

Towards the Fundamental Quantum Limit of Linear Measurements of Classical Signals

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The quantum Cramér-Rao bound (QCRB) sets a fundamental limit for the measurement of classical signals with detectors operating in the quantum regime. Using linear-response theory and the Heisenberg uncertainty relation, we derive a general condition for achieving such a fundamental limit. When applied to classical displacement measurements with a test mass, this condition leads to an explicit connection between the QCRB and the Standard Quantum Limit which arises from a tradeoff between the measurement imprecision and quantum backaction; the QCRB can be viewed as an outcome of a quantum non-demolition measurement with the backaction evaded. Additionally, we show that the test mass is more a resource for improving measurement sensitivity than a victim of the quantum backaction, which suggests a new approach to enhancing the sensitivity of a broad class of sensors. We illustrate these points with laser interferometric gravitational wave detectors.

Introduction.— In high-precision measurements of classical signals, one challenge is to reduce various noise sources so that we can measure the tiny change in the detector state caused by the signal. This is often achieved by minimizing the coupling of the detector to the environment. Eventually, we approach the quantum regime with the dominant noise coming from the statistical nature of the detector quantum state. Maximizing the quantum-limited sensitivity requires proper preparation of the detector state and measurements of its observables—a key task in quantum metrology (cf. the review article by Giovannetti *et al.* [1]). The quantum Cramér-Rao bound (QCRB), derived in the pioneering works of Helstrom [2] and Holevo [3], sets a fundamental limit to the maximum sensitivity for a given detector state. As proved by Braunstein *et al.* [4, 5], this lower bound can be attained only if (i) the detector state is pure and the right observable is measured, so that the quantum Fisher information becomes equal to its classical counterpart, and (ii) the estimator based upon the measurement records is efficient, i.e., the mean squared estimation error saturates the classical Cramér-Rao bound.

In linear measurements, as illustrated in Fig. 1, the detector input port observable, \hat{F} , is linearly coupled to the signal, x . In the case of single-shot detection of a single-parameter signal, this is modelled by the interaction $\hat{H}_{\text{int}} = -\hat{F} x \delta(t)$, and the QCRB for the estimation error, σ_{xx} , is (cf., Chapter 2 of Ref. [6])

$$\sigma_{xx}^{\text{QCRB}} = \frac{\hbar^2}{4\langle\psi|\hat{F}^2|\psi\rangle}, \quad (1)$$

where $|\psi\rangle$ is the initial detector state, and we assume that $\langle\psi|\hat{F}|\psi\rangle = 0$. To attain it, the output-port observable, \hat{Z} , that we measure needs to satisfy [4],

$$\text{Re}[\langle\psi|\hat{\Pi}_z \hat{F}|\psi\rangle] = 0 \quad \forall z, \quad (2)$$

where $\text{Re}[\cdot]$ means taking the real part, and the projection operator $\hat{\Pi}_z$ is defined as $\hat{\Pi}_z \equiv |z\rangle\langle z|$ with $|z\rangle$ being an eigenstate of \hat{Z} and z the measurement outcome. The maximum-likelihood estimator of x , based upon z , will be efficient if $|\psi\rangle$ is Gaussian, or the sample size is large [7].

For detecting signals with multi-dimensional parameters, the QCRB is not as simple [8] as the one shown in Eq. (1). In particular, Tsang *et al.* [9] generalized the QCRB to the linear measurement of a continuous signal $x(t)$ with an infinite-dimensional parameter space (specifically gravitational wave detection using laser interferometers [10, 11]). For time-invariant, linear detectors with $\hat{H}_{\text{int}} = -\hat{F} x(t)$, they showed that the QCRB for estimating the Fourier components, $x(\omega)$, of the signal is

$$\sigma_{xx}^{\text{QCRB}}(\omega) = \frac{\hbar^2}{4\bar{S}_{FF}(\omega)}, \quad (3)$$

where \bar{S}_{FF} is the symmetrized power spectral density that describes the quantum fluctuations (uncertainty) of \hat{F} . Braginsky *et al.* [12] also derived a similar result, in terms of the signal-to-noise ratio.

Until now, it has not been shown generally how the QCRB in Eq. (3) can be achieved. This is, however, crucial for applying the QCRB to guide the design of quantum-limited linear sensors. We fill this gap by showing general conditions for achieving the bound: (1) the detector is at the quantum limit with minimum uncertainty, and (2) the observables \hat{Z} and \hat{F} are uncorrelated (in terms of cross-spectrum):

$$\bar{S}_{ZF}(\omega) = 0. \quad (4)$$

One can find the optimal \hat{Z} satisfying the second condition if the imaginary part of the input susceptibility χ_{FF} vanishes:

$$\text{Im}[\chi_{FF}(\omega)] = 0. \quad (5)$$

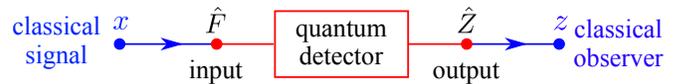


FIG. 1. (color online) A schematic for a quantum measurement of a classical signal using a linear detector. One degree of freedom of the detector is singled out as the input port, for which the observable, \hat{F} , is coupled to the signal, x , and another one as the output port with its observable, \hat{Z} , projectively measured by the observer. The detector is a quantum interface between two classical domains.

When this is not the case and we only have the first condition satisfied, the minimal estimation error will still be bounded:

$$\sigma_{xx}^{\text{QCRB}} \leq \min \sigma_{xx} \leq 2 \sigma_{xx}^{\text{QCRB}}. \quad (6)$$

In deriving the above results, we use the linear-response theory developed by Kubo [13], which has previously been applied to analyze the quantum limited sensitivity of linear detectors [14–17]. Additionally, we apply the recent result on the Heisenberg uncertainty relation for continuous quantum measurements presented in Ref. [18].

