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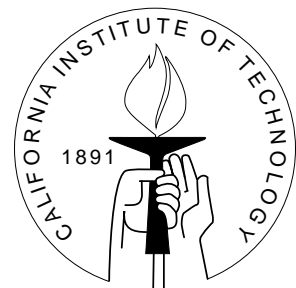
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## STANDARD VOTING POWER INDEXES DON'T WORK: AN EMPIRICAL ANALYSIS

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# Standard Voting Power Indexes Don't Work: An Empirical Analysis

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## Abstract

Voting power indexes such as that of Banzhaf (1965) are derived, explicitly or implicitly, from the assumption that all votes are equally likely (i.e., random voting). That assumption can be generalized to hold that the probability of a vote being decisive in a jurisdiction with  $n$  voters is proportional to  $1/\sqrt{n}$ .

We test—and reject—this hypothesis empirically, using data from several different U.S. and European elections. We find that the probability of a decisive vote is approximately proportional to  $1/n$ . The random voting model (or its generalization, the square-root rule) overestimates the probability of close elections in larger jurisdictions. As a result, classical voting power indexes make voters in large jurisdictions appear more powerful than they really are.

The most important political implication of our result is that proportionally weighted voting systems (that is, each jurisdiction gets a number of votes proportional to  $n$ ) are basically fair. This contradicts the claim in the voting power literature that weights should be approximately proportional to  $\sqrt{n}$ .

Keywords: Banzhaf index, decisive vote, elections, electoral college, Shapley value, voting power

## 1 Introduction

Recent events such as 2000 U.S. Presidential election and the expansion of the European Union have rekindled interest in evaluating electoral systems. Both the U.S. Electoral College system for electing the president and the European Union's Council of Ministers,

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in which the representative from each country gets some specified number of votes, are examples of weighted or asymmetric voting systems. The U.S. Senate can also be considered an asymmetric voting system, since the number of people represented by each senator varies greatly from state to state. In these asymmetric voting systems, voters can have potentially differential impact on electoral outcomes. A natural question that arises, therefore, is whether or not a particular system is politically fair.

In a weighted voting system, two aspects of voting power are of potential interest: (a) the voting power of a particular member of the legislature (or, of a particular state in the Electoral College, or a particular country represented in the E.U.), and (b) the power of an individual voter. The first aspect of voting power is relevant for understanding how the legislature works, and the second aspect relates to the fairness of the system with respect to the goal of representing people equally.

Voting power can be defined and measured in many different ways (see Saari and Sieberg, 1999, and Felsenthal and Machover, 1998). Most these measures are theoretical in the sense that they calculate voting power under a given electoral system by making some simplifying assumption about individuals' voting behavior. All the standard measures of theoretical voting power yield the counterintuitive result that, in a proportional voting system, voters in large districts tend to have disproportionate power. Thus, it has been claimed that voters in large states have more power in the U.S. Electoral College (Banzhaf, 1968), and that, if E.U. countries were to receive votes in the Council of Ministers proportional to their countries' populations, then voters in large countries would have disproportionate power (Felsenthal and Machover 2000).

These claims are controversial and are defended based on mathematical argument. However, these arguments are ultimately based on assumptions that can be checked, and falsified, with real data (as is noted by Heard and Swartz, 1999). In this paper, we perform empirical checks and find both the assumptions underlying voting power measures and the numerical measures themselves to be seriously flawed. In testing the foundations of theoretical voting power measures, we show how to *empirically* estimate voting power from observed elections data. When such data is available, this empirical measure should be preferred since it does not require making these problematic assumptions.

The most important political implication of our findings is that proportional weighting systems are, in fact, basically fair to all voters, and alternative systems that have been recommended in the voting power literature—for example, giving each jurisdiction a vote proportional to the square root of the number of people represented—are unfair to voters in large jurisdictions. This is the same conclusion reached, from a game-theoretic argument, by Snyder, Ting, and Ansolabehere (2001).

Voting power indexes have been criticized before, largely from the direction that they do not capture idiosyncrasies in any given electoral system (e.g., Garrett and Tsebelis, 1999). (By their very nature, standard voting power measures rely only on the mathematical rules of a voting system and not on past or anticipated future patterns of voting within the system.) Voting power measures have also been evaluated in the context of

corporate governance (Leech, 2002). Our paper is new in that it gathers data from a wide variety of elections to show the inappropriateness of the mathematical model underlying the usual measures of voting power. We also explain from a theoretical perspective why voting power measures are inconsistent with modern models of public opinion and electoral politics.

## 2 The mathematical model underlying voting power measures

### 2.1 Defining the power of an individual voter

In this paper we shall consider elections with two parties (A and B), with  $n_j$  voters in jurisdictions  $j = 1, \dots, J$ . Each jurisdiction  $j$  is given  $e_j$  “electoral votes,” the vote in each jurisdiction is chosen winner-take-all, and the total winner is the party with more electoral votes, with ties at all levels broken by coin tosses. We further define  $v_j$  as party A’s share of the vote in jurisdiction  $j$ ;  $E_{-j}$  as the total number of electoral votes, excluding those from jurisdiction  $j$ , that go for party A; and  $E = \sum_{j=1}^J e_j$  as the total number of electoral votes.

Voting power can be defined in various ways, but the definition most relevant to representation of voters is in terms of the probability that a voter is decisive (see Heard and Schwartz, 1999, and Felsenthal and Machover, 1998). At the top level, the voting power of jurisdiction  $j$  is the probability that party A wins if jurisdiction  $j$  goes for party A, minus the probability that party A wins if jurisdiction  $j$  goes for party B. Next, the power of a given *voter* in jurisdiction  $j$  is the probability that party A wins if that voter supports A, minus the probability that A wins if that voter supports B.

