Stabilization of Relative Equilibria

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***Abstract*—This paper discusses the problem of obtaining feedback laws to asymptotically stabilize relative equilibria of mechanical systems with symmetry. We show how to stabilize an internally unstable relative equilibrium using internal actuators. The methodology is that of *potential shaping*, but the system is allowed to be underactuated, i.e., have fewer actuators than the dimension of the shape space. The theory is illustrated with the problem of stabilization of the cowboy relative equilibrium of the double spherical pendulum.**

***Index Terms*—Mechanics, relative equilibria, stabilization.**

I. INTRODUCTION

IN THIS paper, we discuss the problem of stabilization of relative equilibria of a mechanical system with symmetry. We are interested in stabilizing relative equilibria for which the *internal*, or *shape*, configuration of the system is unstable. We shall examine how to stabilize such a relative equilibrium using *partial* internal actuation, by which we mean stabilization using internal actuators, where the number of internal actuators can be less than the dimension of the shape space.

We will explain the theory developed by using the *double spherical pendulum* as an example. This system is pictured in Fig. 1 and, as we shall explain, has an internally unstable relative equilibrium.

This system consists of two spherical pendula in a gravitational field, each modeled as a point mass (the bob) at the end of a rigid massless rod, the first of which is suspended from a fixed point, which we shall take as the origin of our coordinate system, and the second of which is suspended from the bob of the first pendulum. We neglect the axial rotation of the rods, so there are two degrees of freedom for each pendulum. We let q_1 and q_2 be the positions of the bobs of the first and second pendula, respectively, relative to their points of suspension. This system has a rather simple symmetry group $G = S^1$, whose action corresponds to rotation of the system about the vertical axis through the origin. This action leaves invariant the kinetic energy metric and the potential function, and the corresponding conserved momentum map for the system is of course just the angular momentum about the vertical axis.

A *relative equilibrium* of this system is a trajectory that is given by steady motion along the group direction, with the internal configuration or the shape remaining fixed. A relative equilibrium corresponds to a literal equilibrium of the symplectically reduced system, which is obtained by restricting the original system to a constant momentum surface, and then quotienting by the group action.

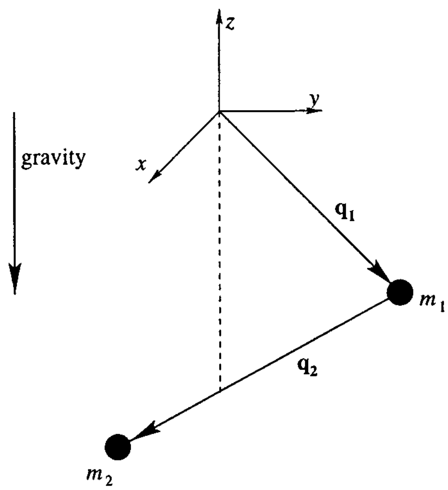

Fig. 1. The double spherical pendulum.

The double spherical pendulum (DSP) system has two relative equilibria, the *straight-stretched out* equilibrium and the *cowboy* equilibrium (see Fig. 2). The straight-stretched out solution is a minimum of the energy-momentum function and is a stable relative equilibrium, whereas the cowboy solution is a saddle point of the energy-momentum function and can be destabilized by a small amount of dissipation. A discussion of reduction of mechanical systems, relative equilibria, and stability analysis of relative equilibria (using the energy-momentum method) can be found in [4]. Also included, as an example, is the computation and stability analysis of the relative equilibria of the DSP, based on [6].

Assume that the system is actuated using *internal* forces only. For this system, this means that there is no external torque about the z-axis. Furthermore, we assume that the number of inputs we have can be strictly less than the dimension of the shape space, which is three. Thus, we will be assuming that we have partial internal actuation. The methods in this paper will show how to find feedback laws that render the cowboy solution an asymptotically stable equilibrium.

The methods can also be used for other interesting tasks as well, such as to effect an *orbit transfer* from the cowboy solution to the straight stretched out solution. This is done by making use of a latent global heteroclinic connection between the two solutions that can be accessed with controls. Results of this sort are proved using the stabilization techniques together with a relative La Salle theorem discussed below. The orbit transfer methods will be the subject of another publication.

To obtain our feedback laws, we will use a combination of the techniques of van der Schaft [10] on the stabilization of

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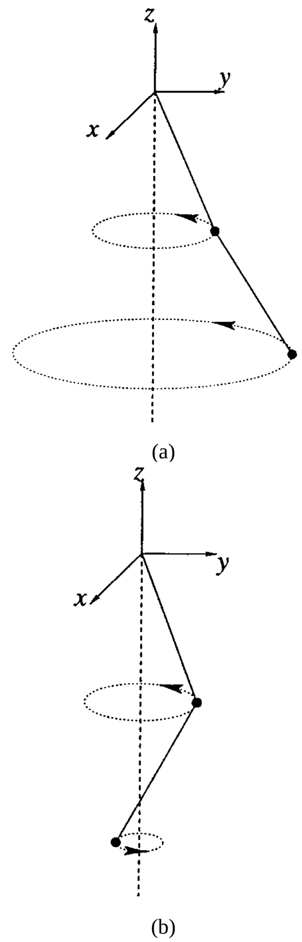

Fig. 2. (a) The stretched out solution and (b) the cowboy solution.

Hamiltonian systems and the energy-momentum method for stability analysis of relative equilibria due to Simo *et al.* [9].

