Supplemental Material for "Towards the Fundamental Quantum Limit of Linear Measurements of Classical Signals"

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I. LINEAR-RESPONSE THEORY

Here we briefly introduce the linear-response theory that has been applied in our analysis. One can refer to Refs. [S1–S4] for more details. Given the model illustrated in Fig. 1 of the main paper, the Hamiltonian for the measurement setup is

$$\hat{H}_{\rm tot} = \hat{H}_{\rm det} + \hat{H}_{\rm int}\,, \tag{S1}$$

where $\hat{H}_{\rm det}$ is the free Hamiltonian for the detector, and $\hat{H}_{\rm int}$ describes the coupling between the classical signal and the detector. We consider the steady state with the coupling turned on at $t = -\infty$. The solution to any operator $\hat{A}$ of the detector at time t in the Heisenberg picture is given by

$$\hat{A}(t) = \hat{U}_I^\dagger(-\infty, t)\hat{A}^{(0)}(t)\hat{U}_I(-\infty, t) \tag{S2}$$

with $\hat{A}^{(0)}(t)$ denoting the operator under the free evolution:

$$\hat{A}^{(0)}(t) \equiv \hat{U}_0^\dagger(-\infty, t)\hat{A}\,\hat{U}_0(-\infty, t)\,. \tag{S3}$$

The unitary operator for the free-evolution part is defined as $\hat{U}_0(-\infty, t) \equiv \mathcal{T}\exp\{-(i/\hbar)\int_{-\infty}^{t}{\rm d}t'\hat{H}_{\rm det}(t')\}$ with $\mathcal{T}$ being the time-ordering, and, for the interaction part, we have defined $\hat{U}_I(-\infty, t) \equiv \mathcal{T}\exp\{-(i/\hbar)\int_{-\infty}^{t}{\rm d}t'\hat{H}^{(0)}_{\rm int}(t')\}$.

For the measurement to be linear, $\hat{H}_{\rm det}$ only involves linear or quadratic functions of canonical coordinates, among which their commutators are classical numbers, i.e., not operators; the interaction $\hat{H}_{\rm int}$ is in the bilinear form:

$$\hat{H}_{\rm int} = -\hat{F}\,x(t)\,. \tag{S4}$$

As a result, Eq. (S2) leads to the following exact solution to the input-port observable $\hat{F}$ and output-port observable $\hat{Z}$:

$$\hat{Z}(t) = \hat{Z}^{(0)}(t) + \int_{-\infty}^{+\infty}{\rm d}t'\chi_{ZF}(t,\,t')\,x(t')\,, \tag{S5}$$

$$\hat{F}(t) = \hat{F}^{(0)}(t) + \int_{-\infty}^{+\infty}{\rm d}t'\chi_{FF}(t,\,t')\,x(t')\,. \tag{S6}$$

The susceptibility χ_{AB} $(A, B = Z, F)$, which describes the detector response to the signal, is defined as

$$\chi_{AB}(t,\,t') \equiv \frac{i}{\hbar}[\hat{A}^{(0)}(t),\,\hat{B}^{(0)}(t')]\Theta(t-t') \tag{S7}$$

with $\Theta(t)$ being the Heaviside function. Notice that the susceptibilities are classical numbers and only involve operators under the free evolution, which are consequences of the detector being linear.

For the measurement to be continuous, we need to be able to projectively measure the output-port observable at different times precisely without introducing additional noise. This can happen only if $\hat{Z}(t)$ commutes with itself at different times, namely,

$$[\hat{Z}(t),\,\hat{Z}(t')] = 0 \quad \forall t, t'\,. \tag{S8}$$

It is called the condition of simultaneous measurability in Ref. [S3] which also shows that it implies

$$[\hat{Z}^{(0)}(t),\,\hat{Z}^{(0)}(t')] = [\hat{F}^{(0)}(t),\,\hat{Z}^{(0)}(t')]\Theta(t-t') = 0\,, \tag{S9}$$

or equivalently,

$$\chi_{ZZ}(t,\,t') = \chi_{FZ}(t,\,t') = 0\,, \tag{S10}$$

which is central to the discussion of continuous, linear quantum measurements.