The probability of a vote being decisive is important directly—it represents your influence on the electoral outcome, and this influence is crucial in a democracy—and also indirectly, because it could influence campaigning. For example, one might expect campaign efforts to be proportional to the probability of a vote being decisive, multiplied by the expected number of votes changed per unit of campaign expense, although there are likely strategic complications since both sides are making campaign decisions (see Brams and Davis 1974). The probability that a single vote is decisive in an election is also relevant in determining the utility of voting, the responsiveness of an electoral system to voter preferences, the efficacy of campaigning efforts, and comparisons of voting power (Riker and Ordeshook, 1968, Ferejohn and Fiorina, 1974, Brams and Davis, 1975, Aldrich, 1993). Perhaps the simplest measure of decisiveness is the (absolute) Banzhaf (1965) index (which has also been proposed by Penrose, 1946, and others; see Felsenthal and Machover, 2000, Section 1.2.3, for a historical overview), which is the probability that an individual vote is decisive under the assumption that all voters are deciding their votes independently and at random, with probabilities 0.5 for each of two parties.

In general, the key step in defining voting power is assigning a probability distribution over all possible voting outcomes. The Banzhaf index and related measures are sometimes defined in terms of game theory and sometimes in terms of set theory, but they can all be interpreted in terms of probability models. For example, suppose that your voting power is defined as the number of coalitions of other voters for which your vote is decisive. This is simply proportional to the probability of decisiveness, under the assumption that all vote outcomes are equally likely. We refer to this as the *random voting model*.

As we discuss below, the random voting model has strong implications for voting power. It also has strong implications for actual votes—and these implications do not fit reality. The later parts of this paper discuss the implications for voting power of that lack of fit.

## 2.2 What does the random voting model imply about voting power?

For an individual in jurisdiction  $j$ , his or her vote is decisive if (a) the  $e_j$  electoral votes of jurisdiction  $j$  are decisive in the larger election, and (b) the individual's vote is decisive in the election within the jurisdiction. Using conditional probability notation, this can be written as,

$$\text{individual voting power} = \Pr(\text{jurisdiction } j\text{'s } e_j \text{ electoral votes are decisive}) \times \\ \times \Pr(\text{a given vote is decisive in jurisdiction } j \mid \text{jurisdiction } j\text{'s } e_j \text{ electoral votes are decisive}).$$

We can write each of these in our notation, keeping in mind that ties are decided by coin flips,

$$\Pr(\text{jurisdiction } j\text{'s } e_j \text{ electoral votes are decisive}) = \\ \Pr(E_{-j} \in (0.5E - e_j, 0.5E)) + \frac{1}{2}\Pr(E_{-j} = 0.5E - e_j) + \frac{1}{2}\Pr(E_{-j} = 0.5E) \\ \Pr(\text{a given vote is decisive in jurisdiction } j) = p_{v_j}(0.5)/n_j, \quad (2)$$

where  $p_{v_j}$  is the continuous probability density assigned to the vote for party A in jurisdiction  $j$ . (We are assuming that  $n_j$  in any district is large enough—greater than 10, say—so that the model for the can be approximated by a continuous distribution. A derivation of this approximation, under general conditions, appears in the Appendix.)

Under the random voting model, the two events (1) and (2) are independent, so we can evaluate the probabilities separately, which we now do.

### 2.2.1 The probability that a jurisdiction is decisive, under the random voting model

Assuming random voting, one can directly calculate the probability that jurisdiction  $j$ 's electoral votes are decisive by assigning a probability  $1/2^{J-1}$  to each of the configurations

of the other  $J - 1$  jurisdictions, which in turn induces a distribution on  $E_{-j}$ , so that (1) can be calculated. In specific cases, the results can reveal important properties of the electoral system. We illustrate with a simple example with  $J = 4$ . Suppose  $(e_1, e_2, e_3, e_4) = (12, 9, 6, 2)$ . Then the fourth jurisdiction has zero voting power—its two electoral votes can never determine the winner. In addition, the first three jurisdictions each have equal voting power of  $1/2$ —any of these jurisdictions will be decisive if the other two are split. In this example, the relation between electoral votes and voting power is far from linear.

If the number of jurisdictions is large, however, with no single jurisdiction dominating, and no unusual patterns (such as in the example above in which all but one of the  $e_j$ 's are divisible by 3), then it is possible to approximate the probabilities in (1) using a continuous distribution. (This would be most simply done using the normal distribution, but other forms such as the scaled beta distribution used by Gelman, King, and Boscardin, 1998, are also possible). One can then approximate (1) by  $\int_{0.5E-e_j}^{0.5E} p(E_{-j})dE_{-j}$ . If the further assumption is made that  $e_j$  is small compared to the uncertainty in  $E_{-j}$ , then  $\Pr$  (jurisdiction  $j$ 's  $e_j$  electoral votes are decisive) will be approximately proportional to  $e_j$ .

For the rest of this paper we shall assume that (1) is proportional to  $e_j$ . Computing (1) more precisely in special cases is potentially important, but such details do not affect the main point of this paper.

### 2.2.2 The probability that a voter is decisive within a jurisdiction, under the random voting model

The key way that the random voting model affects the calculation of voting power is through factor (2), the probability that a vote is decisive within a jurisdiction.

Under random voting, the distribution  $p_{v_j}$  of the vote share among the  $n_j$  voters in jurisdiction  $j$ , is approximately normally distributed with mean 0.5 and standard deviation  $0.5/\sqrt{n_j}$ , hence the approximation,

$$\Pr(\text{a given vote is decisive in jurisdiction } j) = p_{v_j}(0.5)/n_j = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n_j}}.$$

What matters here is not the factor  $\sqrt{2/\pi}$  but the proportionality with  $1/\sqrt{n_j}$ .

