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SUPERMODULARIZABILITY

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Abstract

We study the ordinal content of assuming supermodularity, including conditions under which a binary relation can be represented by a supermodular function. When applied to revealed-preference relations, our results imply that supermodularity is some times not refutable: A consumer's choices can be rationalized with a supermodular utility function if they can be rationalized with a monotonic utility function. Hence, supermodularity is not empirically distinguishable from monotonicity. We present applications to assortative matching, decision under uncertainty, and to testing production technologies.

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1 Introduction

Supermodularity has proved a useful assumption in very different areas in economics. It has been useful because it has strong theoretical implications.[1] Here, we study the strength of supermodularity from a different perspective: When is a finite set of data incompatible with a supermodular structure? We find that supermodularity is typically quite weak, in the sense that it is difficult to refute with a finite set of data.

Supermodularity is a cardinal property of a function defined on a lattice. Quasisupermodularity, introduced by Milgrom and Shannon (1994), is an ordinal property implied by supermodularity. It is a powerful and essential theoretical property in the study of monotone comparative statics, and the economics literature recognizes this. Nevertheless, supermodularity has dominated quasisupermodularity as an assumption in economic applications. This is true even in the case where the function assumed to be supermodular has only ordinal significance. It is therefore surprising that there have been no studies clarifying the ordinal significance of supermodularity. Our work attempts to fill this gap. The results identify exactly the ordinal conditions assumed when supermodularity is assumed.

We proceed to outline our results and their implications for a number of economic environments.

First, consider a consumer with preferences over a finite set of consumption bundles in $\mathbb{R}^n$. We characterize the preferences that can be represented by a supermodular utility function, and we show that any antisymmetric and monotone preference relation can be represented by a supermodular utility function. Hence, if one observes a finite number of a consumer's choices (for example for different prices), supermodular representation is implied by monotonicity. Our result is reminiscent of Afriat's (1967) (see also Varian (1982)) finding that the concavity of utility functions has no testable implications beyond monotonicity. We find that supermodularity has no testable implications beyond monotonicity.

[1]See the books by Topkis (1998) and Vives (1999) and the recent survey by Vives (2005).

Secondly, consider matching in the marriage market (Becker, 1973) where, if woman w and man m marry, they generate surplus $f(m, w)$. In Becker's model of assortative matching, f is a supermodular function. Now, suppose one observes men and women's characteristics, and who matches to whom. We characterize those observations that are compatible with a supermodular surplus function. For the case when individual characteristics are one-dimensional, we prove that the matchings are compatible with supermodularity if and only if they are assortative.

Our third application is to decision-making under uncertainty. Supermodularity is used to model uncertainty aversion in the Choquet expected utility model (Schmeidler, 1989). We show that, in many situations, uncertainty aversion has no testable implications. In particular, uncertainty aversion is not refutable using data on choices over binary acts when each event is perceived as having a different likelihood. It is only refutable using data on more complicated acts, but such acts also entail attitudes towards risk.

Our fourth and final application considers data on a firm's factor demands (as in e.g. Afriat (1972) and Varian (1984)). We present conditions under which the data is compatible with, respectively, monotone and supermodular technologies.

We end with a note on the previous literature. The seminal papers on supermodularity and lattice programming in economics are Topkis (1978), Topkis (1979), Vives (1990) and Milgrom and Roberts (1990). Li-Calzi (1990) presents classes of functions f such that a continuous increasing transformation of f is supermodular. Most of our proofs apply an integer version of the Theorem of the Alternative (Aumann, 1964; Fishburn, 1970). Fostel, Scarf, and Todd (2004) and Chung-Piaw and Vohra (2003) present proofs of Afriat's theorem using similar ideas. For consumer-maximization problems, where constraints take the form of a budget set, supermodularity is not sufficient for monotone comparative statics results. Quah (2004) shows that a concave and supermodular utility implies a class of monotone comparative statics. It seems important, then, to study the testable implications of the joint assumptions of supermodularity and concavity. Neither our nor Afriat's results give an answer to this problem. The problem is that Afriat's separation argument is in the so-called "Afriat numbers," while ours is in the utility indexes themselves.

In Section 2 we present basic definition and preliminary results. In Sections 3 and 4 we present results on the representation of binary relations by supermodular and quasisupermodular functions. We present some implications for a model of decision-making under uncertainty in Section 6. Section 5 has our results on assortative matching. Finally, Section 7 contains results on the refutability of supermodularity in production.

2 Preliminary definitions and results.

Let X be a set. A **preorder** on X is a binary relation that is reflexive and transitive. A **partial order** on X is a preorder that is also antisymmetric. A **partially-ordered**

set is a pair $(X, \preceq)$ where X is a set and $\preceq$ is a partial order on X. A **lattice** is a partially-ordered set $(X, \preceq)$ such that for all $x, y \in X$, there exists a unique greatest lower bound $x \wedge y$ and a unique least upper bound $x \vee y$ according to $\preceq$.

For a lattice $(X, \preceq)$, we write $x \preceq_1 y$ if there exists $z \neq y$ for which $y = x \vee z$, and we write $x \preceq_2 y$ if there exists $z \neq x$ for which $x = y \wedge z$. We will write $x \parallel y$ if neither $x \preceq y$ or $y \preceq x$.

Given a finite lattice $(X, \preceq)$, let R be a preorder on X. Define the binary relation P_R by xP_Ry if xRy and not yRx. A **representation of** R is a function $u : X \to \mathbb{R}$ for which i) for all $x, y \in X$, if xRy, then $u(x) \geq u(y)$, and ii) for all $x, y \in X$, if xP_Ry, then $u(x) > u(y)$. We will be concerned with representations of R satisfying interesting properties.