The methodology is based on *potential shaping*, that is, the selection of controls that, in effect, change the potential energy of the system and add damping to make equilibria asymptotically stable. Using the coupling of the modes as well as the existence of *a priori* stable directions, one needs only *partial actuation*.

This methodology is to be compared with that of Bloch *et al.* [1], [2] on controlled Lagrangians, whose methodology involves the reshaping of the *kinetic energy* of the system to stabilize a relative equilibrium. It is also designed for the stabilization of *balance systems* where there might be no internal actuation at all. Typical applications are the stabilization of inverted spherical pendula and underwater vehicles.

Also of interest is the work of Bullo [3], who considers systems with Abelian symmetry with actuation along both the group and the internal directions. Under appropriate assumptions, it is shown that it is possible to exponentially stabilize the system to a desired relative equilibrium on a desired level surface of the momentum.

The following is an outline of the rest of this paper. In Section II, we discuss LaSalle's principle and how it can be used for the stabilization of partially actuated Hamiltonian systems. In Section III, we discuss the implications of these results for mechanical systems with symmetry. In that section, we also show how our results can be used for asymptotic stabilization of the cowboy solution of the double spherical pendulum. Last, in Section IV, we make some concluding remarks.

II. Stabilization of Hamiltonian Systems

In this section, we give some general results on the asymptotic stabilization of Hamiltonian systems. Our discussion is in the context of Hamiltonian systems on Poisson manifolds, and is a generalization of what van der Schaft [10] proves for mechanical systems with Lagrangian of the form "kinetic minus potential." This added generality will be used later, when we apply the results to reduced mechanical systems.

Let P be a Poisson manifold and let X_H be the Hamiltonian vector field corresponding to the Hamiltonian H on P. Assume that $z_0 \in P$ is an equilibrium for the associated Hamiltonian dynamical system

$$\dot{z} = X_H(z). \tag{2.1}$$

Let $dH(z_0) = 0$, which means that $\delta^2 H(z_0)$, the second derivative of H at z_0, is intrinsically defined.

Now let us modify the above system by adding inputs. Let F_i, $i = 1, \cdots, m$ be real valued functions on P that satisfy $F_i(z_0) = 0$. We add inputs in the following way:

$$\dot{z} = X_H(z) - X_{F_1}(z)u_1 - \cdots - X_{F_m}(z)u_m. \tag{2.2}$$

Consider the set of functions defined by

$$\begin{aligned}\mathcal{C}\colon =&\operatorname{span}\{F_i, \{H, F_i\}, \{H, \{H, F_i\}\}, \cdots\},\\ &i = 1, \cdots, m.\end{aligned}$$

Here, by "span" we mean the collection of all linear combinations with real, constant, coefficients. Then define the codistribution $d\mathcal{C}$, as follows: $d\mathcal{C}(z) := \operatorname{span}\{dg(z) | g \in \mathcal{C}\}$.

We then have the following theorem.

Theorem 2.1: Assume that $\delta^2 H(z_0)$ is positive definite (denoted $\delta^2 H(z_0) > 0$). Thus, z_0 is an isolated equilibrium of X_H and is a strict minimum of H. Let the codistribution $d\mathcal{C}$ be of dimension $\dim P$ on a neighborhood of z_0. Then the feedback $u_i(z) = k_i X_{F_i}[H](z)$, where the k_i are positive constants, makes z_0 an asymptotically stable equilibrium.

Nijmeier and van der Schaft [8] state and prove this result in the special case of mechanical systems with Lagrangian of the form "kinetic minus potential." Since their proof generalizes with only minor modifications to the Poisson manifold setting, we omit it here.

Lemma 2.2: If the functions F_i Poisson commute, i.e., $\{F_i, F_j\} = 0$, the feedback can be expressed in the form $u_i = -k_i \dot{F}_i$.

By $\dot{F}_i$, we mean the time rate of change of F_i for the system *with feedback*. Note that

$$\begin{aligned}\dot{F}_i =& dF_i \cdot (X_H(z) - X_{F_1}(z)u_1 - \cdots - X_{F_m}(z)u_m)\\ =& dF_i \cdot (X_H(z) - X_{F_1}(z)u_1 - \cdots - X_{F_m}(z)u_m)\\ =& \{F_i, H\} - \{F_i, F_1\}u_1 - \cdots - \{F_i, F_m\}u_m\\ =& \{F_i, H\} = -X_{F_i}[H]\end{aligned}$$

which proves the assertion in the lemma above. Since $\dot{F}_i$ does not depend on the inputs, u_i can be defined in terms of $\dot{F}_i$ as in the lemma. This way of expressing the feedback is convenient in situations where we can directly measure the time derivatives of the functions F_i.

Next, we shall discuss what can be done if $\delta^2 H(z_0)$ is not positive definite. Let us consider the system (2.2) with the new feedback $u_i(z) = -c_i F_i(z) + v_i$, where the c_i are positive constants. Note that $c_i F_i(z) X_{F_i}(z) = c_i X_{(1/2)F_i^2}(z)$. Thus with this feedback, the new system obtained is

$$\begin{aligned}\dot{z} &= X_H(z) + \sum_i c_i F_i(z) X_{F_i}(z) - \sum_i X_{F_i}(z) v_i \\ &= X_H(z) + \sum_i c_i X_{(1/2)F_i^2}(z) - \sum_i X_{F_i}(z) v_i \\ &= X_{\tilde{H}}(z) - X_{F_1}(z) v_1 - \cdots - X_{F_m}(z) v_m\end{aligned}$$

