A COMPARISON OF SEVERAL TECHNIQUES FOR DESIGNING CONTROLLERS OF UNCERTAIN DYNAMIC SYSTEMS*

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Abstract

In recent years, a number of techniques have been developed for the design of linear, constant gain feedback controllers for systems with imprecisely known parameters. In this paper, several of these techniques are compared in the context of the design of a lateral autopilot for a rudderless remotely piloted vehicle with uncertain aerodynamic coefficients. Properties of the design techniques on which the comparison is based include closed-loop system performance at nominal and off-nominal parameter values, computational cost and complexity, ease of implementation in a real system, and generality of the parameter uncertainty which can be dealt with.

I. Introduction

In many physical systems, an accurate knowledge of certain parameters is very difficult or very expensive to obtain. The designer of a remotely piloted vehicle flight control system, for example, frequently has available little data regarding aerodynamic coefficients, due to a lack of wind tunnel tests.

The design of feedback controllers for systems with uncertain parameters can be accomplished in several ways. Parameter uncertainty can often be reduced substantially through extensive testing or through use of real-time or nonreal-time system identification techniques ([1] , [2] , for example). Alternately, parameter uncertainties may simply be accepted at their a priori levels, and a control system designed so as to be, in some sense, insensitive to parameter variations. It is the latter approach which is investigated in this paper. In particular, a linear, constant gain feedback controller is sought for a linear system with an uncertain system matrix, such that the closed-loop system behavior is acceptable for all values of the uncertain parameters within specified limits.

Considerable effort has been expended in this area over the years, and a number of controller design methods have been devised. To date, no one technique has received widespread acceptance from control system designers. In [3] , a comparative assessment of seven such methods was made in the context of wing load alleviation for the C-5A, with uncertainties assumed to exist in dynamic pressure, structural damping and frequency, and the stability derivative M_w. The techniques investigated were referred to as the additive noise design, the minimax design [4] , the multiplant design, the sensitivity vector augmentation design ([5] , for example), the state dependent noise design, the mismatch estimation design, and the uncertainty weighting design. Most of the methods were found to be at least somewhat burdensome computationally, and most did not produce control system designs judged to be significant improvements over a standard linear-quadratic synthesis design [6] which assumes precisely known parameters. The uncertainty weighting and minimax techniques were judged to be generally superior to the other approaches. In [7] , the comparative assessment in [3] was extended to include an information matrix approach [8] and a finite dimensional inverse method. The former method was found to perform favorably when compared with the minimax and uncertainty weighting methods.

Other methods proposed in recent years include the random variable method of Hadass and Powell [9] , the expected cost method of Ly and Cannon [10] , [11] and a related method due to Heath and Dillow [12] , the guaranteed cost control method of Chang and Peng [13] , and a modification of the latter approach, the multistep guaranteed cost control method due to Vinkler and Wood [14] .

In this paper, continuous- and discrete-parameter versions of the expected cost method, the guaranteed cost control method, the multistep guaranteed cost control method, the minimax method, the uncertainty weighting method, and the standard linear-quadratic synthesis method are compared. Two problems are considered, a second-order system with two uncertain parameters, and a fifth-order lateral autopilot for a

* This research was supported by Teledyne Ryan Aeronautical.

CH1392-0/78/0000-0031$00.75 © 1978 IEEE

rudderless remotely piloted vehicle with an uncertain aerodynamic coefficient $C_{n_{\delta_a}}$.

