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TESTING MODELS WITH MULTIPLE EQUILIBRIA BY QUANTILE METHODS

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Abstract

This paper proposes a method for testing complementarities between explanatory and dependent variables in a large class of economic models. The proposed test is based on the monotone comparative statics (MCS) property of equilibria. Our main result is that MCS produces testable implications on the (small and large) quantiles of the dependent variable, despite the presence of multiple equilibria. The key features of our approach are: (1) we work with a nonparametric structural model of a continuous dependent variable in which the unobservable is allowed to be correlated with the explanatory variable in a reasonably general way; (2) we do not require the structural function to be known or estimable; (3) we remain fairly agnostic on how an equilibrium is selected. We illustrate the usefulness of our result for policy evaluation within Berry, Levinsohn, and Pakes's (AER, 1999) model.

JEL classification numbers: C1,C5

Key words: Econometrics of Games, Monotone Comparative Statics, Quantile Regression, Multiple Equilibria

Testing Models with Multiple Equilibria by Quantile Methods

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1 Introduction

In many conventional economic models, equilibrium uniqueness comes at a cost of strong and often untenable assumptions. Consider, for example, general equilibrium models: the uniqueness conditions with some natural economic meaning imply the strong weak axiom, which in turn cannot be expected to hold beyond single-agent economies (Arrow and Hahn, 1971). Therefore it is not surprising to find equilibrium multiplicity present in a variety of contexts, ranging from general equilibrium models in microeconomics, oligopoly models and network externalities in industrial organization, to non-convex growth models in macroeconomics or models of statistical discrimination in labor economics.

Performing comparative statics with multiple equilibria is a challenge. How changes in explanatory variables affect dependent variables depends on the way a particular equilibrium is selected. Unfortunately, the theoretical literature offers little guidance on equilibrium selection.[1] As a consequence, policy analysis seems impossible as policy effects may well vary across different equilibria. More to the point, without equilibrium selection, it is hard to identify the structure underlying economic models when multiple equilibria are present. And with no knowledge of the structure, we can say little about general comparative statics effects. We should emphasize that we are concerned with testing for the existence of a comparative statics effect; the counterfactual prediction of the effects of policies remains virtually impossible without substantial information about equilibrium selection.

In this paper, we restrict our attention to economic models that exhibit *complementarities* between explanatory and dependent variables. In such models, despite the possible presence of multiple equilibria, a monotone comparative statics (MCS) prediction holds: there is a smallest and a largest equilibrium, and these change monotonically with explanatory variables (Milgrom and Roberts, 1994; Villas-Boas, 1997). The paper's

[1] Consider Kreps (1990), for example: "*There are ... lots of Nash equilibria to this game. Which one is the 'solution'? I have no idea and, more to the point, game theory isn't any help. Some (important) sorts of games have many equilibria, and the theory is of no help in sorting out whether any one is the 'solution' and, if one is, which one is.*"

main contribution is to show how MCS arguments translate into observable restrictions on the conditional quantiles of the dependent variable.

Our framework is as follows: similar to Jovanovic (1989), we start with an underlying economic model relating dependent and explanatory variables. We disturb the model by adding an unobservable disturbance term that captures individual heterogeneity, or other unaccounted random features. The assumptions we impose on the resulting structure are fairly weak: we allow for unknown structural function, unknown equilibrium selection, and reasonably general correlation between the disturbance and the explanatory variable. Our main result is that MCS produces testable implications on the (small and large) quantiles of the dependent variable.[2] The result does not assume, nor require estimating, an equilibrium selection procedure.[3]

The intuition behind is fairly simple. Consider a model in which there are complementarities between explanatory and dependent variables. When the generated equilibrium is unique, then the model can be globally implicitly solved and the resulting reduced form is such that the dependent variable increases in the explanatory variable. This property translates into first order stochastic dominance among distributions: *all* conditional quantiles of the dependent variable are increasing functions of the explanatory variable. When the model generates multiple equilibria, the above implicit function arguments fail to hold globally. It remains, however, the MCS property of the extremal equilibria. By focusing on regions in which the monotonicity of equilibria holds, we still obtain that *tail* (small and large) conditional quantiles of the endogenous variable increase in the explanatory variable. Testing for complementarities is thus possible by examining the behavior of extreme conditional tails of the dependent variable.

Our method applies to a large class of economic models with continuous dependent variables. These are: models of individual decision making in which the equilibrium

[2]An earlier test for MCS can be found in Athey and Stern (1998) in the context of firms' choice of organizational form. This prior work, however, does not address equilibrium problems.

[3]Understandably, estimating the structural parameters requires additional parametric assumptions on the equilibrium selection. Examples are discrete-choice models that estimate agents' payoff functions (Bjorn and Vuong, 1985; Bajari, Hong, and Ryan, 2004; Sweeting, 2005).

values are the solutions of an extremum problem, and one-dimensional equilibrium models where equilibria are fixed points. Since the dependent variable is continuous, our findings complement those developed by the growing literature on discrete games with multiple equilibria (Bresnahan and Reiss, 1990, 1991; Berry, 1992; Tamer, 2003; Ciliberto and Tamer, 2004; Aguirregabiria and Mira, 2007). Unlike in these papers, however, our methods can only be applied to discuss comparative statics effects, and are silent about other structural features of the model.

The next section discusses equilibrium multiplicity in Berry, Levinsohn, and Pakes's (1999) influential empirical model of price-setting with differentiated products. In Section 3 we introduce a class of structural models, and present our results. We conclude in Section 4 with a discussion and possible extensions of our approach. The supplementary material contains three appendices: Appendix A gives proofs of additional results stated in the text, while Appendix B provides details on the BLP application; finally, Appendix C illustrates the interaction of equilibrium multiplicity and identification in structural models that satisfy conditional moment restrictions.

2 Example

We now present a simplified version of Berry, Levinsohn, and Pakes's (1995) model of price competition with differentiated products. We use this model for two purposes: first, to illustrate the challenges posed by equilibrium multiplicity, even in popular and well-behaved economic models. Second, to argue that our methods provide useful tools for policy analysis in these models. Concretely, we discuss the analysis of the Japanese "Voluntary Export Restraint" (VER) policy for automobile exports published in Berry, Levinsohn, and Pakes (1999) (BLP hereafter).

In our version of the BLP model there are two firms, each producing one good. Firm 1 is foreign and Firm 2 is a home firm. Following BLP, we model the VERs as increases in firms' marginal costs. Firm i sets the price p_i of its product and obtains profits $(p_i - c_i - \lambda \text{VER}_i)D_i(p_i, p_{-i})$, where $D_i(p_i, p_{-i})$ is a demand for Firm i's good, c_i is i's marginal cost, VER_i is a dummy variable for the VER, and λ is the corresponding tax per unit of i's production. The firms' profit functions determine their best-response (reaction) functions. Given the marginal cost c_i and the demand function $D_i(p_i, p_{-i})$, let β_i denote Firm i's best-response function; so $p_i^* = \beta_i(p_{-i}, \text{VER}_i)$ is Firm i's optimal choice of p_i when its competitor sets a price p_{-i}. Only the foreign firm is potentially subject to the VER. Then, the equilibrium choice of the home firm's price p is determined by the fixed-point condition:

$$\beta_2\left(\beta_1(p, \text{VER}), 0\right) - p = 0. \tag{1}$$

Under standard continuity and compactness assumptions, there is at least one equilibrium. It is however difficult to guarantee that this equilibrium is unique. Not only are the known conditions for uniqueness very strong (Gabay and Moulin, 1980; Caplin and Nalebuff, 1991) there is a sense in which games generally tend to have large numbers of equilibria. In a model of randomly generated games, McLennan (2005) shows that the mean number of equilibria grows exponentially with the number of strategies. Games of strategic complements, which are especially relevant for our paper, tend to have particularly large numbers of equilibria (Takahashi, 2005). As pointed out by Berry, Levinsohn, and Pakes (1995, 1999), the BLP model in particular is not guaranteed to have a unique equilibrium. We now illustrate their point through a concrete example.

2.1 BLP Model with Multiple Equilibria

We shall follow BLP in imposing additional structure on D_i and c_i. Demand arises from a random utility specification: $u_{hi} = -\alpha p_i + \xi_i + \varsigma_h$, in which u_{hi} is the utility of product i $(i = 1, 2)$ to individual h $(h = 1, ..., I)$, p_i is the price of product i and ξ_i is its unobserved characteristic; α is the taste parameter on price, assumed constant across individuals, and ς_h is a stochastic term that represents the deviations from an average behavior of agents. The distribution of ς_h is induced by the unobserved characteristics of individual h and their interactions with products' characteristics; we assume that $E(\varsigma_h) = 0$.

Table 1: BLP Equilibrium Prices (p_1, p_2) in a Logit Model with Conditional Heteroskedasticity

$\text{VER}_1 = 0$	$\text{VER}_1 = 1$
$(2.0339, 2.0339)$	$(2.0751, 2.0704)$
$(2.7336, 2.7336)$	$(2.6205, 2.6154)$
$(3.1238, 3.1238)$	$(3.2087, 3.2039)$

NOTE: Model parameters are: $\alpha_0 = 0.3859$, $\gamma_0 = 0$, $\lambda_0 = 0.01$, $\rho_0 = 0.1667$ and $\tau_0 = 10$, with $g(x) = \rho_0 + \exp(-\tau_0/x)$.