Single-shot Measurements.— Before discussing the continuous measurements, we will first illustrate the basic formalism using the example of a single-shot measurement with $\hat{H}_{\text{int}} = -\hat{F} x \delta(t)$. Such an interaction will leave \hat{F} unchanged, but induce a shift on any observable that does not commute with \hat{F} . Specifically, the solution to \hat{Z} reads

$$\hat{Z} = \hat{Z}^{(0)} + (i/\hbar)[\hat{Z}^{(0)}, \hat{F}^{(0)}]x \quad (7)$$

where the superscript (0) denotes evolution under the detector free Hamiltonian \hat{H}_{det} . For linear detectors, the canonical coordinates have classical-number (i.e., not operator) commutators, and \hat{H}_{det} only contains their linear or quadratic functions. The relevant observables, \hat{Z} and \hat{F} , also depend linearly on the canonical coordinates. This justifies application of linear-response theory, in which different quantities are linked by classical-number susceptibilities. A brief introduction to the linear-response theory is in the supplemental material.

In this example, we introduce the following susceptibility:

$$\chi_{ZF} \equiv (i/\hbar)[\hat{Z}^{(0)}, \hat{F}^{(0)}], \quad (8)$$

which quantifies response of the detector output to the signal: $\hat{Z} = \hat{Z}^{(0)} + \chi_{ZF} x$. Given the projective measurement of \hat{Z} , we can construct an unbiased estimator of the signal:

$$\hat{x}_{\text{est}} = \hat{Z}/\chi_{ZF}. \quad (9)$$

The resulting mean squared error σ_{xx} is determined by the quantum uncertainty of $\hat{Z}^{(0)}$, i.e.,

$$\sigma_{xx} \equiv \text{Tr}[\hat{\rho}_{\text{det}}(\hat{x}_{\text{est}} - x)^2] = \sigma_{ZZ}/\chi_{ZF}^2, \quad (10)$$

where $\sigma_{ZZ} \equiv \text{Tr}[\hat{\rho}_{\text{det}}(\hat{Z}^{(0)})^2]$ assuming zero mean and $\hat{\rho}_{\text{det}}$ is the density matrix of the detector initial state. From the general Heisenberg uncertainty relation between $\hat{Z}^{(0)}$ and $\hat{F}^{(0)}$:

$$\sigma_{ZZ}\sigma_{FF} - \sigma_{ZF}^2 \geq (\hbar^2/4)\chi_{ZF}^2 \quad (11)$$

with $\sigma_{ZF} \equiv \text{Tr}[\hat{\rho}_{\text{det}}(\hat{Z}^{(0)}\hat{F}^{(0)} + \hat{F}^{(0)}\hat{Z}^{(0)})/2]$ being their cross correlation, we obtain

$$\sigma_{xx} \geq \frac{\hbar^2}{4\sigma_{FF}} + \frac{\sigma_{ZF}^2}{\sigma_{FF}\chi_{ZF}^2} \geq \frac{\hbar^2}{4\sigma_{FF}} = \sigma_{xx}^{\text{QCRB}}. \quad (12)$$

Achieving the QCRB therefore requires that the detector is at quantum limit with minimum uncertainty, i.e., in a pure Gaussian state with Eq. (11) taking the equal sign, and additionally

$$\sigma_{ZF} = 0. \quad (13)$$

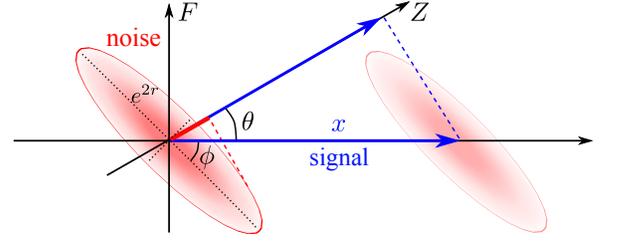


FIG. 2. (color online) Illustration of the single-shot measurement with the detector in a pure, Gaussian, squeezed state (the noise ellipse represents its Wigner function). The optimal observable \hat{Z} to achieve the QCRB is neither the conjugate variable of \hat{F} (along the horizontal axis), which contains the largest signal, nor the one having the minimum noise (parallel to the semi-minor axis of the noise ellipse). Instead it is the one uncorrelated with \hat{F} and $\tan\theta = \sin(2\phi) \sinh(2r)/[\cosh(2r) + \cos(2\phi) \sinh(2r)]$, in which r and ϕ are the squeezing factor and angle, as derived in Ref. [5] using Eq. (2).

Since $\hat{Z} = \int dz \hat{\Pi}_z z$, this condition is equivalent to Eq. (2). When discussing a similar example, Braunstein *et al.* [5] derived the optimal \hat{Z} using Eq. (2), as illustrated in Fig. 2.

Continuous Measurements.— The discussion for the continuous measurements is quite similar to the single-shot case, but with additional complications due to the involvement of many degrees of freedom—the detector is a continuum field. We focus on linear detectors that are time-invariant, i.e., having a time-independent \hat{H}_{det} and being in a stationary state $[\hat{\rho}_{\text{det}}, \hat{H}_{\text{det}}] = 0$, allowing for frequency-domain analysis of both dynamics and noise.

As in Eq. (7), \hat{Z} in the continuous case is given by

$$\hat{Z}(t) = \hat{Z}^{(0)}(t) + \int_{-\infty}^{\infty} dt' \chi_{ZF}(t-t') x(t') \quad (14)$$

with the susceptibility, $\chi_{ZF} \equiv (i/\hbar)[\hat{Z}^{(0)}(t), \hat{F}^{(0)}(t')]\Theta(t-t')$, a function of the time difference $t-t'$. In the frequency domain, it becomes

$$\hat{Z}(\omega) = \hat{Z}^{(0)}(\omega) + \chi_{ZF}(\omega)x(\omega), \quad (15)$$

where $f(\omega) \equiv \int_{-\infty}^{\infty} dt e^{i\omega t} f(t)$. The unbiased estimator of $x(\omega)$, following Eq. (9), is then $\hat{x}_{\text{est}}(\omega) = \hat{Z}(\omega)/\chi_{ZF}(\omega)$.