When the free Hamiltonian for the detector is time-independent, the susceptibility will only depend on the time difference, i.e.,

$$\chi_{AB}(t,\,t') = \chi_{AB}(t-t')\,, \tag{S11}$$

which is the case considered in the main paper. This allows us to move into the frequency domain, and rewrite Eqs. (S5) and (S6) as

$$\hat{Z}(\omega) = \hat{Z}^{(0)}(\omega) + \chi_{ZF}(\omega)\,x(\omega)\,, \tag{S12}$$

$$\hat{F}(\omega) = \hat{F}^{(0)}(\omega) + \chi_{FF}(\omega)\,x(\omega)\,. \tag{S13}$$

in which the Fourier transform $\hat{A}(\omega) \equiv \int_{-\infty}^{+\infty}{\rm d}t\,e^{i\omega t}\hat{A}(t)$. Furthermore, we consider the detector being in a stationary state, i.e., its density matrix $\hat{\rho}_{\rm det}$ commuting with $\hat{H}_{\rm det}$. The statistical property of the relevant operators, which defines the quantum noise of the detector, can then be quantified by using the frequency-domain spectral density, which is given by

$$S_{AB}(\omega) \equiv \int_{-\infty}^{+\infty}{\rm d}t\,e^{i\omega t}{\rm Tr}[\hat{\rho}_{\rm det}\,\hat{A}^{(0)}(t+\tau)\hat{B}^{(0)}(\tau)]\,, \tag{S14}$$

where τ can be arbitrary due to the stationarity, and we have assumed ${\rm Tr}[\hat{\rho}_{\rm det}\hat{A}] = {\rm Tr}[\hat{\rho}_{\rm det}\hat{B}] = 0$ without loss of generality. Or equivalently, the spectral density can also be defined through

$${\rm Tr}[\hat{\rho}_{\rm det}\,\hat{A}^{(0)}(\omega)\hat{B}^{(0)\dagger}(\omega')] \equiv 2\pi\,S_{AB}(\omega)\delta(\omega-\omega')\,. \tag{S15}$$

The corresponding symmetrized version of the previously defined spectral density is

$$\bar{S}_{AB}(\omega) \equiv \frac{1}{2}[S_{AB}(\omega) + S_{BA}(-\omega)]\,, \tag{S16}$$

which is a summation of both the positive-frequency and negative-frequency spectra.

From the definitions of the susceptibility and spectral density, we have a general equality relating them to each other:

$$\chi_{AB}(\omega) - \chi^*_{BA}(\omega) = \frac{i}{\hbar}[S_{AB}(\omega) - S_{BA}(-\omega)]\,. \tag{S17}$$

When applying this to the case with $\hat{A} = \hat{B}$, it leads to the famous Kubo's formula:

$$\mathrm{Im}[\chi_{AA}(\omega)] = \frac{1}{2\hbar}[S_{AA}(\omega) - S_{AA}(-\omega)]\,. \tag{S18}$$

Such an imaginary part of the susceptibility $\mathrm{Im}[\chi_{AA}(\omega)]$ quantifies the dissipation, and, in the thermal equilibrium, it is related to the symmetrized spectral density $\bar{S}_{AA}(\omega)$ through the fluctuation-dissipation theorem. The measurement process is far from the thermal equilibrium, and therefore the usual fluctuation-dissipation theorem cannot be applied. Nevertheless, when the detector is ideal at the quantum limit with minimum uncertainty, we can also find some general relations between the susceptibility and the symmetrized spectral density, e.g., Eq. (18) and Eq. (21) in the main paper, the later of which will be proven in the next section.

II. PROOF OF EQ. (21)

Here we show the proof of Eq. (21) in the main paper. In the continuous, linear measurements, the detector is a continuum field that contains many degrees of freedom which are coupled to each other through the free evolution. The degrees of freedom for the input and output port that we pick are continuously driven by the ingoing part of the continuum field, which is similar to the in field introduced in Ref. [S5]. In the steady state with the initial condition decaying away, their observables $\hat{Z}_{1,2}$ and $\hat{F}$ can be generally represented in terms of the ingoing field:

$$\hat{Z}^{(0)}_{1,2}(t) = \int_{-\infty}^{\infty} \mathrm{d}t'\,\mathscr{Z}_{1,2}(t-t')\hat{d}(t') + \mathrm{h.c.}\,, \tag{S19}$$

$$\hat{F}^{(0)}(t) = \int_{-\infty}^{\infty} \mathrm{d}t'\,\mathscr{F}(t-t')\hat{d}(t') + \mathrm{h.c.}\,. \tag{S20}$$

Here $\mathscr{Z}$ and $\mathscr{F}$ are some complex-valued functions; h.c. denotes Hermitian conjugate; $\hat{d}(t)$ is annihilation operator of the ingoing field that satisfies the following commutator relation:

$$[\hat{d}(t),\, \hat{d}^\dagger(t')] = \delta(t-t')\,. \tag{S21}$$

In the frequency domain, Eqs. (S19) and (S20) can be rewritten as

$$\hat{Z}^{(0)}_{1,2}(\omega) = \mathscr{Z}_{1,2}(\omega)\hat{d}(\omega) + \mathscr{Z}^*_{1,2}(-\omega)\hat{d}^\dagger(-\omega)\,, \tag{S22}$$

$$\hat{F}^{(0)}(\omega) = \mathscr{F}(\omega)\hat{d}(\omega) + \mathscr{F}^*(-\omega)\hat{d}^\dagger(-\omega)\,, \tag{S23}$$

and the commutator for the ingoing field is

$$[\hat{d}(\omega),\, \hat{d}^\dagger(\omega')] = 2\pi\,\delta(\omega-\omega')\,. \tag{S24}$$