$\text{VER}_1 = 0$	$\text{VER}_1 = 1$
$(2.0339, 2.0339)$	$(2.0751, 2.0704)$
$(2.2796, 2.2796)$	$(2.2399, 2.2347)$
$(6.3417, 6.3417)$	$(6.3529, 6.3490)$

NOTE: Model parameters are: $\alpha_1 = 0.2464$, $\gamma_1 = 0.0776$, $\lambda_1 = 0.0099$, $\rho_1 = 0.1074$, and $\tau_1 = 12.9913$ with $g(x) = \rho_1 + \exp(-\tau_1/x)$.

Marginal costs are held constant across firms: $\ln c_i = \gamma$. To fully specify profits requires additional assumptions on ς_h in the random utility model. We assume that the individual deviations ς_h are heteroskedastic in a way that depends on prices: $\varsigma_h = g(p_i p_{-i})\epsilon_h$, where g is a positive real function that is increasing, and ϵ_h are standardized iid random variables. Setting g to be constant yields the baseline specification. For nonconstant cases, we restrict g to be twice continuously differentiable, and such that $0 \leqslant xg'(x)/g(x) \leqslant 1$ for all $x > 0$, so that firms face strategic complementarities, and that demand is decreasing (see Appendix B for details).[4]

As a first illustration, the left panel in Table 1 shows numerical results of the resulting equilibria in the pricing equation (1). There is an equilibrium involving relatively small prices, one involving larger prices, and one middle equilibrium. The table illustrates a general phenomenon: there is always a smallest and a largest equilibrium, and, when $\lambda > 0$, these increase with the VER. The reason is shown on the left in Figure 1. The plotted functions are the pricing game's best-response functions, $p \mapsto \beta_2(\beta_1(p, \text{VER}), 0)$, whose fixed points correspond to the equilibria of the model. Imposing the VER causes the game's best response function to increase point wise, which increases the extremal equilibria and decreases the middle equilibrium.[5]

[4]This specification is numerically convenient; by choosing different g's we can experiment with the equilibrium set.

[5]The middle equilibrium is well known to be locally unstable for many learning dynamics. By considering a different g, we can obtain more equilibria, and only work with the stable ones: nothing in our analysis depends on using an unstable equilibrium. These issues are discussed in some detail in Echenique (2002).

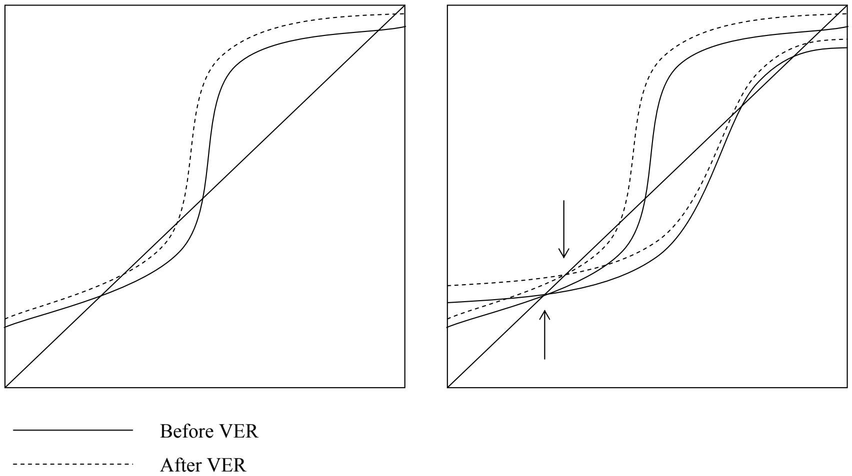

Figure 1: Monotone Comparative Statics of Equilibria.

2.2 Discussion

As already indicated in Berry, Levinsohn, and Pakes (1999), multiple equilibria in the BLP model present problems for policy evaluation.[6] Here, we further analyze the mechanisms by which this equilibrium multiplicity interferes with standard approaches to structural estimation and policy analysis.

2.2.1 Observable Policy Implications

What are the welfare effects of the actions firms take in response to the VER? The answer depends on which of the possible equilibria we expect to appear. When the BLP model has a unique equilibrium, then the effects of the VER are unambiguous: whenever the implicit tax parameter λ is greater than zero, prices will rise with the VER. As a consequence, if $\lambda > 0$, we expect the OLS regression coefficient of prices on the VER dummy to be positive. BLP report such regression results in Table 4 in Berry, Levinsohn, and Pakes (1999).

When the equilibrium is not unique, the implications of $\lambda > 0$ are no longer obvious: imposing the VER raises prices, and therefore lowers consumer surplus, in some of the equilibria, but not others. Therefore both a positive and a negative OLS regression

[6] See Section V (part C) in Berry, Levinsohn, and Pakes (1999).

coefficient on the VER dummy are compatible with $\lambda > 0$. The point should be clear from the left panel in Table 1: if the firms are likely to play one (or both) of the extremal equilibria, then the mean prices will rise with the VER; but if the firms are likely to play the middle equilibrium, then the mean prices will *fall* with the VER.

For example, if the firms select each of the extremal equilibria with probability 0.2, then the OLS regression coefficient of the price set by the home firm on the VER dummy would be -0.04. Interestingly, BLP's reported regression exhibits the (counterintuitive) negative effect of the VER on prices. In light of our comments, equilibrium multiplicity is a possible explanation for such regression results.

2.2.2 Structural Model Estimation

In addition to lack of implications for regressions, we argue that multiplicity poses problems for estimating the structural parameters in the BLP model, including the implicit tax parameter λ. In particular, if the equilibrium selections with and without the VER have their supports in the intersection of the sets of equilibria, then different parameter values may give rise to the same observables, thus causing identification to fail.[7]

The graph on the right in Figure 1 illustrates the point. The smallest equilibrium without the VER (indicated with an arrow pointing up) is identical for the two best-response functions; similarly, the smallest equilibria with the VER (indicated with an arrow pointing down) are the same. All the other equilibria, however, are different. If the firms were to always choose the smallest equilibrium, then the econometrician would not be able to identify any of the structural parameters, including the implicit tax parameter λ in Equation (1).

The right panel in Table 1 gives a numerical example: the smallest equilibria with

[7]Already Berry and Reiss (2007) note (p. 1864): "*empirical models with heterogeneous potential entrants pose thorny conceptual and practical problems for empirical researchers. Chief among them are the possibility that entry models can have multiple equilibria, or worse, no pure strategy equilibria. In such cases, standard approaches to estimating parameters may break down, and indeed key parameters may no longer be identified.*"

and without the VER are the same as those in the left panel of the table, even though the structural parameter values in the two models differ.[8]

2.2.3 Monotone Comparative Statics

We now argue that it is possible to evaluate the impact of the VER on prices without having to estimate the structural parameters of the model. In Equation (1), VER and prices are complements. Hence, the comparative statics effects take the form of a monotone comparative statics (MCS) prediction: if the implicit tax on exports λ is positive, then the VER will cause the extremal price equilibria to increase (Milgrom and Roberts, 1994; Villas-Boas, 1997). Evaluating the price impact of VER in the BLP framework is then equivalent to testing for the presence of MCS.

There are two difficulties in implementing such a test. First, the model that BLP bring to the data contains an additional source of randomness: firm-specific disturbance ω_i added to the average log-cost γ. In presence of such unobserved firm heterogeneity, BLP prices will no longer be discretely distributed over equilibrium sets; instead, they will have mixture distributions (in Section 3 we show how such mixtures arise). The second difficulty in testing for MCS stems from the equilibrium selection being unknown. For example, assume that the price of the home firm's product is like the one in Figure 2. Note that non-extremal equilibria may induce smaller observations with the VER than without it. In addition, the equilibrium selection can work against the positive effect of the VER, making equilibria p_2^1, p_3^1 and p_4^1 quite likely. All we have to work with is the MCS property: that in the presence of the VER the smallest equilibrium and the largest equilibrium have increased.

Our main result will be to provide simple conditions under which the effect of extremal equilibria will prevail *for large and small* values of the dependent variable. The conditions are simple and do not require any knowledge of the equilibrium selection; they come in

[8] The numerical values are obtained by using Matlab statistical software whose precision has been set to its highest value of 1e-16. In Appendix B, we report additional numerical results showing overlapping sets of equilibria for different parameter values in the BLP model.