where $\tilde{H}:= H + \sum_i (1/2) c_i F_i^2$ is called the *modified Hamiltonian*. Since $F_i(z_0) = 0$, it follows that z_0 is a critical point of $\tilde{H}$, and thus an equilibrium of the vector field $X_{\tilde{H}}$. Now, if $\delta^2 \tilde{H}(z_0)$ is positive definite, we can apply Theorem 2.1 to the new system achieve asymptotic stabilization. Thus, we shall consider whether we can choose constants c_i so as to make $\delta^2 \tilde{H}(z_0) > 0$. Some calculation shows that

$$\delta^2 \tilde{H}(z_0) = \delta^2 H(z_0) + (dF(z_0))^T C (dF(z_0))$$

where $F = (F_1, \cdots, F_m): P \to \mathbb{R}^m$, $dF(z_0): T_{z_0}P \to \mathbb{R}^m$ is the differential of F, and $C = \mathrm{diag}(c_1, \cdots, c_m)$. Following van der Schaft [10], we then use the following simple linear algebra result.

Lemma 2.3: Let S be a symmetric $n \times n$ matrix and let P be a surjective $m \times n$ matrix. Then there exists a symmetric $m \times m$ matrix C such that $S + P^T C P > 0$ if S restricted to $\ker P$ is positive definite. Further, if S is positive definite on $\ker P$, C can be chosen to be diagonal.

Using this lemma, we can conclude that if $\delta^2 H(z_0)$ is positive definite on $\ker dF(z_0)$; then we can find positive constants c_i such that $\delta^2 \tilde{H}(z_0)$ is positive definite, thus enabling us to use Theorem 2.1. Note that the set of functions $\mathcal{C}$ used in Theorem 2.1 will now have to be replaced by the set $\tilde{\mathcal{C}}$, defined by

$$\tilde{\mathcal{C}}:= \mathrm{span}\left\{F_i, \left\{\tilde{H}, F_i\right\}, \left\{\tilde{H}, \left\{\tilde{H}, F_i\right\}\right\}, \cdots\right\}, \quad i = 1, \cdots, m.$$

The distribution $d\tilde{\mathcal{C}}$ is defined analogously to the distribution $d\mathcal{C}$, i.e., $d\tilde{\mathcal{C}}(z):= \mathrm{span}\{dg(z) | g \in \tilde{\mathcal{C}}\}$. Now we are ready to state an appropriate generalization of Theorem 2.1.

Theorem 2.4: Consider the following system on a Poisson manifold P:

$$\dot{z} = X_H(z) - X_{F_1}(z) u_1 - \cdots - X_{F_m}(z) u_m.$$

Let z_0 be an equilibrium of the vector field X_H, let $dH(z_0) = 0$, let $F_i(z_0) = 0$ for $i = 1, \cdots, m$, and let $\delta^2 H(z_0)$ be positive definite on $\ker dF(z_0)$. Thus, we can choose constants c_i, $i = 1, \cdots, m$ such that $\delta^2 \tilde{H}(z_0)$ is positive definite. With this choice of constants c_i, assume that the codistribution $d\tilde{\mathcal{C}}$ equals the whole cotangent space (i.e., is of maximal dimension) on some neighborhood of z_0. Further, assume that the functions F_i Poisson commute, i.e., $\{F_i, F_j\} = 0$. Then for any choice of constants $k_i > 0$, the feedback $u_i = -c_i F_i - k_i \dot{F}_i$ makes z_0 an asymptotically stable equilibrium.

Proof: According to the preceding discussion, one can choose constants c_i such that

$$\delta^2 \tilde{H}(z_0) = \delta^2 H(z_0) + (dF(z_0))^T C (dF(z_0)) > 0$$

where $C = \mathrm{diag}(c_1, \cdots, c_m)$ and $\tilde{H}:= H + \sum_i (1/2) c_i F_i^2$. Let $u_i = -c_i F_i + v_i$, which gives us the modified system

$$\dot{z} = X_{\tilde{H}}(z) - X_{F_1}(z) v_1 - \cdots - X_{F_m}(z) v_m.$$

Now apply Theorem 2.1 and Lemma 2.2 to this system: the functions F_i Poisson commute and hence the feedback $v_i = -k_i \dot{F}_i(z)$ makes z_0 an asymptotically stable equilibrium. Thus the feedback law for the original system is $u_i = -c_i F_i(z) - k_i \dot{F}_i(z)$. ■

Remark: To use this theorem, we first have to choose the constants c_i and then check that the codistribution $d\tilde{\mathcal{C}}$ is of maximal dimension on some neighborhood of z_0. Under certain circumstances, it is possible to show that if $d\mathcal{C}$ is of maximal dimension, then $d\tilde{\mathcal{C}}$ is of maximal dimension for all possible values of the constants c_i. Thus we would need to work with only the original Hamiltonian H, and not the modified Hamiltonian $\tilde{H}$. The precise statement is as follows. If the manifold P and functions H, F_i are analytic, and if for all choices of the constants c_i, $d\tilde{\mathcal{C}}$ is of *constant* dimension in a neighborhood of z_0, then the codistribution $d\mathcal{C}$'s being of maximal dimension on a neighborhood of z_0 implies that $d\tilde{\mathcal{C}}$ is of maximal dimension for all choices of the constants c_i. For the proof of this statement, refer to van der Schaft [10]. In this paper, however, we will not use this result due to the difficulty of verifying that $d\tilde{\mathcal{C}}$ is of constant dimension for all values of the constants c_i.

III. Mechanical Systems with Symmetry

This section applies the results of the previous section to the stabilization of relative equilibria of mechanical systems with symmetry.