Given that the detector is in a stationary state, the quantum fluctuation can be quantified by using the spectral density. There is also a Heisenberg uncertainty relation for the continuous measurements in terms of spectral densities and susceptibilities (cf., Chapter VI in Ref. [16] or Ref. [18]):

$$\bar{S}_{ZZ}(\omega)\bar{S}_{FF}(\omega) - |\bar{S}_{ZF}(\omega)|^2 \geq \frac{\hbar^2}{4}|\chi_{ZF}(\omega)|^2 + \hbar \left| \text{Im}[\bar{S}_{ZZ}(\omega)\chi_{FF}(\omega) - \bar{S}_{ZF}^*(\omega)\chi_{ZF}(\omega)] \right|. \quad (16)$$

Here the symmetrized spectral densities \bar{S}_{ZZ} , \bar{S}_{FF} and \bar{S}_{ZF} are defined as $\bar{S}_{AB}(\omega) \equiv [S_{AB}(\omega) + S_{BA}(-\omega)]/2$ with the unsymmetrized one S_{AB} defined by $\text{Tr}[\hat{\rho}_{\text{det}} \hat{A}^{(0)}(\omega) \hat{B}^{(0)\dagger}(\omega')]$ $\equiv 2\pi S_{AB}(\omega)\delta(\omega - \omega')$ [17]; χ_{FF} is defined in the same way as χ_{ZF} in Eq. (14) and with $\hat{Z}^{(0)}$ replaced by $\hat{F}^{(0)}$.

With Eq. (16), the error $\sigma_{xx}(\omega) \equiv \bar{S}_{ZZ}(\omega)/|\chi_{ZF}(\omega)|^2$ for estimating $x(\omega)$ thus satisfies

$$\sigma_{xx}(\omega) \geq \frac{\hbar^2}{4\bar{S}_{FF}} + \frac{|\bar{S}_{ZF}|^2 + \hbar|\text{Im}[\bar{S}_{ZZ}\chi_{FF} - \bar{S}_{ZF}^*\chi_{ZF}]|}{\bar{S}_{FF}|\chi_{ZF}|^2}. \quad (17)$$

As proven in Ref. [18], when the detector is at the quantum limit, i.e., in a pure, stationary, Gaussian state—the multi-mode squeezed state [19], not only does Eq. (16) become an equality, but also we have

$$\text{Im}[\bar{S}_{ZZ}(\omega)\chi_{FF}(\omega) - \bar{S}_{ZF}^*(\omega)\chi_{ZF}(\omega)]|_{\text{quantum limit}} = 0. \quad (18)$$

At this point, we only require Eq. (4) to attain the QCRB—the first term in Eq. (17).

We now show that if Eq. (5) is satisfied, the optimal observable \hat{Z} , which realizes Eq. (4), exists. In general, \hat{Z} is a linear combination of two conjugate variables (denoted by $\hat{Z}_{1,2}$) of the output port, up to some constant:

$$\hat{Z}(\omega) = \hat{Z}_1(\omega) \sin \theta + \hat{Z}_2(\omega) \cos \theta. \quad (19)$$

Eq. (4) can then be realized if there is a real solution to θ :

$$\tan \theta = -\bar{S}_{Z_2F}(\omega)/\bar{S}_{Z_1F}(\omega) \in \text{Reals}, \quad (20)$$

or $\text{Im}[\bar{S}_{Z_1F}(\omega)\bar{S}_{Z_2F}^*(\omega)] = 0$. This turns out to be equivalent to $\text{Im}[\chi_{FF}(\omega)] = 0$ due to the following equality:

$$\text{Im}[\bar{S}_{Z_1F}(\omega)\bar{S}_{Z_2F}^*(\omega)] = (\hbar/4)\text{Im}[\chi_{FF}(\omega)], \quad (21)$$

which is generally valid for detectors at the quantum limit.

If $\text{Im}[\chi_{FF}]$ is nonzero, we will not find the optimal \hat{Z} that exactly achieves the QCRB. Nevertheless, the estimation error σ_{xx} , minimized over all possible θ in Eq. (19), is still bounded as shown in Eq. (6). This is because

$$\min_{\theta} \left| \bar{S}_{ZF}(\omega)/\chi_{ZF}(\omega) \right| \leq \hbar/2. \quad (22)$$

Including Eq. (17), the above inequality implies Eq. (6). The detailed proofs for Eqs. (S32) and (22) are provided in the supplemental material [20].

Classical Displacement Measurements.— The above discussion applies to general linear measurements. Here we specifically look measurements of displacement; the detector often consists of a quantum field and a test mass with its position being displaced by a classical signal, which can be a result of the action of a force signal. The interaction between the field and the test mass leads to an important sensitivity limit—the Standard Quantum Limit (SQL), first derived by Braginsky [16]. Below we show an explicit connection between the SQL and the QCRB, and also discuss the active role of the test mass in enhancing the detector sensitivity.

In terms of a mathematical description, we denote the input port observable of the field as $\hat{\mathcal{F}}$ and the output as $\hat{\mathcal{Z}}$, to distinguish from \hat{F} and \hat{Z} (relevant for the entire detector); $\hat{\mathcal{F}}$ is coupled to the test mass position \hat{q} via the interaction $-\hat{q}\hat{\mathcal{F}}$,

and $\hat{\mathcal{Z}}$ is projectively measured. Solving the detector dynamics leads to (in the frequency domain):

$$\hat{F}^{(0)} = \frac{\hat{\mathcal{F}}^{(0)}}{1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}}, \quad \hat{Z}^{(0)} = \hat{\mathcal{Z}}^{(0)} + \frac{\chi_{ZF}\chi_{qq}\hat{\mathcal{F}}^{(0)}}{1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}}. \quad (23)$$

In the literature, the first term $\hat{\mathcal{Z}}^{(0)}$ of the detector output observable $\hat{Z}^{(0)}$ is referred to as the imprecision noise; the second term, proportional to $\hat{\mathcal{F}}^{(0)}$, is the quantum backaction noise.