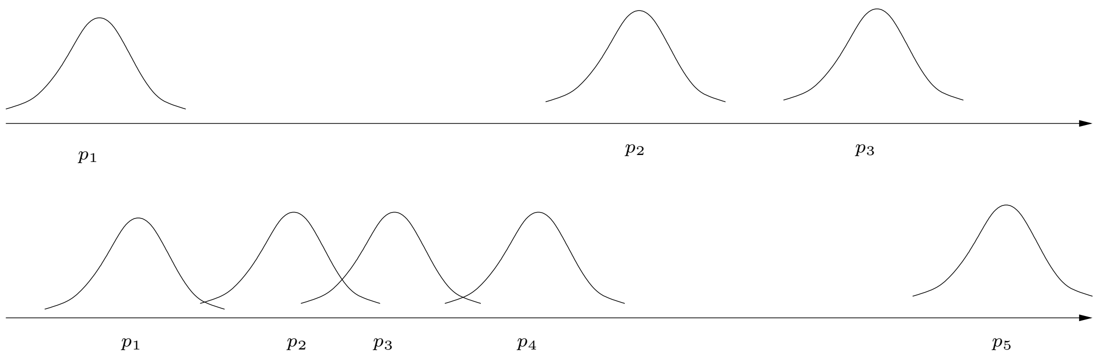

Figure 2: Set of equilibria is $\{p_1^0, p_2^0, p_3^0\}$ before and $\{p_1^1, p_2^1, p_3^1, p_4^1, p_5^1\}$ after VER.

the form of restrictions on the distribution tails of the disturbance. If they are satisfied, then the MCS effects that the VER has on extremal equilibria translate into testable implications on some large and small enough quantile of the conditional distribution of prices.

3 Structural Model and Results

3.1 Structure

We consider a structural equation given by

$$r(Y, X) = U, \tag{2}$$

where $r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is specified by economic theory.[9] The variables that enter the structural model in (2) are: a dependent variable $Y \in \mathbb{R}$, an explanatory variable $X \in \mathbb{R}$, and a disturbance to the system $U \in \mathbb{R}$. When the structural function r is parameterized by a finite dimensional parameter θ in Θ, one can write $r(Y, X, \theta) = U$ in Equation (2). We assume that X and Y are observable, but U is not; U can be thought of as unaccounted heterogeneity in the model.

[9]Following Matzkin (1994, 2005), we consider structural equations in which U is additively separable. The methods developed here are not suited for the non-separable problem $\tilde{r}(Y, X, U) = 0$. In such cases the framework in **?** may still be applied.

We have in mind the structural equations derived from two classes of economic models. One class predicts equilibrium values y based on a first-order condition $r(y, x) = 0$; these are single-person decision models, such as models solved by a social planner. A second class predicts equilibrium values y based on a fixed-point condition $r(y, x) = \rho(y, x) - y = 0$. The BLP model in Section 2 belongs to the latter class.

Given the function r, the structural econometric model is built by introducing the disturbance term U in the underlying economic model. Different realizations of U induce values of Y that deviate from the equilibria predicted by the economic model. The disturbance U has a clear interpretation as the extent to which a realized Y violates the exact (undisturbed) equilibrium condition.

When Equation (2) determines Y as a function $Y = m(X, U)$, the distribution of the disturbance U conditional on the explanatory variable X, denoted $F_{U|X}$, determines unambiguously the conditional distribution of Y, denoted $F_{Y|X}$. We say that $F_{Y|X}$ is generated by the *structure* $S = (r, F_{U|X})$. On the other hand, when Equation (2) has multiple solutions, a complete specification of the structure must include a rule that selects a particular realization y from the set of solutions. Such an *equilibrium-selection rule* can depend on the realized values of X and U.

Hereafter, we let $\mathcal{X} \subseteq \mathbb{R}$ denote the support of X, and assume that for any $x \in \mathcal{X}$, $F_{U|X=x}$ has a strictly positive density $f_{U|X=x}$ on $\mathbb{R}$. The explanatory variable X can be discrete or continuous. We define the equilibrium set as the set of solutions to Equation (2) when $X = x$ and $U = u$: let $(x, u) \in \mathcal{X} \times \mathbb{R}$ and $\mathcal{E}_{xu} = \{y \in \mathbb{R} : r(y, x) = u\}$. We shall work with the following assumption.

Assumption S1. (i) The function $r(y, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous; (ii) for any $x \in \mathcal{X}$, $\lim_{y \to -\infty} r(y, x) = +\infty$ and $\lim_{y \to +\infty} r(y, x) = -\infty$; (iii) for any $(x, u) \in \mathcal{X} \times \mathbb{R}$, $\mathcal{E}_{xu}$ is a finite set. We write $\mathcal{E}_{xu} = \{\xi_{1xu}, ..., \xi_{n_x xu}\}$ where $\xi_{1xu} \leqslant ... \leqslant \xi_{n_x xu}$ and $n_x = \text{Card}(\mathcal{E}_{xu})$.

Assumptions S1.i and S1.ii are standard. S1.ii is akin to an Inada condition; in particular, S1.i and S1.ii imply, by the Intermediate Value Theorem, that a solution to the structural equation in (2) always exists. Assumption S1.iii requires r not to be constant over any subintervals. By using suitable arguments from differential topology, S1.iii can be shown to hold generically (see Mas-Colell, Whinston, and Green (1995) for examples of these arguments). That the number of equilibria only depends on the explanatory variable X is not a serious restriction; it can simply be satisfied by duplicating elements of the equilibrium set until the cardinality of the latter no longer depends on U.

We specify the selection rule as follows: let $\mathcal{P}_{xu}$ be a probability distribution over $\mathcal{E}_{xu}$, which assigns probabilities $\{\pi_{1x}, ..., \pi_{n_x x}\}$ to outcomes $\{\xi_{1xu}, ..., \xi_{n_x xu}\}$, such that $\pi_{1x} > 0$ and $\pi_{n_x x} > 0$. For a given x, different realizations u can affect the support of $\mathcal{P}_{xu}$, but not the probabilities assigned to different outcomes in the support. For example, $\mathcal{P}_{xu}$ might assign equal probabilities across all elements of $\mathcal{E}_{xu}$. The conditional distribution of Y is then obtained as follows.

Proposition 1. *Assume S1 holds, and fix a selection rule $\mathcal{P}_{XU}$. Then, for any $x \in \mathcal{X}$ there are distribution functions $F_{iY|X=x}(y) = \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leqslant y) f_{U|X=x}(u)du$, for $1 \leqslant i \leqslant n_x$, such that, $j \geqslant i$ implies that $F_{jY|X=x}$ first-order stochastically dominates $F_{iY|X=x}$. And, for any $y \in \mathbb{R}$, $F_{Y|X=x}(y) = \sum_{i=1}^{n_x} \pi_{ix} F_{iY|X=x}(y)$.*

When multiple equilibria exist, $F_{Y|X}$ is generated by the structure $S = (r, F_{U|X}, \mathcal{P}_{XU})$, which now includes the additional element $\mathcal{P}_{XU}$. Proposition 1 shows that under S the conditional distribution of the dependent variable has a mixture form. When equilibrium is unique, the results of Proposition 1 reduce to the usual expression of the image distribution $F_{Y|X}$ of Y given X: $F_{Y|X=x}(y) = F_{U|X=x}\big(r(y,x)\big)$.

In general, the structure S may not be known. We work with a class of structures that share a qualitative feature: they exhibit *complementarities* between the explanatory variable X and the dependent variable Y.

Assumption S2. $r(y, x)$ is monotone increasing in x on $\mathbb{R}$.

Assumption S2 says that x and y are complements. Such complementarity usually follows from a supermodularity property of the primitive model. The key feature of S2 is that it implies a MCS property: the extremal equilibria of $r(y, x) = 0$ increase with x (Milgrom and Roberts, 1994; Villas-Boas, 1997). See Figure 1 for an illustration. We now review briefly some of the many economic models that satisfy Assumptions S1 and S2.

3.1.1 Individual decision maker

Consider models of individual decision making, in which the dependent variable is one-dimensional, and determined through the first-order condition of an optimization problem. An important class of such models are the ones solved by a social-planning problem, such as growth and macroeconomic models in Barro and Sala-I-Martin (2003) and Ljungqvist and Sargent (2004). Other examples include models of firms' investment choices used for testing if investment is sensitive to Tobin's q (Hayashi, 1982; Hayashi and Inoue, 1991).

3.1.2 One-dimensional equilibrium

Consider one-dimensional equilibrium models where equilibria are fixed points. For example, in a two-player game one can compose the two players' best-response functions, similarly to how we dealt with BLP's model in Section 2. As a consequence, duopoly models generally have the structure we need. Cournot n-firm oligopoly models also reduce to a one-dimensional equilibrium model by an aggregation procedure as described by Amir (1996). One can thus examine if entry of additional firms to a market causes a decrease in prices as in Amir and Lambson (2000). Additional examples can be found in overlapping-generations models, such as the ones in Ljungqvist and Sargent (2004), and two-good general equilibrium models.

3.2 Main Result

The presence of complementarities between X and Y is the basis of our main result: we show that an increase in x implies an increase in all the sufficiently large (and small) quantiles of $F_{Y|X=x}$. The result will follow from combining Assumption S2 with restrictions on U.