### 2.2.3 The power of an individual voter, under the random voting model

Now that the two factors (1) and (2) have been approximated assuming random voting, they can be multiplied to yield a voting power approximately proportional to  $e_j/\sqrt{n_j}$  for any individual in jurisdiction  $j$ .

Under the natural weighting system in which the electoral votes  $e_j$  are set proportional to  $n_j$ , an individual's voting power is then proportional to  $\sqrt{n_j}$ , and the sum of the voting powers of the  $n_j$  voters in the jurisdiction is proportional to  $n_j^{3/2}$ . Hence the titles of the papers by Banzhaf (1968) and Brams and Davis (1974).

Conversely, a suggested reform (Penrose, 1946, Felsenthal and Machover, 2000) is to set  $e_j$  proportional to  $\sqrt{n_j}$ , so that individual voting power (assuming the random voting model) is approximately the same across jurisdictions. Even with large  $n_j$ 's, this is only an approximation (see Section 2.2.1), but it captures the essential implications of the random voting model.

### 2.3 Probability models for voting: going beyond the square-root rule

As has been noted by many researchers (e.g., Beck, 1975, Margolis, 1977, Merrill, 1978, and Chamberlain and Rothchild, 1981), there are theoretical and practical problems with a model that models votes as independent coin flips (or, equivalently, that counts all possible arrangements of preferences equally). The simplest model extension is to assume votes are independent but with probability  $p$  of going for party A, with some uncertainty about  $p$  (for example,  $p$  could have a normal distribution with mean 0.50 and standard deviation 0.05). However, this model is still too limited to describe actual electoral systems. In particular, the parameter  $p$  must realistically be allowed to vary, and modeling this varying  $p$  is no easier than modeling vote outcomes directly. Actual election results can be modeled using regressions (e.g., Campbell, 1992), in which case the predictive distributions of the election outcomes can be used to estimate the probability of decisive votes. Gelman, King, and Boscardin (1998) argue that, for modeling voting decisions, it is appropriate to use probabilities from forecasts, since these represent the information available to the voter before the election occurs. For retrospective analysis, it may also be interesting to use models based on perturbations of actual elections as in Gelman and King (1994).

At this point, the idea of modeling vote outcomes seems daunting—there are so many different possible models, and such a wealth of empirical data, that it would seem impossible to make any general recommendations. Hence, researchers have argued in favor of the random voting model as a reasonable—or perhaps the only possible—*a priori* choice.

However, general *a priori* models other than random voting are possible. The key is to realize that, as discussed in the previous section, what is important for voting power is how  $p_{v_j}(0.5)$ , the probability density of the vote proportion at 0.5, varies with  $n_j$ .

In particular, Good and Mayer (1975) and Chamberlain and Rothchild (1981) derive a  $1/n_j$  rule—that is, a model where the probability of decisiveness is inversely proportional to the number of voters. This model arises by assuming that votes are binomial

distributed, but with binomial probabilities  $p$  that themselves vary over the  $n_j$  voters in jurisdiction. Then, for large or even moderate values of  $n_j$ , the distribution  $p_{v_j}$  of the vote proportion  $v_j$  is essentially fixed (not depending on  $n_j$ ), so that  $p_{v_j}(0.5)$  is a constant, and the probability of a vote being decisive is proportional to  $1/n_j$  (from (2)). The proportionality constant depends on the conditions of the election—but for computing voting power, the only thing that matters is the proportionality with  $1/n_j$ .

A similar result arises if the probability distribution of votes is obtained from forecasts (whether from a regression-type model, subjective forecasters, or some combination of the two). For example, in a two-party election with 10,000 voters, if one party is forecast to get 52% of the vote with a standard error of 3%, then the probability that an individual vote is decisive is then approximately  $\frac{1}{\sqrt{2\pi}(0.03)} \exp(-\frac{1}{2}(0.05/0.03)^2)/n_j = 1.84/n_j$ .

## 2.4 Voting power and the closeness of elections

To summarize, standard voting power measures are based on the assumption of random voting—or, more specifically, on the assumption that the probability of decisiveness within a jurisdiction is proportional to  $1/\sqrt{n_j}$ . This in turn corresponds to the assumption that  $p_{v_j}(0.5)$ , the probability density function of the vote proportion  $v_j$  near 0.5, is proportional to  $\sqrt{n_j}$ . Or, to say it another way, the key assumption is that elections are much more likely to be close (in percentage terms) when  $n_j$  is large.

In contrast, it is usual in forecasting elections to model the vote proportion  $v_j$  directly, with no dependence on  $n_j$ , in which case  $p_{v_j}(0.5)$  does not depend on  $n_j$  and so the probability of decisiveness is proportional to  $1/n_j$ .

The models for votes have strong implications for voting power. The  $1/\sqrt{n_j}$  model leads to the normative recommendation that, to equalize the voting power of all individuals, electoral votes  $e_j$  should be set roughly proportional to  $\sqrt{n_j}$ . In contrast, the model in which vote proportions do not depend on  $n_j$  implies that it is basically fair to set  $e_j$  roughly proportional to  $n_j$ .

Now that we have isolated the key question—how does the probability of a close election depend on  $n_j$ —we can explore it empirically (in Sections 3 and 4, following the work of Mulligan and Hunter, 2001) and theoretically (in Section 5). The square-root rule has important practical implications. Such an assumption can and should be checked with actual data, not simply asserted.