A. The Setting

Let $\pi: Q \to Q/G =: S$ be a principal G-bundle, so that Q is a configuration manifold and the Lie group G acts freely and properly on Q. Following standard practice, we refer to the quotient space S as the *shape space*. When dealing with a local trivialization in which Q is diffeomorphic to $G \times S$, we write $q \in Q$ as $(g, r) \in G \times S$.

Let $\langle\langle \cdot, \cdot \rangle\rangle$ be a G-invariant metric on Q. Let V be a G-invariant potential on Q. Thus L defined by

$$L(v_q) = \frac{1}{2}\langle\langle v_q, v_q \rangle\rangle - V(q)$$

is a G-invariant Lagrangian on Q. The equations of motion of the system in local coordinates $q = (q^1, \cdots, q^n)$, subject to a generalized force τ, are as follows:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(q(t), \dot{q}(t)) - \frac{\partial L}{\partial q^i}(q(t), \dot{q}(t)) = \tau_i(t) \quad i = 1, \cdots, n$$

which we shall write for short as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau. \tag{3.1}$$

At each instant t, $\tau(t)$ is regarded as an element of $T^*_{q(t)}Q$.

In a local trivialization, $T^*_{q(t)}Q$ is identified with $T^*_{g(t)}G \times T^*_{r(t)}S$, and we write $\tau = (\tau_g, \tau_r) \in T^*_g G \times T^*_r S$. We will suppose that we are only allowed to have internal actuation, i.e., no external forces or torques are allowed, i.e., $\tau_g = 0$. This condition of *internal actuation* is intrinsic, independent of the chosen local trivialization; it means that τ is *horizontal*; i.e., for each $v \in T_q Q$ such that $T\pi \cdot v = 0$, we have $\tau(v) = 0$.

In fact, τ will be assumed to be of the form

$$\tau = dF_1(q)u_1 + \cdots + dF_m(q)u_m \tag{3.2}$$

where the F_i are independent real-valued G-invariant functions on Q and the u_i are real-valued control inputs. The one-forms dF_i on Q annihilate the vertical subspace of T_qQ, and thus are internal forces in the preceding sense.

The system is said to be *underactuated* if m is strictly less than the dimension of the shape space. The Legendre transform $\mathbb{F}L$: $(q, \dot{q})(q, \partial L/\partial \dot{q})$ is a diffeomorphism from TQ to T^*Q. Let (q, p) be cotangent bundle coordinates on T^*Q. Let H be the corresponding Hamiltonian function on T^*Q given by the push-forward by $\mathbb{F}L$ of the energy function E on TQ, defined by $E(q, \dot{q}) = \langle \mathbb{F}L(\dot{q}), \dot{q}\rangle - L(q, \dot{q})$. If, in coordinates, L is of the form $(1/2)\dot{q}^T M(q)\dot{q} - V(q)$, then the coordinate expression for H is: $H(q, p) = K(q, p) + V(q) = (1/2)p^T M^{-1}(q)p + V(q)$.

We begin with the following lemma.

Lemma 3.1: For hyperregular Lagrangians (i.e., $\mathbb{F}L$ is a diffeomorphism), the equations on T^*Q obtained by pushing forward, by the map $\mathbb{F}L$, the vector field on TQ defined by the Euler-Lagrange equations (with forcing) (3.1) are

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} + \tau. \end{aligned} \tag{3.3}$$

This is a standard computation and so we omit the details.

Using the form for τ assumed in (3.2), the preceding equations on T^*Q can be rewritten as

$$\dot{z} = X_H(z) + \begin{bmatrix} 0 \\ dF_1(q) \end{bmatrix} u_1 + \cdots + \begin{bmatrix} 0 \\ dF_m(q) \end{bmatrix} u_m.$$

Let us use the natural projection of T^*Q onto Q to lift the function F_i on Q to get a function on T^*Q. Abusing notation slightly, let us use the name F_i for the lifted function also. Thus X_{F_i} is a Hamiltonian vector field on T^*Q

$$X_{F_i} = \begin{bmatrix} \dfrac{\partial F_i}{\partial p} \\ -\dfrac{\partial F_i}{\partial q} \end{bmatrix} = \begin{bmatrix} 0 \\ -dF_i(q) \end{bmatrix}.$$

Thus the equations on T^*Q can be written as follows:

$$\dot{z} = X_H(z) - X_{F_1}(z)u_1 - \cdots - X_{Fm}(z)u_m. \tag{3.4}$$

Now let J: $T^*Q \to \mathfrak{g}^*$ be the standard equivariant cotangent bundle momentum map for the cotangent lifted action of G on T^*Q (see, for example, [5]).

Choose a momentum value $\mu \in \mathfrak{g}^*$. Since the functions H, F_i on T^*Q are G-invariant, their restrictions to $J^{-1}(\mu)$ drop to the quotient space $(T^*Q)_\mu := J^{-1}(\mu)/G_\mu$. Here, G_μ is the isotropy subroup of G that leaves the momentum μ invariant under the left action of G on $\mathfrak{g}^*$, and is the subgroup of G that acts on $J^{-1}(\mu)$. Let these functions on $J^{-1}(\mu)/G_\mu$ be called H_μ, $F_{i\mu}$, respectively. The function H_μ is called the *reduced Hamiltonian*.

