P_{-\ell} \cup \{i\})]), \quad (4)$$

for each  $P_\ell$  and  $i \in P_\ell$ .

A party structure together with specifications of (equilibrium) nominations for that party structure and all adjacent ones is in equilibrium if no member of one party wishes to switch to the other party, anticipating the equilibrium that would ensue.

### 4.1 Endogenous Parties and Nomination by Voting

We first revisit nominations by party voting. Consider the following example.

**EXAMPLE 8** *Every Equilibrium Outcome is the Median with Endogenous Parties, but not with Exogenous Parties*

There are seven voters,  $N = \{1, \dots, 7\}$ , and two parties that partition  $N$  as follows:  $P_1 = \{1, 2, 3, 7\}$  and  $P_2 = \{4, 5, 6\}$ . Let 6 beat 3 in an election. One equilibrium *when these are exogenous parties* is  $(3, 5)$ , with candidate 5 winning. This is not, however, part of an equilibrium with endogenous parties. Candidate 4, the median, can join  $P_1$ . With the new lineup of  $P'_1 = \{1, 2, 3, 4, 7\}$  and  $P'_2 = \{5, 6\}$ ,  $(4, 5)$  is an equilibrium (with either exogenous or endogenous parties). Let us check that  $P'_1 = \{1, 2, 3, 4, 7\}$  and  $P'_2 = \{5, 6\}$ ,  $(4, 5)$  is part of an equilibrium with endogenous parties). Clearly, candidate 4 would not wish to switch, as 4 wins the election. Candidates 1, 2, 3, and 7 would have no effect on the outcome by switching to  $P_2$  as it is still an equilibrium to have 4 nominated by  $P_1$  against 5 from  $P_2$ ; and candidates 5 and 6 would have no effect on the outcome by switching to  $P_1$  as it is then still an equilibrium to have 4 nominated against the remaining candidate in  $P_2$ .

This feature that the median is the winner is not just an artifact of this example, but is true of all equilibria under nominations by voting when parties are endogenous.

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<sup>15</sup>In the case where one of the parties is empty, then its nomination is ignored, and the other party's nominee wins the election by default.

**PROPOSITION 7** *When nominations are by votes, then in every equilibrium with endogenous parties  $W[\text{nom}(P_1), \text{nom}(P_2)] = M$ . Moreover, such an equilibrium exists.*

While the outcome is necessarily the median once parties are endogenized under nominations by voting, the parties can still have a variety of configurations. For instance, it could be that the equilibrium is to have the median alone in one party, or instead at the other extreme to have all voters in the same party. What is tied down is that unless one of the nominees is the median, then the party structure will turn out to be unstable. This emphasizes that the *equilibrium party structure cannot be separated from what the equilibrium nominees are*. It could be that parties are stable with one pair of nominees, but not with another.

## 4.2 Endogenous Parties and Nomination by Spending Competition

We now turn to endogenizing parties under spending competition. Here, it turns out that non-median outcomes are possible, as we now show.

**EXAMPLE 9** *Existence of a Non-Median Equilibrium Outcome*

There are five voters  $N = \{1, \dots, 5\}$ , and two parties that partition  $N$  as follows:  $P_1 = \{1, 3\}$  and  $P_2 = \{2, 4, 5\}$ . Voters' ideal points are ordered by their labels. Moreover, assume that  $d_1(2, 3) > d_3(3, 2)$ , and  $d_2(i, j) > d_h(k, t)$  for all  $h \in \{3, 4, 5\}$  and all  $i, j$  such that  $2 \geq i > j$ . Also, let 3 prefer 2 to 4.