Consider x_1 and x_2 in $\mathcal{X}$ with $x_1 < x_2$. Let $n_1 = n_{x_1}$ and $n_2 = n_{x_2}$. How does the MCS property translate into observable implications on $F_{Y|X=x_1}$ and $F_{Y|X=x_2}$? Denote $\bar{F}_{Y|X} = 1 - F_{Y|X}$ the conditional distribution tail of Y. Let $\pi_{1i} = \pi_{ix_1}$ and $\pi_{2j} = \pi_{jx_2}$. Using the mixture result in Proposition 1 and focusing on the largest equilibria, we then have:

$$\begin{aligned}\frac{\bar{F}_{Y|X=x_1}(y)}{\bar{F}_{Y|X=x_2}(y)} &= \frac{\bar{F}_{n_1Y|X=x_1}(y)}{\bar{F}_{n_2Y|X=x_2}(y)}\frac{\sum_{i=1}^{n_1}\pi_{1i}[\bar{F}_{iY|X=x_1}(y)/\bar{F}_{n_2Y|X=x_2}(y)]}{\sum_{j=1}^{n_2}\pi_{2j}[\bar{F}_{jY|X=x_2}(y)/\bar{F}_{n_2Y|X=x_2}(y)]} \\ &\leqslant \frac{\bar{F}_{n_1Y|X=x_1}(y)}{\bar{F}_{n_2Y|X=x_2}(y)}\frac{1}{\pi_{2n_2}}, \end{aligned} \tag{3}$$

where the second inequality follows because $\pi_{2n_2} > 0$, $\bar{F}_{jY|X=x_2}(y)/\bar{F}_{n_2Y|X=x_2}(y) > 0$, and because stochastic dominance implies $\bar{F}_{iY|X=x}(y) \leqslant \bar{F}_{n_1Y|X=x_1}(y)$.

The upper bound in Equation 3 involves the probability of the largest equilibrium π_{2n_2}—on which we place no restrictions other than being positive—as well as the ratio of the distributions $\bar{F}_{n_1Y|X=x_1}$ and $\bar{F}_{n_2Y|X=x_2}$. These distributions are unknown and depend on the locations of the largest equilibria; hence they are difficult to control. A careful change of variables, however, transforms the problem so that (in the limit) the behavior of their ratio depends solely on the properties of r and $F_{U|X}$.

Lemma 2. *Under S1 and S2, and given $(y_0, x) \in \mathbb{R} \times \mathcal{X}$, we have $\mathbb{1}(y \leqslant \xi_{n_x x u}) = \mathbb{1}(u \leqslant r^e(y,x))$ for any $y \geqslant y_0$, where $r^e(y,x)$ is the non-increasing envelope of $r(y,x)$ on $[y_0, +\infty)$, i.e. $r^e(y,x) = \inf\{q(y) : q$ is non-increasing on $[y_0, +\infty)$ and $q(y) \geqslant r(y,x)$ for all $y \in [y_0, +\infty)\}$.*

The idea in Lemma 2 is to consider a non-increasing transformation r^{e} which coincides with r around the largest equilibrium (Figure 3). For $y \geqslant y_0$ then

$$\frac{\bar{F}_{n_1 Y|X=x_1}(y)}{\bar{F}_{n_2 Y|X=x_2}(y)} = \frac{\int_{-\infty}^{r^{\mathrm{e}}(y,x_1)} f_{U|X=x_1}(u)du}{\int_{-\infty}^{r^{\mathrm{e}}(y,x_2)} f_{U|X=x_2}(u)du} = \frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_1))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))}. \tag{4}$$

Now, how the increase in the largest equilibria translates into $F_{Y|X=x_1}$ and $F_{Y|X=x_2}$, depends on two factors: (i) the limit behavior or $r^{\mathrm{e}}(y,x_1)$ and $r^{\mathrm{e}}(y,x_2)$ as y grows, and (ii) the limit behavior of the distribution $F_{U|X}$. On (i), recall that, by S2, $r(y,x)$ is monotone increasing in x. Hence $r(y,x_1) \leqslant r(y,x_2)$, which given the continuity and limit conditions in S1 implies $r^{\mathrm{e}}(y,x_1) \leqslant r^{\mathrm{e}}(y,x_2)$ for all $y \in [y_0,+\infty)$. We allow for two cases. In the first, we control the limit ratio of $r(y,x_1)$ to $r(y,x_2)$ (e.g. as when $r(y,x) = -|\theta x|y$). In the second, we control the difference between $r(y,x_1)$ and $r(y,x_2)$ (e.g. as in $r(y,x) = \theta x - y$). Each case requires an assumption on $F_{U|X}$.

Assumption S3. (i) $\lim_{y\to+\infty}[r(y,x_1)/r(y,x_2)] = \lambda$, where $\lambda \in [0,1)$; (ii) $F_{U|X}$ is rapidly varying at $-\infty$; (iii) $\left[F_{U|X=x_1}(u)/F_{U|X=x_2}(u)\right]$ is bounded as u goes to $-\infty$.
Assumption S3'. (i) $\lim_{y\to+\infty}[r(y,x_1) - r(y,x_2)] = \delta$ with $\delta < 0$; (ii) $F_{U|X} \circ \ln$ is rapidly varying at 0; (iii) $\left[F_{U|X=x_1}(u)/F_{U|X=x_2}(u)\right]$ is bounded as u goes to $-\infty$.

Take Assumption S3: property S3.ii requires that the tail of the distribution $F_{U|X}$ is not too heavy. Recall that an increasing positive real function F is *rapidly varying at a* if $\lim_{x\to a} F(\lambda x)/F(x)$ is 0 for $\lambda \in (0,1)$ and $+\infty$ for $\lambda > 1$. Rapid variation is a well-known condition in the statistics of extreme values, which is satisfied by most distributions familiar to practitioners. Property S3.iii, on the other hand, ensures that $F_{U|X=x_2}$ does not decrease towards 0 faster than $F_{U|X=x_1}$. This property is trivially satisfied when U is independent of X, and accommodates some interesting cases where U and X are dependent (see Section 4.1.2). Assumption S3 implies that the last term in Equation (4) converges to 0 as y grows. Indeed, let $\lambda_1 \in (\lambda,1)$: then by S3.i there is $y_1 \in \mathbb{R}$ such that $r(y,x_1) \leqslant \lambda_1 r(y,x_2)$ whenever $y \geqslant y_1$ and hence $r^{\mathrm{e}}(y,x_1) \leqslant \lambda_1 r^{\mathrm{e}}(y,x_2)$.

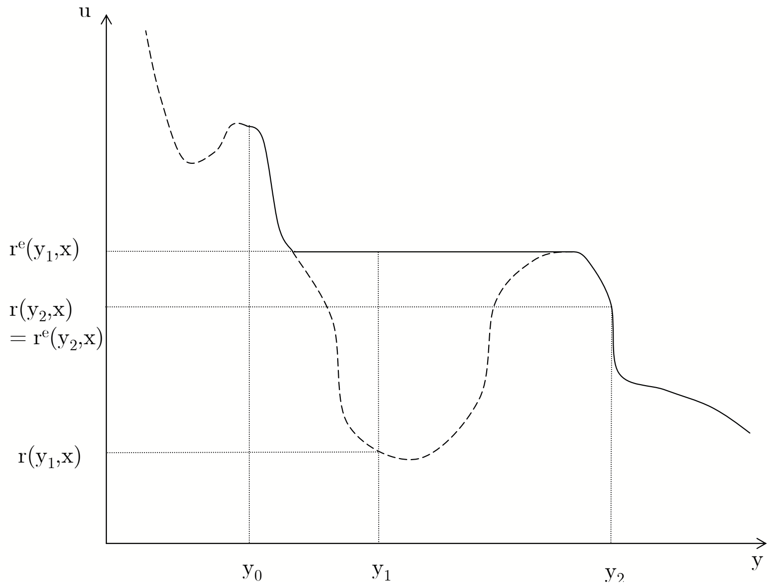

Figure 3: Plots of $y \mapsto r(y, x)$ (dashed line) and $y \mapsto r^{\mathrm{e}}(y, x)$ (solid line) with x fixed.

As $F_{U|X}$ is increasing, we have:

$$\begin{aligned}
\lim_{y\to+\infty} \quad & \frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_1))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))} \\
\leqslant \quad & \lim_{y\to+\infty} \frac{F_{U|X=x_1}(\lambda_1 r^{\mathrm{e}}(y,x_2))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))} \\
= \quad & \lim_{y\to+\infty} \underbrace{\left[\frac{F_{U|X=x_1}(\lambda_1 r^{\mathrm{e}}(y,x_2))}{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_2))}\right]}_{A} \underbrace{\left[\frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_2)))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))}\right]}_{B}.
\end{aligned} \tag{5}$$

By S1.ii, r goes to $-\infty$ as y gets large, and so does its envelope r^e; hence B in (5) remains bounded as y increases. Using the rapid variation in S3.ii, the term A goes to 0 as y increases, and so does the product $A \times B$. As a result,

$$\lim_{y\to+\infty} \frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_1))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))} = 0. \tag{6}$$

Similarly, under Assumption S3'.i, $d(y) = r(y, x_1) - r(y, x_2)$ converges to $\delta < 0$. Let $\delta_1 \in (\delta, 0)$ and $y_1' \in \mathbb{R}$ be such that, for $y \geqslant y_1'$, we have $d(y) < \delta_1$. Hence, $d^{\mathrm{e}}(y) \leqslant \delta_1$ where $d^{\mathrm{e}}(y) = r^{\mathrm{e}}(y, x_1) - r^{\mathrm{e}}(y, x_1)$. Applying the previous reasoning to $\exp[d^{\mathrm{e}}(y)]$, we again get the limit result in Equation (6). Combining the latter with Equations (3) and (4) then shows that:

$$\lim_{y\to+\infty} \frac{\bar{F}_{Y|X=x_1}(y)}{\bar{F}_{Y|X=x_2}(y)} = 0. \tag{7}$$