For any $z \in J^{-1}(\mu)$, let $[z]$ denote the G_μ orbit through z. Thus $[z] = G_\mu \cdot z \in J^{-1}(\mu)/G_\mu$. Note that if $z \in T^*_q Q$ lies on $J^{-1}(\mu)$, and if $\zeta = [z]$, we have $F_i(q) = F_i(z) = F_{i\mu}(\zeta)$. (Recall F_i represents both a function on Q as well as its pull back to T^*Q.)

By Noether's theorem, the vector fields X_H,X_{F_i} are tangent to $J^{-1}(\mu)$. By the theory of symplectic reduction [7], $(T^*Q)_\mu$ is a symplectic manifold, and the restrictions of the vector fields X_H,X_{F_i} to $J^{-1}(\mu)$ drop to the Hamiltonian vector fields X_{H_μ},$X_{F_{i\mu}}$ on $(T^*Q)_\mu$. Thus we get a reduced system of equations on $(T^*Q)_\mu$ which we shall write as follows:

$$\dot{\zeta} = X_{H_\mu}(\zeta) - X_{F_{1\mu}}(\zeta)u_1 - \cdots - X_{F_{m\mu}}(\zeta)u_m. \tag{3.5}$$

Recall that $z_e \in J^{-1}(\mu)$ is a *relative equilibrium* of the system $\dot{z} = X_H(z)$ when $\zeta_e := [z_e] \in (T^*Q)_\mu$ is an equilibrium of the reduced system $\dot{\zeta} = X_{H_\mu}(\zeta)$. For each $q \in Q$, let $\mathbb{I}_q$ denote the *locked inertia tensor* at q (see, for example, [4] for the definition). The *amended potential* V_μ, which is a real-valued function on Q, is defined as $V_\mu(q) := V(q) + (1/2)\langle \mu, \mathbb{I}_q^{-1}\mu\rangle$. It is a fact that if $z_e \in T^*_{q_e}Q$ lies on $J^{-1}(\mu)$, z_e is a relative equilibrium iff q_e is a critical point of V_μ, and z_e is of the form $\alpha_\mu(q_e)$ where α_μ is the one-form on Q defined by $\langle \alpha_\mu(q), v_q\rangle = \langle \mu, \mathfrak{A}(v_q)\rangle$, where $\mathfrak{A}$: $TQ \to \mathfrak{g}$ is the *mechanical connection* on Q (again, see [4] for the definition).

B. Statement of the Problem

The problem we solve in this section is the following. Given a relative equilibrium $z_e \in J^{-1}(\mu)$, we wish to find feedback laws $u_i = u_i(\zeta)$ such that a) ζ_e remains an equilibrium of the reduced closed-loop system [note that since the feedback laws are of the form $u_i = u_i(\zeta)$, $\zeta \in (T^*Q)_\mu$, the closed-loop system does indeed drop to $(T^*Q)_\mu$] and b) ζ_e is an asymptotically stable equilibrium of the reduced closed-loop system.

C. Block-Diagonal Form for $\delta^2 H_\mu(\zeta_e)$

Since ζ_e is an equilibrium of X_{H_μ}, ζ_e is a critical point of H_μ, and thus the second derivative $\delta^2 H_\mu(\zeta_e)$ can be intrinsically defined. By the energy-momentum method, it is possible to choose a basis of $T_{\zeta_e}(T^*Q)_\mu$ such that $\delta^2 H_\mu(\zeta_e)$ has a convenient block-diagonal form. In order to describe this form, we will need some more definitions and constructions. (Our discussion of the energy-momentum construction here is brief—for a complete account, see [4].)

Let $\mathfrak{g}_\mu$ be the Lie algebra of the isotropy subgroup G_μ. The subspace $\mathcal{V} \subset \mathcal{T}_{q_e}Q$ is defined as the orthogonal complement of the tangent space to the G_μ-orbit through q_e. The metric on Q is used for defining the orthogonal complement. Thus $\mathcal{V} = (\mathfrak{g}_\mu \cdot q_e)^\perp$. Let $\mathcal{V}_{\rm RIG}\colon = (\mathfrak{g}_\mu)^\perp \cdot q_e \subset \mathcal{V}$, where the orthogonal complement of $\mathfrak{g}_\mu$ is computed using the inner product on $\mathfrak{g}$ defined by the locked inertia tensor at q_e, $\mathbb{I}_{q_e}$. $\mathcal{V}_{\rm RIG}$ is the subspace of $\mathcal{V}$ consisting of *rigid* velocities, i.e., the intersection of $\mathcal{V}$ with the tangent space to the group orbit through q_e. Since z_e is a relative equilibrium, q_e is a critical point of V_μ. The second derivative $\delta^2 V_\mu(q_e)$ is a symmetric two-form on $T_{q_e}Q$. Now let $\mathcal{V}_{\rm INT}$ be a complement of $\mathcal{V}_{\rm RIG}$ in $\mathcal{V}$, chosen in such a way that the restriction of $\delta^2 V_\mu(q_e)$ to $\mathcal{V}$ block-diagonalizes with respect to the splitting $\mathcal{V} = \mathcal{V}_{\rm RIG} \oplus \mathcal{V}_{\rm INT}$. $\mathcal{V}_{\rm INT}$ represents the space of *internal* velocities at q_e.