For  $P_1$  and  $P_2$  above,  $(1, 2)$  is a pair of nominations that form an equilibrium where the general winner is voter 2. Let us check that there is some specification of equilibria for each possible switching of some voter, so that no voter would desire to switch parties. If voter 1 switches party then  $P_1$  only consists of voter 3, the median. In this case, regardless of the nominee from  $P_2$ , the final winner is voter 3, and voter 1 is made worse off. If instead voter 3 switched parties, then voter 1 would become the only possible nomination in  $P_1$ . In  $P_2$ , voter 2 outbids any member, so she is nominated as part of any equilibrium. Voter 3 is not strictly better off since voter 2 is still the general winner. It is clear that voter 2 will not gain by switching parties, regardless of the equilibrium specification. So, we are left only to consider what happens if voter 4 (or 5) switches parties. Here,  $(1, 2)$  is still an equilibrium because then 4 (5) does not want to outspend 1 as they would still lose to 2 (and 3 still does not want to outspend 1 given that  $d_1(2, 3) > d_3(3, 2)$ ); and voter 2 continues outbids the members of her party.

Example 9 shows that nomination by spending can provide non-median outcomes that are robust to party switching, in contrast to nominations by voting.

Again, with nominations by spending competition there are issues of equilibrium existence. However, the directional party condition is again sufficient to guarantee existence.

**PROPOSITION 8** *Suppose that nominations are by spending competition. If preferences satisfy the directional party condition and are in the same direction for each party, and  $N \geq 5$ , then an equilibrium with endogenous parties exists.*<sup>16</sup>

The proof of the proposition involves an explicit construction of the two parties and nominations, putting the two most extreme voters (in terms of the directional preference) in different parties. For instance, if lowest indexed voters are those who have stronger preferences under the directional preference, then the constructed equilibrium parties would have 1 and 3 together in one party and 2 and 4 together in the other, with any allocation of the remaining voters between the parties. 1 and 2 are nominated and 2 wins the election. None of the remaining voters can switch the outcome by changing parties. 2 clearly has no gain from changing, and if 1 changes parties, then 3 wins the nomination and the election, which cannot be improving for 1.

Example 9 and the proof of Proposition 8 show us that even with endogenous parties, it is possible to have extreme outcomes under nomination by spending competition. This makes the point that how nominations are conducted can have a big impact on outcome, and that if spending plays a substantial role in the nomination process, then outcomes can differ dramatically from a pure voting setting.

## 5 Concluding Remarks

We have seen that the nomination process is important in determining the outcome of elections, even in a simple single-peaked world. Non-median outcomes can emerge from an election even when parties vote over their nominations for some fixed configurations of parties, but not when parties are endogenous. More extreme outcomes are possible under nominations by spending competition and those persist even when parties are endogenous. Our analysis thus provides insight into why non-median outcomes occur in settings where the election is well ordered on one dimension. This suggests that it is important to model nomination processes in order to understand electoral outcomes, even in the starkest settings.

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<sup>16</sup>In the case where  $N = 3$ , there need not always exist an equilibrium. For instance, suppose that 1 cares most, then 2, then 3, where 2 is the median. Suppose also that 1 beats 3 in an election. If 1 and 2 are in the same party, then the nomination of that party must be 1 (regardless of whether 3 is present). That is not stable as then 2 would rather switch parties and win the nomination and then the election. It is also not stable to have 1 and 2 in separate parties, as then 1 would like to join the party that 2 is in, to win that nomination and the overall election.

There is much room for further research, and important ways in which the analysis should be extended. We close with the mention a few of the most obvious directions for further study.

First, we have modeled extreme versions of nomination processes, where either there are party leaders, there is a vote among party members, or there is simply a spending competition among party members. Reality is, of course, more complex, and involves combinations of these three elements. Party leadership has some discretion in identifying potential nominees, the electorate has substantial input, and spending by potential nominees can also clearly have an effect. Identifying how these different influences interact is of interest.

Second, our analysis has been confined to elections of single representatives or officials from two party settings. While this has wide application (even beyond the U.S.), it is also important to understand nomination processes in multiparty systems, as well as things like selections of party lists and platform design and their influence on electoral competition.

Third, general forms of stability with endogenous parties, where one allows either more than two parties or more than one voter to change at a time, face substantial existence hurdles. Nonetheless this needs to be investigated, as in situations where two parties are nominating extreme candidates, there are strong incentives for centrist voters to split off and form their own party. This again points to an interest in the modeling of multiple party systems, even for the understanding of two party systems.<sup>17</sup> Although modeling party formation has generally been a difficult task and there is a paucity of workable models; it is such a important aspect of electoral competition that it begs for further analysis.