For the special case when the input susceptibility of the field is zero: $\chi_{\mathcal{F}\mathcal{F}} = 0$, the resulting estimation error is

$$\sigma_{xx} = \frac{\bar{S}_{ZZ}}{|\chi_{ZF}|^2} + 2\text{Re} \left[\chi_{qq}^* \frac{\bar{S}_{ZF}}{\chi_{ZF}} \right] + |\chi_{qq}|^2 \bar{S}_{\mathcal{F}\mathcal{F}}. \quad (24)$$

If the imprecision noise and the backaction noise are uncorrelated, i.e. $\bar{S}_{ZF} = 0$, its lower bound will be the SQL:

$$\sigma_{xx} = \frac{\bar{S}_{ZZ}}{|\chi_{ZF}|^2} + |\chi_{qq}|^2 \bar{S}_{\mathcal{F}\mathcal{F}} \geq \hbar|\chi_{qq}| \equiv \sigma_{xx}^{\text{SQL}}. \quad (25)$$

The SQL can be surpassed by using quantum non-demolition (QND) measurements [21]: e.g., coherent noise cancellation schemes [22] or equivalently, optimal readout schemes [23] which cancel the backaction noise. In particular, optimal readout schemes utilize quantum correlations \bar{S}_{ZF} , and can be understood by applying the uncertainty relation $\bar{S}_{ZZ}\bar{S}_{\mathcal{F}\mathcal{F}} \geq |\bar{S}_{ZF}|^2 + \hbar^2|\chi_{ZF}|^2/4$ to rewrite Eq. (24) as

$$\sigma_{xx} \geq \frac{\hbar^2}{4\bar{S}_{\mathcal{F}\mathcal{F}}} + \left| \frac{\bar{S}_{ZF}}{\chi_{ZF}} + \chi_{qq}\bar{S}_{\mathcal{F}\mathcal{F}} \right|^2 \geq \frac{\hbar^2}{4\bar{S}_{\mathcal{F}\mathcal{F}}}. \quad (26)$$

The ultimate bound will be the QCRB if we read out the optimal output observable satisfying $\bar{S}_{ZF}/\chi_{ZF} + \chi_{qq}\bar{S}_{\mathcal{F}\mathcal{F}} = 0$, which, from Eq. (23), is equivalent to Eq. (4) shown earlier. The SQL can therefore be viewed as arising from a suboptimal readout scheme.

In cases where $\chi_{\mathcal{F}\mathcal{F}}$ is not zero, one can similarly show that the estimation error is again bounded by the QCRB:

$$\sigma_{xx} \geq \frac{\hbar^2}{4\bar{S}_{FF}} = \frac{\hbar^2}{4\bar{S}_{\mathcal{F}\mathcal{F}}} |1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}|^2. \quad (27)$$

In contrast to Eq. (26), here we have a factor of $|1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}|^2$, which can be smaller than unity. There are two equivalent interpretations: (1) the test mass response is modified by the quantum field:

$$\chi_{qq}^{\text{eff}} = \frac{\chi_{qq}}{1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}}; \quad (28)$$

and (2) the quantum fluctuations of the field are modified by the test mass, as manifested by the relation between $\hat{F}^{(0)}$ and $\hat{\mathcal{F}}^{(0)}$ in Eq. (23). The latter highlights the active (and enhancing) role of the test mass, rather than being a victim of the quantum backaction. Below, we illustrate this using gravitational wave (GW) detection with laser interferometers as an example.

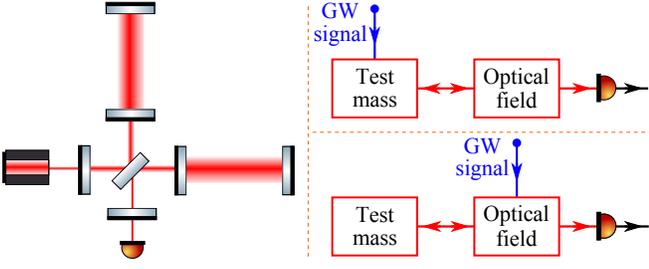


FIG. 3. (color online) Schematic diagram of a LIGO-like interferometer (left). Two approximately equivalent physical pictures for the detection principle (right).

Gravitational-wave Detection.— A typical GW detector, such as LIGO [24], is shown schematically in Fig. 3. This is an interferometer with Fabry-Pérot arm cavities formed by suspended mirrors (test masses). The usual picture of the detection principle envisions the GW as a tidal force on the test masses, and the resulting differential motion being probed by the optical field. Another picture is to view the GW as a strain directly coupled to the optical field [25, 26]. The latter is more appropriate when the GW wavelength is comparable to or shorter than the interferometer arm length, otherwise it is approximately equivalent to the former. We will apply it in later discussions to highlight the active role of the test mass mentioned earlier.

Putting the GW detection under the general framework, the classical signal is

$$x = L_{\text{arm}} h_{\text{GW}}, \quad (29)$$

where L_{arm} is the arm length, and h_{GW} is the GW strain. The test mass motion that we care about is the differential mode of the four mirrors in the two arms, with the susceptibility:

$$\chi_{qq} = -4/(M\omega^2), \quad (30)$$

where M is the mirror mass. The quantum field is the optical field, coupled to the test mass via the radiation pressure.

