APPLICATION OF MULTIVARIABLE MEASUREMENT AND CONTROL STRATEGIES TO A PROBLEM IN HEAT CONDUCTION

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Summary

A detailed analysis of the boundary control of a one-dimensional heat conduction system is performed using recent results in multivariable feedback theory. Non-interacting control, optimal control, inverse Nyquist array and characteristic locus techniques were applied to the system. An analysis is made of the potential improvements in performance through the use of extra measurements and through the change in measurement location. Results show the advantages and disadvantages of the techniques and point to the flexibility of the characteristic locus method with and without the squaring down of extra measurements.

Introduction

This paper describes an in-depth analysis of a practical, multivariable, distributed parameter system using both frequency-domain and time-domain analysis. As a typical distributed system, a one-dimensional, heat conduction problem is considered. The normalized temperature distribution $u(z,t)$ is described by the diffusion equation

$$\frac{\partial^2 u(z,t)}{\partial z^2} = \frac{\partial u(z,t)}{\partial t} \quad (1)$$

The controls $v_1(t)$ and $v_2(t)$ are taken to be the heat fluxes at $z = 0$ and $z = 1$. Thus the initial and boundary conditions are

$$u(z, 0) = u_0(z)$$

$$\left. \frac{\partial u(z,t)}{\partial z} \right|_{z=0} = -v_1(t), \quad \left. \frac{\partial u(z,t)}{\partial z} \right|_{z=1} = v_2(t). \quad (2)$$

Though this heat conduction process is a relatively simple control problem, the analysis leads to conclusions that can be extended to general, multivariable control theory. In particular, the two-input system is studied in regard to model reduction and controllability, and feedback design is considered using multivariable root-locus techniques, optimal and modal control, non-interacting control, the inverse Nyquist array method and the characteristic locus procedure. Furthermore, the role of the number of measurements and their location is analyzed.

Model Reduction

Difficulties arise in the control system design of distributed parameter systems because of state variations in both time and space. The thrust of many of the feedback control design techniques for distributed systems is to reduce the system to a lumped one and then to

take advantage of the many theories available for lumped parameter control design. However, problems arise in that all of the analysis performed on the lumped system is dependent on the method and accuracy of the reduction. Though considerable model reduction is necessary to reduce computational complexities in the design procedures, excessive or inaccurate reduction can lead to a system whose behavior is quite different from that of the original system.

To obtain the lumped parameter model for a system described by partial differential equations, many efficient techniques are described in the literature. In particular, much work has been published on various lumping strategies for linear diffusion equations. A particularly useful means of treating both linear and nonlinear P.D.E. systems is the method of weighted residuals (m.w.r.) along with other pseudo-modal techniques, such as finite element methods [1] or the use of spline functions [2]. Though these techniques are quite powerful, solutions using the m.w.r. often require considerable effort to determine the set of orthogonal coordinate functions and a high-order lumped model for accurate results. To eliminate these difficulties, spatial discretization techniques are often quite useful for linear diffusion systems, since they retain the physical characteristics of the system. However, they too often lead to high-order lumped models.

Thus to improve the accuracy and reduce the order of the model, it may be best to use an exact reduction technique. Since the heat equation is governed by a parabolic equation, exact lumping can be performed using a Laplace transform in time or through a modal analysis. The latter, which is simply an application of the separation of variable solution procedure, is quite attractive for systems which can be made self-adjoint, since the technique leads directly to the eigenvalues and eigenfunctions (modes) of the system. If the eigenvalues are real, discrete and well spaced, the modal representation is a convenient method to reduce the order of the system, since only the dominant modes need be retained for design purposes.

Both the Laplace transform and modal analysis techniques were applied to the one-dimensional heat conduction problem. Though the methods can be shown to lead to equivalent results, modal analysis leads directly to a lumped, state-space representation. The problem is redefined using the Dirac Delta function:

$$\begin{aligned} \frac{\partial u(z,t)}{\partial t} &= \frac{\partial^2 u(z,t)}{\partial z^2} + \delta(z-0)v_1(t) + \delta(z-1)v_2(t) \\ z &= 0^-, \quad z = 1^+ \quad \frac{\partial u(z,t)}{\partial z} = 0. \end{aligned} \quad (3)$$