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<sup>17</sup>There are economies of scale and other aspects of parties (branding, reputation, etc.) that may make it hard to form new parties (or even to switch parties), and so our endogenous party equilibrium analysis may still be a good starting point. But understanding party formation more generally is clearly important.

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## 7 Appendix

### 7.1 Proofs of the Propositions

**Proof of Proposition 1:** Let  $D_\ell$  and  $D_{-\ell}$  respectively be the leaders of parties  $\ell$  and  $-\ell$ . Denote by  $(Nom(P_\ell), Nom(P_{-\ell}))$  the pairs of nominations. Without loss of generality, assume  $M \in P_\ell$ .

Suppose  $D_\ell \geq M$ . First, we show that the winning candidate in equilibrium lies in  $[M, D_\ell]$ . By way of contradiction, suppose the winner, call it  $W^*$ , is to the left of (less than)  $M$ . If  $D_\ell$  nominates  $M$ , then  $W[W^*, M] = M$  and so  $D_\ell$  is strictly better off by single-peakedness. Outcome  $W^*$  could not be supported in equilibrium, a contradiction. If  $W^* > D_\ell$ , then from a similar argument,  $D_\ell$  is better off nominating herself because  $W[W^*, D_\ell] = D_\ell$ , a contradiction.

Secondly, we prove existence. If  $D_\ell = M$ , then it is always an equilibrium for  $D_\ell$  to nominate herself and for  $D_{-\ell}$  to choose arbitrarily a nominee in  $P_{-\ell}$ . If  $D_\ell > M$ , then take  $\hat{x}$  which is defined as the closest point to  $D_\ell$  in  $P_\ell \cap [M, D_\ell]$  such that  $W[y, \hat{x}] = \hat{x}$  for all  $y \in P_{-\ell}$ . If  $\hat{x} = D_\ell$ , then  $(D_\ell, y)$  with any  $y \in P_{-\ell}$  is an equilibrium. If  $\hat{x} \neq D_\ell$ , then for all  $x \in P_\ell \cap (\hat{x}, D_\ell]$ , there exists  $y \in P_{-\ell}$  such that  $W[x, y] = y$  (for if this were not true,  $x$  would be closer to  $D_\ell$  which violates the definition of  $\hat{x}$ ). Define  $x^* \equiv \min(P_\ell \cap (\hat{x}, D_\ell])$ . Let  $y^* \in P_{-\ell}$  be the closest point to  $D_{-\ell}$  in  $P_{-\ell}$  such that  $W[x^*, y^*] = y^*$ . Note that  $W[x, y^*] = y^*$  for all  $x \in (\hat{x}, D_\ell]$ . Now, if  $y^* \in (\hat{x}, D_\ell]$ , then  $(x^*, y^*)$  is an equilibrium because the candidates in  $P_\ell$  that could defeat  $y^*$  would make  $D_\ell$  strictly worse off, and so  $x^*$  is a best-response for  $D_\ell$ . By definition,  $y^*$  is the best nomination for  $D_{-\ell}$  when  $Nom(P_\ell) = x^*$ . But, if  $y^* < \hat{x}$ , then  $(\hat{x}, y^*)$  is an equilibrium because  $D_{-\ell}$  is indifferent between all the alternatives in  $P_{-\ell}$  while  $\hat{x}$  is  $D_\ell$ 's best choice when facing  $y^*$ .

Now suppose  $D_\ell < M$  and let  $\mathcal{P}$  be the set of voters' peaks. Consider the dual  $(\mathcal{P}', >')$  of  $(\mathcal{P}, >)$  where  $i$ 's peak in  $\mathcal{P}'$  is greater than  $j$ 's if and only if it is smaller than  $j$ 's in  $\mathcal{P}$ . The above argument completes the proof as  $D_\ell >' M$  in  $\mathcal{P}'$ . ■

**Proof of Proposition 3:** First, we prove that a pair of nominations is an equilibrium under a vote by party members if and only if this pair is an equilibrium with nomination by medians as party leaders. Then we show existence and conclude.