## 2.5 Voting power and representation

Another way to look at voting power in a two-stage electoral system is in terms of the net number of voters whose opinions are carried by the representative. For example, if parties A and B receive 51% and 49% of the vote, respectively, then party A has a net



support of 2% of the voters in that district. That is, the number of electoral votes for jurisdiction  $j$  (or, more generally, its voting power) should be set proportional to the absolute difference in votes between the two parties, which in our notation is  $2n_j|v_j-0.5|$ . The vote  $v_j$  (and, to a lesser extent, the turnout,  $n_j$ ) are random variables that are unknown after the election, and so it is natural to work with the expected net voters for the winning candidate in the jurisdiction,  $E(2n_j|v_j-0.5|)$ .

Under the random voting model,  $v_j$  has a mean of 0.5 and a standard deviation proportional to  $1/\sqrt{n_j}$ , and so the expected vote differential,  $E(2|v_j-0.5|n_j)$ , is proportional to  $\sqrt{n_j}$ . Penrose (1946) and Felsenthal and Machover (2000, Section 2.3) use this reasoning to support the claim that large jurisdictions are overrepresented in proportional weighting.

Conversely, if the proportional vote margin is independent of  $n_j$ , this supports proportional weighting and suggests a problem with the Banzhaf index and related voting power measures. We study the empirical relation between  $E(|v_j-0.5|)$  and  $n_j$  in Sections 3 and 4.

### 3 Data from the U.S. electoral college

Perhaps the most frequently-considered example of voting power in elections (as distinguished from voting within a legislature) is for the President of the United States. The general conclusion is that the Electoral College benefits voters in large states. For example, Banzhaf (1968) claims to offer “a mathematical demonstration” that it “discriminates against voters in the small and middle-sized states by giving the citizens of the large states an excessive amount of voting power,” and Brams and Davis (1974) claim that the voter in a large state “has on balance greater potential voting power . . . than a voter in a small state.” Mann and Shapley (1960), Owen (1975), and Rabinowitz and Macdonald (1986) come to similar conclusions. This impression has also made its way into the popular press; for example, Noah (2000) states, “the distortions of the Electoral College . . . favor big states more than they do little ones.”

As discussed in the previous section, these claims—which are particularly counter-intuitive given that small states are *overrepresented* in electoral votes, because of the extra two votes given to each state, no matter what size—arise directly from the square-root assumption embedded in the standard power indexes. In order to see whether this assumption is reasonable, it is necessary to look at the data.

There are four major factors affecting the probability of a decisive vote in the Electoral College. We have already discussed two of these factors: the number of voters in the state and the number of electoral votes. The third factor is the closeness of the national vote—if this is not close (as, for example, in 1984 or 1996), then the vote in any given state will be irrelevant. The fourth factor is the relative position of the state politically. For example, it is highly unlikely that voters in Utah will be decisive: if the national

election is close enough that Utah’s electoral votes will be relevant, then Utah will almost certainly go strongly toward the Republicans.

How can or should these factors be used to determine voting power? We consider two analyses. In Section 3.1, we look only at the number of voters and the number of electoral votes—that is, the “structure” of the electoral system. As discussed in the previous section, the voting power will then depend on the dependence of  $p_{v_j}(0.5)$  on  $n_j$ , which we can study empirically. Section 3.2 examines estimates of the probability of decisiveness from the political science literature that use a range of forecasting information, including the relative positions of the states, and then see empirically how voting power depends on the size of the state.

### 3.1 Closeness of the election as a function of the number of voters

As Banzhaf (1968), Brams and Davis (1974; 1975, p. 155), and Owen (1975, p. 953), make clear, the power-index results for the Electoral College are consequence of the claim that elections in large states are more likely to be close. More precisely, the random voting model implies that the standard deviation of the difference in vote proportions between the two parties will be inversely proportional to the square root of the number of voters.

In fact, however, this is not the case, or at least not to the extent implied by the square-root rule. To analyze this systematically, we extend an analysis of Colantoni, Levesque, and Ordeshook (1975, pp. 144–145) and display in Figure 1 the vote differentials as a function of number of voters for all states (excluding the District of Columbia) for all elections from 1960 to 2000.

We test the square-root hypothesis by fitting three different regression lines to  $|v_j - 0.5|$  as a function of  $n_j$ . First, we use the lowess procedure (Cleveland, 1979) to fit a nonparametric regression line. Second, we fit a curve of the form  $y = c/\sqrt{n_j}$ , using least-squares to find the best-fitting value of  $c$ . Third, we find the best-fitting curve of the form  $y = cn_j^\alpha$ . If the best-fitting  $\alpha$  is near  $-0.5$ , this would support the square-root rule and the claims of the voting power literature. But if the best-fitting  $\alpha$  is near 0, then this provides evidence *rejecting* the voting power model in favor of the model in which the vote differential does *not* systematically vary with  $n_j$ .

We do not mean to imply by this analysis that state size is the only factor or even the most important factor determining the closeness of elections. Rather, we are giving insight into the fact that the power indexes of Banzhaf (and others) rely on an assumption which does not fit the data.