Say a function $u : X \to \mathbb{R}$ is **supermodular** if for all $x, y \in X$, $u(x \vee y) + u(x \wedge y) \geq u(x) + u(y)$, and **strictly supermodular** if the inequalities are strict for all $x, y \in X$ for which $x \parallel y$. A function $u : X \to \mathbb{R}$ is **quasisupermodular** if for all $x, y \in X$, $u(x) \geq u(x \wedge y)$ implies $u(x \vee y) \geq u(y)$ and $u(x) > u(x \wedge y)$ implies $u(x \vee y) > u(y)$. Say a function $u : X \to \mathbb{R}$ is **monotonic** if for all $x, y \in X$, $x \preceq y$ implies $u(x) \leq u(y)$, and **strictly monotonic** if the inequality holds strictly for all $x, y \in X$ for which $x \preceq y$ and $x \neq y$.

For a binary relation R, an **R-cycle** is a set $\{x_1, ..., x_K\}$, where $K > 1$ for which for all $i = 1, ..., K-1$, x_iRx_{i+1} and x_KRx_1.[2]

The following proposition is simple, and its proof illustrates a method we use extensively in the paper.

Proposition 1 *Let $(X, \preceq)$ be a finite lattice. Then there exists a strictly supermodular $u : X \to \mathbb{R}$.*

Proof. We may view a function $u : X \to \mathbb{R}$ as an element of $\mathbb{R}^X$. The existence of a strictly supermodular u is equivalent to the existence of $u \in \mathbb{R}^X$ for which for all $x, y \in X$ such that $x \parallel y$, $(1_{x\wedge y} + 1_{x\vee y} - 1_x - 1_y) \cdot u > 0$. By the integer version of the Theorem of the Alternative (Aumann, 1964; Fishburn, 1970), such a u does not exist if and only if for each $\{x, y\}$ for which $x \parallel y$, there exists $n_{\{x,y\}} \in \mathbb{Z}_+$, at least one of which is strictly positive, for which

$$\sum_{\{\{x,y\}:x\parallel y\}} n_{\{x,y\}} (1_{x\vee y} + 1_{x\wedge y} - 1_x - 1_y) = 0. \tag{1}$$

We shall construct an infinite sequence $\{x_k\}$ of elements of X such that, for all x_k, there exists some y for which $n_{\{x_k,y\}} > 0$, $x_{k+1} \neq x_k$ and $x_{k+1} \succeq x_k$. Such a sequence is unbounded, which contradicts the finiteness of X together with the antisymmetry and transitivity of $\preceq$. There exists $n_{\{x,y\}} > 0$, so define $x_1 \equiv x$. Now, suppose that x_k has

[2] This definition is somewhat nonstandard, as it permits cycles of length two.

been defined. Then there exists some y for which, without loss of generality, $n_{\{x_k,y\}} > 0$. Define $x_{k+1} \equiv x_k \vee y$. Then $x_{k+1} \neq x_k$ and $x_{k+1} \succeq x_k$. Lastly, as (1) is satisfied and $n_{\{x_k,y\}} > 0$, there must exist some z for which $n_{\{z,x_{k+1}\}} > 0$. This completes the induction step. We have therefore derived a contradiction by constructing an unbounded sequence. Hence, a supermodular u must exist. ■

Our next result is a characterization of monotonic representation. The result is simple and rather basic to utility theory, but apparently it is new. We shall use it to establish Corollary 4 of Theorem 3.

Define the binary relation T by xTy if either $y \preceq x$ or xRy.

Proposition 2 *There exists a monotonic representation u of R if and only if for all $\{x_1, ..., x_K\} \subseteq X$, if $x_i T x_{i+1}$ for all $i = 1, ..., K-1$, then $x_K P_R x_1$ is false.*

Proof. The existence of such a function is equivalent to the existence of a vector $u \in \mathbb{R}^X$ satisfying the following three properties: *i*) for all $x, y \in X$ for which $x \succeq y$ and $x \neq y$, $(1_x - 1_y) \cdot u \geq 0$, *ii*) for all $x, y \in X$ for which xRy and $x \neq y$, $(1_x - 1_y) \cdot u \geq 0$, and *iii*) for all $x, y \in X$ for which xP_Ry, $(1_x - 1_y) \cdot u > 0$. Clearly, these inequalities are satisfied if and only if for all x, y for which xTy and $x \neq y$, $(1_x - 1_y) \cdot u \geq 0$, with the inequality strict in the case of xP_Ry. By the Theorem of the Alternative, such a u does not exist if and only if for all $x, y \in X$ for which xTy and $x \neq y$, there exists $n_{x,y} \in \mathbb{Z}_+$ so that

$$\sum_{\{(x,y):xTy\}} n_{x,y} (1_x - 1_y) = 0,$$

where there exists a pair xP_Ry for which $n_{x,y} > 0$. Modifying results of Diestel (2000) 1.9.7, or Berge (2001) (15.5), demonstrate that such equality can be true if and only if there exist a collection of T-cycles, say

$$\left\{x_1^1, ..., x_{K_1}^1\right\}, ..., \left\{x_1^N, ..., x_{K_N}^N\right\}$$

for which $n_{x,y}$ is the number of times xTy appears in one of the T-cycles. In other words, the equality can only hold if and only if there exists a set $\{x_1, ..., x_K\} \subseteq X$ for which for all $i = 1, ..., K-1$, $x_i T x_{i+1}$ and $x_K P_R x_1$. ■

3 Supermodular Representation.

Let $(X, \preceq)$ be a finite lattice and R be a preorder on X. We present results on when R has a supermodular representation. We first present a simple sufficient condition which implies that a monotonic representation must yield a supermodular representation. We then provide a necessary and sufficient condition for supermodular representation, and discuss the notion of quasi-supermodularity.