Let us first show that if a pair of nominations is an equilibrium with medians as party leaders, then it is an equilibrium under nomination by voting. So, let (one of) the medians of each party be a party leader:  $D_\ell = M_\ell$  and  $D_{-\ell} = M_{-\ell}$ . Suppose  $(Nom(P_\ell), Nom(P_{-\ell})) = (i, j)$  is an equilibrium with medians as party leaders. This means that  $W[i, j] \succeq_{M_\ell} W[x, j]$  for all  $x \in P_\ell$ . If  $W[M_\ell, j] = M_\ell$ , then it must be that  $i = M_\ell$ . In that case, regardless of  $x$ , since  $M_\ell$  is a median of the party and preferences are single peaked, there is not a strict majority of the party that prefers  $x$  to a median of the party, and so it remains an equilibrium nomination for  $\ell$  under voting. So consider the case where  $W[M_\ell, j] \neq M_\ell$ . In that case, it must be that either  $j$  lies between the overall median and  $M_\ell$ , or on the other side of the median from  $M_\ell$ . This means that for any  $x$  (including  $i$ ),  $W[x, j]$  lies to the same side of  $M_\ell$  as  $j$ . In that case, a (weak) majority has the same preferences as  $M_\ell$  over the pair  $W[i, j]$  and  $W[x, j]$ . Thus, if  $W[i, j] \succeq_{M_\ell} W[x, j]$ , then this is true for at least a weak majority of member of party  $P_\ell$  and so no other nominee would defeat  $i$  as a nominee.

Since  $\ell$  was arbitrary, any  $(i, j)$  which is an equilibrium with medians as party leaders is an equilibrium under a nomination by voting.

To see the converse, consider an equilibrium  $(i, j)$  under nomination by voting. Suppose that this is not an equilibrium either any choice of medians as party leaders. So, there exists a party  $\ell$  such that  $i$  would not be the choice of the party median(s) in response to  $j$ . Thus,  $i$  cannot be a party median. As argued above, the only possible outcomes as a function of the nominations of party  $\ell$  either include at least one of the medians, or all lie on the same side of party median(s) as  $j$ . Consider the latter case where neither median would win against  $j$ . There, all of the party members to the opposite side of the party median(s) to  $j$  have the same preferences as the party median(s) over all the possible outcomes since all possible outcomes are to one side of the party median(s). In that case, it must be that if  $i$  is not defeated by a strict majority, then there is no other nomination that the median (or either median if there is more than one) would prefer to  $i$ . So, it must be that at least one party median would defeat  $j$ . In particular, it must be that if there are two party medians, then the median closest to  $W[i, j]$  would defeat  $j$  (since the set of winners against any candidate is a connected set). However, this means that the median closest to  $W[i, j]$  would also be preferred by a strict majority of party  $\ell$  to  $W[i, j]$ , as all voters to the opposite side of that median would prefer that median to  $W[i, j]$ . This is a contradiction, and so our supposition was wrong and the claim follows.

By Proposition 1, we know that there exists an equilibrium under nominations by any pair of leaders, and so there exists one where the medians are party leaders. Therefore, by the first part of the proof, an equilibrium exists under vote by party members. The third part of our claim follows immediately from Proposition 1. ■

**Proof of Proposition 4:** Without loss of generality, let  $P_2$  be the party containing the median. Suppose to the contrary of the proposition, that the winner  $j$  was from  $P_1$ . Let  $k$  be the member of  $P_2$  closest to  $P_1$  and let  $i = \text{Nom}(P_2)$ . Then  $d_k(k, j) > d_i(j, k)$ , as it must be that  $d_i(j, k) < 0$  and  $d_k(k, j) > 0$ . Thus, it could not have been an equilibrium to nominate  $i$ .