II. Statement of the Control Problem

Using state space notation, one may describe the dynamics of a linear, time-invariant system with several uncertain, but constant, parameters by

$$\dot{x}(t) = A(q)x(t) + Bu(t) \quad , \quad t \geq 0 \tag{1}$$

where

$x(t)$ = state vector (nx1)
$u(t)$ = control vector (mx1)
q = vector of uncertain parameters referenced to their nominal values (n'x1)
A = open-loop dynamics matrix (nxn)
B = control distribution matrix (nxm)
t = independent variable

The vector q is assumed to lie somewhere within a closed, bounded region $\Omega \subset R^{n'}$. For simplicity, it is assumed that Ω is rectangular in shape and includes the origin, so that each component of q is bounded above and below as follows:

$$\begin{aligned} a_i &\leq q_i \leq b_i \\ a_i &\leq 0, \quad b_i \geq 0 \end{aligned} \qquad i = 1,\ldots,n' \tag{2}$$

$q = 0$ corresponds to the nominal design condition. The structure of $A(q)$ is assumed to be as follows:

$$A(q) = A_0 + \sum_{i=1}^{n'} q_i A_i \tag{3}$$

where A_i, $i = 0,\ldots,n'$, are constant nxn matrices, i.e., the uncertain parameters are assumed to enter A in a linear fashion. It is assumed that $(A(q),B)$ form a controllable pair for all $q\varepsilon\Omega$.

A quadratic cost functional J is defined as follows:

$$J = \int_0^{\infty} (x^T Q x + u^T R u)dt \tag{4}$$

where Q is an nxn positive semidefinite matrix and R is an mxm positive definite matrix. It is assumed that $(A(q),Q^{1/2})$ form an observable pair for all $q\varepsilon\Omega$, where $Q^{1/2}$ denotes any square root of the matrix Q $(Q^{1/2}(Q^{1/2})^T = Q)$.

A linear, constant gain feedback control law of the form

$$u(t) = -Cx(t) \tag{5}$$

is sought - all state variables are assumed available for feedback.

Each of the design methods can handle a more general problem than that described above. However, the more general situations which can be handled differ from method to method. The above problem formulation is about the most general which can be handled by all the methods.

III. Description of the Controller Design Methods

Standard Linear-Quadratic Regulator Design Method

The standard linear-quadratic regulator design, as described in [6], bases the choice of controller feedback gains on nominal parameter values. Parameter uncertainties are not explicitly taken into account in the design process. The controller gains are determined by finding the positive definite solution to the algebraic Riccati equation

$$0 = SA_0 + A_0^T S - SBR^{-1}B^T S + Q \tag{6}$$

and then evaluating

$$C = R^{-1}B^T S \tag{7}$$

Under the controllability, observability, and positivity assumptions made above, a unique positive definite solution to the algebraic Riccati equation is guaranteed to exist [15]. It is easily determined using the method of eigenvector decomposition [16]. The resulting closed-loop system is guaranteed to be stable when the uncertain parameters lie within some neighborhood of their nominal values. In many situations, though, this neighborhood does not include all of Ω.

Guaranteed Cost Control Method

With the guaranteed cost control design method [13], controller gains are determined by solving a matrix algebraic equation similar to the Riccati equation, but with an additional term which bounds the effects of parameter uncertainty. Specifically, one finds the positive definite solution (if it exists) to the algebraic equation

$$0 = SA_0 + A_0^T S - SBR^{-1}B^T S + Q + U(S) \tag{8}$$

with

$$U = \sum_{i=1}^{n'} N_i E_i N_i^T \tag{9}$$

where N_i is the orthogonal transformation which diagonalizes the symmetric matrix $(SA_i+A_i^TS)$:

$$N_i^T(SA_i + A_i^T S)N_i = \Lambda_i \tag{10}$$

(Λ_i is diagonal and contains the eigenvalues of $SA_i+A_i^TS$, i.e., $(\Lambda_i)_{kk}=(\lambda_i)_k$, $(\Lambda_i)_{kj}=0$, $k\neq j$, where $(\lambda_i)_k$ is the k^{th} eigenvalue of $SA_i+A_i^TS$.