Our results have implications for the refutability of supermodularity on finite sets of data: With data on prices and consumption vectors $x \in X$, one can let R be the revealed preference relation (Afriat, 1967). Our results give conditions on when revealed preference can falsify a supermodular utility. By Corollary 4 below, supermodularity is not distinguishable from monotonicity using finite data on consumption.

Define the binary relation T_1 as xT_1y if either $x \succeq_1 y$ or xRy, and define T_2 by xT_2y if either $x \succeq_2 y$ or xRy.

Theorem 3 *Suppose that R is antisymmetric. Suppose that for all $\{x_1, ..., x_K\}$, if $x_iT_1x_{i+1}$ for all $i = 1, ..., K-1$, then $x_KP_Rx_1$ is false. Then there exists a supermodular representation u of R. The preceding statement also holds if T_1 is replaced by T_2.*

Proof. We will only show that the first statement is true; the second will follow by symmetric arguments. As in previous proofs, the existence of such a representation u is equivalent to the existence of a vector $u \in \mathbb{R}^X$ for which $i)$ for all $x, y \in X$ for which $x \parallel y$, $(1_{x\vee y} + 1_{x\wedge y} - 1_x - 1_y) \cdot u \geq 0$, and $ii)$ for all $x, y \in X$ for which xP_Ry, $(1_x - 1_y) \cdot u > 0$. If such a u does not exist, there exist a pair of collections, $n_{\{x,y\}}$ and $n_{x,y}$, each in $\mathbb{Z}_+$, such that

$$\sum_{\{\{x,y\}:x\parallel y\}} n_{\{x,y\}} (1_{x\vee y} + 1_{x\wedge y} - 1_x - 1_y) + \sum_{\{(x,y):xP_Ry\}} n_{x,y} (1_x - 1_y) = 0, \tag{2}$$

where some $n_{x,y} > 0$. We will construct a sequence of elements ranked according to T_1.

We will construct a sequence such that $y_{k+1}T_1y_k$, $y_{k+1} \neq y_k$, and there exists some z for which either $n_{\{z,y_k\}} > 0$ or $n_{(z,y_k)} > 0$. Since there exists some $n_{x,y} > 0$, let $y_1 = y$. Now suppose that y_k is defined. Then there exists z for which either $n_{\{z,y_k\}} > 0$ or $n_{(z,y_k)} > 0$. In the first case, $(z \vee y_k) T_1y_k$, so define $y_{k+1} = z \vee y_k$. In the second case, zP_Ry_k, so define $y_{k+1} = z$. Then $y_{k+1} \neq y_k$ and $y_{k+1}T_1y_k$. In either case, the fact that (2) is satisfied implies that there exists a w for which either $n_{\{w,y_{k+1}\}} > 0$ or $n_{(w,y_{k+1})} > 0$. This completes the construction of the sequence.

As X is finite, this demonstrates the existence of a T_1-cycle. Since $\preceq$ is asymmetric and transitive, we conclude that there must exist a set $\{x_1, ..., x_K\} \subseteq X$ for which for all $i = 1, ..., K-1$, $x_iT_1x_{i+1}$, and $x_KP_Rx_1$. ■

Corollary 4 *Let R be antisymmetric. If R has a monotonic representation, then it has a supermodular representation.*

Proof. Note that $\succeq_1$ is a coarser order than $\succeq$. So the inexistence of $\{x_1, ..., x_K\} \subseteq X$ with x_iTx_{i+1} for all $i = 1, ..., K-1$ and $x_KP_Rx_1$ implies the sufficient condition in Theorem 3. The result follows then from Proposition 2. ■

To see why we require R to be antisymmetric in Theorem 3, consider the following example.

Example 5 *Let $X = \{0,1\}^2$ with the usual ordering. Let R be representable by the function $u : X \to \mathbb{R}$ for which $u((0,0)) = 0$, and $u((0,1)) = u((1,0)) = u((1,1)) = 1$.*

Clearly, R cannot be represented by a supermodular function (any such function v would require $v((0,1)) = v((1,0)) = v((1,1)) > v((0,0))$, so that $v((0,0)) + v((1,1)) < v((1,0)) + v((0,1))$. Nevertheless, the relation P_R consists of $(1,1)\, P_R\, (0,0)$, $(0,1)\, P_R\, (0,0)$, and $(1,0)\, P_R\, (0,0)$. For all $x \neq (0,0)$, $(0,0)\, T_1 x$ is false, so that the hypotheses of Theorem 1 are satisfied (except for antisymmetry).

To see that the condition in Theorem 3 is not necessary (even under the assumption of antisymmetry of R), consider the following example.