Thus, with respect to a basis of $\mathcal{V}$ that is the union of a basis of $\mathcal{V}_{\rm RIG}$ and a basis of $\mathcal{V}_{\rm INT}$, the matrix representation of $\delta^2 V_\mu(q_e)|\mathcal{V}$ has the form

$$\begin{bmatrix} A_\mu & 0 \\ 0 & B_\mu \end{bmatrix} \tag{3.6}$$

where $A_\mu = \delta^2 V_\mu(q_e)|\mathcal{V}_{\rm RIG}$ and $B_\mu = \delta^2 V_\mu(q_e)|\mathcal{V}_{\rm INT}$. The energy-momentum method tells us that with respect to an appropriately chosen basis, the matrix of $\delta^2 H_\mu(\zeta_e)$ is

$$\delta^2 H_\mu(\zeta_e) = \begin{bmatrix} A_\mu & 0 & 0 \\ 0 & B_\mu & 0 \\ 0 & 0 & K_\mu \end{bmatrix}. \tag{3.7}$$

The matrices A_μ, B_μ have been defined earlier; and the matrix K_μ is a matrix of size $\dim\ S \times \dim\ S$ that depends on the kinetic energy metric only and is known to be positive definite. If A_μ is positive definite, the rigid motion of the system is stable—i.e., if we lock up the internal joints of the system, them the system will rotate stably. If B_μ is positive definite, the internal configuration of the system is stable.

D. Feedback Stabilization of the Reduced System

Now we shall examine how we can use Theorem 2.4 of the previous section to derive feedback laws to asymptotically stabilize relative equilibria of the reduced system. We will examine each condition in the statement of Theorem 2.4 and see what it means in the present setting.

1) Shaping the Reduced Hamiltonian: Now suppose the initial condition $z(0)$ of the system (3.4) lies in $J^{-1}(\mu)$. Since the vector fields X_H and X_{F_i} are (by Noether's theorem) tangent to $J^{-1}(\mu)$, the trajectory $z(t)$ remains on $J^{-1}(\mu)$, and $\zeta(t)\colon = [z(t)] = G_\mu \cdot z(t) \in (T^*Q)_\mu$, is a trajectory of the reduced system (3.5). If $z \in T_q^*Q \cap J^{-1}(\mu)$, and $\zeta = [z]$, then $F_{i\mu}(\zeta) = F_i(z) = F_i(q)$. Let us set

$$u_i = -c_i F_{i\mu}(\zeta) + v_i = -c_i F_i(q) + v_i$$

where the v_i are regarded as new inputs. To preserve the relative equilibrium, we shall assume that the functions F_i are such that $F_i(q_e) = F_{i\mu}(\zeta_e) = 0$. Thus, the Euler–Lagrange equations (with forcing) are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = \sum_i dF_i(q)(-c_i F_i(q) + v_i).$$

We now obtain the reduced equations corresponding to this system. Since $L = K - V$, where K is the kinetic energy and V is the potential, we can rewrite the above equation as follows:

$$\begin{aligned}\frac{d}{dt}\frac{\partial K}{\partial \dot q} - \frac{\partial K}{\partial q} &= -\frac{\partial V}{\partial q} - \sum_i dF_i(q)c_iF_i(q) + dF_1(q)v_1 \\ &\quad + \cdots + dF_m(q)v_m \\ &= -\frac{\partial}{\partial q}\left(V + \sum_i \frac{1}{2}c_iF_i^2\right) + dF_1(q)v_q \\ &\quad + \cdots + dF_m(q)v_m \\ &= -\frac{\partial \tilde V}{\partial q} + dF_1(q)v_1 + \cdots + dF_m(q)v_m\end{aligned}$$

where $\tilde V = V + (1/2)\sum_i c_iF_i^2$. We call $\tilde L\colon = K - \tilde V$ the *modified Lagrangian*. Then the equations become

$$\frac{d}{dt}\frac{\partial \tilde L}{\partial \dot q} - \frac{\partial \tilde L}{\partial q} = dF_1(q)v_1 + \cdots + dF_m(q)v_m.$$

The modified Hamiltonian corresponding to $\tilde L$ is

$$\tilde H = K + \tilde V = H + \tfrac{1}{2}\sum_i c_iF_i^2.$$

As in (3.4), we can obtain the corresponding Hamiltonian equations on T^*Q. They are

$$\dot z = X_{\tilde H}(z) - X_{F_1}(z)v_1 - \cdots - X_{Fm}(z)v_m. \tag{3.8}$$

As in (3.5), we can obtain a reduced system on $(T^*Q)_\mu$

$$\dot\zeta = X_{\tilde H_\mu}(\zeta) - X_{F_{1\mu}}(\zeta)v_1 - \cdots - X_{Fm\mu}(\zeta)v_m. \tag{3.9}$$

Here $\tilde H_\mu$ is obtained by restricting $\tilde H$ to $J^{-1}(\mu)$ and then dropping it to $J^{-1}(\mu)/G_\mu = (T^*Q)_\mu$. Note that $\tilde H_\mu = H_\mu + (1/2)\sum_i c_iF_{i\mu}^2$.

Now

$$\tilde V_\mu(q) = \tilde V(q) + \frac{1}{2}\langle \mu, \mathbb{I}_q^{-1}\mu\rangle = V_\mu + \frac{1}{2}\sum_i c_iF_i^2$$

and $F_i(q_e) = 0$. We can verify that $\tilde V_\mu(q_e) = 0$. Thus, $z_e = \alpha_\mu(q_e)$ remains a relative equilibrium for the system $X_{\tilde H}$, and therefore $\zeta_e = [z_e]$ is an equilibrium of $X_{\tilde H_\mu}$.