The statement in (7) is crucial. It says that for large enough values of y, $x_1 \leqslant x_2$ implies that the corresponding conditional distributions are ordered.[10] This is equivalent to an ordering of large enough conditional quantiles. We have thus shown:

Theorem 3. *Assume S1, S2, and either S3 or S3' hold. Fix a selection rule $\mathcal{P}_{XU}$. Let $(x_1, \ldots, x_N) \in \mathcal{X}^N$ be such that: $x_1 \leqslant \ldots \leqslant x_N$. Then, there exists $\bar{y}_N \in \mathbb{R}$ such that for all $y \geqslant \bar{y}_N$, $\bar{F}_{Y|X=x_1}(y) \leqslant \ldots \leqslant \bar{F}_{Y|X=x_N}(y)$. Equivalently, there exists $\bar{\alpha}_N \in (0,1)$ such that for all $\alpha \in [\bar{\alpha}_N, 1)$, $F^{-1}_{Y|X=x_1}(\alpha) \leqslant \ldots \leqslant F^{-1}_{Y|X=x_N}(\alpha)$.*

The previous analysis has focused on quantiles with probabilities close to one, but an analogous result continues to hold for probabilities close to zero.

4 Discussion

Theorem 3 derives observable implications of models with complementarities between the dependent variable Y and the explanatory variable X, which are valid despite the possible presence of multiple equilibria. These implications come in the form of order restrictions on the extreme (high and low) quantiles of Y conditional on X. We now discuss important features and possible limitations of the results of Theorem 3 when used for testing the presence of MCS; we also compare and contrast our approach with alternative testing methods.

4.1 Main Features

We first discuss the applicability of our results.

4.1.1 Robustness to Identification Failures

We have shown that the MCS property has implications for the conditional quantiles of Y given X. Given a sample of observations on the dependent and explanatory variables,

[10] When $x_1 = x_2$ the conditional distributions of Y are identical.

these quantiles are by definition identified and consistently estimable using standard nonparametric methods. In particular, no additional restrictions on the structural function r are needed for estimation. Consequently, the results of Theorem 3 can be used to test for complementarities whether or not the structural function r is identified.

It is in general difficult to provide primitive conditions for identification, so most models merely impose it as an assumption. There are, however, important results that analyze identification under additional restrictions on the structure. Say that the equilibrium in Equation (2) is *unique*. When the explanatory variable X and the latent disturbance U are known to be independent, primitive conditions for identification of the structural function r can be found in Matzkin (2005), for example. When U is suspected to be endogenous (e.g. if X is a policy treatment, the deviations U may be more likely under some policies than others), it is often assumed that a mean independence condition $E(U|Z) = 0$ a.s. holds with respect to an instrument $Z \in \mathcal{Z}$. In models in which the reduced form is separable $Y = m(X) + U$ for example, Newey and Powell (2003) characterize identification of m in terms of the completeness of the conditional distribution of X given Z. Treatment of the nonseparable case $Y = m(X, U)$ requires additional monotonicity assumptions on m as well as independence of U and Z (Chesher, 2003; Blundell and Powell, 2003). Extending those results to frameworks in which multiple equilibria exist is still an open question.

4.1.2 Unobserved Heterogeneity

Given that Theorem 3 does not require the structural function r to be identified or estimable, its results are fairly robust to departures from independence or mean independence conditions between the latent disturbance U and the explanatory variable X. The implications of MCS remain valid even in absence of an instrument Z, which, due to the nonlinearity of the structural model in Equation (2) may be difficult to construct.

As a consequence, our testable implications apply even in models in which U is *endogenous*. In particular, under the assumptions of Theorem 3, the individual heterogeneity U can be correlated with the explanatory variable X in a reasonably general way. Say that conditional on X, U is normally distributed with mean $\mu(X)$ and variance σ^2. When $\mu(x)$ is non-increasing in x, a simple application of L'Hôpital's rule shows that Assumptions S3.ii and S3.iii hold. A simple example would be the one in which X and U are jointly normally distributed with a non-positive correlation coefficient. A positive correlation between X and U, under which Assumption S3.iii fails, prevents the econometrician from learning anything about MCS property. The intuition behind is simple: following an increase in X, U can in those cases increase so as to decrease the extremal equilibria.

In addition to being correlated with the explanatory variable, we allow U to be *heteroskedastic* conditional on X. Say that given X, U is normally distributed with mean 0 and variance $\sigma^2(X)$. If $\sigma^2(x)$ is non-decreasing in x, then Assumption S3.iii holds. Therefore a normal disturbance whose conditional variance increases with the equilibrium level satisfies our Assumption S3.

4.2 Limitations

We now caution for possible limitations of our approach.

4.2.1 Tail Observations and Robustness to Outliers

Theorem 3 suggests that one can use observations from the extreme (high and low) quantiles of Y conditional on X in order to test for the presence of MCS. Such a test shall obviously be affected by the presence of outliers. When the latter are caused by mismeasurements, methods proposed in Chen, Hong, and Tamer (2005), for example, can be used to filter the errors prior to applying the test. Unless outliers are easy to detect, one should be careful when considering very large (or small) quantiles of the dependent variable. In particular, the results of Theorem 3 lend themselves to the study of cases where X can take some relatively small number of values for which large numbers of observations of Y are available. Evaluations of policy effects, such as those following the

VER, are one such example: typically X then takes on two values.

4.2.2 Continuous Explanatory Variable

The cutoff level $\bar{y}_N$ in Theorem 3 is conditional on a realization of the sample of explanatory variables, $(x_1, \ldots, x_N) \in \mathcal{X}^N$. This is not a problem in applications in which the explanatory variables are treated as given. In some situations, however, an unconditional version of Theorem 3 is needed. The latter follows easily when the explanatory variables are discrete: it suffices to apply the reasoning in Section 3.2 to all the points in $\mathcal{X}$. When the explanatory variables are *continuous*, we need to include an extra step which will ensure that x's do not get too close: given a random sample $(X_1, \ldots, X_N)$ drawn from F_X, consider the joint distribution of the $N-1$ spacings between the consecutive order statistics $(X_1^N, \ldots, X_N^N)$. Fix any $\varepsilon > 0$, and let $\delta_N > 0$ be such that the probability of all spacings being greater than δ_N, is greater or equal than $1-\varepsilon$. Applying the reasoning in Section 3.2 to x and $x + \delta_N$ we get the following corollary to Theorem 3:

Corollary 4. *Assume S1, S2, and either S3 or S3' hold. Fix a selection rule $\mathcal{P}_{XU}$. Given $\varepsilon > 0$, there exists $\bar{y}_N \in \mathbb{R}$ such that for all $y \geqslant \bar{y}_N$, $\Pr\{\bar{F}_{Y|X_1^N}(y) \leqslant \ldots \leqslant \bar{F}_{Y|X_N^N}(y)\} \geqslant 1-\varepsilon$. Equivalently, there exists $\bar{\alpha}_N \in (0,1)$ such that for all $\alpha \in [\bar{\alpha}_N, 1)$, $\Pr\{F^{-1}_{Y|X_1^N}(\alpha) \leqslant \ldots \leqslant F^{-1}_{Y|X_N^N}(\alpha)\} \geqslant 1-\varepsilon$.*

In a sense, Corollary 4 gives a stochastic version of the orderings in Theorem 3.

4.2.3 Test Implementation

Finally, the conditional distributions (and quantiles) of the dependent variable are typically *unknown* and need to be estimated from the data. A statistical test of the orderings in Theorem 3 and its Corollary 4 can then be constructed by deriving the asymptotic distribution of the conditional quantile estimators—the key is to derive the latter by imposing assumptions on the distributions $F_{U|X}$ while maintaining our agnosticism about the equilibrium selection $\mathcal{P}_{XU}$. When using the asymptotics, however, one needs to control the speed at which the probability $\bar{\alpha}_N$ increases (or decreases) relative to the sample size N. See **?** for results, albeit in a somewhat different framework.

4.3 Comparison with Direct Testing Methods

The conditional quantile implications of Theorem 3 are derived under Assumptions S3 or S3', which restrict the tail behavior of the conditional distribution of U given X. That only tails of $F_{U|X}$ are concerned is not surprising: recall that Theorem 3 uses the MCS property, which holds only for extremal equilibria. In standard econometric practice, on the other hand, the focus is rather on the conditional mean of U. A routine assumption is that of mean independence between U and an instrument Z. We now discuss how such moment conditions may be exploited to construct alternative tests for complementarities. We shall focus on direct methods which consist in consistently estimating r, then testing whether the monotonic increasing property in Assumption S2 holds.

4.3.1 Moment Conditions

Say that in addition to Assumptions S1 and S2, the structural model in Equation (2) is known to satisfy:

Assumption S3". There exists an observable $Z \in \mathcal{Z}$ such that $E(U|Z) = 0$ a.s.