Example 6 *Let $X = \{0,1\}^2$ with the usual ordering. Let R be representable by the function $u : X \to \mathbb{R}$ for which $u((0,0)) = 0$, $u((0,1)) = -1$, $u((1,0)) = 2$, and $u((1,1)) = 1.5$. Clearly, u is strictly supermodular. However, note that $(1,1)\, T_1\, (1,0)$, yet $(1,0)\, P_R\, (1,1)$. Moreover, $(0,0)\, T_2\, (1,0)$, yet $(1,0)\, P_R\, (0,0)$.*

In light of Example 6, one may ask for a necessary and sufficient condition for supermodular representation. We obtain one from an application of the Theorem of the Alternative:

Theorem 7 *Let $(X, \preceq)$ be a finite lattice. There exists a supermodular $u : X \to \mathbb{R}$ which represents R if and only if for all $N, K \in \mathbb{N}$, for all $\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N$, $\{z_l\}_{l=1}^K$, $\{w_l\}_{l=1}^K \subseteq X$ for which for all $l = 1, ..., K-1$, $z_l R w_l$ and for which*

$$\sum_{i=1}^{N} (1_{x_i \vee y_i} + 1_{x_i \wedge y_i}) + \sum_{l=1}^{K} 1_{z_l} = \sum_{i=1}^{N} (1_{x_i} + 1_{y_i}) + \sum_{l=1}^{K} 1_{w_l},$$

$w_K P_R z_K$ does not hold.

Proof. As in previous proofs, the existence of such a representation u is equivalent to the existence of a vector $u \in \mathbb{R}^X$ for which *i*) for all $x, y \in X$ for which $x \parallel y$, $(1_{x\vee y} + 1_{x\wedge y} - 1_x - 1_y) \cdot u \geq 0$, and *ii*) for all $x, y \in X$ for which $x P_R y$, $(1_x - 1_y) \cdot u > 0$. By the integer version of the Theorem of the Alternative, this statement is false if and only if for all $x, y \in X$ for which $x \parallel y$, there exists some $n_{\{x,y\}} \in \mathbb{Z}_+$ and for all $x, y \in X$ for which xRy, there exists some $n_{x,y} \in \mathbb{Z}_+$, and there exists at least one $n_{x,y} > 0$ (for which $x P_R y$) such that

$$\sum_{\{\{x,y\}:x\parallel y\}} n_{\{x,y\}} (1_{x\vee y} + 1_{x\wedge y} - 1_x - 1_y) + \sum_{\{(x,y):xRy\}} n_{x,y} (1_x - 1_y) = 0.$$

Separating terms, we obtain

$$\begin{aligned} & \sum_{\{\{x,y\}:x\parallel y\}} n_{\{x,y\}} (1_{x\vee y} + 1_{x\wedge y}) + \sum_{\{(x,y):xRy\}} n_{x,y} 1_x \\ = & \sum_{\{\{x,y\}:x\parallel y\}} n_{\{x,y\}} (1_x + 1_y) + \sum_{\{(x,y):xRy\}} n_{x,y} 1_y. \end{aligned}$$

It is easy to see that this is equivalent to the existence of $N, K \in \mathbb{N}$, $\{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N$, $\{z_l\}_{l=1}^K$, $\{w_l\}_{l=1}^K \subseteq X$ such that for all $l = 1, ..., K-1$, $z_l R w_l$ and for which

$$\sum_{i=1}^{N} (1_{x_i \vee y_i} + 1_{x_i \wedge y_i}) + \sum_{l=1}^{K} 1_{z_l} = \sum_{i=1}^{N} (1_{x_i} + 1_{y_i}) + \sum_{l=1}^{K} 1_{w_l},$$

and $z_K P_R w_K$. ■

4 Quasisupermodular Representation.

Milgrom and Shannon (1994) introduce the notion of quasisupermodularity as an ordinal generalization of supermodularity. They show that quasisupermodularity is necessary and sufficient for a class of monotone comparative statics. Here we state a simple characterization of when R can be represented by a quasisupermodular function. The purpose of the result is mainly as a comparison with Theorem 7.

Note that the condition in Theorem 7 implies that there are no R-cycles with P_R-elements, a well-known necessary and sufficient condition for representability (see e.g. Richter (1966)). The analogous condition for quasisupermodularity will have no such implication; so we assume outright that R is complete and transitive.

Theorem 8 *Let $(X, \preceq)$ be a finite lattice, and that R is complete and transitive. Then there exists a quasisupermodular $u : X \to \mathbb{R}$ which represents R if and only if for all x, y, $\{z_l\}_{l=1}^2$, $\{w_l\}_{l=1}^2 \subseteq X$ for which $z_1 R w_1$, $z_l R w_l$ and for which*

$$(1_{x\vee y} + 1_{x\wedge y}) + \sum_{l=1}^{2} 1_{z_l} = (1_x + 1_y) + \sum_{l=1}^{2} 1_{w_l},$$

$w_2 P_R z_2$ does not hold.

Proof. R is representable by a quasisupermodular function u if and only if there exists a function $u : X \to \mathbb{R}$ that represents R. This is equivalent to the statement that $xR(x\wedge y)$ implies $(x\vee y)Ry$ and $xP_R(x\wedge y)$ implies $(x\vee y)P_R y$. This is equivalent to the condition displayed in the statement of the Proposition. ■

Milgrom and Shannon (1994) present an example of a quasisupermodular function for which no monotonic transformation yields a supermodular function. We show how their example fails the condition in Theorem 7.

