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HOW TO CUT A CAKE HEALTHILY

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Abstract

The Sliding Knife procedure as well as the Cake Cutting and Fair Border existence theorems, stated for additive evaluations, hold unchanged for concave ones.

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1 Introduction

The problem of dividing an object among some people so that everybody is satisfied of the received share is an ancient one, some interesting cases are reported in the Bible (e.g. Numbers 33:54) and in the Babylonian Talmud (e.g. Kethubot 93a). Mathematical issues on the existence and construction of a solution evolved and grew at an impressive speed after the seminal work of Steinhaus (1949), see for example Brams and Taylor (1996) or Robertson and Webb (1998) and the references therein.

Building on earlier contributions of Banach, Knaster, and Steinhaus, in a classic paper, Dubins and Spanier (1961) devised a procedure to cut a cake for n individuals so that each of them receives a slice she evaluates at least $1/n$th of the entire cake.

> "...A knife is slowly moved at constant speed parallel to itself over the top of the cake. At each instant the knife is poised so that it could cut a unique slice of the cake. As times goes by the potential slice increases monotonely from nothing until it becomes the entire cake. The first person to indicate satisfaction with the slice then determined by the position of the knife receives that slice and is eliminated from further distribution of the cake. (If two or more participants simultaneously indicate satisfaction with the slice, it is given to any one of them.) The process is repeated with the other $n-1$ participants and with that remains of the cake...The method described above is equally applicable for the division of any object provided only that (1) the value assigned by any participant to any part of the object equals the sum of the values of the subparts when the part is subdivided into any finite number of subparts; and (2) the value to each participant of the potential slice varies in a continuous fashion as the knife is moved over the object..."

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Though the procedure is claimed to work under assumptions (1) and (2), it is easy to see that it does not require the full strength of (1). As a matter of fact, it is enough to assume that: (1') the value assigned by any participant to any part of the object is *not greater than* the sum of the values of the subparts when the part is subdivided into any finite number of subparts.

Starting from this observation, we decided to consider the cake cutting problem in a more general setting in which the participants' evaluations of the slices are represented by submodular set functions rather than additive measures. The motivation for this generalization is quite natural: after eating a first slice of a cake, the next slice is likely to be less desirable, because of satiety, dietary and health concerns, etc.

In the economics jargon, we are moving from constant to decreasing marginal evaluations, which is the usual assumption in consumer theory. To the best of our knowledge, the only attempt in this sense was made by Berliant, Dunz, and Thomson (1992). Our approach seems simpler and more intuitive (see the Concluding Remarks).

2 The Result

Let us introduce some notions. Given a measurable space (S, Σ), a *capacity* on Σ is a set function $\nu : \Sigma \to [0, 1]$ such that

(a) $\nu(\emptyset) = 0$ and $\nu(S) = 1$,

(b) $\nu(A) \leq \nu(B)$ for all $A, B \in \Sigma$ such that $A \subseteq B$,

(c) $\nu(A_n) \downarrow 0$ for any monotone sequence $\{A_n\} \subseteq \Sigma$ with $A_n \downarrow \emptyset$.

A capacity ν is *concave* (*submodular*) if

(d) $\nu(A \cup B) \leq \nu(A) + \nu(B) - \nu(A \cap B)$ for all $A, B \in \Sigma$;

while it is a *probability measure* if

(e) $\nu(A \cup B) = \nu(A) + \nu(B) - \nu(A \cap B)$ for all $A, B \in \Sigma$.

Finally, ν is *nonatomic* if

(f) for each set A such that $\nu(A) > 0$ there exists $B \subseteq A$ such that $0 < \nu(B) < \nu(A)$.

Our contribution is to consider nonatomic concave capacities rather than nonatomic probability measures. To see why property (d) captures the idea of decreasing marginal evaluations, consider a nonatomic probability measure μ and a continuous strictly increasing function $u : [0,1] \to [0,1]$ such that $u(0) = 0$ and $u(1) = 1$. It is easy to check that the set function defined, for all $A \in \Sigma$, by

$$\nu(A) = u(\mu(A))$$

is a nonatomic concave capacity if and only if u is concave.[1] Therefore, if μ represents an ideal "frictionless" evaluation, without any concern for satiation, diets and health, etc., the capacity ν adds this obviously important concerns into the picture by assigning a decreasing value (utility) to the extra pieces of the good the person is getting.