Assumption S3" says that U is mean independent of an instrument Z.[11] In order to derive its implications on the observables, we need the conditional distribution of Y given both Z and X. The latter is easily obtained using the same reasoning as in Proposition 1: $F_{Y|X=x,Z=z}(y) = \sum_{i=1}^{n_x} \pi_{ixz} F_{iY|X=x,Z=z}(y)$ for any $(y, x, z) \in \mathbb{R} \times \mathcal{X} \times \mathcal{Z}$, where $F_{iY|X=x,Z=z}(y) \equiv \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leqslant y) f_{U|X=x,Z=z}(u)du$, and $f_{U|X,Z}$ is the conditional

[11] It is worth pointing out that Assumptions S3.iii (or S3'.iii) and S3" are not mutually exclusive; in Section 4.1.2 we have given an example of a disturbance U that verifies S3.iii and is also mean independent of X.

density of U given (X, Z), which we assume to be strictly positive on $\mathbb{R}$. Then,

$$\begin{aligned} E[r(Y,X)|Z=z] &= E\left[\int_{-\infty}^{+\infty} r(y,x)dF_{Y|X=x,Z=z}(y) \middle| Z=z\right] \\ &= E\left[\sum_{i=1}^{n_x} \pi_{ixz} \int_{-\infty}^{+\infty} r(y,x)dF_{iY|X=x,Z=z}(y) \middle| Z=z\right] \\ &= E\left[\sum_{i=1}^{n_x} \pi_{ixz} \int_{-\infty}^{+\infty} u f_{U|X=x,Z=z}(u)du \middle| Z=z\right] = 0, \end{aligned} \tag{8}$$

where the third equality uses the fact that $\xi_{ixu} \mapsto u$ is a one-to-one mapping, and the fourth combines the mean independence condition on U with $\sum_{i=1}^{n_x} \pi_{ixz} = 1$ for every $(x, z) \in \mathcal{X} \times \mathcal{Z}$. Despite the presence of multiple equilibria, Assumption S3" induces a familiar conditional moment restriction on the observables Y, X and Z.

4.3.2 Identification

Whether or not Assumption S3" may be used for estimation depends on the ability of the restriction $E[r(Y, X)|Z] = 0$ a.s. to identify the structural function r.[12] We now argue that, while Equation (8) remains unaffected by multiplicity, the same cannot be said about its solutions (in r).

For this, consider a simple parametric structural model $r(Y, X, \theta) = \theta^2 Y + \theta Y^2 - Y^3 + \theta X$, in which X and Y are complements if and only if the scalar θ is positive. Now, multiple equilibria for Y pose a basic problem: the solutions to the moment equation $E[r(Y, X, \theta)|Z] = 0$ a.s. are impossible to compute without specifying an equilibrium selection procedure. As a result, (1) different selection procedures lead to different solutions (in θ) to this equation, and (2) under some procedures the solution is not unique, leading to non-identification. In Appendix C we give numerical examples of two equilibrium selections correlated with X, for which the set of solutions to $E[r(Y, X, \theta_0)|X] = 0$, when $\theta_0 = 1$, is either $\{1\}$ or $\{-1, 1\}$ (see Appendix C for details).[13] Thus, even when

[12] Chernozhukov, Imbens, and Newey (2007) provide conditions under which the restriction $E[r(Y, X)|Z] = 0$ a.s. suffices to identify r, however, only locally.

[13] On the other hand, correlation between X and the equilibrium selection is to be expected in many applications. For example, Bohnet, Frey, and Huck (2001), Cooter (2002), and Funk (2007) argue that the main effect of certain policies is to affect equilibrium selections.

X is exogenous, and the structural parameter θ_0 locally identified, one cannot identify the property of complementarities based on the conditional moment restriction alone. One could expect identification to be even more difficult in structural models that are nonparametric.

4.3.3 Estimation

In Section 4.3.2 we have emphasized how equilibrium multiplicity may lead to lack of identification. Here we assume that, nevertheless, the structural function r is identified, say, by imposing assumptions regarding equilibrium selection procedure in addition to S3".

Despite identification, estimating r based on the conditional moment restriction $E[r(Y, X)|Z] = 0$ a.s. is likely to suffer, as pointed out by Chernozhukov, Imbens, and Newey (2007), from an ill-posed inverse problem. In particular, the nonparametric estimator $\hat{r}$ of the structural function r is likely to be discontinuous, so that small differences in the realizations of Y, X and Z are likely to lead to large deviations in the estimator $\hat{r}$. Consistent estimation of r may in such cases require additional restrictions on the set of possible structural functions, similar to those used in Blundell and Powell (2003) and Newey and Powell (2003), for example. It is worth pointing out that even with a consistent estimator in hand, testing for monotonicity of the structural function remains a difficult problem.

In models in which r is finitely parameterized by θ, it may be the case that Assumption S2 reduces to a simple restriction on the structural parameter. When the latter is identified, consistent estimation based on $E[r(Y, X, \theta)|Z] = 0$ a.s. can be done along the lines discussed in Ai and Chen (2003), for example. Testing for complementarities can then be carried out using standard techniques.

4.3.4 Inference on Parameter Sets

In Section 4.1 we have emphasized the robustness of our approach to departures from identifiability of r, as well as independence and/or mean independence of the unobservable U. In some models, however, additional information on the structure may be available which can be used for estimation and inference when the structural function is only *set identified*. For example, in parametric models, Chernozhukov, Hong, and Tamer (2007) construct estimators and confidence regions for the identified set, and develop an inference method. Extending their methods to nonparametric settings is, however, an open problem. In nonparametric regression models, Santos (2007) proposed methods which could be used to test whether a weak monotonicity property in S2 holds on the identified set.

While robust to identification failures, the methods based on set identification suffer from another problem: the monotonicity in Assumption S2 may not hold globally on the identified set. Section 4.3.2 and Appendix C contain one such example: for one identified value of the structural parameter, X and Y are complements, while for the other they are substitutes. In such cases, testing for complementarities between X and Y is still possible based on the conditional quantile implications derived in Theorem 3.

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SUPPLEMENTARY APPENDICES

Appendix A Proofs

Proof of Proposition 1. For any $(y,x) \in \mathbb{R} \times \mathcal{X}$, $F_{Y|X=x}(y) = \int_{-\infty}^{+\infty} \mathcal{P}_{xu}(y) f_{U|X=x}(u)du$ with $\mathcal{P}_{xu}(y) = \sum_{i=1}^{n_x} \pi_{ix} \mathbb{1}(\xi_{ixu} \leqslant y)$, where $\mathbb{1}$ denotes the standard indicator function: For any event A in $\mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$, $\mathbb{1}(A) = 1$ if A is true, and 0 otherwise. Combining all of the above we get:

$$F_{Y|X=x}(y) = \sum_{i=1}^{n_x} \pi_{ix} \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leqslant y) f_{U|X=x}(u)du.$$

For any $x \in \mathcal{X}$ and any $1 \leqslant i \leqslant n_x$, let $F_{iY|X=x}(y) = \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leqslant y) f_{U|X=x}(u)du$ for all $y \in \mathbb{R}$. Then $F_{iY|X=x}(y) : \mathbb{R} \to \mathbb{R}$ is right-continuous, $\lim_{y \to -\infty} F_{iY|X=x}(y) = 0$, $\lim_{y \to +\infty} F_{iY|X=x}(y) = 1$, and $F_{iY|X=x}$ is nondecreasing in y. Hence, $F_{iY|X=x}$'s are distribution functions and the conditional distribution of the dependent variable can be written as in Proposition 1. Moreover, for any $(y,x) \in \mathbb{R} \times \mathcal{X}$ we have $F_{iY|X=x}(y) - F_{jY|X=x}(y) = \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leqslant y < \xi_{jxu}) f_{U|X=x}(u)du \geqslant 0$ whenever $\xi_{jxu} \geqslant \xi_{ixu}$, i.e. $F_{jY|X=x}(y) \leqslant F_{iY|X=x}(y)$ whenever $j \geqslant i$. So, $F_{jY|X=x}$ first-order stochastically dominates $F_{iY|X=x}$ for any $j \geqslant i$. □