A natural choice for the output port is the outgoing part of the continuum field, similar to the out field in Ref. [S5], which guarantees that the condition in Eq. (S8) can be fulfilled due to causality. Its two conjugate variables $\hat{Z}_{1,2}$ satisfies

$$[\hat{Z}_k(t),\, \hat{Z}_l(t')] = -\sigma_y^{kl}\delta(t-t')\,, \tag{S25}$$

where $k, l = 1, 2$ and σ_y is the Pauli matrix. In the frequency domain, the above commutator reads

$$[\hat{Z}_k(\omega),\, \hat{Z}^\dagger_l(\omega')] = -2\pi\,\sigma_y^{kl}\delta(\omega-\omega')\,. \tag{S26}$$

Together with Eq. (S24), this implies the following constraint on those functions in Eq. (S22):

$$\mathscr{Z}_k(\omega)\mathscr{Z}^*_l(\omega) - \mathscr{Z}^*_k(-\omega)\mathscr{Z}_l(-\omega) = -\sigma_y^{kl}\,, \tag{S27}$$

which is an important equality for the proof.

We first prove Eq. (21) in the case when the detector is in the vacuum state, i.e.,

$$\hat{\rho}_{\mathrm{det}} = |0\rangle\langle 0|\,. \tag{S28}$$

Correspondingly, we have $\mathrm{Tr}[\hat{\rho}_{\mathrm{det}}\,\hat{d}(\omega)\hat{d}^\dagger(\omega')] = 2\pi\,\delta(\omega-\omega')$ and $\mathrm{Tr}[\hat{\rho}_{\mathrm{det}}\,\hat{d}^\dagger(\omega)\hat{d}(\omega')] = 0$, which are equivalent to

$$S_{\hat{d}\hat{d}^\dagger}(\omega) = 1\,, \quad S_{\hat{d}^\dagger\hat{d}}(\omega) = 0\,. \tag{S29}$$

From Eqs. (S22) and (S23), the above spectral density for $\hat{d}$ leads to

$$S_{Z_{1,2}F}(\omega) = \mathscr{Z}_{1,2}(\omega)\mathscr{F}^*(\omega)\,, \tag{S30}$$

$$S_{FF}(\omega) = |\mathscr{F}(\omega)|^2\,. \tag{S31}$$

Using the constraint in Eq. (S27) and the definition of symmetrized spectral density Eq. (S16), we find

$$\mathrm{Im}[\bar{S}_{Z_1F}(\omega)\bar{S}^*_{Z_2F}(\omega)] = \frac{1}{8}[S_{FF}(\omega) - S_{FF}(-\omega)]\,. \tag{S32}$$

With the Kubo's formula Eq. (S18):

$$\mathrm{Im}[\chi_{FF}(\omega)] = \frac{1}{2\hbar}[S_{FF}(\omega) - S_{FF}(-\omega)]\,, \tag{S33}$$

finally it gives rise to Eq. (21) in the main paper, i.e.,

$$\mathrm{Im}[\bar{S}_{Z_1F}(\omega)\bar{S}^*_{Z_2F}(\omega)] = \frac{\hbar}{4}\mathrm{Im}[\chi_{FF}(\omega)]\,. \tag{S34}$$

We can further show that Eq. (S34) also holds for the general, stationary, pure Gaussian state—multi-mode squeezed state $\hat{\rho}_{\mathrm{det}} = \hat{\mathcal{S}}|0\rangle\langle 0|\hat{\mathcal{S}}^\dagger$, in which the squeezing operator $\hat{\mathcal{S}}$ is defined as [S6]

$$\hat{\mathcal{S}} \equiv \exp\left\{\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}[\xi(\omega)\hat{d}^\dagger(\omega)\hat{d}^\dagger(-\omega) - \mathrm{h.c.}]\right\} \tag{S35}$$

with $\xi(\omega) = \xi(-\omega)$. This is because $\hat{\mathcal{S}}$ only makes a Bogoliubov transformation of $\hat{d}$. The spectral densities in Eqs. (S30) and (S31) are in the same form as in the case of vacuum state, after replacing $\mathscr{Z}_{1,2}$ by $\mathscr{Z}'_{1,2}$ and $\mathscr{F}$ by $\mathscr{F}'$:

$$\mathscr{Z}'_{1,2}(\omega) \equiv \mathscr{Z}_{1,2}(\omega)\cosh r_s + e^{-i\phi_s}\mathscr{Z}^*_{1,2}(-\omega)\sinh r_s\,, \tag{S36}$$

$$\mathscr{F}'(\omega) \equiv \mathscr{F}(\omega)\cosh r_s + e^{-i\phi_s}\mathscr{F}^*(-\omega)\sinh r_s\,, \tag{S37}$$

where the real-valued functions r_s and ϕ_s are defined through $\xi(\omega) \equiv r_s(\omega)e^{i\phi(\omega)}$. Such a transform will leave Eq. (S34) unchanged.