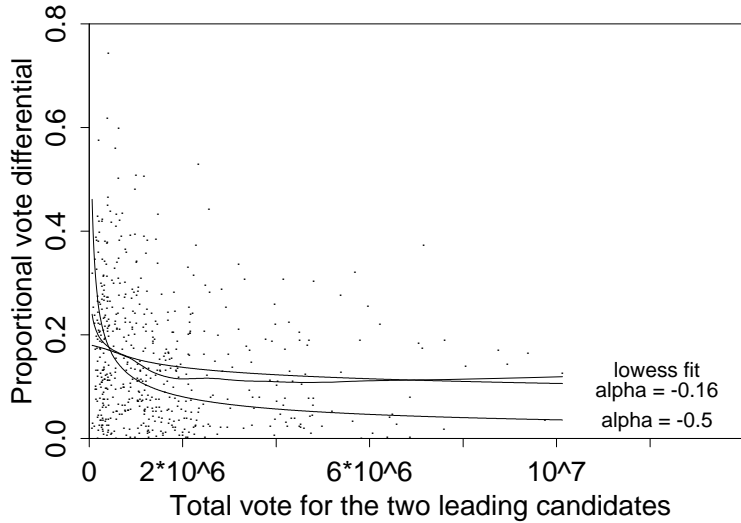


Figure 1: *The margin in state votes for President as a function of the number of voters  $n_j$  in the state: each dot represents a different state and election year from 1960–2000. The margins are proportional; for example, a state vote of 400,000 for the Democratic candidate and 600,000 for the Republican would be recorded as 0.2. Lines show the lowess (nonparametric regression) fit, the best-fit line proportional to  $1/\sqrt{n_j}$ , and the best fit line of the form  $cn_j^\alpha$ . As shown by the lowess line, the proportional vote differentials show only very weak dependence on  $n_j$ . The  $1/\sqrt{n_j}$  line, implied by standard voting power measures, does not fit the data.*

### 3.2 Using election forecasts to estimate the probability of a decisive vote

Another way to study voting power is to estimate the probability of casting a decisive vote in each state, using all available information, and then studying the dependence of this probability on state size. This was done by Gelman, King, and Boscardin (1998), using a hierarchical regression model with error terms at the national, regional, and state levels. The model, based on that of Campbell (1992), was fairly accurate, with state-level errors of about 3.5%. Figure 2 displays the resulting estimates of probabilities of decisive vote, for the Presidential elections between 1948 and 1992. The relation between the probability of a vote being decisive and the size of the state is very weak.

The claim in the voting power literature that large states benefit from the Electoral College were mistaken because of their implicit assumption that elections in larger states would be much closer than those in small states. Although this square-root rule does *not* empirically apply to Presidential elections, might it hold in other elections or decision-making settings, in which case results such as Banzhaf (1965) could be reasonable? We consider this next.

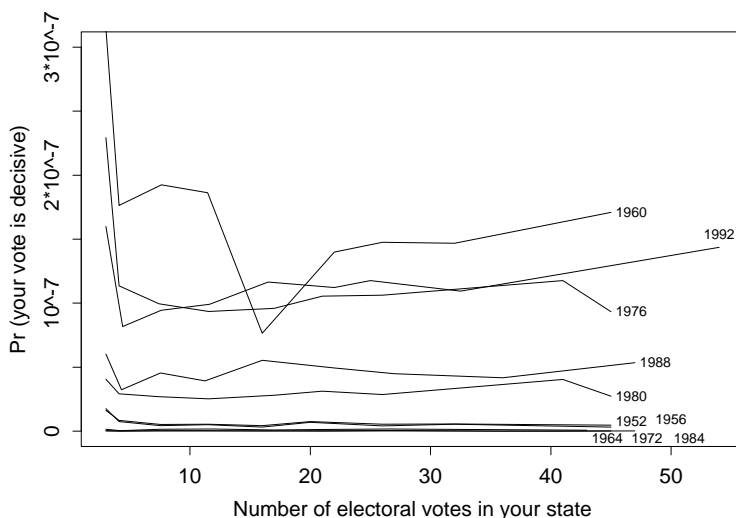


Figure 2: *The average probability of a decisive vote as a function of the number of electoral votes in the voter’s state, for each U.S. Presidential election from 1952–1992 (excluding 1968, when a third party won in some states). The probabilities are calculated based on a forecasting model that uses information available two months before the election. This figure is adapted from Gelman, King, and Boscardin (1998). The most notable features of this figure are: first, that the probabilities are all very low; and second, that the probabilities vary little with state size, with the most notable pattern being that voters in the very smallest states are a bit more likely to be decisive.*

## 4 Data from other electoral systems

We examine the dependence of closeness of elections as a function of number of voters for various electoral systems in the United States and Europe. From (2), the probability that a single vote is decisive is  $p_{v_j}(0.5)/n_j$  or, more generally,  $1/n_j$  times the probability that  $2|v_j - 0.5|$ , the proportional vote difference between the two leading parties, is within some specified distance of 0. Standard voting power indexes are based on a model that implies that the standard deviation of  $v_j$  is proportional to  $1/\sqrt{n_j}$ , so that as  $n_j$  increases, elections are more likely to be close.

We replicate the analysis in Section 3.1 for these various electoral systems. The graphs in Figure 3 show, for each of several electoral systems, the absolute value of the proportional vote differential vs. the number of voters for the two leading parties in the election. (We have excluded as uncontested any election in which the losing party received less than 10% of the vote.) As with the Electoral College (shown in Figure 1), we see in some cases a very slight decrease in the proportional vote differential as a function of the number of voters—but this decline is much less than predicted by the square-root rule. Each graph displays the lowess (nonparametric regression) line, the best-fit  $1/\sqrt{n_j}$  line, and the best-fit line of the form  $cn_j^\alpha$ . In each case, the lowess line is much closer to horizontal than to  $1/\sqrt{n_j}$ , and the best-fit parameters  $\alpha$  are all closer to 0 than to  $-0.5$ ,