Proof of Lemma 2. Fix $(y_0, x) \in \mathbb{R} \times \mathcal{X}$: continuity and limit conditions on $r(y,x)$ in S1 then ensure that the envelope $r^{\mathrm{e}}(y,x)$ is well defined on $[y_0, +\infty)$. Now consider $y \geqslant y_0$. That $\mathbb{1}(y \leqslant \xi_{n_x xu}) = \mathbb{1}(u \leqslant r^{\mathrm{e}}(y,x))$ follows from showing that $r^{\mathrm{e}}(\xi_{n_x xu}, x) = r(\xi_{n_x xu}, x)$, as r^e is non-increasing and $\xi_{n_x xu}$ is the largest equilibrium. We proceed in two steps. First, we show that for all $y > \xi_{n_x xu}$ we have $r(\xi_{n_x xu}, x) > r(y,x)$. If that were not the case then there would exist a $y' > \xi_{n_x xu}$ such that $r(\xi_{n_x xu}, x) \leqslant r(y', x)$. But this is incompatible with $\xi_{n_x xu}$ being the largest equilibrium: we would have $u \leqslant r(y', x)$, so given the limit condition S1.ii on r at $+\infty$ there would be an equilibrium larger that $\xi_{n_x xu}$. Second, we show that $r^{\mathrm{e}}(\xi_{n_x xu}, x) = r(\xi_{n_x xu}, x)$. By definition of r^e, we have $r^{\mathrm{e}}(\xi_{n_x xu}, x) \geqslant r(\xi_{n_x xu}, x)$, so we need to rule out that the strict inequality holds. We again reason by contradiction: assume that $r^{\mathrm{e}}(\xi_{n_x xu}, x) > r(\xi_{n_x xu}, x)$. From the first step we know that $r(\xi_{n_x xu}, x) > r(y,x)$ for all $y > \xi_{n_x xu}$. Then, consider the function which coincides with $r^{\mathrm{e}}(y,x)$ for $y < \xi_{n_x xu}$ and with $\min\{r^e(y,x), r(y,x)\}$ for $y \geqslant \xi_{n_x xu}$. This function is non-increasing, larger than r, and smaller than r^e at $\xi_{n_x x u}$, which is impossible by the definition of r^e. □

Appendix B Details on the BLP Example

This appendix contains a detailed discussion of the simplified BLP model considered in Section 2; extensions to many firms and products are discussed in the end of the appendix. On the demand side, we use a random utility specification:

$$u_{hi} = -\alpha p_i + \xi_i + \varsigma_h, \tag{9}$$

in which u_{hi} is the utility of product i ($i = 1, 2$) to individual h ($i = 1, ..., I$), p_i and ξ_i are respectively the price and the unobserved characteristic of product i, $-\alpha$ ($\alpha > 0$) is the taste parameter on price assumed constant across individuals, and ς_h is a stochastic term that represents the deviations from an average behavior of agents and whose distribution is induced by unobserved characteristics of the individual h, unobserved characteristics of product i, and their interactions.

A baseline specification of the random utility in Equation (9) moreover assumes that ς_h are iid across products i and individuals h. We depart from the iid assumption and assume that the individual deviations from the average utility depend on the level of prices through a variance term. Specifically, we model ς_h as:

$$\varsigma_h = g(p_i p_{-i})\epsilon_h, \tag{10}$$

where the function $g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is twice continuously differentiable and such that $g'(x) \geqslant 0$ and $0 \leqslant xg'(x)/g(x) \leqslant 1$ for all $x > 0$. The innovations ϵ_h are iid across products i and individuals h with a distribution function F that satisfies $E(\epsilon_h) = 0$ and $var(\epsilon_h) = 1$, so that $g^2(p_i p_{-i})$ corresponds to the variance of the disturbance ς_h in the random utility specification (9). For example, assuming that ϵ_h's are (standardized) Gumbel random variables, the resulting individual choice model is logit with conditional heteroskedasticity.[14]

Under the assumptions on ς_h, the market share of Firm i, denoted s_i, equals $F(\Delta)$, where we have let $\Delta = \alpha(p_{-i} - p_i)/g(p_i p_{-i})$. The first derivative of s_i with respect to p_i is

$$\frac{\partial s_i}{\partial p_i} = -f(\Delta)\frac{\alpha}{g(p_i p_{-i})}\left[1 + \frac{p_i p_{-i} g'(p_i p_{-i})}{g(p_i p_{-i})}\left(\frac{p_{-i}}{p_i} - 1\right)\right],$$

which given the conditions imposed on g is always negative. Hence, the demand for product i, $D_i(p_i, p_{-i}) = M s_i$ where M is the market size, is decreasing in p_i which is as we would expect.

On the supply side, we assume complete specialization and constant marginal costs across firms:[15]

$$\ln c_i = \gamma. \tag{11}$$

The presence of the VER—implemented as an implicit tax on firms' production—is modeled by a dummy variable VER_i. Firms' profits, denoted Π_i, are then given by: $\Pi_i = (p_i - c_i - \lambda \mathrm{VER}_i) D_i(p_i, p_{-1})$, where λ is the implicit tax parameter. Each firm is assumed to choose prices which maximize its profits. Then, the first order condition of Firm i's profit maximization $\partial \Pi_i / \partial p_i = 0$ can be written as:

$$F(\Delta) - \frac{f(\Delta)(p_i - c_i - \lambda \mathrm{VER}_i)\alpha}{g(p_i p_{-i})}\left[1 + \frac{p_i p_{-i} g'(p_i p_{-i})}{g(p_i p_{-i})}\left(\frac{p_{-i}}{p_i} - 1\right)\right] = 0. \tag{12}$$

Equilibrium prices p_1 and p_2 necessarily satisfy the system of first order conditions relative

[14] The probability density of a standardized Gumbel random variable is: $f(\epsilon) = \sigma^{-1}\exp[-(\epsilon-\mu)/\sigma - \exp(-(\epsilon-\mu)/\sigma)]$ for $\epsilon > 0$, where $\sigma = \sqrt{6}/\pi$ and $\mu = -\gamma\sigma$, with γ denoting Euler's constant, $\gamma \simeq 0.577$. The corresponding distribution function is $F(\epsilon) = [1 + \exp(-\epsilon/\sigma)]^{-1}$.

[15] In the model Berry, Levinsohn, and Pakes (1999) bring to the data, the log-costs also contain an additive unobserved random cost component ω_i which is such that $E(\omega_i) = 0$.

to Firm 1 and Firm 2.[16] Combining Equations (11) and (12) we then get:

$$\ln(p_i - b(p_i, p_{-i}, \alpha) - \lambda \text{VER}_i) - \gamma = 0, \tag{14}$$

in which the markups b are given by:

$$b(p_i, p_{-i}, \alpha) = \frac{F(\Delta)}{f(\Delta)} \frac{g(p_i p_{-i})}{\alpha} \left[1 + \frac{p_i p_{-i} g'(p_i p_{-i})}{g(p_i p_{-i})} \left(\frac{p_{-i}}{p_i} - 1 \right) \right]^{-1}.$$

Note that the markups b are nonlinear functions of the demand parameter α, and the prices p_i and p_{-i}; the nonlinearities in b are induced by both g and F.

In what follows, we assume that only Firm 1 (the foreign firm) is subject to the VER. Writing Equation (14) for Firm 1, and combining it with its equivalent for Firm 2 (the home firm), we then obtain the following system:

$$\ln(p_1 - b(p_1, p_2, \alpha) - \lambda \text{VER}_1) - \gamma = 0 \tag{15}$$

$$\ln(p_2 - b(p_2, p_1, \alpha)) - \gamma = 0, \tag{16}$$

The pricing equation (1) is then obtained by substituting $p_1 = \beta_2(p_1, \text{VER}_1)$ obtained from Equation (15), into $p_2 = \beta_2(p_1, 0)$ in Equation (16). Note that in Equation (1) we let $p = p_2$ denote the home firm's (Firm 2's) price, while $\text{VER} = \text{VER}_1$ is the VER dummy on the foreign firm (Firm 1). Table 2 gives numerical examples of BLP equilibrium prices obtained by solving the system of Equations (15)-(16), under the assumptions that F is standardized Gumbel, and that the heteroskedasticity g is of the form: $g(x) = \rho + \exp(-\tau/x)$. These are meant to complement the numerical results presented in the body of the paper.

[16] In order to check that a solution to Equation (12) maximizes Firm i's profits, we need to check that the second order condition $\partial^2 \Pi_i / \partial p_i^2 < 0$ holds. At the optimum, we have

$$\frac{\partial^2 \Pi_i}{\partial p_i^2} = -\frac{F(\Delta)}{p_i - c_i - \lambda \text{VER}_i} \left\{ 2\left[1 + \left(1 - \frac{c_i + \lambda \text{VER}_i}{p_i}\right) \frac{p_i p_{-i} g'(p_i p_{-i})}{g(p_i p_{-i})} \right] - \Delta(p_i - c_i - \lambda \text{VER}_i) \frac{p_{-i}^2 g''(p_i p_{-i})}{\alpha + \Delta p_{-i} g'(p_i p_{-i})} - \frac{f'(\Delta) F(\Delta)}{f^2(\Delta)} \right\}. \tag{13}$$

Given our assumptions on g, the second order conditions are satisfied for all symmetric equilibria as we then have $\Delta = 0$ so $f'(\Delta) = 0$. For all other equilibria, we check numerically that the second order condition is satisfied.

Table 2: BLP Equilibrium Prices (p_1, p_2) in a Logit Model with Conditional Heteroskedasticity

$\text{VER}_1 = 0$	$\text{VER}_1 = 1$
$(2.0339, 2.0339)$	$(2.0494, 2.0455)$
$(3.2623, 3.2623)$	$(3.2087, 3.2039)$
$(3.8982, 3.8982)$	$(3.9482, 3.9438)$

NOTE: Model parameters are $\alpha_2 = 0.2748$, $\gamma_2 = 0.0789$, $\lambda_2 = 0.0087$, $\rho_2 = 0.1411$, and $\tau_2 = 14.9974$ with $g(x) = \rho_2 + \exp(-\tau_2/x)$.