A solution of the form

$$u(z, t) = \sum_{n=0}^{\infty} a_n(t) \varphi_n(z) \quad (4)$$

$$\delta(z-0)v_1(t) + \delta(z-1)v_2(t) = \sum_{n=0}^{\infty} b_n(t) \varphi_n(z).$$

can then be assumed. After substituting into (3), choosing the separation constant as $-\lambda_n$ and applying the boundary conditions, the equations can be solved to yield

$$\varphi_n(z) = \begin{cases} 1.0 & n=0 \\ \sqrt{2} \cos n\pi z & n=1, 2, \dots \end{cases} \quad (5)$$

Application of the orthogonality of the eigenfunctions, $\varphi_n(z)$, then leads to the coefficients $a_n(t)$ and $b_n(t)$:

$$a_n(t) = \int_0^1 \varphi_n(z) u(z, t) dz,$$

$$b_n(t) = \begin{cases} v_1(t) + v_2(t) & n=0 \\ \sqrt{2} v_1(t) + (-1)^n \sqrt{2} v_2(t) & n=1, 2, \dots \end{cases} \quad (6)$$

Since the eigenvalues, $\lambda_n = n^2\pi^2$, increase rapidly with increasing n , the system can be accurately represented by the first few eigenfunctions, $n = 0, 1, \dots, N$. The process model can then be obtained as an N th-order lumped state-space representation with the $N+1$ states $a_0, \dots, a_N$:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{v}(t), \quad u(z, t) = \mathbf{C}\mathbf{x}(t), \quad (7)$$

where $\mathbf{x}^T(t) = [a_0(t), \dots, a_N(t)]$,

$$\mathbf{A} = \text{diag}(-n^2\pi^2), \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \\ \vdots & \vdots \\ \sqrt{2} & (-1)^N \sqrt{2} \end{bmatrix},$$

$$\mathbf{C} = [\varphi_0(z), \varphi_1(z), \dots, \varphi_N(z)].$$

In theory, this representation requires the complete temperature profile, $u(z, t)$, of the system. However, this difficulty can be circumvented by using an optimal smoothing technique or state estimation or by measuring $u(z_i, t)$ at M spatial points where $M < N+1$ and using output control on the measurements. For the latter case, the system should be controlled by using set points on the outputs rather than on the states, since the $N+1$ th-order state vector $\mathbf{x}(t)$ cannot be obtained uniquely with $M < N+1$ measurements.

Regardless of the technique for estimating $u(z, t)$, the appropriate transfer function representation can be obtained by converting to the Laplace domain. With $u_0(z) = 0$,

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\mathbf{v}(s), \quad u(z, s) = \mathbf{C}\mathbf{x}(s), \quad (8)$$

or with measurements at M distinct points:

$$u(s) = \mathbf{C}\mathbf{x}(s), \quad \mathbf{C} = \begin{bmatrix} \varphi_0(z_1) & \dots & \varphi_N(z_1) \\ \vdots & & \vdots \\ \varphi_0(z_M) & \dots & \varphi_N(z_M) \end{bmatrix} \quad (9)$$

Thus the lumped parameter system process transfer function is given by $G_p(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$.

Root-Locus

The concept of root-locus analysis is basic to classical control system design for single-input, single-output processes. The root-locus diagram is advantageous since it describes the character of the response as the gain of the controller is continuously changed, by allowing rapid determination of the roots of the characteristic equation. Though scalar root-locus techniques are well-known, the multivariable root-locus problem is relatively new. Kouvaritakis and Macfarlane [3,4] describe the technique and discuss the analysis of system zeros.

The objective of the root-locus method is to investigate the behavior of the closed-loop characteristic frequencies when the feedback gain matrix has the form $G_c = k\mathbf{I}$. The root-loci for the modal lumped representation of the heat conduction system are the loci of the roots of the characteristic equation

$$|\mathbf{sI} - \mathbf{A}| |\mathbf{I} + k\mathbf{G}_p| = 0 \quad (10)$$

as k varies from 0 to ∞ . Obviously, the poles ($k = 0$) of the system are at $s = 0, -\pi^2$ and $-4\pi^2$ independent of the measurements.