III. MINIMUM OF $|\bar{S}_{ZF}/\chi_{ZF}|$

Here we prove Eq. (22) of the main paper. Given the output-port observable $\hat{Z} = \hat{Z}_1 \sin\theta + \hat{Z}_2 \cos\theta$, we have

$$\bar{S}_{ZF}(\omega) = \bar{S}_{Z_1F}(\omega)\sin\theta + \bar{S}_{Z_2F}(\omega)\cos\theta\,, \tag{S38}$$
$$\chi_{ZF}(\omega) = \chi_{Z_1F}(\omega)\sin\theta + \chi_{Z_2F}(\omega)\cos\theta\,. \tag{S39}$$

The absolute value of their ratio is simply, for $\theta \neq 0$,

$$\mathcal{R} \equiv \left|\frac{\bar{S}_{ZF}(\omega)}{\chi_{ZF}(\omega)}\right| = \left|\frac{\bar{S}_{Z_1F}(\omega) + \bar{S}_{Z_2F}(\omega)\cot\theta}{\chi_{Z_1F}(\omega) + \chi_{Z_2F}(\omega)\cot\theta}\right|\,. \tag{S40}$$

Using Eqs. (S10) and (S17), we can express the susceptibility $\chi_{Z_{1,2}F}$ in terms of the unsymmetrized spectral density:

$$\chi_{Z_{1,2}F}(\omega) = \frac{i}{\hbar}[S_{Z_{1,2}F}(\omega) - S_{FZ_{1,2}}(-\omega)]\,. \tag{S41}$$

Form the expressions for $S_{Z_{1,2}F}$ shown in Eq. (S30), the above ratio can be rewritten as

$$\mathcal{R} = \frac{\hbar}{2}\left|\frac{1+\alpha\beta}{1-\alpha\beta}\right|\,, \tag{S42}$$

where we have defined

$$\alpha \equiv \frac{\mathscr{Z}_1^*(-\omega) + \mathscr{Z}_2^*(-\omega)\cot\theta}{\mathscr{Z}_1(\omega) + \mathscr{Z}_2(\omega)\cot\theta}\,, \tag{S43}$$
$$\beta \equiv \frac{\mathscr{F}(-\omega)}{\mathscr{F}^*(\omega)}\,. \tag{S44}$$

With the constraint Eq. (S27), one can show that

$$|\alpha| = 1\,. \tag{S45}$$

We can therefore write α as $e^{i\phi_\alpha}$ with ϕ_α being real, and obtain

$$\mathcal{R} = \frac{\hbar}{2}\left[\frac{1+|\beta|^2 - 2|\beta|\sin\phi'_\alpha}{1+|\beta|^2 + 2|\beta|\sin\phi'_\alpha}\right]^{1/2}\,, \tag{S46}$$

in which we have introduced

$$\phi'_\alpha \equiv \phi_\alpha + \arctan[\mathrm{Re}(\beta)/\mathrm{Im}(\beta)]\,. \tag{S47}$$

Due to the one-to-one mapping between θ and ϕ'_α, minimizing $\mathcal{R}$ over θ is therefore equivalent to that over ϕ'_α. The minimum of $\mathcal{R}$ is achieved when $\phi'_\alpha = \pi/2$ and

$$\mathcal{R}_{\min} = \frac{\hbar}{2}\left|\frac{1-|\beta|}{1+|\beta|}\right|\,. \tag{S48}$$

It is always smaller than $\hbar/2$, i.e.,

$$\mathcal{R}_{\min} \leq \frac{\hbar}{2}\,, \tag{S49}$$

and reaches the equal sign when either

$$|\beta| = 0 \quad \text{or} \quad |\beta| \to \infty\,. \tag{S50}$$

From the definition of β Eq. (S44), this corresponds to either $\mathscr{F}(-\omega) = 0$ or $\mathscr{F}(\omega) = 0$, which is equivalent to

$$S_{FF}(-\omega) = 0 \quad \text{or} \quad S_{FF}(\omega) = 0\,, \tag{S51}$$

according to Eq. (S31). With the same argument as the one presented in the previous section, the above conclusion is not conditional on whether the detector is in the vacuum state or in the general, stationary, pure Gaussian state.

Q.E.D.

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