E_i is defined by:

$$\begin{cases} (E_i)_{kk} = \begin{cases} a_i(\lambda_i)_k \quad , & (\lambda_i)_k < 0 \\ b_i(\lambda_i)_k \quad , & (\lambda_i)_k \geq 0 \end{cases} \\ (E_i)_{kj} = 0 \quad , \quad k \neq j \end{cases} \tag{11}$$

This expression for U(S), suggested in [13], is an upper bound on the quantity

$$\sum_{i=1}^{n'} q_i(SA_i + A_i^T S)$$

in the sense that

$$x^T U(S) x \geq x^T \left[\sum_{i=1}^{n'} q_i(SA_i + A_i^T S) \right] x \tag{12}$$

for all $q \epsilon \Omega$ and $x \epsilon R^n$. Different upper bounds are suggested in [17] and [18].

The controller feedback gains are determined according to Eq. (7). The value of the performance criterion J, as defined by Eq. (4), achieved using a feedback controller designed by this technique is guaranteed to be less than

$$x^T(0) S x(0)$$

Hence, the name guaranteed cost control. The resulting controller is, in a certain sense, an optimal design for all $q \epsilon \Omega$ [14], with the various desirable properties of an optimally designed system described in [15], [19], [20].

Certain assumptions made in the problem formulation are needed only for the guaranteed cost control and multistep guaranteed cost control methods, specifically, the assumptions that A is a linear function of each uncertain parameter, that the region of parameter uncertainty is rectangular, and that B is independent of the uncertain parameters. (Actually, a certain limited form of parameter dependence in B can be accommodated with these methods, but, for simplicity, it is assumed here that B is independent of q.) On the other hand, variations of q with time can be treated with these methods, but not with the others. As long as q is bounded, it need not be constant.

Equation (8) is solved here by an extension of Kleinman's iterative technique for solving an algebraic Riccati equation [21].

Multistep Guaranteed Cost Control Method

One drawback to the guaranteed cost control approach is its tendency to produce relatively large controller feedback gains, relatively large control effort, and overdamped dominant closed-loop poles. The multistep guaranteed cost control method was suggested in [14] as a means of avoiding these difficulties. This technique involves constructing the controller in several steps, rather than in a single step, as in [13]. The technique is as follows:

1. Select weighting matrices Q and R and the scalar μ, $0 < \mu \leq 1$.

2. Determine the positive-definite solution S_0 to the following algebraic Riccati equation

$$S_0 A_0 + A_0^T S_0 - S_0 B R^{-1} B^T S_0 + Q = 0 \tag{13}$$

by eigenvector decomposition [16].

3. Evaluate the controller feedback gain matrix

$$C_0 = R^{-1} B^T S_0 \tag{14}$$

4. Evaluate $U(S_0)$ in accordance with Eqs. (9)-(11). Determine the value of ρ_0 for which the matrix

$$Q_0 = Q - \rho_0 U(S_0) \tag{15}$$

becomes indefinite. Set j equal to one.

5. Evaluate

$$F_{j-1} = A_0 - BC_{j-1} \tag{16}$$

6. Obtain the positive-definite solution S_j to the following equation:

$$S_j F_{j-1} + F_{j-1}^T S_j - S_j B R^{-1} B^T S_j + \mu Q + \rho_{j-1} U(S_j) = 0 \tag{17}$$

by an extension of Kleinman's method [21].

7. Evaluate the feedback gain matrix

$$C_j = C_{j-1} + R^{-1} B^T S_j \tag{18}$$

8. Evaluate $U(S_j)$ in accordance with Eqs. (9)-(11). Determine the value of ρ_j for which the matrix

$$Q_j = \mu Q - (\rho_j - \rho_{j-1}) U(S_j) \tag{19}$$

becomes indefinite. If $\rho_j \geq 1$, a successful control system design has been achieved. If $\rho_j < 1$, but j = maximum desired number of iterations, design process terminates unsuccessfully. Otherwise, increase j by one and go to Step 5.