For symmetric measurement placement ($z_2 = 1 - z_1$), the root-loci show that the system remains stable at high gains for $z_1 \leq 0.33$; whereas, it becomes unstable at high gains for $0.33 < z_1 < 0.50$. This results from the large lag time between the control action at $z = 0$ and its effect on the output. For the situation where $z_1 > 0.50$, a permutation matrix should be used to interchange the inputs and outputs to reduce the difficulties caused by the large lag times. Additionally, the root-loci analysis predicts that the responses for symmetric measurements with $z_1 < 0.25$ are non-oscillatory; whereas, for other cases, oscillations are expected at moderate to high gains. For unsymmetric measurement placement, stability is also enhanced if lag times are reduced by placing measurements near the controls.

Time-Domain Analysis

The time-domain approach for control system analysis utilizes the differential or difference equations directly rather than using transfer functions, as in the frequency-domain analysis. Of the many time-domain procedures, optimal control and modal control are the most common techniques and thus have been studied extensively for both lumped and distributed parameter systems. Because of the great deal of literature available on these theories and on their application to heat conduction systems, only a cursory examination of the techniques will be presented. Additionally, other methods such as adaptive control and state estimation will not be considered here, though they may be quite useful in practical applications.

Linear-quadratic optimal control can readily be applied to the heat conduction system to obtain the feedback control law using either the lumped or distributed parameter model. By combining state estimation with a linear-quadratic control strategy, excellent feedback control is possible. However since this paper is intended as a comparison of theoretical analysis techniques in relation to measurement problems, the optimal control analysis of the system will not be considered.

Modal control is another time domain technique applicable to both the lumped and distributed system [6]. Modal analysis is based on the postulate that the transient behavior of a process is primarily determined by the modes associated with the smallest eigenvalues and that the response of the system can be improved by using a control design to increase these eigenvalues. Though Rosenbrock's approach implied the possibility of altering each eigenvalue separately without limit, this is impractical due to limitations on the number of controls and measurements and their locations. For this two-control process, the first two eigenvalues can be altered without affecting the higher $N-2$ eigenvalues; however, the first two eigenvectors are also altered slightly since the matrix B is fixed. Thus a disturbance in either of the first two modes will cause a disturbance in the higher modes. However since these modes decay rapidly, little difficulty should result.

Modal control can also be applied directly to the distributed parameter system. Gould and Murray-Lasso [6,7] present a detailed discussion of modal control for linear, distributed systems. However, using these techniques and considering the limitations imposed by taking only a finite number of discrete measurements and by manipulating only two controls, the analysis becomes equivalent to using modal lumping and applying lumped modal analysis.

Though modal analysis allows exact placing of the poles, the procedure leads to a simple proportional controller and cannot give any guidance as to the selection of additional control action. More importantly, the technique uses no information concerning the zeros of the transfer functions. Problems can easily arise if the zeros of the closed-loop transmittances move into the right-half plane. Furthermore, the method gives the best results only if all the states of the system are accessible. Though this does not cause any difficulty for the heat conduction process, it is not practical for most systems.

Non-Interacting Control

The major difficulty in multivariable, feedback control design arises from the steady-state and dynamic interactions which occur between the various input and output variables. In most systems, it is desired that one specific output $y_i(s)$ responds to input $r_i(s)$, and all other outputs remain unchanged. The term interaction can then be used to refer to the effects that a particular input $r_i(s)$ has on the output $y_j(s)$, $j \neq i$. Thus low interaction is usually desirable. In fact if it is possible to eliminate all the coupling between variables and if the number of controllable inputs and outputs is equal, the multivariable system can be treated as a combination of single-loop systems, and classical techniques can be used to tune each loop. Unfortunately, most multivariable systems have considerable interaction, thus leading to problems in control system design. Therefore, several

techniques have been devised to eliminate or at least reduce the interaction to an acceptable level.

For the heat conduction process, analysis of the system transfer function shows that interaction is high and cannot be eliminated by simple selection of the measurement locations. Thus compensators that reduce or eliminate interaction may be useful. Several design techniques including perfect non-interacting compensation, steady-state decoupling and set point compensation are available for this purpose. Additionally, we have developed a technique that eliminates interaction by using extra measurements within an inner loop, leading to a relatively simple control strategy.