As shown in Refs. [27, 28], the entire interferometer can be mapped to a single-cavity-mode optomechanical device, described by the standard cavity optomechanics [29]. The input observable $\hat{\mathcal{F}}$ is the time-varying part of the radiation pressure, which is proportional to the amplitude quadrature \hat{X} of the cavity mode:

$$\hat{\mathcal{F}} = 2P_{\text{cav}}/c = \hbar g \hat{X}, \quad (31)$$

of which the relevant susceptibility is given by [27]:

$$\chi_{\mathcal{F}\mathcal{F}} = \frac{\hbar g^2 \Delta}{(\omega - \Delta + i\gamma)(\omega + \Delta + i\gamma)}. \quad (32)$$

Here $g \equiv 2\sqrt{\bar{P}_{\text{cav}}\omega_{\text{cav}}}/(\hbar L_{\text{arm}}c)$ with \bar{P}_{cav} the average optical power inside the cavity and ω_{cav} the cavity resonant frequency; $\Delta = \omega_0 - \omega_{\text{cav}}$ is the detuning of the laser frequency ω_0 ; γ is the cavity bandwidth. The output observable $\hat{\mathcal{Z}}$ is a

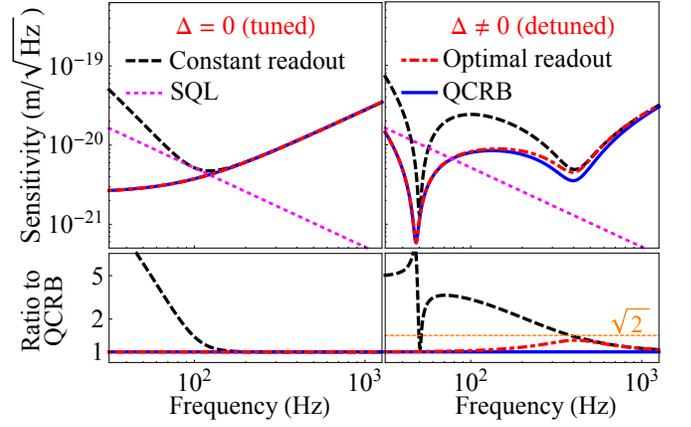


FIG. 4. (color online) The top row shows the QCRB (solid curve) for LIGO-type GW detector with detuning frequency $\Delta = 0$ (left) and $\Delta/(2\pi) = 400$ Hz (right), and various sensitivity curves for comparison: (i) dash curve—constant phase quadrature readout, (ii) dash-dot curve—readout quadrature optimized to maximize sensitivity at each frequency, and (iii) dot curve—the SQL $\sqrt{4\hbar/(M\omega^2)}$. The bottom row shows the ratio to the QCRB for selected curves. Other relevant parameters are: $M = 40$ kg, $P_{\text{cav}} = 800$ kW, $L_{\text{arm}} = 4$ km, $\gamma/(2\pi) \approx 100$ Hz, and laser frequency $\omega_0/(2\pi) \approx 3 \times 10^{14}$ Hz.

linear combination of the amplitude and phase quadrature of the outgoing field at the dark (differential) port.

In Fig. 4, we plot the resulting QCRB for the two cases: $\Delta = 0$ (tuned) and $\Delta \neq 0$ (detuned), assuming other parameters similar to LIGO. In comparison, we have also included the SQL, and the estimation error $\sigma_{xx}^{1/2}$, i.e. the sensitivity, for the phase quadrature readout and the optimal readout. The tuned case having $\chi_{\mathcal{F}\mathcal{F}} = 0$ provides a concrete example of Eq. (25) and Eq. (26). Indeed, the optimal readout, which surpasses the SQL by canceling the backaction noise, leads to a sensitivity exactly equal to the QCRB.

In the detuned case with $\chi_{\mathcal{F}\mathcal{F}} \neq 0$, the first point we want to highlight is that the maximum difference between the optimal-readout sensitivity, considered by Harms *et al.* [30], and the QCRB is at most $\sqrt{2}$ in amplitude, in accordance with our general result Eq. (6). The second point is that there are two noticeable dips in the QCRB. They imply that the amplitude quadrature of the cavity mode has higher fluctuations around these dips than other frequencies. Both can be interpreted as arising from positive feedback induced optical resonance. The higher frequency one coincides with the detuning frequency, which is at the cavity resonance. The low frequency one provides an example of the extra factor $|1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}|^2$ in Eq. (27). Physically, this has to do with the ponderomotive squeezing (or amplification) effect [23, 31], which recently has been demonstrated experimentally [32–34]. The test mass acts as a Kerr-type nonlinear medium converting the amplitude fluctuations into the phase fluctuations, which in turn, feeds back to the amplitude quadrature due to the cavity detuning. Since the test mass susceptibility goes as $1/\omega^2$, cf. Eq. (30), the feedback gain is frequency dependent, resulting in the sharp reso-

nance feature. The underlying physics is similar to the intracavity squeezing studied theoretically by Peano *et al.* [35] and experimentally by Korobko *et al.* [36].

An equivalent interpretation of the low frequency dip was presented in Refs. [27, 37]. It was attributed to the so-called optical spring effect, an example of Eq. (28)—the optomechanical interaction changes the test mass dynamics by creating a new mechanical resonance, around which the response to GWs is enhanced. The previous optical feedback interpretation, however, removes the distinction between optics and mechanics—the role of the latter also modifies the quantum fluctuations of the optical field. This suggests a new approach to designing optomechanical sensors. We can add proper optical filters in the feedback loop, together with the internal ponderomotive squeezing, to shape the optical feedback gain, so that the quantum fluctuation of the field is enhanced in the frequency band of interest. Since the sensitivity using the optimal readout is bounded, cf. Eq. (6), this will result in high detector sensitivity at relevant frequencies, with limitations only coming from the losses. Incorporating the effect of losses is critical and the subject of future work.