An alternative to Step 8 which is sometimes useful is the following:

8'. If $\rho_{j-1} \geq 1$, a successful design has been achieved. Otherwise, let $\rho_j = \rho_{j-1} + \Delta\rho$, where $\Delta\rho$ is a positive quantity input to the routine either in advance or in an interactive mode as the computations proceed. Increase j by one and go to Step 5.

Step 8 tends to produce fairly small changes in ρ from one iteration to the next when μ is small.

The iterative process can be speeded up by using Step 8'.

This multistep version of the guaranteed cost control technique, though requiring more computation than the method described in [13], avoids the unnecessarily tight control associated with that method. A rationale for the various steps of this method is presented in [14]. A useful feature of this design method is the fact that the range of parameter uncertainty over which a given set of feedback gains produce a stable (in fact, an optimal) closed-loop control system is easily determined.

Expected Cost Methods

The single-step and multistep guaranteed cost control methods assume that the parameter vector q lies with probability one inside the closed, bounded region Ω of R^n. No use is made of any more quantitative statistical information about q. The expected cost method, as presented in [10] and [11], can make use of such knowledge.

The value of J, as given by Eq. (4), achieved using the constant gain, linear feedback control law (5), is

$$J[x(0),q,C] = x^T(0)S(q,C)x(0) \tag{20}$$

where S(q,C) is the positive definite solution to the algebraic Lyapunov equation

$$S(q,C)[A(q)-BC] + [A(q)-BC]^T S(q,C) + Q + C^T RC = 0 \tag{21}$$

This equation has a positive definite solution if A(q)-BC is a stability matrix.

Given a probability density function p(q) which describes the distribution of q within Ω, define

$$J^*[x(0),C] = \int_{q\epsilon\Omega} J[x(0),q,C]p(q)dq$$

$$= x^T(0)[\int_{q\epsilon\Omega} S(q,C)p(q)dq]x(0) \tag{22}$$

(i.e., $J^*[x(0),C]$ is the expected value of the cost $J[x(0),q,C]$ over all $q\epsilon\Omega$.) Let us assume that there exists a nonempty subset $\mathcal{C}$ of R^{mxn} such that A(q)-BC is stable for all $q\epsilon\Omega$ and all $C\epsilon\mathcal{C}$. Equation (22) is meaningful only for $C\epsilon\mathcal{C}$. If $J^*[x(0),C]$ is minimized with respect to C, the result is dependent upon x(0). This dependence can be eliminated by first averaging $J^*[x(0),C]$ over all possible x(0), i.e., by minimizing

$$\hat{J}(C) = Tr[S^*(C)X_0] \tag{23}$$

where

$$X_0 = E[x(0)x^T(0)] \tag{24}$$

$$S^*(C) = \int_{q\epsilon\Omega} S(q,C)p(q)dq \tag{25}$$

Determining C so as to minimize $\hat{J}(C)$ is not a practical design technique if a continuous probability density function is assumed, due to excessive computational requirements [10], [11]. Except in fairly simple problems, such as the second-order example below, an analytical expression cannot be obtained for $\hat{J}(C)$. Thus, numerical techniques must be used to minimize the scalar $\hat{J}$ with respect to the nm elements of C. Each evaluation of this quantity requires an integration over all $q\epsilon\Omega$. Each integration step requires the solution of an n-th order algebraic Lyapunov equation.

These computational obstacles can be reduced to a manageable level, however, if the point of view is taken that the inclusion of all $q\epsilon\Omega$ in the definition of $\hat{J}(C)$ is unnecessary in practical terms. If only a fairly small number of points in Ω, which are in some sense representative of the entire region, are considered (for example, the corners and selected intermediate points), much computational effort can be avoided at little cost in performance, it turns out. In effect, the continuous probability density function p(q) in Eq. (25) is replaced by a collection of Dirac delta functions throughout Ω, whose amplitudes add to unity.