Example 9 *Let $X = \{1,2\} \times \{1,2,3,4\}$ and $f : X \to \mathbb{R}$ be $f(1,1) = f(1,4) = 1$, $f(1,2) = f(1,3) = 2$, $f(2,1) = f(2,4) = 3$, $f(2,2) = 4$, and $f(2,3) = 5$.*

Let $x_1 = (2,1)$, $y_1 = (1,2)$, and $x_2 = (2,3)$, $y_2 = (1,4)$. Then the vector

$$\sum_{i=1}^{2} (1_{x_i\vee y_i} + 1_{x_i\wedge y_i}) - \sum_{i=1}^{2} (1_{x_i} + 1_{y_i})$$

is

	1	2	3	4
2	−1	1	−1	1
1	1	−1	1	−1

(3)

Let R be the order on X induced by f. So R is the order that a supermodular transformation of f should represent.

Let $w_1 = x_2 \vee y_2 = (2,4)$ and $z_1 = x_1$; so $z_1 R w_1$. Let $w_2 = x_1 \wedge y_1 = (1,1)$ and $z_2 = y_2$; so $z_2 R w_2$. Let $w_3 = x_2 \wedge y_2 = (1,3)$ and $z_3 = y_1$; so $z_3 R w_3$. Let $w_4 = x_1 \vee y_1 = (2,2)$ and $z_4 = x_2$; so $z_4 P w_4$. It is clear that (3) equals

$$\sum_{k=1}^{4} \left(1_{z_k} - 1_{w_k}\right).$$

The theorem thus implies that there is no supermodular function that represents R.

5 Assortative matching.

We now consider a model of marriage where supermodularity predicts which types of matches one will observe. The model was formulated by Becker (1973); it is based on Shapley and Shubik's (1972) assignment game.[3] Let $M = \{m_1, \ldots m_n\}$ be a set of men, and $W = \{w_1, \ldots w_n\}$ a set of women. Suppose that the marriage of a man and a woman generates a surplus that only depends on certain measurable characteristics of the pair. Becker assumes that the surplus is a supermodular function of the couples' characteristics.

Here, we study a finite set of data on who matches with whom, and on their individual characteristics. First, following Becker's (1973) first model, we consider general multidimensional characteristics. We present a condition that guarantees the data are compatible with a supermodular surplus. Second, we follow Becker in specializing to one-dimensional characteristics and present a necessary and sufficient condition for the data to be compatible with supermodular surplus. The condition is that all matchings in the data must be "assortative," i.e. high-value women must match to high-value men.

A **matching** is a subset μ of $M \times W$ such that for each m there is exactly one w such that $(m, w) \in \mu$, and for two distinct m and m' there is no w with $(m, w) \in \mu$ and $(m', w) \in \mu$. So a matching is a bijection from M onto W. Note that we ignore the possibility that an agent remains unmatched.

Let K be a finite set of observations, indexed by k. An observation consists of how men and women match, and their characteristics. Concretely, consider collections

(a) of vectors $(\theta^k_m)_{m \in M}$ and $(\pi^k_w)_{w \in W}$, where $\theta^k_m \in \mathbb{R}^{d_M}$ and $\pi^k_w \in \mathbb{R}^{d_W}$,

(b) and for all $k = 1, ..., K$, matchings μ^k.

Here d_W and $d_M$is the number of dimensions on, respectively, men and women's characteristics. We assume that the vectors θ^k_m are different, for different m and k; similarly for the vectors π^k_w. Formally, $(m, k) \neq (m', k')$ implies $\theta^k_m \neq \theta^{k'}_{m'}$ and $(w, k) \neq (w', k')$ implies $\pi^k_w \neq \pi^{k'}_{w'}$.

[3]Assortative matching is referred to by Becker as *assortive mating.*

Let $X = \left\{(\theta^k_m, \pi^k_w) : k = 1, ..., K \text{ and } (m,w) \in M \times W\right\}$. We assume that X is a lattice under the usual order.

A function $f : X \to \mathbb{R}_+$ assigns a value to each possible matching, for each observed vector of characteristics. The **value** of a matching μ in observation k is given by

$$V^k_f(\mu) = \sum_{(m,w)\in\mu} f(\theta^k_m, \pi^k_w).$$

Say that f **rationalizes** the observations in (a) and (b) if, for all k, $\mu^k \neq \mu'$ implies that $V^k_f(\mu^k) > V^k_f(\mu')$.

Define the binary relation R on X by xRy if and only if there is k, m, m', w and w' such that

- $x = (\theta^k_m, \pi^k_w)$, $y = (\theta^k_{m'}, \pi^k_{w'})$,
- $(m,w) \in \mu^k$ and $(m', w') \notin \mu^k$.

The following result follows from Theorem 3.

Proposition 10 *Suppose that for all $\{x_1, ..., x_K\}$, if $x_i T_1 x_{i+1}$ for all $i = 1, ..., K-1$, then $x_K P_R x_1$ is false. Then there is a supermodular function that rationalizes the observations in (a) and (b).*

Becker (1973) introduces the model above, for general d_M and d_W but presents no results. Our condition in Proposition 10 is, to the best of our knowledge, the first positive implication of Becker's general model. Now let $d_M = d_W = 1$. Say that a matching μ is **assortative** if, for any (m,w) and (m',w') in μ, $\theta_m < \theta_{m'}$ implies that $\pi_w < \pi_{w'}$.

Proposition 11 *Suppose that $d_M = d_W = 1$. Then there is a supermodular function that rationalizes the observations in (a) and (b) if and only if the matchings μ^k are assortative.*

Proof. (Only if) was shown by Becker (1973).[4] We shall prove (if).