We are now ready to state our result. For the sake of clarity we provide a detailed proof.

Theorem 1 *Let $\nu_1, \nu_2, ..., \nu_n$ be nonatomic concave capacities on a measurable space (S, Σ). Then, given any $\alpha_1, \alpha_2, ..., \alpha_n \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, there exists a partition $\{A_1, A_2, ..., A_n\}$ of S in Σ such that*

$$\nu_i(A_i) \geq \alpha_i$$

for each $i = 1, 2, ..., n$. Moreover, if $\nu_j \neq \nu_k$ for some $j \neq k$ and $\alpha_1, \alpha_2, ..., \alpha_n > 0$, the partition $\{A_1, A_2, ..., A_n\}$ can be chosen to satisfy

$$\nu_i(A_i) > \alpha_i$$

for each $i = 1, 2, ..., n$.

Proof. Let $i \in \{1, 2, ..., n\}$ and $\mathcal{M}(\nu_i)$ be the set of all probability measures μ such that $\mu(A) \leq \nu_i(A)$ for all $A \in \Sigma$.

Claim 1. *For all $A \in \Sigma$,*

$$\nu_i(A) = \max_{\mu \in \mathcal{M}(\nu_i)} \mu(A),$$

and there exists $\mu_i^* \in \mathcal{M}(\nu_i)$ such that each $\mu \in \mathcal{M}(\nu_i)$ is absolutely continuous with respect to μ_i^*.[2]

Proof. By Theorem 14 of Kelley (1959), since ν_i satisfies (a), (b), and (d), then for any subalgebra Σ' of Σ and any finitely additive probability μ' on Σ' such that $\mu'(A') \leq \nu_i(A')$ for every $A' \in \Sigma'$,[3] there exists an extension μ of μ' to Σ which is a finitely additive

[1] Notice that the capacities of the form $u(\mu(\cdot))$ are only a small subset of the class of nonatomic concave capacities.

[2] μ is *absolutely continuous* with respect to μ_i^* if $A \in \Sigma$ and $\mu_i^*(A) = 0$ imply $\mu(A) = 0$.

[3] A *finitely additive probability* is a set function μ' satysfying (a), (b) and (e).

probability such that $\mu(A) \leq \nu_i(A)$ for every $A \in \Sigma$. As a consequence, if $\mathcal{E}(\nu_i)$ is the set of all finitely additive probabilities μ such that $\mu(A) \leq \nu_i(A)$ for all $A \in \Sigma$, we have

$$\nu_i(A) = \max_{\mu \in \mathcal{E}(\nu_i)} \mu(A) \tag{1}$$

for all $A \in \Sigma$. In fact, for every $B \in \Sigma$, $1 = \nu_i(B \cup B^c) \leq \nu_i(B) + \nu_i(B^c)$, hence $1 - \nu_i(B) \leq \nu_i(B^c)$. Consider the finitely additive probability on $\Sigma' = \{\emptyset, B, B^c, S\}$ defined by $\mu'_B(\emptyset) = \nu_i(\emptyset) = 0$, $\mu'_B(B) = \nu_i(B)$, $\mu'_B(B^c) = 1 - \nu_i(B) \leq \nu_i(B^c)$, $\mu'_B(S) = \nu_i(S) = 1$. There exists a finitely additive extension μ_B of μ'_B to Σ such that $\mu_B(A) \leq \nu_i(A)$ for every $A \in \Sigma$. Hence, $\mathcal{E}(\nu_i)$ is nonempty and, for every $A \in \Sigma$, there exists $\mu_A \in \mathcal{E}(\nu_i)$ such that $\nu_i(A) = \mu_A(A) \leq \sup_{\mu \in \mathcal{E}(\nu_i)} \mu(A) \leq \nu_i(A)$, that is $\nu_i(A) = \max_{\mu \in \mathcal{E}(\nu_i)} \mu(A)$ for all $A \in \Sigma$.