The following algorithm was used to determine the feedback gains which minimize $\hat{J}(C)$:

(1) Select weighting matrices Q and R.

(2) Select points $q_1,\ldots,q_N \epsilon\Omega$ to be used in approximating the integral in Eq. (25) by a finite sum. Select or evaluate corresponding probability mass functions.

(3) Choose $C\epsilon\mathcal{C}$ to initialize the minimization algorithm, by an alternate design technique, or other means.

(4) Use a quasi-Newton method to find the value of C which minimizes $\hat{J}(C)$. Function and gradient evaluations require solution of Eq. (21) and the Lyapunov equation

$$L(q,C)[A(q)-BC]^T + [A(q)-BC]L(q,C)+I = 0$$

where I is the nxn identity matrix, for $q=q_1,\ldots,q_N$. The gradient of $\hat{J}$ with respect to C is then evaluated according to an expression given in [11]. The algorithm of Bartels and Stewart [22] is used to solve the Lyapunov equations above.

This discrete-valued uncertain parameter version of the expected cost method is similar in concept to the multiplant design method of [3] and the design method described in [12]. A drawback of the expected cost methods is the fact that a feedback gain matrix C which makes A(q)-BC stable for all $q\epsilon\Omega$ must be known before the method can be started. The other methods discussed in this paper do not require prior knowledge of such a gain matrix. The expected cost methods, however, can handle output feedback, rather than full state feedback, with no additional computational effort, which the other methods cannot do. The output distribution matrix can itself be uncertain.

Minimax Method

The goal of the minimax design method [3,4] is to choose feedback gains so as to optimize the performance when the uncertain parameters assume their most unfavorable possible values. Specifically, we seek the gain set C^* such that

$$Tr[X_0 S(q_w, C^*)] \le Tr[X_0 S(q_w, C)] \qquad (26)$$

for all C, where $q_w \varepsilon \Omega$ is such that

$$Tr[X_0 S(q_w, C^*)] \ge Tr[X_0 S(q, C^*)] \qquad (27)$$

for all $q \varepsilon \Omega$. If q_w is known, determination of C^* is quite simple - the problem reduces to a standard linear-quadratic regulator design with q equated to q_w. Determination of q_w is quite expensive computationally, in general. Fortunately, for the problem under consideration, with a rectangular region of parameter uncertainty and A a linear function of q, q_w (if it exists at all) will lie at one of the corners of Ω [3]. Thus, only a finite number of points need be considered when searching for q_w.

A drawback to the minimax approach is the fact that there may be no $q_w \varepsilon \Omega$ consistent with Eqs. (26) and (27), even when C is nonempty. This fact is demonstrated in the fifth-order example considered below.

Both the minimax and standard linear-quadratic regulator methods are, in a sense, special cases of the expected cost method of [10,11], with the probability density or weighting function, p(q), equal to a Dirac delta function, centered at either q_w or the origin (the nominal design point).

Uncertainty Weighting Method

With the uncertainty weighting method, q_w, the most unfavorable point in parameter space, is determined as in the minimax method. A standard linear-quadratic regulator design is then carried out with q equated to 0, except that the state weighting matrix, Q, used in the design process, is replaced by the matrix

$$Q + \lambda[A(q_w) - A(0)]^T Q[A(q_w) - A(0)]$$

where $\lambda > 0$ is a design parameter to be selected. A rationale for this modification of the state weighting matrix is presented in [3].