$\text{VER}_1 = 0$	$\text{VER}_1 = 1$
–	–
–	–
$(3.1238, 3.1238)$	$(3.2087, 3.2039)$

NOTE: Model parameters are $\alpha_3 = 0.3841$, $\gamma_3 = 0.0923$, $\lambda_3 = 0.0010$, $\rho_3 = 0.1755$, and $\tau_3 = 11.0009$ with $g(x) = \rho_3 + \exp(-\tau_3/x)$.

We end our appendix on BLP with a discussion of the version of the model with many firms and products. Our ideas can be used on models with multidimensional dependent variables, as long as the model has complementarities among the different dimensions. Complementarities ensure that extremal equilibria change monotonically with the explanatory variable. Typical examples of models with complementarities are games of strategic complements (Vives, 1990; Milgrom and Roberts, 1990; Milgrom and Shannon, 1994).

¿From Berry, Levinsohn, and Pakes's (1995) estimates, BLP conclude that prices were not strategic complements in the automotive industry in the period they analyze, a result they regard as surprising. Even if this poses a problem for replicating BLP's analysis, the most interesting use of our methods is with new data, where strategic complements seem likely to be the norm. There is a clear intuitive idea that rivals' increases in price makes one want to increase, not decrease, prices; Milgrom and Shannon (1994) show formally how strategic complements in models of price competition arise under mild conditions on demand (see also Vives (1999)).

Finally, BLP's rejection of strategic complementarities is based on the results obtained under detailed parametric assumptions on the underlying structure. Given the difficulties related to the estimation of structural parameters when multiple equilibria are present (see Section 2.2.2), it may be sensible to use our approach as an alternative: it can be interpreted as a joint test of strategic complementarities and monotone comparative statics.

Appendix C Multiple Equilibria and Identification

We exhibit a simple example which illustrates how in models with multiple equilibria, the equilibrium selection procedure may interfere with identification of finite dimensional parameters in structural models with conditional moment restrictions.

Say that the structural model is given by:

$$\theta^2 Y + \theta Y^2 - Y^3 + \theta X = U \tag{17}$$

where Y is a scalar dependent variable, θ is a real structural parameter of interest, X is a scalar explanatory variable and U is a scalar disturbance term assumed unobserved. The conditional distribution $F_{Y|X}$ of Y given X is induced by the structure $S \equiv (\theta_0, F_{U|X}, \mathcal{P}_{XU})$, in which $\mathcal{P}_{XU}$ denotes the selection procedure according to which a particular equilibrium for Y is chosen from the set of solutions (given $\theta = \theta_0$, X and U) to the structural equation (17).

We consider the case in which there are complementarities between the explanatory and dependent variables, so $\theta_0 > 0$. Suppose that $F_{U|X}$ satisfies:

$$E(U|X) = 0 \text{ a.s.} \tag{18}$$

The above mean independence condition implies the following moment condition on the observables:

$$\theta_0^2 E(Y|X) + \theta_0 \big[E(Y^2|X) + X\big] - E(Y^3|X) = 0 \text{ a.s.} \tag{19}$$

The left hand side of Equation (19) defines a quadratic function g of θ, $g(\theta) \equiv \theta^2 E(Y|X) + \theta\big[E(Y^2|X) + X\big] - E(Y^3|X)$; a parameter value θ_0 of θ is then said to be identified if for any θ, $g(\theta) = 0$ a.s. implies $\theta = \theta_0$. We now show that different equilibrium selection procedures lead to different identification outcomes.

For concreteness, consider the case in which the true value of the structural parameter in Equation (17) is $\theta_0 = 1$, X and U are independent and their respective distributions are as follows: X can take only two values, $x = -8/27$ and $x = -14/27$ with equal probability, while U is normally distributed with mean 0 and standard deviation 8/27.

Figure 4 plots the structural function $r(y, x, \theta_0) \equiv \theta_0^2 y + \theta_0 \left[y^2 + x\right] - y^3$ as a function of y, with x fixed and $\theta_0 = 1$. The mapping $y \mapsto r(y, x, 1)$ is nonlinear and for any $u \in [-5/27 + x, 1 + x]$, the structural equation $r(y, x, 1) = u$ admits multiple equilibria for y. When $u < -5/27 + x$ or $u > 1 + x$ on the other hand, the equilibrium is unique.

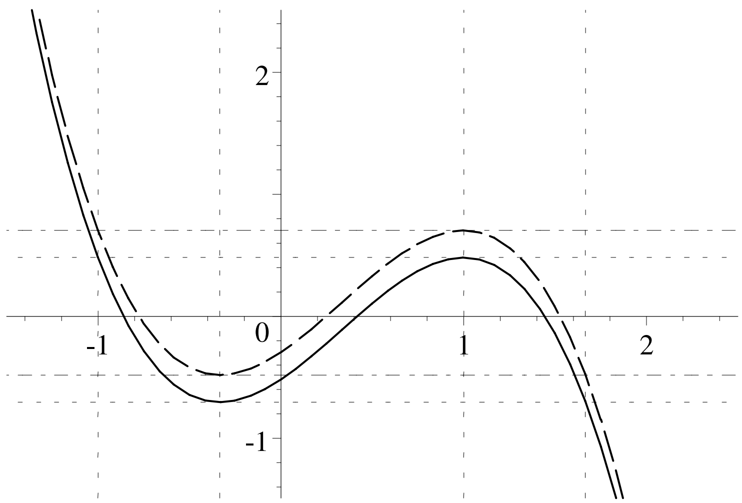

Figure 4: Plot of $y \mapsto r(y, x, 1)$ with $x = -14/27$ (solid line) and $x = -8/271$ (dashed line). Horizontal lines are at $-5/27 + x$ and $1 + x$ with $x = -14/27$ (dotted) and $x = -8/27$ (dot-dashed). Vertical (dotted) lines are at -1, $-1/3$, 1 and 5/3.

Table 3 gives the values of the first three non-centered conditional moments of Y given X, under three different equilibrium selection procedures: (1) $\mathcal{P}^{\text{high}}$ chooses the largest equilibrium with probability one; (2) $\mathcal{P}^{\text{mid}}$ chooses the middle equilibrium with probability one; and (3) $\mathcal{P}^{\text{low}}$ chooses the smallest equilibrium with probability one.

First, consider an equilibrium selection procedure $\mathcal{P}^{\text{a}}_{XU}$ which would make the conditional expectation of Y equal to zero: $E[Y|X] = 0$ a.s. Specifically, using the values from Table 3, one such a procedure is given by mixing $\mathcal{P}^{\text{high}}$ and $\mathcal{P}^{\text{low}}$ with probabilities that depend on x and are given by:

Table 3: Conditional moments of Y given X under different equilibrium selection procedures.

	$x=-8/27$			$x=-14/27$		
	$\mathcal{P}^{\text{high}}$	$\mathcal{P}^{\text{mid}}$	$\mathcal{P}^{\text{low}}$	$\mathcal{P}^{\text{high}}$	$\mathcal{P}^{\text{mid}}$	$\mathcal{P}^{\text{low}}$
$E[Y\|X=x]$	1.4941	0.3260	−0.6429	1.3095	0.3407	−0.8274
$E[Y^2\|X=x]$	2.2999	0.2672	0.7277	2.0293	0.2770	0.7522
$E[Y^3\|X=x]$	3.4977	0.2969	−0.2115	2.8203	0.0992	−0.5938

$\mathcal{P}^{\text{a}}_{xU}$	$\mathcal{P}^{\text{high}}$	$\mathcal{P}^{\text{mid}}$	$\mathcal{P}^{\text{low}}$
$x=-8/27$	30.083%	0	69.917%
$x=-14/27$	38.721%	0	61.279%

Under $\mathcal{P}^{\text{a}}_{XU}$, the equation $g(\theta)=0$ a.s. reduces to $\theta\big[E(Y^2|X)+X\big]-E(Y^3|X)=0$ a.s. which is linear in θ and has a unique solution $\theta_0=1$. In other words, the mean independence restriction in (18) suffices to identify $\theta_0=1$.

On the other hand, consider an equilibrium selection procedure $\mathcal{P}^{\text{b}}_{XU}$ which is such that $E[Y^2|X]+X=0$ a.s. For example, let $\mathcal{P}^{\text{b}}_{XU}$ be given by:

$\mathcal{P}^{\text{b}}_{xU}$	$\mathcal{P}^{\text{high}}$	$\mathcal{P}^{\text{mid}}$	$\mathcal{P}^{\text{low}}$
$x=-8/27$	1.431%	0	98.569%
$x=-14/27$	13.783%	0	86.217%

Under $\mathcal{P}^{\text{b}}_{XU}$, the equation $g(\theta)=0$ a.s. becomes $\theta^2E(Y|X)-E(Y^3|X)=0$ a.s., whose solutions are $\theta_0=\{-1,1\}$. In other words, $\theta_0=1$ is not identified under $\mathcal{P}^{\text{b}}_{XU}$. Rather, under $\mathcal{P}^{\text{b}}_{XU}$ it holds that the mean independence restriction in (18) identifies the set $\theta_0=\{-1,1\}$. However, as the elements of the identified set have opposite signs, it is impossible to conclude under $\mathcal{P}^{\text{b}}_{XU}$ whether X and Y are complements.