For any $\mu \in \mathcal{E}(\nu_i)$ and any monotone sequence $\{A_n\} \subseteq \Sigma$ with $A_n \downarrow \emptyset$, $\mu(A_n) \leq \nu_i(A_n) \to 0$. Hence:

(i) μ is a probability measure, and $\mathcal{E}(\nu_i) = \mathcal{M}(\nu_i)$;

(ii) the continuity of μ at $\emptyset$ is uniform with respect to μ in $\mathcal{M}(\nu_i)$.[4]

By Theorems IV.9.1 and IV.9.2 of Dunford and Schwartz (1958), points (i) and (ii) along with the convexity of $\mathcal{M}(\nu_i)$, guarantee the existence of a μ_i^* in $\mathcal{M}(\nu_i)$ such that every $\mu \in \mathcal{M}(\nu_i)$ is absolutely continuous with respect to μ_i^* (see also Schmeidler, 1972, p. 221 and Delbaen, 1974, p. 226). □

<u>Claim 2.</u> $\mathcal{M}(\nu_i)$ *consists of nonatomic probability measures.*

Proof. Assume that A is an atom for μ_i^*.[5] Then $\mu_i^*(A) > 0$ and $\nu_i(A) > 0$. Let $\Sigma \ni B \subseteq A$, we have either $\mu_i^*(B) = 0$ or $\mu_i^*(B) = \mu_i^*(A)$. If $\mu_i^*(B) = 0$, then $\mu(B) = 0$ for every $\mu \in \mathcal{M}(\nu_i)$, thus $\nu_i(B) = 0$. Else $\mu_i^*(B) = \mu_i^*(A)$, then $\mu_i^*(A - B) = 0$, and $\mu(A - B) = 0$ for every $\mu \in \mathcal{M}(\nu_i)$, so $\mu(A) = \mu(B)$ for every $\mu \in \mathcal{M}(\nu_i)$, and $\nu_i(B) = \nu_i(A)$. Then A is an atom for ν_i, a contradiction. Therefore μ_i^* is nonatomic. Next we show that this implies the nonatomicity of all elements μ in $\mathcal{M}(\nu_i)$. Suppose there exists a μ in $\mathcal{M}(\nu_i)$ having an atom A. Since $\mu(A) > 0$, we have $\mu_i^*(A) > 0$. Let $\{A_1, B_1\}$ be a partition of A such that $\mu_i^*(A_1) = \mu_i^*(B_1) = \frac{1}{2}\mu_i^*(A)$. It must be either $\mu(A_1) = \mu(A)$ or $\mu(B_1) = \mu(A)$. Without loss of generality, assume $\mu(A_1) = \mu(A)$, A_1 is an atom for μ. Let $\{A_2, B_2\}$ be a partition of A_1 such that $\mu_i^*(A_2) = \mu_i^*(B_2) = \frac{1}{2}\mu_i^*(A_1) = \frac{1}{2^2}\mu_i^*(A)$. It must be either $\mu(A_2) = \mu(A_1) = \mu(A)$ or $\mu(B_2) = \mu(A_1) = \mu(A)$. Without loss of generality, assume $\mu(A_2) = \mu(A)$. Proceeding in this way, we can construct a decreasing sequence $\{A_n\} \subseteq \Sigma$ such that $\mu_i^*(A_n) = \frac{1}{2^n}\mu_i^*(A)$ and $\mu(A_n) = \mu(A)$ for all $n \geq 1$. Hence, $\mu_i^*\left(\bigcap_{n=1}^\infty A_n\right) = 0$ and $\mu\left(\bigcap_{n=1}^\infty A_n\right) = \mu(A) > 0$, a contradiction. □

[4] That is, for any monotone sequence $\{A_n\} \subseteq \Sigma$ with $A_n \downarrow \emptyset$ and any $\varepsilon > 0$ there exists $N \geq 1$ such that $\mu(A_n) \leq \varepsilon$ for every $n \geq N$ and every $\mu \in \mathcal{M}(\nu_i)$.

[5] $A \in \Sigma$ is an *atom* for ν if $\nu(A) > 0$, and for any $B \in \Sigma$ such that $B \subseteq A$, either $\nu(B) = 0$ or $\nu(B) = \nu(A)$, clearly ν is nonatomic if and only if it has no atoms.