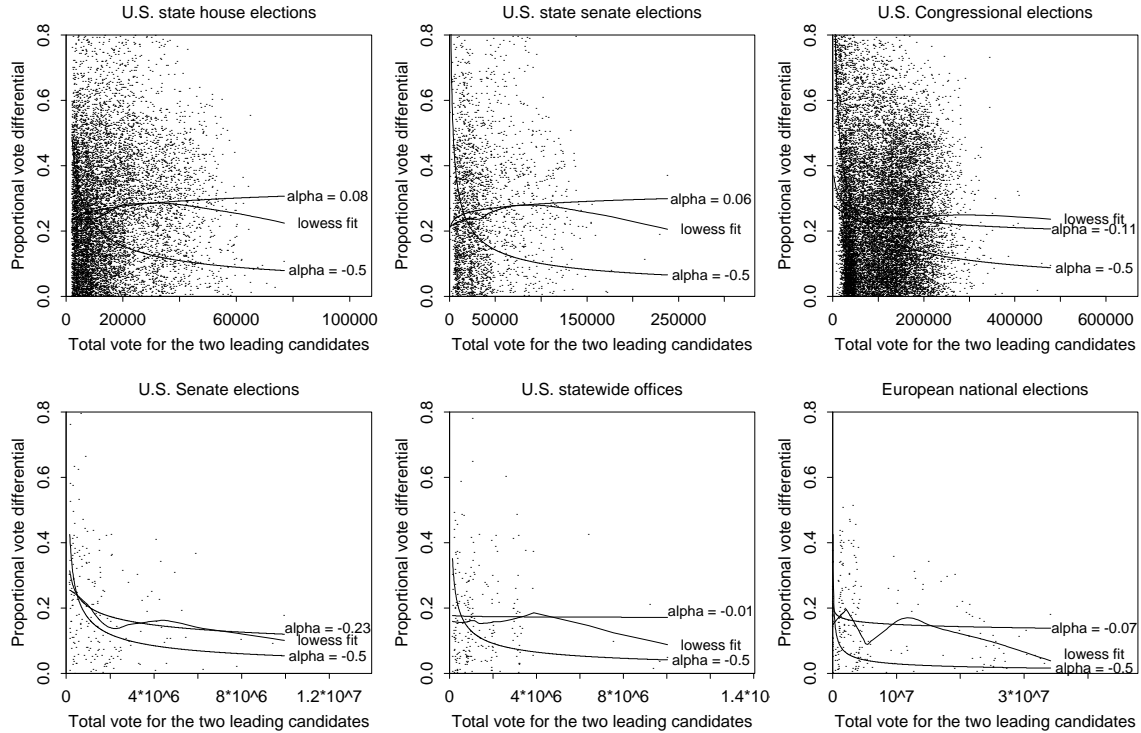


Figure 3: *Proportional vote differential vs. number of major-party voters  $n_j$ , for contested elections in the following electoral systems: (a) lower houses of U.S. state legislatures with single-member districts, 1984–1990, (b) U.S. state senates, 1984–1990, (c) U.S. House of Representatives, 1896–1992, (d) U.S. Senate, 1988–1996, (e) European national elections, 1950–1998. Each plot includes lines showing the lowess (nonparametric regression) fit, the best-fit line proportional to  $1/\sqrt{n_j}$ , and the best fit line of the form  $cn_j^\alpha$ . As shown by the lowess line, the proportional vote differentials show only very weak dependence on  $n_j$ . The  $1/\sqrt{n_j}$  lines, implied by standard voting power measures, do not fit the data.*

which would correspond to the square-root rule.

The lowess and best-fit power-law curves in Figures 1 and 3 are essentially flat. Or it could be said that they decline slightly with  $n$ , perhaps proportional to  $n_j^{-0.1}$ . This implies that the probability of a decisive vote is proportional to  $n_j^{-0.9}$ , which is far closer to  $1/n$  (as in the election forecasting literature) than to  $1/\sqrt{n}$  (as in the voting power literature).

## 5 Theoretical arguments

Sections 3 and 4 give empirical evidence that the distribution of the vote share  $v_j$  is approximately independent of the number of voters,  $n_j$ , at least for reasonably large  $n_j$ . How can we understand this theoretically?

## 5.1 Understanding the results based on local, regional, and national swings

Politically, the reason why the square-root rule does *not* hold is that elections are affected by local, state, regional, and national swings. Such swings have been found in election forecasting models (e.g., Gelman, King, and Boscardin, 1998) and in studies of shifts in public opinion (Page and Shapiro, 1992).

Here, we can appeal to standard theories of public opinion, in which an individual voter’s preference between two parties depends on the voter’s ideological position, the parties’ ideologies, and the voter’s positive or negative impressions of the parties on non-ideological “valence” issues (such as competence as a manager and personal character). To start with, the variation in ideology among the voters induces a spread in the distribution of voters’ probabilities. Next, general changes in impressions about the valence issues—as caused, for example, by a recession or a scandal—shift the entire distribution of the probabilities, so that they will not necessarily be centered at 0.5 (as would otherwise be implied by a Downs-like theory of political competition).

Another way to understand the distribution of vote differentials is to compare to the square-root model, which implies that elections will be extremely close when  $n_j$  is large. This does not happen because of national swings which can shift the mean, in any particular election, away from 0.5.

More generally, swings in public opinion and votes can occur at many levels, from local to national and even internationally. The result of this multilevel or fractal variation is that the standard deviation  $v_j$  will be expected to decline as a function of  $n_j$ , but at a slower rate than  $1/\sqrt{n_j}$  (Whittle, 1956). In fact, the data are consistent with a power law with an exponent of  $\alpha = -0.1$  (see Figures 1 and 3), which could be used to infer something about the fractal nature of voting patterns. For the purposes of this paper, however, we are focusing on the fact that  $\alpha$  is closer to 0 than to  $-0.5$  for a wide variety of electoral systems, and this is consistent with modern models of public opinion (which consider large-scale vote swings) and sociology (in which individuals are connected in complex networks, as in Watts, Dodds, and Newman, 2002).