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Supplemental Material

I. LINEAR-RESPONSE THEORY

Here we briefly introduce the linear-response theory that has been applied in our analysis. One can refer to Refs. [S1–S4] for more details. Given the model illustrated in Fig. 1 of the main paper, the Hamiltonian for the measurement setup is

$$\hat{H}_{\text{tot}} = \hat{H}_{\text{det}} + \hat{H}_{\text{int}}, \quad (\text{S1})$$

where \hat{H}_{det} is the free Hamiltonian for the detector, and \hat{H}_{int} describes the coupling between the classical signal and the detector. We consider the steady state with the coupling turned on at $t = -\infty$. The solution to any operator \hat{A} of the detector at time t in the Heisenberg picture is given by

$$\hat{A}(t) = \hat{U}_I^\dagger(-\infty, t) \hat{A}^{(0)}(t) \hat{U}_I(-\infty, t) \quad (\text{S2})$$

with $\hat{A}^{(0)}(t)$ denoting the operator under the free evolution:

$$\hat{A}^{(0)}(t) \equiv \hat{U}_0^\dagger(-\infty, t) \hat{A} \hat{U}_0(-\infty, t). \quad (\text{S3})$$

The unitary operator for the free-evolution part is defined as $\hat{U}_0(-\infty, t) \equiv \mathcal{T} \exp\{-(i/\hbar) \int_{-\infty}^t dt' \hat{H}_{\text{det}}(t')\}$ with \mathcal{T} being the time-ordering, and, for the interaction part, we have defined $\hat{U}_I(-\infty, t) \equiv \mathcal{T} \exp\{-(i/\hbar) \int_{-\infty}^t dt' \hat{H}_{\text{int}}^{(0)}(t')\}$.

For the measurement to be linear, \hat{H}_{det} only involves linear or quadratic functions of canonical coordinates, among which their commutators are classical numbers, i.e., not operators; the interaction \hat{H}_{int} is in the bilinear form:

$$\hat{H}_{\text{int}} = -\hat{F}x(t). \quad (\text{S4})$$

As a result, Eq. (S2) leads to the following exact solution to the input-port observable \hat{F} and output-port observable \hat{Z} :

$$\hat{Z}(t) = \hat{Z}^{(0)}(t) + \int_{-\infty}^{+\infty} dt' \chi_{ZF}(t, t') x(t'), \quad (\text{S5})$$

$$\hat{F}(t) = \hat{F}^{(0)}(t) + \int_{-\infty}^{+\infty} dt' \chi_{FF}(t, t') x(t'). \quad (\text{S6})$$

The susceptibility χ_{AB} ($A, B = Z, F$), which describes the detector response to the signal, is defined as

$$\chi_{AB}(t, t') \equiv \frac{i}{\hbar} [\hat{A}^{(0)}(t), \hat{B}^{(0)}(t')] \Theta(t - t') \quad (\text{S7})$$

with $\Theta(t)$ being the Heaviside function. Notice that the susceptibilities are classical numbers and only involve operators under the free evolution, which are consequences of the detector being linear.

For the measurement to be continuous, we need to be able to projectively measure the output-port observable at different times precisely without introducing additional noise. This can happen only if $\hat{Z}(t)$ commutes with itself at different times, namely,

$$[\hat{Z}(t), \hat{Z}(t')] = 0 \quad \forall t, t'. \quad (\text{S8})$$

It is called the condition of simultaneous measurability in Ref. [S3] which also shows that it implies

$$[\hat{Z}^{(0)}(t), \hat{Z}^{(0)}(t')] = [\hat{F}^{(0)}(t), \hat{Z}^{(0)}(t')] \Theta(t - t') = 0, \quad (\text{S9})$$

or equivalently,

$$\chi_{ZZ}(t, t') = \chi_{FZ}(t, t') = 0, \quad (\text{S10})$$

which is central to the discussion of continuous, linear quantum measurements.

When the free Hamiltonian for the detector is time-independent, the susceptibility will only depend on the time difference, i.e.,

$$\chi_{AB}(t, t') = \chi_{AB}(t - t'), \quad (\text{S11})$$

which is the case considered in the main paper. This allows us to move into the frequency domain, and rewrite Eqs. (S5) and (S6) as

$$\hat{Z}(\omega) = \hat{Z}^{(0)}(\omega) + \chi_{ZF}(\omega) x(\omega), \quad (\text{S12})$$

$$\hat{F}(\omega) = \hat{F}^{(0)}(\omega) + \chi_{FF}(\omega) x(\omega). \quad (\text{S13})$$

in which the Fourier transform $\hat{A}(\omega) \equiv \int_{-\infty}^{+\infty} dt e^{i\omega t} \hat{A}(t)$. Furthermore, we consider the detector being in a stationary state, i.e., its density matrix $\hat{\rho}_{\text{det}}$ commuting with \hat{H}_{det} . The statistical property of the relevant operators, which defines the quantum noise of the detector, can then be quantified by using the frequency-domain spectral density, which is given by

$$S_{AB}(\omega) \equiv \int_{-\infty}^{+\infty} dt e^{i\omega t} \text{Tr}[\hat{\rho}_{\text{det}} \hat{A}^{(0)}(t + \tau) \hat{B}^{(0)}(\tau)], \quad (\text{S14})$$

where τ can be arbitrary due to the stationarity, and we have assumed $\text{Tr}[\hat{\rho}_{\text{det}} \hat{A}] = \text{Tr}[\hat{\rho}_{\text{det}} \hat{B}] = 0$ without loss of generality. Or equivalently, the spectral density can also be defined through

$$\text{Tr}[\hat{\rho}_{\text{det}} \hat{A}^{(0)}(\omega) \hat{B}^{(0)\dagger}(\omega')] \equiv 2\pi S_{AB}(\omega) \delta(\omega - \omega'). \quad (\text{S15})$$

The corresponding symmetrized version of the previously defined spectral density is

$$\bar{S}_{AB}(\omega) \equiv \frac{1}{2} [S_{AB}(\omega) + S_{BA}(-\omega)], \quad (\text{S16})$$

which is a summation of both the positive-frequency and negative-frequency spectra.