Corollary 1.1 of Dubins and Spanier (1961) guarantees the existence of a partition $\{A_1, A_2, ..., A_n\}$ of S in Σ such that

$$\nu_i(A_i) \geq \mu_i^* (A_i) \geq \alpha_i$$

for each $i = 1, 2, ..., n$.

If $\nu_j \neq \nu_k$, it must be $\mathcal{M}(\nu_j) \neq \mathcal{M}(\nu_k)$. Choose $\mu_j \in \mathcal{M}(\nu_j)$ and $\mu_k \in \mathcal{M}(\nu_k)$ such that $\mu_j \neq \mu_k$ and $\mu_i \in \mathcal{M}(\nu_i)$ arbitrarily, if $i \neq j, k$. If $\alpha_1, \alpha_2, ..., \alpha_n > 0$, by Corollary 1.2 of Dubins and Spanier (1961), there exists a partition $\{A_1, A_2, ..., A_n\}$ of S in Σ such that

$$\nu_i(A_i) \geq \mu_i (A_i) > \alpha_i$$

for each $i = 1, 2, ..., n$. Q.E.D.

3 Concluding Remarks

The elements $A_1, ..., A_n$ of the partition in Theorem 1 can be nastily shaped and it can be important to know whether it is possible to guarantee to each participant a true slice of the cake rather than a bunch of crumbles. This problem can be especially relevant in territorial disputes where, for example, n countries have to partition a land bordering each of them.

Hill (1983) solves the problem for evaluations represented by nonatomic probability measures. The next theorem extends the result to evaluations represented by nonatomic concave capacities. The proof, which is similar to that of Theorem 1, is omitted.

Theorem 2 *Let $L, C_1, C_2, ..., C_n$ be open connected subsets in $\mathbb{R}^2$ with C_i adjacent to L for all $i = 1, 2, ..., n$.*[6] *If $\nu_1, \nu_2, ..., \nu_n$ are nonatomic continuous concave (Borel) capacities on L, and $\alpha_1, \alpha_2, ..., \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$. Then there exist disjoint open connected subsets $A_1, A_2, ..., A_n$ of L such that*

- *A_i is adjacent to C_i for all $i = 1, 2, ..., n$,*
- *$\nu_i(A_i) \geq \alpha_i$ for all $i = 1, 2, ..., n$, and*
- *$\overline{\bigcup_{i=1}^n A_i} = L$.*

Moreover, if $\nu_j \neq \nu_k$ for some $j \neq k$ and $\alpha_1, \alpha_2, ..., \alpha_n > 0$, $A_1, A_2, ..., A_n$ can be chosen to satisfy

$$\nu_i(A_i) > \alpha_i$$

for each $i = 1, 2, ..., n$.

[6] Open connected subsets A and B of $\mathbb{R}^2$ are *adjacent* if $\partial A \cap \partial B$ contains an open arc γ (homeomorphic image of $(0, 1)$) such that $A \cup B \cup \gamma$ is open and connected.

Like the original result, this theorem holds for subsets of $\mathbb{R}^k$, and, dropping all adjacency requirements, this yields the existence of a fair division of a connected cake into connected slices.

Finally, Berliant, Dunz, and Thomson (1992) proved that Theorem 2 holds for the following class of set functions defined on the Borel σ-algebra $\mathcal{B}$ of an open subset L of $\mathbb{R}^k$:

> "...Let m be Lebesgue measure on $\mathbb{R}^k$...The function $u_i : \mathcal{B} \to \mathbb{R}_+$ is *concave* if there exists a function $h_i : \{(x, B) \in L \times \mathcal{B} : x \in B\} \to \mathbb{R}_+$ such that
> (i) for all $B \in \mathcal{B}$, $h_i(\cdot, B)$ is integrable,
> (ii) for all $B, B' \in \mathcal{B}$ with $B' \subseteq B$, for all $x \in B'$, $h_i(x, B') \geq h_i(x, B)$, and
> (iii) for all $B \in \mathcal{B}$, $u_i(B) = \int_B h_i(x, B)\, dm(x)$..."

Relative to nonatomic concave capacities, this seems to be a more complicated and less intuitive class of evaluations.

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