For perfect dynamic compensation of the heat conduction system, the compensator transfer function G_1 is selected to be $G_1 = G_p^{-1}$. This selection leads to a total decoupling for set point changes and even though each disturbance can influence all the outputs, its effect on output u_i is damped by a single controller $g_{ic}(s)$. Though this method is simple and seemingly eliminates all the complications inherent in multivariable, feedback control design, it has several major disadvantages. Perfect compensation requires a perfect transfer function model, can lead to an unnecessarily complicated compensator and fails if the determinant of G_p has right-half plane zeros or if the transfer functions are not square. Thus other techniques may lead to a simpler control scheme which is easier to tune on-line and less sensitive to model inaccuracies.

Though perfect compensation may be impossible or merely impractical, steady-state decoupling is usually quite simple and can be implemented before applying other design techniques such as the inverse Nyquist array or the characteristic locus method. Steady-state decoupling uses a compensator

$$G_1 = \lim_{s \rightarrow 0} G_p^{-1}(s) \quad (11)$$

to eliminate steady-state interaction, but it cannot improve dynamic behavior. Unfortunately, due to the perfect symmetry of the heat conduction system, even this method leads to difficulty. Introduction of this compensator into the control loop leads to dependent control action and is therefore not advisable, since it results in excessive dynamic interaction and poor closed-loop performance.

We have developed another technique, inner-loop decoupling, for eliminating interaction and have applied it to the heat conduction system. Several current references [8,9] discuss squaring down extra measurements in an inner loop to obtain a system that shows better control properties from the viewpoint of the outer loop. Usually the inner loop is used to adjust the poles of the system, and then other frequency response techniques are used to design the outer loop control for the improved process. However, the new technique uses the additional degrees of freedom inherent in adding an inner loop to eliminate or minimize the interaction, with only simple proportional gain compensators and without severely limiting the design freedom available for the outer loop.

Consider the process shown in Figure 1. The outer loop uses two outputs and two controls and is identical to the system used by the previous methods. However using the 3rd-order model for the process, a third independent measurement can be made. The three

Figure 1. Inner Loop Analysis

measurements are denoted by y . The following relationships describe the control scheme.

$$u = LC'x = Cx, \quad y_F = F'C'x = Fx$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} 1 & \varphi_1(z_1) & \varphi_2(z_1) \\ 1 & \varphi_1(z_2) & \varphi_2(z_2) \\ 1 & \varphi_1(z_3) & \varphi_2(z_3) \end{bmatrix}$$

Let G_p be the transfer function from α to u , or essentially the transfer function of the process seen by the outer loop. The objective of the technique is to select the elements of F to reduce or eliminate interactions, i.e., select F so that G_p is nearly diagonal.

To eliminate interaction, there are eight degrees of freedom available--the locations of the two major measurements and the six elements of the squaring down compensator F . The location of the final measurement can be left to the discretion of the designer, since it will only affect the components of F' once F is specified. Obviously, z_3 must be selected different from z_1 and z_2 so that C' is nonsingular. The locations z_1 and z_2 and the six elements of F should be selected so that the (1,2) and (2,1) terms of G_p are identically zero. Of these eight available degrees of freedom, six are used to eliminate interaction, and the remaining two can be used to improve the performance (move the poles) of each non-interacting loop. Simple algebra shows that z_1 must equal 0.2 or 0.6 and z_2 must equal 0.4 or 0.6 for decoupling of the loops. These rules lead to considerable simplification in the transfer functions. For example, for the two situations (0.2,0.6) and (0.6,0.4)

$$G_p = \begin{bmatrix} \frac{2\sqrt{2}C_{12}}{s+2\sqrt{2}C_{12}F_{12}} & 0 \\ 0 & \frac{2\sqrt{2}C_{12}}{s+2\sqrt{2}C_{12}F_{21}} \end{bmatrix} \quad (12)$$

Thus not only has interaction been eliminated by using simple gain in an inner loop, the new process is quite simple, and two degrees of freedom, along with the location of z_3 , remain at the disposal of the designer. The selection of F_{11} and F_{21} can easily be used to move the poles of each decoupled loop.

The final design then involves a squaring down compensator F' , which, together with the measurement

matrix C' , leads to the desired F . The fact that F' is a constant matrix is of great importance. If more degrees of freedom are desired than the two available with this technique, F can be designed as a function of s . However, this leads to an unnecessarily complicated feedback compensator.