Next, suppose that  $i = \text{Nom}(P_2)$  and that  $i$  is beaten by some member of  $P_1$ . A similar argument as the one just given reaches a contradiction. ■

**Proof of Proposition 5:** Without loss of generality, let  $P_1$  contain the median and lie to the left, and order voters by their labels. Let  $k$  be the minimal labeled voter in  $P_2$ . Let  $S_1$  be the subset of voters in  $P_1$  who would beat  $k$  in the election (and this set is non-empty given that the median is in this set). Let  $k = \text{Nom}(P_2)$ . Note that all voters in  $P_1 \setminus S_1$  prefer any nominee from  $S_1$  to  $k$  and so will not wish to outbid any nominee in  $S_1$ ; and changing the nominee from  $P_2$  (given that  $\text{Nom}(P_1) \in S_1$ ) will not change the outcome. Thus, to complete the specification of an equilibrium, it is enough to find a nominee from  $S_1$  that

would not be outbid by any other nominee from  $S_1$ . Consider the two extreme candidates from  $S_1$ , and label them  $i$  and  $j$ . If  $d_i(i, j) \geq d_j(j, i)$ , then set  $Nom(P_1) = i$  and otherwise set  $Nom(P_1) = j$ . ■

**Proof of Proposition 6:** With directional parties, there are two cases: either (I) preference intensities for both parties (weakly) increase in the same direction, or (II) preference intensities for the parties increase in opposite directions.

We show that for both cases an equilibrium can be found.

**Case I.** Without loss of generality, assume that preference intensity in both parties (weakly) increases as the candidates move leftward. Now, choose  $1 \equiv \min P_\ell$  and  $2 \equiv \min P_{-\ell}$ , the leftmost candidates from each party. If  $1 \leq M$  and  $2 \leq M$ , then it is straightforward to check that  $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 2)$  is an equilibrium. However if (say)  $2 > M$ , then pick the leftmost candidate from Party  $\ell$  who can defeat candidate 2 in a pairwise election. In this case,  $M \in P_\ell$  and so such a candidate exists. Call this candidate 3. It is straightforward to check that  $(Nom(P_\ell) = 3, Nom(P_{-\ell}) = 2)$  is an equilibrium.

**Case II.** Let  $C_\ell$  be the direction set of party  $\ell$ , which contains all candidates on the side of the median corresponding to the direction of that party's increasing preferences. Wlog, assume that party  $\ell$ 's preference intensities increase for candidates to the right. Formally,  $C_\ell = \{i \in P_\ell : i \geq M\}$ . Furthermore, let  $\overleftarrow{C}_\ell = \{i \in P_\ell : i < M\}$ . Wlog, assume that preferences are increasing to the left for party  $-\ell$  and  $M \in P_\ell$ .

Case IIa:  $\overleftarrow{C}_{-\ell} = \emptyset$ . Let  $2 = \min C_{-\ell}$  be the candidate from  $C_{-\ell}$  that is closest to the median. If  $C_\ell \setminus [M, 2] \neq \emptyset$ , then choose the candidate closest to 2 in that set and call her 1. Then  $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 2)$  is an equilibrium. Otherwise, if  $C_\ell \setminus [M, 2] = \emptyset$ , then choose the candidate from  $C_\ell$  that is closest to 2, call her 1, and notice  $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 2)$  is an equilibrium.

Case IIb:  $\overleftarrow{C}_{-\ell} \neq \emptyset$ . Let  $2 = \max \overleftarrow{C}_{-\ell}$  be the candidate from  $C_{-\ell}$  that is closest to the median. Denote by 1 the candidate from  $C_\ell$  that is furthest from the median *and* can defeat candidate 2.<sup>18</sup> Now, if  $C_{-\ell} \cap [M, 1] = \emptyset$ , then  $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 2)$  is an equilibrium; otherwise, let  $3 = \max C_{-\ell} \cap [M, 1]$  and note  $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 3)$  is an equilibrium. ■

**Proof of Proposition 7:** We prove that for every equilibrium partition into parties  $(P_1, P_2)$  it must be that  $W[Nom(P_1), Nom(P_2)] = M$

Suppose that  $W^* = W[Nom(P_1), Nom(P_2)] \neq M$  in equilibrium. Without loss of generality, suppose that  $M \in P_1$ . The possible alignments for  $W^*$  can be divided into two distinct cases.