IV. Application of the Design Methods to a Second-Order Example

Consider a second-order system of the form (1) with

$$A(q) = \begin{bmatrix} 0 & 1 \\ -2+q_1 & 1+q_2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (28)$$

where

$$-1 \le q_1 \le 1$$
$$-1 \le q_2 \le 1.5 \qquad (29)$$

This problem was considered in [10,11,14]. The matrices Q, R, A_0, A_1, and A_2 are those in [14]. The controller gain matrices

$$C = [c_{11} \quad c_{12}] \qquad (30)$$

calculated using each of the design methods, along with the trace of S(0,C) (evaluated using Eq. (21)) and the required CPU time (using an IBM 370/3032) are presented in Table 1. The abbreviations LQR, GCC, MGCC, EC, MM, and UW denote linear-quadratic regulator, guaranteed cost control, multistep guaranteed cost control, expected cost, minimax, and uncertainty weighting, respectively. EC(30), EC(99), and EC(∞) denote the expected cost method using 30, 99, and an infinite number of equally spaced and equally weighted points in the region of parameter uncertainty Ω. An analytical expression for J(C) can be obtained in this particular example if p(q) is assumed uniform. Numerical techniques are needed only for minimizing this analytically defined quantity with respect to C. This approach was used to generate the data labeled EC(∞).

Table 1. Controller Feedback Gains, Trace of Cost Matrix for q = 0, and CPU Time for the Various Design Methods - Second-Order Example

Design Method	c_{11}	c_{12}	Tr[S(0,C)]	CPU Time (Seconds)
LQR	0.02	2.07	62	0.6
GCC	1.36	6.42	97	0.7
MGCC	0.33	3.52	73	1.7
EC(∞)	0.38	3.69	74	N/A
EC(99)	0.42	3.84	76	3.9
EC(30)	0.45	3.97	77	1.6
MM	0.02	5.03	95	0.6
UW	0.26	2.97	67	0.6

In Table 2, the eigenvalues of the closed-loop system matrix

$$A(q) - BC$$

are evaluated for five values of the uncertain parameter vector q:

$$q^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad q^1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad q^2 = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}$$

$$q^3 = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}, \quad q^4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad (31)$$

Table 2. Closed-Loop Eigenvalues for Various Parameter Values Using Feedback Gains Calculated by the Various Methods - Second-Order Example

Design Method	$q = q^0$	$q = q^1$	$q = q^2$	$q = q^3$	$q = q^4$
LQR	-.54 ± 1.32j	-1.04 ± 1.4j	+.21 ± 1.73j	+.21 ± .99j	-.82, -1.26
GCC	-.71, -4.70	-.77, -5.65	-1.96 ± .72j	-.74, -3.18	-.39, -6.03
MGCC	-1.26 ± .86j	-1.76 ± .48j	-.51 ± 1.75j	-.51 ± 1.03j	-.43, -3.09
EC(∞)	-1.34 ± .76j	-1.66, -2.02	-.59 ± 1.74j	-.59 ± 1.01j	-.42, -3.27
EC(99)	-1.42 ± .63j	-1.40, -2.44	-.67 ± 1.72j	-.67 ± .98j	-.41, -3.43
EC(30)	-1.49 ± .49j	-1.28, -2.69	-.74 ± 1.70j	-.74 ± .95j	-.41, -3.57
MM	-.59, -3.44	-.70, -4.33	-1.26 ± 1.19j	-.50, -2.02	-.21, -4.82
UW	-.98 ± 1.14j	-1.48 ± 1.03j	-.23 ± 1.79j	-.23 ± 1.10j	-.51, -2.45

The standard linear-quadratic regulator approach produces an unstable closed-loop system for $q = q^2$ or q^3. The other approaches all produce closed-loop systems which are stable at the nominal and extreme parameter values.

Note the relatively large feedback gains and the real, overdamped closed-loop eigenvalues produced by the guaranteed cost control approach.

In carrying out the multistep guaranteed cost control approach, the parameter μ in Eqs. (17), (19) was chosen to be 0.01. Step 8' of the algorithm was used instead of step 8. The sequence $\{\rho_j, j = 1,\dots,4\}$ was chosen to be 0.4, 0.6, 0.8, 1.0. The multistep guaranteed cost control design avoids the undesirable properties of the guaranteed cost control design noted above. The CPU time, though larger, is still negligible.