Inner-loop decoupling is an improvement over other non-interacting control schemes. Though all the methods can easily be performed, the inner-loop decoupling strategy leads to a very simple control structure for processes where extra measurements can readily be obtained and leaves considerable design freedom available for tuning the dynamic behavior of the overall system. The only restriction on the extra measurements is that all outputs must be linearly independent. Additionally, the method still suffers from the apparent necessity of an accurate process model. However, simulations show that though the design is conducted on the approximate, 3rd-order model for the heat conduction process, interaction is still nearly eliminated for the actual system. In fact, simulations also show low sensitivity to the measurement locations. Thus the inner-loop scheme seems to be superior to conventional, non-interacting control methods.

Inverse Nyquist Array

Since classical frequency methods proved so useful for designing single-loop control systems, much work has been devoted to extending these techniques to multivariable systems. The inverse Nyquist array (I.N.A.) technique, introduced by Rosenbrock [10], is a useful extension of scalar Nyquist array methods to the design of multivariable control systems. It allows for considerable flexibility, is insensitive to model inaccuracies and reduces to traditional methods in the single-loop case. Also, the technique is able to handle models specified only in terms of a limited amount of directly obtained experimental data [11] and can be easily incorporated into a computer-aided design package.

The I.N.A. design method involves adding controllers to make the system's open-loop transfer function diagonally-dominant. The remainder of the design is then completed on the basis of a set of individual single loops using conventional single-loop inverse Nyquist techniques. The I.N.A. method thus leads to a stable system which has high integrity and low interaction, when diagonal dominance is imposed.

The 3rd-order model of the heat conduction process was analyzed using the I.N.A. procedure with various measurement locations. The I.N.A. analysis, using the modified Nyquist contour suggested by MacFarlane and Postlethwaite [12] for systems with a pole on the imaginary axis, verifies the root-locus results that stability is insured at high gains for the symmetric case if $z_1 < 0.33$.

The problem then is to design a compensator, $G_c(s)$, so that the system's open-loop transfer function $Q(s)$ is diagonally-dominant when $0.33 < z_1 < 0.50$. Using a general 2×2 compensator, the restrictions on the elements of $G_c(s)$ can be derived. Specifically, consider the case with measurements (0.4,0.6). It can be proven that due to the symmetry of the system it is not possible to design a constant matrix G_c to make the system diagonally-dominant for all s on the modified Nyquist contour, without forming dependent control. A pre-compensator $G_c(s)$ can be designed to insure diagonal dominance; however, the encirclement criterion cannot be satisfied. This was verified using root-locus and characteristic locus analyses. Attempts at eliminating the instability were futile, and further analysis showed that the stability problems result from an unstable finite zero. Unfortunately, it can be shown that dependent control action is necessary to move this zero to the left-half plane. However, dependent control leads to very bad closed-loop response and is thus not desirable. Therefore, the best that can be obtained is a diagonally-dominant system for which single-loop theory can be applied to two loops that remain stable only for moderate gains.

Finally, it is instructive to consider the case where the measurements are not placed symmetrically, i.e., $z_2 \neq 1 - z_1$. The case (0.4,0.8) is diagonally-dominant and stable; thus, arbitrarily high gains can be applied, leading to a system with small interaction. Other cases that were considered are (0.4,0.7) and (0.3,0.8). Only the first of these requires compensation.

Characteristic Locus Method

The characteristic locus method (C.L.M.) is an extension of classical Bode-Nyquist, frequency-response theories and state-space techniques for the design of general, multivariable feedback control schemes. The method combines the essential features of both approaches by using the properties of linear vector spaces defined over base fields of complex functions. As in the I.N.A. technique, the C.L.M. requires the use of computer-aided graphic display and can be incorporated into a computer-aided design package. Though the I.N.A. technique defines approximate conditions for stability using bands on inverse Nyquist plots, the C.L.M. gives an exact indication of stability and a systematic technique for choosing the best controller matrix in terms of system stability, integrity, interaction and accuracy, rather than simply stressing diagonal dominance and single-loop design.