From the definitions of the susceptibility and spectral density, we have a general equality relating them to each other:

$$\chi_{AB}(\omega) - \chi_{BA}^*(\omega) = \frac{i}{\hbar} [S_{AB}(\omega) - S_{BA}(-\omega)]. \quad (\text{S17})$$

When applying this to the case with $\hat{A} = \hat{B}$, it leads to the famous Kubo's formula:

$$\text{Im}[\chi_{AA}(\omega)] = \frac{1}{2\hbar} [S_{AA}(\omega) - S_{AA}(-\omega)]. \quad (\text{S18})$$

Such an imaginary part of the susceptibility $\text{Im}[\chi_{AA}(\omega)]$ quantifies the dissipation, and, in the thermal equilibrium, it is related to the symmetrized spectral density $\bar{S}_{AA}(\omega)$ through the fluctuation-dissipation theorem. The measurement process is far from the thermal equilibrium, and therefore the usual fluctuation-dissipation theorem cannot be applied. Nevertheless, when the detector is ideal at the quantum limit with minimum uncertainty, we can also find some general relations between the susceptibility and the symmetrized spectral density, e.g., Eq. (18) and Eq. (21) in the main paper, the later of which will be proven in the next section.

II. PROOF OF EQ. (21)

Here we show the proof of Eq. (21) in the main paper. In the continuous, linear measurements, the detector is a continuum field that contains many degrees of freedom which are coupled to each other through the free evolution. The degrees of freedom for the input and output port that we pick are continuously driven by the ingoing part of the continuum field, which is similar to the in field introduced in Ref. [S5]. In the steady state with the initial condition decaying away, their observables $\hat{Z}_{1,2}$ and \hat{F} can be generally represented in terms of the ingoing field:

$$\hat{Z}_{1,2}^{(0)}(t) = \int_{-\infty}^{\infty} dt' \mathcal{Z}_{1,2}(t-t')\hat{d}(t') + \text{h.c.}, \quad (\text{S19})$$

$$\hat{F}^{(0)}(t) = \int_{-\infty}^{\infty} dt' \mathcal{F}(t-t')\hat{d}(t') + \text{h.c.}. \quad (\text{S20})$$

Here \mathcal{Z} and \mathcal{F} are some complex-valued functions; h.c. denotes Hermitian conjugate; $\hat{d}(t)$ is annihilation operator of the ingoing field that satisfies the following commutator relation:

$$[\hat{d}(t), \hat{d}^\dagger(t')] = \delta(t-t'). \quad (\text{S21})$$

In the frequency domain, Eqs. (S19) and (S20) can be rewritten as

$$\hat{Z}_{1,2}^{(0)}(\omega) = \mathcal{Z}_{1,2}(\omega)\hat{d}(\omega) + \mathcal{Z}_{1,2}^*(-\omega)\hat{d}^\dagger(-\omega), \quad (\text{S22})$$

$$\hat{F}^{(0)}(\omega) = \mathcal{F}(\omega)\hat{d}(\omega) + \mathcal{F}^*(-\omega)\hat{d}^\dagger(-\omega), \quad (\text{S23})$$

and the commutator for the ingoing field is

$$[\hat{d}(\omega), \hat{d}^\dagger(\omega')] = 2\pi\delta(\omega-\omega'). \quad (\text{S24})$$

A natural choice for the output port is the outgoing part of the continuum field, similar to the out field in Ref. [S5], which guarantees that the condition in Eq. (S8) can be fulfilled due to causality. Its two conjugate variables $\hat{Z}_{1,2}$ satisfies

$$[\hat{Z}_k(t), \hat{Z}_l(t')] = -\sigma_y^{kl}\delta(t-t'), \quad (\text{S25})$$

where $k, l = 1, 2$ and σ_y is the Pauli matrix. In the frequency domain, the above commutator reads

$$[\hat{Z}_k(\omega), \hat{Z}_l^\dagger(\omega')] = -2\pi\sigma_y^{kl}\delta(\omega-\omega'). \quad (\text{S26})$$

Together with Eq. (S24), this implies the following constraint on those functions in Eq. (S22):

$$\mathcal{Z}_k(\omega)\mathcal{Z}_l^*(\omega) - \mathcal{Z}_k^*(-\omega)\mathcal{Z}_l(-\omega) = -\sigma_y^{kl}, \quad (\text{S27})$$

which is an important equality for the proof.

We first prove Eq. (21) in the case when the detector is in the vacuum state, i.e.,

$$\hat{\rho}_{\text{det}} = |0\rangle\langle 0|. \quad (\text{S28})$$

Correspondingly, we have $\text{Tr}[\hat{\rho}_{\text{det}}\hat{d}(\omega)\hat{d}^\dagger(\omega')] = 2\pi\delta(\omega-\omega')$ and $\text{Tr}[\hat{\rho}_{\text{det}}\hat{d}^\dagger(\omega)\hat{d}(\omega')] = 0$, which are equivalent to

$$S_{\hat{d}\hat{d}^\dagger}(\omega) = 1, \quad S_{\hat{d}^\dagger\hat{d}}(\omega) = 0. \quad (\text{S29})$$

From Eqs. (S22) and (S23), the above spectral density for \hat{d} leads to

$$S_{Z_{1,2}F}(\omega) = \mathcal{Z}_{1,2}(\omega)\mathcal{F}^*(\omega), \quad (\text{S30})$$

$$S_{FF}(\omega) = |\mathcal{F}(\omega)|^2. \quad (\text{S31})$$

Using the constraint in Eq. (S27) and the definition of symmetrized spectral density Eq. (S16), we find

$$\text{Im}[\bar{S}_{Z_1F}(\omega)\bar{S}_{Z_2F}^*(\omega)] = \frac{1}{8}[S_{FF}(\omega) - S_{FF}(-\omega)]. \quad (\text{S32})$$

With the Kubo's formula Eq. (S18):

$$\text{Im}[\chi_{FF}(\omega)] = \frac{1}{2\hbar}[S_{FF}(\omega) - S_{FF}(-\omega)], \quad (\text{S33})$$

finally it gives rise to Eq. (21) in the main paper, i.e.,

$$\text{Im}[\bar{S}_{Z_1F}(\omega)\bar{S}_{Z_2F}^*(\omega)] = \frac{\hbar}{4}\text{Im}[\chi_{FF}(\omega)]. \quad (\text{S34})$$