The existence of f rationalizing the observations is equivalent to the existence of a vector $u \in \mathbb{R}^X$ such that $x \parallel y$ implies that $(1_{x\vee y} + 1_{x\wedge y} - 1_x - 1_y)\cdot u \geq 0$, and such that, for each μ^k and $\mu \neq \mu^k$

$$\left(1_{(\theta^k_m,\pi^k_w):(m,w)\in\mu^k} - 1_{(\theta^k_m,\pi^k_w):(m,w)\in\mu}\right)\cdot u > 0.$$

Note that the last statement is equivalent to

$$\left(1_{(\theta^k_m,\pi^k_w):(m,w)\in\mu^k\setminus\mu} - 1_{(\theta^k_m,\pi^k_w):(m,w)\in\mu\setminus\mu^k}\right)\cdot u > 0.$$

[4]Becker attributes the proof to William Brock.

Suppose there is k such that μ^k is not assortative. Let (m,w) and (m',w') in μ^k be such that $\theta^k_m < \theta^k_{m'}$ but $\pi^k_w > \pi^k_{w'}$. Let μ be the matching obtained from μ^k by adding (m,w') and (m',w), and subtracting (m,w) and (m,w).

Then $(\theta^k_{m'},\pi^k_w) = (\theta^k_m,\pi^k_w) \vee (\theta^k_{m'},\pi^k_{w'})$ and $(\theta^k_m,\pi^k_{w'}) = (\theta^k_m,\pi^k_w) \wedge (\theta^k_{m'},\pi^k_{w'})$. So,

$$\begin{aligned} & 1_{(\theta^k_m,\pi^k_w):(m,w)\in\mu^k\setminus\mu} && -1_{(\theta^k_m,\pi^k_w):(m,w)\in\mu\setminus\mu^k} \\ - & \left(1_{(\theta^k_m,\pi^k_w)} + 1_{(\theta^k_{m'},\pi^k_{w'})}\right) && + \left(1_{(\theta^k_m,\pi^k_w)\vee(\theta^k_{m'},\pi^k_{w'})} + 1_{(\theta^k_m,\pi^k_w)\wedge(\theta^k_{m'},\pi^k_{w'})}\right) \\ & && = 0 \end{aligned}$$

By the Theorem of the Alternative, there can not be a rationalizing f. ■

6 Uncertainty aversion and the Choquet expected utility model.

We now turn to a model of decision under uncertainty where supermodularity models uncertainty aversion.

For a finite measurable space, $\left(\Omega, 2^\Omega\right)$, a **capacity** is a function $\nu : 2^\Omega \to \mathbb{R}$ for which $\nu(\varnothing) = 0$, $\nu(\Omega) = 1$, and $A \subseteq B$ implies $\nu(A) \leq \nu(B)$. A capacity is supermodular if it is supermodular when 2^Ω is endowed with the set-inclusion order.

For some outcome space X, the set of **acts** is the set of functions $f : \Omega \to X$. Denote the set of acts by $\mathcal{F}$. A binary relation R over $\mathcal{F}$ conforms to the **Choquet expected utility model** if there exists some $u : X \to \mathbb{R}$ and a capacity ν on Ω for which the function $U : \mathcal{F} \to \mathbb{R}$ represents R, where

$$U(f) \equiv \int_\Omega u(f(\omega))\, d\nu(\omega);[5] \tag{3}$$

Schmeidler (1989) introduces this model and axiomatizes those R conforming to it in an Anscombe and Aumann (1963) framework (such an environment allows him to obtain uniqueness of the representing ν). Let us suppose therefore that $X = \Delta(Y)$ for some finite set Y, and that $u : \Delta(Y) \to \mathbb{R}$ is expected utility. A binary relation R

[5] The Choquet integral with respect to ν is defined as:

$$\begin{aligned} & \int_\Omega g(\omega)\, d\nu(\omega) \\ = & \int_0^{+\infty} \nu(\{\omega : g(\omega) > t\})\, dt + \int_{-\infty}^0 [\nu(\{\omega : g(\omega) > t\}) - 1]\, dt \end{aligned}$$

which conforms to the Choquet expected utility model exhibits **Schmeidler uncertainty aversion** if and only if ν is supermodular.[6]

For a given binary relation R over $\mathcal{F}$ conforming to the Choquet expected utility model, define the likelihood relation R^* over 2^Ω by ER^*F if there exist $x, y \in X$ for which xP_Ry[7] and

$$\left[\begin{array}{l} x \text{ if } \omega \in E \\ y \text{ if } \omega \notin E \end{array}\right] R \left[\begin{array}{l} x \text{ if } \omega \in F \\ y \text{ if } \omega \notin F \end{array}\right].$$

Note that for the Choquet expected utility model, this relation is complete. R^* reflects a "willingness to bet" relation.

Corollary 12 *Suppose that R conforms to the Choquet expected utility model. Then if the likelihood relation R^* is antisymmetric, it is consistent with Schmeidler uncertainty aversion.*

This result implies that in many situations, there are no testable implications of Schmeidler uncertainty aversion from a purely ordinal standpoint. Preferences over binary acts are not enough to refute the hypothesis of Schmeidler uncertainty aversion. Preferences over more complicated acts must be observed, but preferences over more complicated acts also involve attitudes toward risk. Thus, the Schmeidler definition of uncertainty aversion requires observing preferences over risky acts. Of course, this is already clear from his definition. Other theories of uncertainty aversion are due to Epstein (1999) and Ghirardato and Marinacci (2002). Epstein discusses the issue of observability of uncertainty aversion from the likelihood relation at length. We believe our result illustrates just how difficult Schmeidler uncertainty aversion is to refute. The theories of Epstein and Ghirardato and Marinacci are more general and are based on comparative notions of uncertainty aversion, an idea due to Epstein. The differences in the two theories are as to what they take to be the benchmark of "uncertainty neutral." Both theories have implications for the Choquet expected utility model. Epstein uncertainty aversion turns out to be characterized by the likelihood relation when adapted to this framework, whereas Ghirardato-Marinacci uncertainty aversion is not.