The C.L.M. uses the frequency dependent properties of the eigenvalues and eigenvectors of the open-loop transfer function $Q(s) = G_p(s)G_c(s)$ to assess the closed-loop properties of the system. The design technique attempts to compensate and modify the system's response by adjusting the eigenproperties of $Q(s)$. The analysis for multivariable systems leads to a set of criteria for achieving a compromise between the basic objectives of stability, integrity, interaction and accu-

racy [13]. The basis of the technique is the use of the characteristic loci, which are the paths in the complex plane drawn by the eigenvalues of $Q(s)$ as s traverses the standard Nyquist contour. Closed-loop stability can be assured by selecting the compensator $G_c(s)$ such that the net sum of the counterclockwise encirclements of the critical point $(-1,0,0)$ by the characteristic loci is equal to the number of open-loop unstable poles p_0 . For the 3rd-order model of the heat conduction process, $p_0 = 0$. Furthermore, integrity against transducer and error-monitoring channel failures in all possible combinations of the loops can be assured if the characteristic loci of all the principal submatrices of $Q(s)$ satisfy the encirclement criterion. For integrity against actuator failures, similar considerations apply to the matrix $G_c(s)G_p(s)$.

It can be shown that at low frequencies, interaction can be suppressed by imposing large moduli on the characteristic loci, and at high frequencies, by selecting $G_c(s)$ so that the eigenvectors of $Q(s)$ are nearly as possible aligned with the standard base set. If the measure of alignment is taken as the angles between the eigenvectors and the base vectors e_i , the objective is to reduce the misalignment angles. MacFarlane and Belletrutti [13] show that, though these criteria lead to a reduction in interaction, they do not necessarily result in a $Q(s)$ being nearly diagonal or diagonally-dominant, and thus a feedback system can be made nearly non-interacting without imposing diagonally-dominance. Finally, system accuracy will be high provided that the characteristic loci have large moduli at low frequency.

The objective of the characteristic locus method is then to select a controller $G_c(s)$ so that the characteristic loci of $Q(s)$ satisfy the stability criteria and have high gains at low frequencies and low misalignment angles at high frequencies. Additionally, $G_c(s)$ should be selected so that its elements are rational functions of s , so that $|G_c(s)|$ is nonsingular and has no right-half plane zeros and so that all the poles of $G_c(s)$ are in the open left-half complex plane. MacFarlane and Kouvaritakis [14] present a systematic approach for using the C.L.M.. In particular, a means of manipulating the characteristic loci and characteristic directions is developed so that the phase of the loci can be adjusted to achieve acceptable stability and integrity and so that the directions can be aligned and the gains balanced to reduce interaction.

For the symmetric cases with $z_1 < 0.33$, the encirclement criterion is satisfied for all gains. As was also shown using root-loci and I.N.A. analyses, these cases will remain stable with increasing gain. Also as expected, high gains lead to instability for $0.33 < z_1 < 0.50$. In particular, a maximum gain of 32.0 would be allowed before the critical point is encircled with $z_1 = 0.4$. Thus for the symmetric cases, compensation is required for stability of the system at high gains, when $0.33 < z_1 < 0.50$. Additionally, regardless of the measurement location one eigenvalue has large magnitude at low frequencies while the second has very small magnitude, and the misalignment angles are both 45° at high frequencies. Thus some compensation is desired to improve integrity, interaction and accuracy of the system. Since the cases (0.3,0.7), (0.4,0.6) and (0.4,0.8) are representative of the problem, they were analyzed in detail using the C.L.M..

Though the case (0.4,0.8) is stable for all gains without compensation, a compensator is useful for balancing the magnitudes of the loci at low frequency

and for reducing misalignment angles at high frequency. For instance, with the compensator

$$G_c(s) = \frac{0.8}{s} \begin{bmatrix} 4.348 & -4.348 \\ -4.348 & 4.348 \end{bmatrix} + \begin{bmatrix} 1.0 & -0.371 \\ 0 & 0.929 \end{bmatrix}, \quad (13)$$

the characteristic loci have nearly identical moduli at low frequencies, and the misalignment angles are quite small at high frequencies. Thus the C.L.M. seemingly leads to a stable system with high integrity, low interaction and high accuracy. Furthermore the method concludes that this design has the best compromise between these qualities, since the compensators were designed using approximately commutative theories. Similar results are obtained with other measurements.