We will now determine a block-diagonal representation for the quadratic form $\delta^2\tilde H_\mu(\zeta_e)$. Analogous to (3.7), we will get a 3×3 block-diagonal matrix. Note that the differences, if any, between the matrix we get and the matrix in (3.7) will only be in the (1, 1) and (2, 2) blocks; the (3, 3) block depends on the kinetic energy metric, whereas $\tilde H$ differs from H only in the potential term.

We first need to consider $\delta^2\tilde V_\mu(q_e)|\mathcal{V}$, where the vector space $\mathcal{V}$ is as defined earlier. Since $\tilde V_\mu = V_\mu + (1/2)\sum_i c_iF_i^2$, we get

$$\delta^2\tilde V_\mu(q_e) = \delta^2 V_\mu(q_e) + \tfrac{1}{2}\delta^2\left(\sum_i c_iF_i^2\right).$$

We have already seen that $\mathcal{V} = \mathcal{V}_{\rm RIG} \oplus \mathcal{V}_{\rm INT}$, and that the matrix of $\delta^2 V_\mu(q_e)$ with respect to this splitting of $\mathcal{V}$ has the form given in (3.6). Now consider the term $\delta^2(\sum_i c_iF_i^2)$. Let

$F\colon = (F_1, \cdots, F_m)\colon Q \to \mathbb{R}^m$. Note that $F(q_e) = 0$. Using this, it is fairly easy to check that

$$\delta^2 \left(\sum_i c_i F_i^2 \right) = (dF(q_e))^T C dF(q_e)$$

where $dF(q_e)\colon T_{q_e}q \to \mathbb{R}^m$ is the derivative of F, and $C = \mathrm{diag}\{c_1, \cdots, c_m\}$. Note that $dF(q_e)$ annihilates all vertical vectors as F is G-invariant. In particular, $dF(q_e) = 0$ on $\mathcal{V}_{\mathrm{RIG}}$. Thus, with respect to the splitting $\mathcal{V} = \mathcal{V}_{\mathrm{RIG}} \oplus \mathcal{V}_{\mathrm{INT}}$, the matrix of $\delta^2(\sum_i c_i F_i^2)|\mathcal{V}$ is

$$\begin{bmatrix} 0 & 0 \\ 0 & K^T C K \end{bmatrix}$$

where K is the matrix of $dF(q_e)\colon \mathcal{V}_{\mathrm{INT}} \to \mathbb{R}^m$. Therefore, the matrix of $\delta^2 \tilde{V}_\mu(q_e)|\mathcal{V}$ is

$$\begin{bmatrix} A_\mu & 0 \\ 0 & B_\mu + K^T C K \end{bmatrix}.$$

The matrix of $\delta^2 \tilde{H}_\mu(\zeta_e)$ can now be given

$$\delta^2 \tilde{H}_\mu(\zeta_e) = \begin{bmatrix} A_\mu & 0 & 0 \\ 0 & B_\mu + K^T C K & 0 \\ 0 & 0 & K_\mu \end{bmatrix}.$$

As in the statement of Theorem 2.4, we choose the constants c_i such that $\delta^2 \tilde{H}_\mu(\zeta_e)$ is positive definite. We seek conditions under which $\delta^2 \tilde{H}_\mu(\zeta_e)$ is positive definite. For this to be the case, each block on the diagonal of the block-diagonal matrix for be positive definite. We know that K_μ is always positive definite. Second, we shall *assume* that A_μ is positive definite. Finally, we need $B_\mu + K^T C K$ to be positive definite. By Theorem 2.3 in the previous section, we can find $C := \mathrm{diag}\{c_1, \cdots, c_m\}$ such that $B_\mu + K^T C K$ is positive definite if and only if B_μ is positive definite on $\ker K$.

2) Checking the Rank of the Codistribution $d\tilde{\mathcal{C}}_\mu$: Theorem 2.1, applied to the reduced system, requires that the codistribution $d\tilde{\mathcal{C}}_\mu$ be of maximal dimension, i.e., be equal to the whole cotangent space of $(T^*Q)_\mu$, on a neighborhood of the equilibrium ζ_e, where

$$\tilde{\mathcal{C}}_\mu\colon = \mathrm{span}\left\{F_{i\mu}, \left\{\tilde{H}_\mu, F_{i\mu}\right\}, \left\{\tilde{H}_\mu, \left\{\tilde{H}_\mu, F_{i\mu}\right\}\right\}, \cdots\right\}, \quad i = 1, \cdots, m \tag{3.10}$$

and $d\tilde{\mathcal{C}}_\mu$ is defined in a manner similar to that in the previous section, i.e., $d\tilde{\mathcal{C}}_\mu = \mathrm{span}\{dg(z) | g \in \tilde{\mathcal{C}}_\mu\}$.

It is possible to obtain, in terms of functions on T^*Q, a sufficient condition that ensures that $d\tilde{\mathcal{C}}_\mu$ is of maximal dimension. Let

$$\tilde{\mathcal{C}}\colon = \mathrm{span}\left\{F_i, \left\{\tilde{H}, F_i\right\}, \left\{\tilde{H}, \left\{\tilde{H}, F_i\right\}\right\}, \cdots\right\}, \quad i = 1, \cdots, m.$$

We define $d\tilde{\mathcal{C}}$ in the usual manner, and $\ker d\tilde{\mathcal{C}}(z)$ is defined to be the subspace of $T_z(T^*Q)$ that is annihilated by $d\tilde{\mathcal{C}}(z)$. Note that since the functions in $\tilde{\mathcal{C}}$ are all G-invariant, we know that $\ker d\tilde{\mathcal{C}}(z) \supset \mathfrak{g} \cdot z$ for all z in a neighborhood of z_e. We will say that $d\tilde{\mathcal{C}}(z)$ is of maximal dimension if $\ker d\tilde{\mathcal{C}}(z)$ is *equal* to $\mathfrak{g} \cdot z$.