7 Supermodular production technology.

We now study when a supermodular technology can rationalize data on factor demands. This exercise follows Afriat (1972), who studied the refutability of concave and monotone production functions. Afriat assumes that both factor demands and production output are observable. When output is observable, one can directly test if the output is supermodular in the factors used. We assume instead that only factors and their prices are observed, and we ask when these can be rationalized using a supermodular technology. Varian (1984) makes the same assumption when testing for cost minimization.

[6]Schmeidler's definition in terms of R states that for any two acts $f, g \in \mathcal{F}$, and $\alpha \in [0,1]$, if fRg, then $\alpha f + (1-\alpha)\, gRg$.

[7]Here we are abusing notation by identifying a constant act with the value that constant act takes.

The data consists of pairs (w, z), where w is a vector of factor prices and z is a vector of factor demands at prices w. Concretely, assume a collection of K observations (w^k, z^k), $k = 1, \ldots K$, such that, for every k, $w_k \in \mathbb{Q}^n$ and $z_k \in \mathbb{Q}^n$. We require the observations to be rational because of our use of the integral version of the Theorem of the Alternative. When convenient, we write w_k as w_{z_k}. Let $X = \{z_k : k = 1, \ldots K\}$ be a finite lattice under the usual order on $\mathbb{R}^n$.

We denote the revealed-preference binary relation on X by P, so $z_k P z$ for all $z \neq z_k$, and all k.

Say that a function $f : X \to \mathbb{R}_+$ **rationalizes** the data $(w_k, z_k)_{k=1}^K$ if, for each k,

$$f(z_k) - w_k \cdot z_k \geq f(z) - w_k \cdot z,$$

for all $z \in X$. Interpret $f(z)$ as the revenue the firm receives when it uses factors z; if the firm is competitive, $f(z)$ is proportional to its production function.

We first show that monotonicity has no testable implications using data on factor demands. This result does not concern supermodularity, but it is apparently new and the condition for representation (statement 3 in Proposition 13) can be compared to the condition for supermodular representation. The result is a formalization of the notion that a profit-maximizing firm will not operate where its revenue function is decreasing.

Define the binary relation R on X by zRz' if either $z \preceq_2 z'$ or zPz'.

Proposition 13 *The following statements are equivalent.*

1. *There is a rationalizing function.*
2. *There is a monotone increasing rationalizing function.*
3. *For any P-cycle $\{z_1, ..., z_N\}$, we have*

$$\sum_{n=1}^{N} w_{z_n}(z_n - z_{n+1}) \leq 0$$

(read $N + 1$ as 1).

Proof. The existence of a rationalizing function is equivalent to the existence of a vector $u \in \mathbb{R}_+^X$ that satisfies the following $K \times |X|$ conditions: for each k and $z \in X$,

$$(1_{z_k} - 1_z) \geq w_k \cdot (z_k - z) \tag{4}$$

The equivalence of 1 and 3 follows directly from the integer version of the theorem of the alternative, by a similar argument to the proof of Theorem 3. Trivially, 2 implies 1.

We now prove that 3 implies 2, which finishes the proof of the proposition. The existence of a monotone increasing rationalizing function is equivalent to the existence of a vector $u \in \mathbb{R}_{+}^{X}$ satisfying the conditions in (4) and, in addition, that for each $z, z' \in X$ with $z > z'$, $(1_z - 1_{z'}) \geq 0$. Suppose there is no monotone rationalizing function. By the integer version of the theorem of the alternative, there are collections of non-negative integers, $(\eta_{k,z})$ and $(\eta_{z,z'})$ such that

$$\sum_{k,z} \eta_{k,z} (1_{z_k} - 1_z) + \sum_{\{(z,z'):z>z'\}} \eta_{z,z'} (1_z - 1_{z'}) = 0 \tag{5}$$

and

$$\sum_{k,z} \eta_{k,z} w_k \cdot (z_k - z) > 0.$$

Define the collection $(\eta'_{k,z})$ by

$$\eta'_{k,z} = \eta_{k,z} + \sum_{\{z':z_k>z'\}} \eta_{z_k,z'}.$$

Then $\sum_{k,z} \eta'_{k,z} (1_{z_k} - 1_z) = 0$ and, since $z_k > z$ implies that $w_{z_k}(z_k - z) \geq 0$, we have that

$$\sum_{k,z} \eta'_{k,z} w_k \cdot (z_k - z) = \sum_{k,z} \eta_{k,z} w_k \cdot (z_k - z) + \sum_{\{(k,z):z_k>z\}} \eta_{z_k,z} w_{z_k} \cdot (z_k - z) > 0.$$

There is then a P-cycle violating the condition in 3. ■

Proposition 14 *There is a supermodular rationalizing function if, for any R-cycle* $\{z_1, ..., z_N\}$, *we have*

$$\sum_{\{n:z_n P z_{n+1}\}} w_{z_n}(z_n - z_{n+1}) \leq 0$$

(read $N+1$ *as* 1*).*

Proof. The existence of a supermodular rationalizing function is equivalent to the existence of a vector $u \in \mathbb{R}_{+}^{X}$ satisfying the conditions in (4) in the proof of Proposition 13, and, in addition, that for each $z, z' \in X$ with $z \parallel z'$ $(1_{z \vee z'} + 1_{z \wedge z'} - 1_z - 1_{z'}) \geq 0$.