The use of easily available measurements in excess of the commanded outputs (the commanded outputs are those directly needed for comparison with the set points) is considered, so as to make more efficient use of the gain. The procedure involves squaring down the extra measurements within an inner loop, in order to form a new set of outputs equal in number to the number of inputs. Instead of using the extra degrees of freedom to reduce interaction as was done previously, the objective is to use the inner loop to suitably place the poles of the outer loop so that the C.L.M. (or I.N.A. method) can be applied to a 'better' system as seen from a control point of view. The inner-loop design is based on the placement of the finite zeros and the manipulation of the root-loci asymptotes so that by finally setting the inner-loop gain at a suitable value, the poles of the outer loop are pulled into better locations in the complex frequency plane.

The case (0.4,0.6) previously was found to be unstable for high gains due to an unstable finite zero, and it was shown that only dependent control would lead to stability since such control action was necessary to move the finite zero. Unfortunately, simple algebraic calculations show that an inner loop has no effect on the finite zeros of the system, and therefore the stability arguments are unchanged from before. Nevertheless, a compensator can be designed to improve the system by reducing interaction and increasing accuracy, if the values of the gain are restricted so as to assure stability.

Since the C.L.M. design for the case (0.4,0.8) without an inner loop led to a compensator that perfectly balanced the gain at low frequency and significantly reduced the misalignment angles at high frequency, there is little or no incentive to add an inner loop; however, the characteristic loci for the case (0.3,0.7) with an inner loop are much better balanced than without an inner loop, and the misalignment angles are greatly reduced for moderate frequencies. Thus it can be concluded that, though the inner loop is often beneficial, it does not always improve the overall design. If the criteria for a good C.L.M. design can be satisfied without an inner loop, then further analysis may not be necessary. However if the misalignment angles cannot be reduced or the gains balanced without an inner loop, the inner loop may be useful.

Control System Performance

The control structure analysis for the heat conduction system was performed on the 3rd-order, lumped model using various frequency-response techniques. These include non-interacting control methods, root-loci

analysis, inverse Nyquist array and the characteristic locus method. Additionally, the time-domain techniques of optimal and modal control were applied to both the lumped and distributed models. Since all the schemes provide a design for the control structure, a comparison among the system responses is needed. Additionally since most of the designs were performed using the 3rd-order lumped model, actual system responses are necessary to assess the effectiveness of the designed controllers.

In general, simulations show that the process behaves qualitatively similarly to the 3rd-order model and that high gain significantly reduces interaction for certain cases. However, proportional control leads to considerable offset. Even if only one of the two set points is changed, there is offset in both outputs. This is obviously a result of the steady-state interactions in the system. Though steady-state decoupling would seem beneficial, previous analysis showed that such compensation leads to a dependent control system. Thus even though the root-loci and I.N.A. analyses showed that arbitrarily high gains would reduce interaction without leading to instability in certain cases, very high gains are often needed to sufficiently reduce the offset and interaction. Such high gains may be impractical or may lead to large overshoot, and even when steady-state interactions are reduced by high gains, considerable dynamic interaction may still be present.

Though simple introduction of high gain into both control loops may be adequate, the control structure can often be improved by a slightly more complicated compensator. From classical theory, it is obvious that some form of integral action should be introduced to eliminate the offset. The C.L.M. determined the best compensator of the proportional-integral (PI) form to reduce interaction, to increase system integrity and accuracy and to insure stability. However, simple analysis shows that, due to the integral action, the closed-loop responses become oscillatory. Simulations were performed using the C.L.M. designs with and without the inner loop. They show that system responses can be improved with integral action and that appropriate design of the inner loop can eliminate the oscillations. For the case (0.3,0.7), simulations show that for low gains considerable interaction and offset are present along with long response times; whereas, for high gains (Figure 2a), interaction is reduced and the system becomes oscillatory. As predicted, the responses (Figure 2b) for the system with an inner loop and the outer-loop compensator designed using the C.L.M. show tremendous improvement. The inner-loop design is excellent, since even at a relatively low overall gain of 20, oscillations and interaction are minor and response times are short.