We can further show that Eq. (S34) also holds for the general, stationary, pure Gaussian state—multi-mode squeezed state $\hat{\rho}_{\text{det}} = \hat{S}|0\rangle\langle 0|\hat{S}^\dagger$, in which the squeezing operator \hat{S} is defined as [S6]

$$\hat{S} \equiv \exp\left\{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [\xi(\omega)\hat{d}^\dagger(\omega)\hat{d}^\dagger(-\omega) - \text{h.c.}]\right\} \quad (\text{S35})$$

with $\xi(\omega) = \xi(-\omega)$. This is because \hat{S} only makes a Bogoliubov transformation of \hat{d} . The spectral densities in Eqs. (S30) and (S31) are in the same form as in the case of vacuum state, after replacing $\mathcal{Z}_{1,2}$ by $\mathcal{Z}'_{1,2}$ and \mathcal{F} by \mathcal{F}' :

$$\mathcal{Z}'_{1,2}(\omega) \equiv \mathcal{Z}_{1,2}(\omega)\cosh r_s + e^{-i\phi_s}\mathcal{Z}_{1,2}^*(-\omega)\sinh r_s, \quad (\text{S36})$$

$$\mathcal{F}'(\omega) \equiv \mathcal{F}(\omega)\cosh r_s + e^{-i\phi_s}\mathcal{F}^*(-\omega)\sinh r_s, \quad (\text{S37})$$

where the real-valued functions r_s and ϕ_s are defined through $\xi(\omega) \equiv r_s(\omega)e^{i\phi_s(\omega)}$. Such a transform will leave Eq. (S34) unchanged.

III. MINIMUM OF $|\bar{S}_{ZF}/\chi_{ZF}|$

Here we prove Eq. (22) of the main paper. Given the output-port observable $\hat{Z} = \hat{Z}_1 \sin \theta + \hat{Z}_2 \cos \theta$, we have

$$\bar{S}_{ZF}(\omega) = \bar{S}_{Z_1F}(\omega) \sin \theta + \bar{S}_{Z_2F}(\omega) \cos \theta, \quad (\text{S38})$$

$$\chi_{ZF}(\omega) = \chi_{Z_1F}(\omega) \sin \theta + \chi_{Z_2F}(\omega) \cos \theta. \quad (\text{S39})$$

The absolute value of their ratio is simply, for $\theta \neq 0$,

$$\mathcal{R} \equiv \left| \frac{\bar{S}_{ZF}(\omega)}{\chi_{ZF}(\omega)} \right| = \left| \frac{\bar{S}_{Z_1F}(\omega) + \bar{S}_{Z_2F}(\omega) \cot \theta}{\chi_{Z_1F}(\omega) + \chi_{Z_2F}(\omega) \cot \theta} \right|. \quad (\text{S40})$$

Using Eqs. (S10) and (S17), we can express the susceptibility $\chi_{Z_{1,2}F}$ in terms of the unsymmetrized spectral density:

$$\chi_{Z_{1,2}F}(\omega) = \frac{i}{\hbar} [S_{Z_{1,2}F}(\omega) - S_{FZ_{1,2}}(-\omega)]. \quad (\text{S41})$$

Form the expressions for $S_{Z_{1,2}F}$ shown in Eq. (S30), the above ratio can be rewritten as

$$\mathcal{R} = \frac{\hbar}{2} \left| \frac{1 + \alpha\beta}{1 - \alpha\beta} \right|, \quad (\text{S42})$$

where we have defined

$$\alpha \equiv \frac{\mathcal{L}_1^*(-\omega) + \mathcal{L}_2^*(-\omega) \cot \theta}{\mathcal{L}_1(\omega) + \mathcal{L}_2(\omega) \cot \theta}, \quad (\text{S43})$$

$$\beta \equiv \frac{\mathcal{F}(-\omega)}{\mathcal{F}^*(\omega)}. \quad (\text{S44})$$

With the constraint Eq. (S27), one can show that

$$|\alpha| = 1. \quad (\text{S45})$$

We can therefore write α as $e^{i\phi_\alpha}$ with ϕ_α being real, and obtain

$$\mathcal{R} = \frac{\hbar}{2} \left[\frac{1 + |\beta|^2 - 2|\beta| \sin \phi'_\alpha}{1 + |\beta|^2 + 2|\beta| \sin \phi'_\alpha} \right]^{1/2}, \quad (\text{S46})$$

in which we have introduced

$$\phi'_\alpha \equiv \phi_\alpha + \arctan[\text{Re}(\beta)/\text{Im}(\beta)]. \quad (\text{S47})$$

Due to the one-to-one mapping between θ and ϕ'_α , minimizing \mathcal{R} over θ is therefore equivalent to that over ϕ'_α . The minimum of \mathcal{R} is achieved when $\phi'_\alpha = \pi/2$ and

$$\mathcal{R}_{\min} = \frac{\hbar}{2} \left| \frac{1 - |\beta|}{1 + |\beta|} \right|. \quad (\text{S48})$$

It is always smaller than $\hbar/2$, i.e.,

$$\mathcal{R}_{\min} \leq \frac{\hbar}{2}, \quad (\text{S49})$$

and reaches the equal sign when either

$$|\beta| = 0 \quad \text{or} \quad |\beta| \rightarrow \infty. \quad (\text{S50})$$

From the definition of β Eq. (S44), this corresponds to either $\mathcal{F}(-\omega) = 0$ or $\mathcal{F}(\omega) = 0$, which is equivalent to

$$S_{FF}(-\omega) = 0 \quad \text{or} \quad S_{FF}(\omega) = 0, \quad (\text{S51})$$

according to Eq. (S31). With the same argument as the one presented in the previous section, the above conclusion is not conditional on whether the detector is in the vacuum state or in the general, stationary, pure Gaussian state.

Q.E.D.

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