Now suppose there is no supermodular rationalizing function. By the integer version of the theorem of the alternative, there are collections of non-negative integers, $(\eta_{k,z})$ and $(\eta_{z,z'})$ such that

$$\sum_{k,z} \eta_{k,z} (1_{z_k} - 1_z) + \sum_{z \parallel z'} \eta_{z,z'} (1_{z \vee z'} + 1_{z \wedge z'} - 1_z - 1_{z'}) = 0 \tag{6}$$

and

$$\sum_{k,z} \eta_{k,z} w_k \cdot (z_k - z) > 0.$$

Define the collections $(\eta'_{k,z})$ and $(\eta'_{z,z'})$ by

$$\eta'_{k,z} = \eta_{k,z} + \sum_{\{\{z,z'\}:z_k = z\vee z'\}} \eta_{z,z'}$$

and by $\eta'_{z,z'} = \sum_{\{\{z',z''\}:z=z'\wedge z''\}} \eta_{z',z''}$.

We decompose each summand in the sum on the right of (5) by

$$\eta_{z,z'}\left(1_{z\vee z'} + 1_{z\wedge z'} - 1_z - 1_{z'}\right) = \eta_{z,z'}\left(1_{z_k} - 1_z\right) + \eta_{z,z'}\left(1_{z\wedge z'} - 1_{z'}\right),$$

where k is such that $z \vee z' = z_k$. Hence, (5) and the definition of $(\eta'_{k,z})$ and $(\eta'_{z,z'})$ imply that

$$\sum_{k,z} \eta_{k,z}\left(1_{z_k} - 1_z\right) + \sum_{\{\{z,z'\}:\exists z'' z=z''\wedge z'\}\eta_{z,z'}} \left(1_{z\vee z'} + 1_{z\wedge z'} - 1_z - 1_{z'}\right) = 0$$

Note that, for each k and z such that $z_k = z \vee z'$ for some z', we have that $w_k(z_k - z) \geq 0$. So $\sum_{k,z} \eta'_{k,z} w_k(z_k - z) > 0$, as we have only added non-negative terms. ■

References

Afriat, S. (1972): "Efficiency Estimation of Production Functions," *International Economic Review*, 13(3), 568–598.

Afriat, S. N. (1967): "The Construction of Utility Functions from Expenditure Data," *International Economic Review*, 8(1), 67–77.

Anscombe, F., and R. Aumann (1963): "A Definition of Subjective Probability," *Annals of Mathematical Statistics*, 34, 199–205.

Aumann, R. J. (1964): "Subjective Programming," in *Human Judgments and Optimality*, ed. by M. Shelley, and G. L. Bryan. Wiley. New York.

Becker, G. S. (1973): "A Theory of Marriage: Part I," *Journal of Political Economy*, 81(4), 813–846.

Berge, C. (2001): *Theory of Graphs*. Dover.

Chung-Piaw, T., and R. Vohra (2003): "Afriat's Theorem and Negative Cycles," Mimeo, Northwestern University.

Diestel, R. (2000): *Graph Theory*. Springer.

Epstein, L. G. (1999): "A Definition of Uncertainty Aversion," *Review of Economic Studies*, 66(3), 579–608.

Fishburn, P. C. (1970): *Utility Theory for Decision Making*. Wiley, New York.

FOSTEL, A., H. SCARF, AND M. TODD (2004): "Two New Proofs of Afriat's Theorem," *Economic Theory*, 24(1), 211–219.

GHIRARDATO, P., AND M. MARINACCI (2002): "Ambiguity Made Precise: A Comparative Foundation," *Journal of Economic Theory*, 102(2), 251–289.

LI-CALZI, M. (1990): "Generalized Symmetric Supermodular Functions," Mimeo, Stanford University.

MILGROM, P., AND J. ROBERTS (1990): "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58(6), 1255–1277.

MILGROM, P., AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62(1), 157–180.

QUAH, J. K.-H. (2004): "The Comparative Statics of Constrained Optimization Problems," Mimeo, Oxford University.

RICHTER, M. K. (1966): "Revealed Preference Theory," *Econometrica*, 34(3), 635–645.

SCHMEIDLER, D. (1989): "Subjective probability and expected utility without additivity," *Econometrica*, 57(3), 571–587.

SHAPLEY, L. S., AND M. SHUBIK (1972): "The Assignment Game I: The Core," *International Journal of Game Theory*, 1(1), 111–130.

TOPKIS, D. M. (1978): "Minimizing a Submodular Function on a Lattice," *Operations Research*, 26(2), 305–321.

TOPKIS, D. M. (1979): "Equilibrium Points in Nonzero-Sum n-Person Submodular Games," *SIAM Journal of Control and Optimization*, 17(6), 773–787.

——— (1998): *Supermodularity and Complementarity*. Princeton University Press, Princeton, New Jersey.

VARIAN, H. R. (1982): "The Nonparametric Approach to Demand Analysis," *Econometrica*, 50(4), 945–974.

——— (1984): "The Nonparametric Approach to Production Analysis," *Econometrica*, 52(3), 579–598.

VIVES, X. (1990): "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics*, 19(3), 305–321.

——— (1999): *Oligopoly Pricing*. MIT Press, Cambridge, Massachusetts.

——— (2005): "Complementarities and Games: New Developments," *Journal of Economic Literature*, 43(2), 437–479.