Obviously in a heat conduction problem, another difficulty in designing a control scheme is the lag time between a control action and when the outputs respond to it. For the heat conduction process, this lag leads to large overshoot when the feedback loop has high gains, but will not lead to instability as long as the measurements are placed within the first and last third of the system. If the measurements are very near the edges of the system, even high gains will not lead to overshoot; whereas, high gains leads to system instability if the measurements are within the center third of the system. Thus it would seem appropriate to add some sort of anticipatory control, i.e., add some derivative action. This should reduce oscillations and keep the system stable up

Figure 2. System response for unit set point change at $z_1 = 0.3$ and no change in set point at $z_2 = 0.7$

(a) Feedback gain 20.0 (b) C.L.M. compensation with Inner Loop, overall gain 20.0

to higher gain. None of the common multivariable control design techniques incorporate the use of derivative action, since it is often difficult to physically incorporate into the process and can lead to difficulties for step input changes and for noisy systems.

Additionally, the inner-loop decoupling control strategy was considered. Theoretically, it led to a perfectly non-interacting system for several measurement locations, but the design was based on the 3rd-order model and could be useless for the actual system. Simulations show, however, that it is an excellent control scheme for the heat conduction process, even when the measurement locations are not those specified by the technique. Previous analysis showed that the case (0.2,0.8) is stable and diagonally dominant. Thus high gains should reduce

offset, interaction and response time. Though interaction is small at a gain of 50, such gain is quite high, and steady-state offset is still about 5%. Using extra measurements and an appropriate inner-loop compensator, the 3rd-order model was shown to reduce to a completely non-interacting system for which classical methods could be used to complete the design. Third-order simulations verify the results predicted from the model. Interaction is completely eliminated and the response is non-oscillatory. Additionally with $F_{11}=F_{21}=0$, proportional control does not result in steady-state offset. The actual system behaves qualitatively as expected. Figures 3a and 3b show the improvement obtained using inner-loop decoupling.

Figure 3. System responses for unit set point change at $z_1 = 0.2$ and no change in set point at $z_2 = 0.8$

a) Feedback gain 1.0 (b) Inner Loop Decoupling, overall gain 1.0

Finally, perfect non-interacting compensation and steady-state decoupling are considered. The latter is obviously undesirable since it leads to dependent control. Unfortunately, perfect non-interacting control also has little practical application, though in theory it is an excellent strategy. For the 3rd-order, heat conduction system, the required compensation is a proportional-integral-derivative (PID) controller with large derivative action, which leads to excessive control for system noise or disturbances and difficulties for set point changes since the derivative of the error becomes infinite. Additionally, perfect compensation sacrifices closed-loop performance to non-interaction and thus leads to poor dynamic behavior.

Conclusions

Though much of the analysis and design in this project is an application of published results, the work provides an insight into the current state of multivariable, feedback control theory. We have performed a complete analysis of a distributed control problem using various currently available methods. In particular, an extensive study of time-domain analysis, frequency-domain design and non-interacting control has been conducted for a two-input, heat conduction system. Though heat conduction systems have previously been studied in relation to optimal and modal control, the multiple-input problem has been relatively neglected.

Additionally, the role of the number of measurements and their location was studied. Due to the ease of taking temperature measurements for the heat conduction problem, outputs can be obtained as needed by the control system. Thus the complete system state could easily be approximated using many measurements and optimal smoothing or fewer measurements and state estimation. However since such flexibility is not available in many practical systems, the control scheme must work on a limited amount of output information. Thus the analysis of the heat conduction system was performed with a finite number of measurements, leading to the problems of measurement placement and feedback loop interaction and instability. Since most design techniques require feedback of outputs equal in number to the inputs, analysis of the use of 'extra' measurements for improving system response, by reducing interaction or moving system poles, was performed. This work led to the new technique of inner-loop decoupling, which shows excellent results for the highly-interacting, highly-symmetric, heat conduction system. In cases where extra outputs are available or easily accessible, they should be considered and may simplify the control structure significantly. However since a simple inner loop cannot move system zeros, stability problems due to unstable finite zeros cannot easily be eliminated with extra measurements. Unfortunately, no good technique is currently available to insure stability of such systems.

Finally we have considered the role of measurement location. Each choice of measurement location leads to a completely different optimal design and significantly affects the usefulness of each design technique, since the locations determine the extent of interaction, transportation lag and system symmetry. As expected, problems are minimized with measurements near the edges of the system, since interaction and transportation lags are reduced.

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