The initial state covariance matrix, X_0, in Eq. (23) was chosen to be the identity matrix. The EC(30), EC(99), and EC(∞) designs do not produce significantly different results. Thus, in this example, a modest number of points, equally spaced throughout Ω, characterize the entire region quite well.

The point $q_w \varepsilon \Omega$ needed in the minimax and uncertainty weighting approaches was found to be q^2. The parameter λ used in the uncertainty weighting approach was chosen to be 10. Note that the minimax approach tends to exhibit overcontrolled behavior, here, like the guaranteed cost control method, though to a lesser degree. The gain C_{12} is relatively large, as is the trace of $S(0,C)$, and the closed-loop eigenvalues are generally real and overdamped. The uncertainty weighting approach does not exhibit these adverse characteristics.

In summary, the standard linear-quadratic design failed in this example, and the guaranteed cost control and minimax designs produced somewhat overcontrolled behavior. The uncertainty weighting, multistep guaranteed cost control, and expected cost methods, in order of increasing computational cost, all performed very well. The computational costs were very small in all cases, as would be hoped for a second-order system.

V. Design of a Fifth-Order Lateral Autopilot for a Remotely Piloted Vehicle

Now consider a system having the form of Eqs. (1)-(3), with

$$x^T = [v, p, r, \phi, \delta_a], u = \delta_{a_c}, q = \frac{C_{n_{\delta_a}} - 1.99}{1.99}$$

where

v = component of vehicle velocity parallel to pitch axis

p = vehicle roll rate

r = vehicle yaw rate

ϕ = vehicle roll angle

δ_a = aileron deflection

δ_{a_c} = commanded aileron deflection

$C_{n_{\delta_a}}$ = dimensionless partial derivative of moment about vehicle yaw axis with respect to aileron deflection

and

$$A_0 = \begin{bmatrix} -.85 & 25.47 & -979.5 & 32.14 & 0 \\ -.339 & -8.789 & 1.765 & 0 & 59.89 \\ .021 & -.547 & -1.407 & 0 & 6.477 \\ 0 & 1 & .0256 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.71 \\ 0 & 0 & 0 & 0 & 3.22 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-1.5 \le q \le 0.5$$

The above equations describe the lateral dynamics of a particular remotely piloted vehicle which has no rudder. The aerodynamic derivative $C_{n_{\delta_a}}$ is assumed uncertain in the range

$$-.99 \le C_{n_{\delta_a}} \le 2.99$$

Its nominal value is 1.99 ($q = 0$). The weighting matrices Q and R are chosen to be the same as in [14].

The controller gain matrices

$$C = [C_{11} \quad C_{12} \quad C_{13} \quad C_{14} \quad C_{15}]$$

calculated using each of the design methods, along with the required CPU time, are presented in Table 3. EC(21) and EC(6) denote the expected cost method using 21 and 6 equally spaced (a distance of 0.1 and 0.4 apart) and equally weighted points in the region of parameter uncertainty Ω. $\bar{J}(C)$ cannot be evaluated analytically in this example; hence, there is no EC(∞) entry in Table 3.

Table 3. Controller Feedback Gains and CPU Time for the Various Design Methods - Fifth-Order Example

Design Method	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	CPU Time (Sec.)
LQR	-.0029	-.084	1.80	.013	.34	0.8
GCC	-.017	-.10	2.84	-.023	4.96	3.7
MGCC	-.0065	-.075	1.83	.012	1.24	33.3
EC(21)	-.0091	.093	.95	.081	.26	16.4
EC(6)	-.0090	.11	.84	.083	.28	7.3
MM		No Solution				
UW		No Solution				

Closed-loop eigenvalues and the trace of the cost matrix when q assumes its nominal value ($q = 0$) associated with the various design methods are presented in Table 4. In Table 5, the three dominant closed-loop eigenvalues are given for the extreme values of q, for the various design methods.