(1)  $W^* < M$ . Note first that there is no candidate in  $P_2 \cap (M, M_2]$  that can beat

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<sup>18</sup>Such a candidate can always be found since it is possible to choose the median.

$Nom(P_1)$ , for if there were then  $Nom(P_2)$  would not be nominated in equilibrium. Now, let  $P_1 = P_1/\{M\}$  and  $P_2 = P_2 \cup \{M\}$ . Since  $M_2 > M > W^*$ , a majority in  $P_2$  will prefer the result  $W[Nom(P_1), M] = M$ , and among the candidates that can win  $Nom(P_1)$ ,  $M$  is the closest to  $M_2$  and thus it is  $P_2$ 's best-response. Clearly,  $M$  will also prefer this result. Thus,  $W^* = W[Nom(P_1), Nom(P_2)]$  is not an equilibrium because  $M$  will switch parties.

(2)  $W^* > M$ . Here, the same reasoning applies.  $W^* \in P_1$  and  $W^* > M$ . Since  $M_1 < M < W^*$  a majority in  $P_1$  will prefer to nominate  $M$  and get the result  $W[M, Nom(P_2)] = M$ . Thus,  $W^* = W[Nom(P_1), Nom(P_2)]$  is not an equilibrium.

Note also that there exists a partition into parties with this outcome. To see this, choose parties with no overlap such that the median is the most extreme voter in one of the parties. Let  $h$  be the voter immediately to the right of the median and  $t$  be the voter immediately to the left of the median. If  $h$  defeats  $t$ , then have the median be in the party that contains  $t$  (and nominations be  $M$  and  $h$ ), and otherwise have the median be in the party that contains  $h$  (and nominations be  $M$  and  $t$ ). ■

**Proof of Proposition 8:** Without loss of generality, suppose that preference intensity increases leftwards (left directional parties). Since  $N \geq 5$ , there exists a partition of  $N$  into  $(P_1^*, P_2^*)$  such that  $\min P_1^* < \min P_2^* < M$  and no  $i \in N$  is such that  $\min P_1^* < i < \min P_2^*$ . Let  $m_1 = \min P_1^*$  and  $m_2 = \min P_2^*$ . By the algorithm in the proof of Proposition 6,  $(m_1, m_2)$  is an equilibrium of the nomination process and so  $((P_1^*, P_2^*), (m_1, m_2))$  may be an equilibrium with endogenous parties. We prove next that it actually is an equilibrium. First, take any voter  $x > m_2$ . If  $x$  switches party, then the algorithm predicts that  $(m_1, m_2)$  is still an equilibrium. Therefore,  $x$  cannot be strictly better off in all the equilibria of the game with partition  $(P_x \setminus \{x\}, P_{-x} \cup \{x\})$ . Secondly, if  $m_1$  changes party, then  $Nom(P_1^* \setminus \{m_1\}) > m_2$  because  $m_1$  and  $m_2$  are the leftmost candidates in each party. Since  $m_1 < m_2$ , this cannot benefit  $m_1$  by single-peakedness as it could only push the final winner to the right. Finally,  $W[m_1, m_2] = m_2$  and thus there is no equilibrium that could make  $m_2$  strictly better off after switching. ■

## 7.2 Nomination by Party Leaders and Strong Equilibria

A *strong equilibrium* in the case of nominations by a vote of party leaders is a pair of nominations  $Nom(P_1) \in P_1$  and  $Nom(P_2) \in P_2$  such that:

- (1) The pair is an equilibrium in the case of nominations by a voter of party leaders.
- (2) There does not exist any pair of nominees  $(i, j)$  where  $i \in P_1$  and  $j \in P_2$  such that  $W[i, j]$  is preferred to  $W[Nom(P_1), Nom(P_2)]$  by the leader of  $P_1$  and the leader of  $P_2$ .