## 5.2 A simulation study based on Presidential votes by Congressional district

We can also understand the distribution of votes in terms of sums of random variables. For example, California in the 2000 Presidential election had 38 times as many voters as Vermont. If we could think of California as a sum of 38 independent Vermont-sized pieces, then we would expect its  $v_j$  to have  $1/\sqrt{38} = 1/6.2$  of the standard deviation of Vermont’s. But this is not the case, either in terms of the closeness of the actual election result or in terms of forecasting. Uncertainties about  $v_j$  are not appreciably smaller in large than in small states.

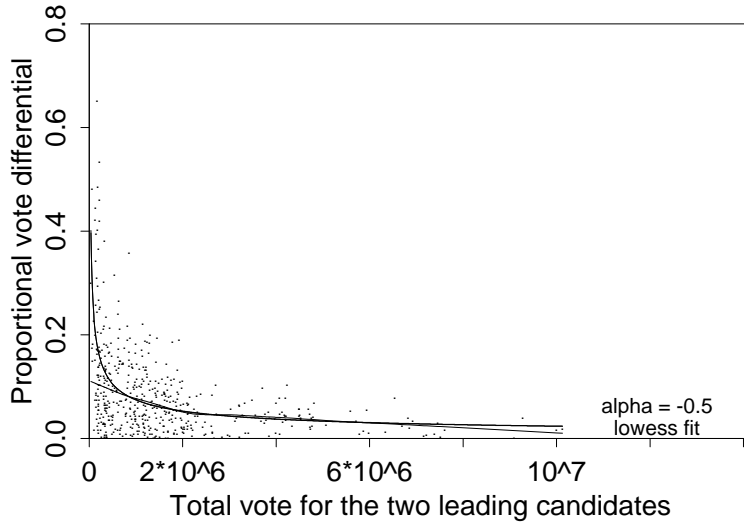


Figure 4: *Proportional vote differential vs. number of voters  $n_j$ , for a random simulation of the electoral college based on Presidential elections by Congressional district from 1960–2000. For each year, votes in all the districts were shifted so that the total vote was 50% for each party. The districts were then permuted at random within each year so that, for example, “Alabama in 1960” was constructed from 9 randomly-chosen Congressional districts in that election year, “Alaska in 1960” was a different Congressional district chosen at random, and so forth. Lines display the lowess fit and the best-fit line of the form  $cn_j^\alpha$ . The best fit is  $\alpha = -0.5$ , which makes sense since the “states” were formed by combining districts at random, eliminating state, regional, and national swings.*

To get a sense of what would happen if there were no state, regional, or national opinion swings, we performed a simulation study using the Electoral College data displayed in Figure 1. For each election year, we took the votes for President by Congressional district and subtracted off a constant so that the national mean was 0.5 (thus removing nationwide swings). We then permuted the 435 districts in each year and reallocated them to states based on the number of Congressional districts in each “state”; this random permutation removed all correlations associated with state and regional swings. Finally, we plotted the absolute vote differential vs. turnout for these simulated elections and computed the best-fit line of the form  $cn_j^\alpha$ .

Figure 4 shows the results. With these simulated data, the square-root rule fits very well (in fact, the best-fit power  $\alpha$  was  $-0.494$ , essentially equal to a theoretical value of  $-0.5$ ). This graph shows that classical voting power measures would be appropriate if elections had no state, regional, or national swings. The comparison to Figure 1 shows the inappropriateness of the voting power model for actual Presidential elections.

## 6 Discussion

### 6.1 Mathematical results and normative claims

Proponents of voting power measures make strong claims—not just mathematical statements, but normative recommendations. For example, Penrose (1946) asserts, “A nation of 400 million people should, therefore, have ten times as many votes (or members) on an international assembly as a nation of 4 million people,” and later writers have made similarly confident pronouncements (see the titles of Banzhaf’s two papers and the quotations at the beginning of Section 3).

We showed (in Sections 3 and 4) that the “square-root rule” for closeness of elections, which underlies standard voting power measures, is inappropriate for data from a wide range of elections, a result that is consistent with the findings of Mulligan and Hunter (2001). Section 5 discussed theoretical reasons why the square-root rule does not hold.

One justification for voting power measures, even when they do not fit actual electoral data, is that they are *a priori* rules to be used in general, without reference to details of any particular elections (see, for example, Felsenthal and Machover, 1998, p. 12). We have no problem with the concept of *a priori* rules. After all, it seems quite reasonable for electoral votes to be assigned based on structural features such as the rules for voting and number of voters, and not on transient patterns of political preferences. For example, nobody is suggesting that Utah and Massachusetts get extra electoral votes to make up for their lost voting power due to being far from the national median.

However, it does seem reasonable to demand that an *a priori* rule be appropriate, on average, in the real world. At some point, the burden of proof has to be on the proposers of any rule to justify that it is empirically reasonable, in its general patterns if not in all details. As we have shown in Section 5, the random voting model is not consistent with accepted models of swings in public opinion. Snyder, Ting, and Ansolabehere (2001) have demonstrated similar problems of voting power indexes with game theory.

Voting power measures are based on an empirically falsified and theoretically unjustified model. A more realistic and reasonable model allows votes to be affected by local, regional, and national swings (“parallel publics,” in the terminology of Page and Shapiro, 1992), with the result that large elections are *not* necessarily close, and that proportional weights in a legislature are approximately fair.

### 6.2 Population and turnout

Throughout this paper we have made no distinction between the voters and the persons represented in an election. In reality, however, voter turnout varies dramatically between countries and between areas within a country. In addition, children and noncitizens are represented in a democracy even though they do not have the right to vote. Thus, it is



standard for weights in voting systems to be set in terms of the population, rather than the number of voters, in each jurisdiction.