Table 4. Closed-Loop Eigenvalues and Trace of Cost Matrix at Nominal q for the Various Design Methods

Design Method	Closed-Loop Eigenvalues	Tr[S(0,C)]
LQR	-19.74, -10.96, -3.30±5.23j,-.62	1660
GCC	-117.1, -9.90 , -1.57±6.51j,-.10	2359
MGCC	-41.28, -10.05, -2.14±6.22j,-.29	1803
EC(21)	-15.92±10.60j , -2.00±7.31j,-.36	1998
EC(6)	-16.27±10.78j , -1.86±7.38j,-.34	2096

The standard linear-quadratic regulator approach again produces an unstable closed-loop system at an extreme value of the uncertain parameter. The guaranteed cost control, multistep guaranteed cost control, and expected cost methods produce closed-loop systems which are stable at the nominal and extreme parameter values. The guaranteed cost control approach produces larger feedback gains than the other methods and requires a very fast actuator (pole at -117.1). Step 8' was used instead of step 8 in carrying out the multistep guaranteed cost control approach. The parameter μ was chosen to be 0.0001. The sequence $\{\rho_j,\ j = 1,\ldots,6\}$ was chosen to be 0.375, 0.75, 0.8125, 0.875, 0.9375, 1.0. Again, the undesirable features of the guaranteed cost control design were avoided using the multistep guaranteed cost control approach. The expected cost designs based upon 6 and 21 points were both quite satisfactory and not significantly different from each other. The expected cost methods were less costly computationally than the multistep guaranteed cost control approach in this example. The 33 seconds of CPU time associated with the latter approach is still not particularly expensive, though. The minimax and uncertainty weighting methods failed when applied to the problem, since there is no $q_w \epsilon \Omega$ consistent with Eqs. (26), (27).

Table 5. Dominant Closed-Loop Eigenvalues at Extreme Values of q for the Various Design Methods

Design Method	Dominant Closed-Loop Eigenvalues $q = -1.5$	$q = 0.5$
LQR	+.13±5.08j, -.38	-5.17±4.27j, -.70
GCC	-.48±5.65j, -.08	-1.94±6.74j, -.11
MGCC	-.34±5.40j, -.21	-2.86±6.36j, -.30
EC(21)	-.91±5.10j, -.39	-2.41±7.87j, -.35
EC(6)	-.97±5.17j, -.36	-2.19±7.95j, -.33

VI. Conclusion

Six methods for designing constant gain feedback controllers for linear systems with uncertain parameters have been compared. All methods, other than the standard linear-quadratic approach, have been suggested elsewhere as suitable techniques for producing a stable closed-loop control system over a large range of parameter uncertainty. The minimax and uncertainty weighting methods, which were found in [3] to be superior to a number of other design methods (none of which were considered here, except for the multiplant method, which is a special case of the expected cost method), failed when applied to the fifth-order example, because there is no point in the specified region of parameter uncertainty with the desired minimax property. It was found that only a modest number of discrete parameter values need be used in conjunction with the expected cost method in order to obtain satisfactory results, thereby resolving the questions raised in [10,11] as to its practicality.

Before any truly definitive statements about the relative merits of these design methods can be made, they should be tested on additional, and more complex, examples. However, it would seem at this point that the expected cost and multistep guaranteed cost control methods are the methods of choice. While they required more computation time than the other methods, the time required was by no means objectionable, and the results were quite good in both problems considered. The expected cost method can handle a more general form of uncertainty in the various matrices describing the system than can the other methods. It can also produce a controller design based upon output feedback just as easily as one based upon full state feedback, which the other methods cannot. Its most serious drawback may be the fact that it is not self-starting, requiring, perhaps, a set of feedback gains obtained by a different method for initialization.

Acknowledgment

The authors would like to thank Professors Robert H. Cannon, Jr. and Homer J. Stewart for many helpful discussions during the course of this research and Thomas Bauer for his programming support.

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