The idea is that the party leaders cannot get a better outcome by agreeing to change strategies.

Returning to Example 2, there are seven voters,  $N = \{1, \dots, 7\}$ , and two parties that

partition  $N$  as follows:  $P_1 = \{2, 3, 6\}$  and  $P_2 = \{1, 4, 5, 7\}$ . The voters' ideal points are ordered by their labels. The party leaders are 6 and 7. Let preferences be such that  $W[i, 5] = i$  unless  $i = 6$  or  $i = 7$ .

The equilibria are  $(6, 7)$  and  $(3, 4)$ . However,  $(3, 4)$  is not a strong equilibrium because both party leaders prefer  $W[6, 7] = 6$  to  $W[3, 4] = 4$ .

**PROPOSITION 9** *If the pairs of nominees  $(i, j)$  and  $(i', j')$  are both strong equilibria in the case of nominations by a vote of party leaders, then  $W[i, j] = W[i', j']$ .*

**Proof of Proposition 9:** The possible locations of party leaders can be divided into two cases.

(1) Party leaders are on the same side of the median. Let  $D_\ell$  and  $D_{-\ell}$  respectively be the leaders of parties  $\ell$  and  $-\ell$ . Without loss of generality, assume that  $M \in P_\ell$ , and  $D_{-\ell} < D_\ell \leq M$ . We know that  $W[i, D_\ell] = D_\ell$  is an equilibrium outcome whenever  $i < D_\ell$ , and we will show that  $D_\ell$  is the only strong equilibrium outcome. Suppose that  $W^*$  is a strong equilibrium outcome different from  $D_\ell$ . Then  $W^* \in [D_\ell, M]$ , since whenever  $W^* < D_\ell$ ,  $D_\ell$  can improve the outcome by nominating himself, and whenever  $W^* > M$ ,  $D_\ell$  can improve the outcome by nominating  $M$ . So  $W^* \in [D_\ell, M]$ . But, then both  $D_{-\ell}$  and  $D_\ell$  would prefer that  $i < D_\ell$  and  $D_\ell$  are their respective parties' nominees. Thus, the outcome  $W^* \neq D_\ell$  is not supportable as a strong equilibrium, which is a contradiction.

(2) Party leaders are on opposite sides of the median. Without loss of generality, assume that  $M \in P_\ell$ , and  $D_{-\ell} < M < D_\ell$ . We will show that whenever  $D_{-\ell} < M < D_\ell$ , there is always exactly one equilibrium outcome, and hence only one strong equilibrium outcome. Recall, from the proof of Proposition 1, that  $\hat{x}$  is defined as the closest candidate to  $D_\ell$  in  $P_\ell \cap [M, D_\ell]$  such that  $W[\hat{x}, y] = \hat{x}$  for all  $y \in P_{-\ell}$ . First of all, we know that for any equilibrium outcome  $W^*$ ,  $W^* \in [\hat{x}, D_\ell]$ ; otherwise  $D_\ell$  could strictly improve the outcome. Trivially, if  $\hat{x} = D_\ell$ , then the only possible equilibrium outcome is  $W[D_\ell, \text{nom}(P_{-\ell})] = D_\ell$ . Now, let  $\hat{x} \neq D_\ell$ , and (as in the proof of Proposition 1), define  $x^* \equiv \min(P_\ell \cap (\hat{x}, D_\ell])$  and  $y^* \in P_{-\ell}$  as the closest point to  $D_{-\ell}$  in  $P_{-\ell}$  such that  $W[x^*, y^*] = y^*$ . Whenever  $y^* \in [D_{-\ell}, M]$ ,  $D_\ell$ 's best-response is to nominate  $\hat{x}$  which, by definition, defeats all of  $P_{-\ell}$ . So, in this case, the only equilibrium outcome is  $W^* = \hat{x}$ . Suppose instead that  $y^* \in [\hat{x}, D_\ell]$ . Then,  $W[x^*, y^*] = y^*$  is the only possible equilibrium outcome. ■