We agree that it is reasonable to set weights in terms of population but note that this inherently leads to variation in voting power: as the percentage voter turnout declines in a jurisdiction, the power of each remaining voter increases. We consider this acceptable because these voters represent that entire jurisdiction—but we recognize that this goes beyond the simple calculation of probability of decisiveness. The reasoning is closer to the representativeness argument of Section 2.5.

### 6.3 A .9-power rule?

As noted at the end of Section 4, our empirical analyses (and also those of Mulligan and Hunter, 2001) are consistent with a probability of decisive vote proportional to  $n_j^{-0.1}$ . This implies that a fair allocation of electoral votes is in proportion to the 0.9th power of the number of voters (or, as discussed in Section 6.2, of the population in the jurisdiction).

We hesitate to make a recommendation of the .9-power rule since it lacks the Platonic appeal of the proportional and square-root rules. The proportional rule is close to the .9 power and is simpler to explain. However, if probabilistic assumptions were to be used in computing voting power of jurisdictions (as in Section 2.2.1), then it might be reasonable to use the .9 power to estimate the power of individual voters.

### 6.4 Conclusion

It is often claimed that, in a proportionally weighted electoral system, voters in large jurisdictions have disproportionate “voting power.” This statement is only correct if elections in large jurisdictions are much closer than in small jurisdictions: the “square root rule.” Empirically, this rule does not hold—in several electoral systems for which we have gathered data, the probability that an election is close is much more like a constant than proportional to  $1/\sqrt{n_j}$ .

From a theoretical perspective, our result—that large elections are not appreciably closer than small elections—makes sense because election results are characterized by national and regional swings. Voting power measures go wrong by assuming that the  $n_j$  voters are acting independently (or, more generally, that they are divided into independent groups, where the number of groups is proportional to  $n_j$ ).

We hope that this explication will allow researchers to better understand the limits of current theoretical methods used in evaluating electoral systems.

## Appendix A The probability of affecting the election outcome, if an individual vote is never a decisive event

As illustrated by the Presidential election in Florida in 2000, an election can be disputed even if the votes are not exactly tied. This may seem to call into question the very concept of a decisive vote. Given that elections can be contested and recounted, it seems naive to suppose that the difference between winning and losing is no more than the change in a vote margin from  $-1$  to  $+1$ , which is what we have been assuming.

In fact, our decisive-vote calculations are reasonable, even for real elections with disputed votes, recounts, and so forth. We show this by setting up a more elaborate model that allows for a gray area in vote counting, and then demonstrating that the simpler model of decisive votes is a reasonable approximation.

Consider a two-party election and label  $v$  as the proportion of the  $n$  votes received by party A. We model vote-count errors, disputes, etc., by defining  $\pi(v)$  as the probability that party A wins, given a true proportion  $v$ . With perfect voting,  $\pi(v) = 0$  if  $v < 0.5$ , 1 if  $v > 0.5$ , or 0.5 if  $v = 0.5$ . More realistically,  $\pi(v)$  is a function of  $v$  that equals 0 if  $v$  is clearly less than 0.5 (e.g.,  $v < -0.499$ ), 1 if  $v$  is clearly greater than 0.5, and is between 0 and 1 if  $v$  is near 0.5.

In that case, the probability that your vote determines the outcome of the election, conditional on  $v$  (defined now as the proportion in favor of candidate A, excluding your potential vote), is  $\pi(v + 1/n) - \pi(v)$ . If your uncertainty about  $v$  is summarized by a probability distribution,  $p(v)$ , then your probability of decisiveness is,

$$\begin{aligned} \Pr(\text{decisive vote}) &= \mathbb{E}[\pi(v + 1/n) - \pi(v)] \\ &= \sum_v [\pi(v + 1/n) - \pi(v)] p(v). \end{aligned} \tag{3}$$

At this point, we make two approximations, both of which are completely reasonable in practice. First, we assume that the election will only be contested for a small range of vote proportions, which will lie near 0.5: thus, there is some small  $\epsilon$  such that  $\pi(v) = 0$  for all  $v < 0.5 - \epsilon$  and  $\pi(v) = 1$  for all  $v > 0.5 + \epsilon$ . Second, we assume that the probability density  $p(v)$  for the election outcome has an uncertainty that is greater than  $\epsilon$  (for example, perhaps  $\epsilon = 0.001$  and  $v$  can be anticipated to an accuracy of 2%, or 0.02). Then we can approximate  $p(v)$  in the range  $0.5 \pm \epsilon$  by the constant  $p(0.5)$ . Expression (3)

can then be written as,

$$\begin{aligned}
\Pr(\text{decisive vote}) &= \int_{0.5-\epsilon}^{0.5+\epsilon} [\pi(v + 1/n) - \pi(v)] p(0.5) dv \\
&= p(0.5) \int_{0.5-\epsilon}^{0.5+\epsilon} [\pi(v + 1/n) - \pi(v)] dv \\
&= p(0.5) \left[ \int_{0.5-\epsilon+1/n}^{0.5+\epsilon+1/n} \pi(v) dv - \int_{0.5-\epsilon}^{0.5+\epsilon} \pi(v) dv \right] \\
&= p(0.5) \left[ \int_{0.5+\epsilon}^{0.5+\epsilon+1/n} \pi(v) dv - \int_{0.5-\epsilon}^{0.5-\epsilon+1/n} \pi(v) dv \right] \\
&= p(0.5) [1/n \cdot 1 - 1/n \cdot 0] \\
&= p(0.5)/n,
\end{aligned}$$

which is the same probability of decisiveness as calculated assuming all votes are recorded correctly (see also Good and Meyer, 1975, and Margolis, 1983).

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