Theorem 3.2: Suppose that for all z in a neighborhood of z_e, we have $\ker d\tilde{\mathcal{C}}(z) = \mathfrak{g} \cdot z$. Then $d\tilde{\mathcal{C}}_\mu$ is of maximal dimension on a neighborhood of ζ_e.

Proof: If B is a G-invariant function on T^*Q, then let $\overline{B}$ be the function on $J^{-1}(\mu)$ obtained by restricting B, and let B_μ be the function obtained by dropping $\overline{B}$ to $(T^*Q)_\mu$. Note that $B \in \tilde{\mathcal{C}}$ implies that $B_\mu \in \tilde{\mathcal{C}}_\mu$.

Let

$$\pi_\mu\colon J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu = (\boldsymbol{T}^*Q)_\mu$$

be the projection. Assume that for all z in a neighborhood U of z_e, $\ker d\tilde{\mathcal{C}}(z) = \mathfrak{g} \cdot z$. Note that $W\colon = \pi_\mu(J^{-1}(\mu) \cap U)$ is an open neighborhood of ζ_e in $(T^*Q)_\mu$. Let $\zeta \in W$. To show that $d\tilde{\mathcal{C}}_\mu$ is of maximal dimension at ζ, we need to show that for all $v \in \ker d\tilde{\mathcal{C}}_\mu(\zeta)$, $v = 0$. Let $v \in \ker d\tilde{\mathcal{C}}_\mu(\zeta)$. Choose $z \in J^{-1}(\mu) \cap U, \overline{v} \in T_z(J^{-1}(\mu))$, such that $T\pi_\mu \cdot \overline{v} = v$. Now for $B \in \mathcal{C}$

$$dB \cdot \overline{v} = d\overline{B} \cdot \overline{v} = dB_\mu \cdot T\pi_\mu \cdot \overline{v} = dB_\mu \cdot v = 0.$$

Hence $\overline{v} \in \ker d\mathcal{C}(z) = \mathfrak{g} \cdot z$. This implies that

$$\overline{v} \in T_z(J^{-1}(\mu)) \cap \mathfrak{g} \cdot z = \mathfrak{g}_\mu \cdot z$$

and thus $T\pi_\mu \cdot \overline{v} = v = 0$. ■

Thus, for the purposes of applying Theorem 2.4, it is sufficient to verify that the codistribution $d\tilde{\mathcal{C}}$ satisfies the following rank condition:

$$\ker d\tilde{\mathcal{C}}(z) = \mathfrak{g} \cdot z \tag{3.11}$$

for all z in a neighborhood of z_e.

3) Poisson Commutativity of the Functions $F_{i\mu}$: It follows from the theory of symplectic reduction that for any G-invariant functions B_1, B_2 on (T^*Q), $\{B_1, B_2\}$ is a G-invariant function and $\{B_1, B_2\}_\mu = \{B_{1\mu}, B_{2\mu}\}$, where $B_{i\mu}$ and $\{B_1, B_2\}_\mu$ are obtained by restricting B_i, $\{B_1, B_2\}$ respectively to $J^{-1}(\mu)$ and then dropping to $(T^*Q)_\mu$. If (q, p) are cotangent bundle coordinates on T^*Q, then the functions F_i on (T^*Q) depend on q alone. Thus $\{F_i, F_j\} = 0, i, j = 1, \cdots, m$, and so $\{F_{i\mu}, F_{j\mu}\} = \{F_i, F_j\}_\mu = 0$.

Extension of Theorem 2.4: Now we can state the extension of Theorem 2.4 to the case of mechanical systems with symmetry.

Theorem 3.3: Let $\pi\colon Q \to Q/G =\colon S$ be a principal G-bundle. Let $\langle\langle \cdot, \cdot \rangle\rangle$ be a G-invariant metric on Q. Let V be a G-invariant potential on Q, so that L defined by $L(v_q) = (1/2)\langle\langle v_q, v_q \rangle\rangle - V(q)$ is a G-invariant Lagrangian on Q. Let there be a generalized force $\tau = dF_1(q)u_1 + \cdots + dF_m(q)u_m$ acting on the system, where the functions F_i are G-invariant functions on Q. Thus the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = dF_1(q)u_1 + \cdots + dF_m(q)u_m.$$

Let $z_e \in T^*_{q_e}Q$ be a relative equilibrium. Let $J(z_e) = \mu$. Assume that:

1) $A_\mu = \delta^2 V_\mu(q_e)|\mathcal{V}_{\mathrm{RIG}}$ is positive definite;
2) $B_\mu = \delta^2 V_\mu(q_e)|\mathcal{V}_{\mathrm{INT}}$ is positive definite on $\ker dF(q_e) \cap \mathcal{V}_{\mathrm{INT}}$, and $C := \mathrm{diag}\{c_1, \cdots, c_m\}$ is chosen such that

$B_\mu + K^T C K$ is positive definite. (As before, $F := (F_1, \cdots, F_m)$: $Q \to \mathbb{R}^m$.)

Let $\tilde{\mathcal{C}}$ be defined by

$$\tilde{\mathcal{C}}: = \operatorname{span}\left\{F_i, \left\{\tilde{H}, F_i\right\}, \left\{\tilde{H}, \left\{\tilde{H}, F_i\right\}\right\}, \cdots\right\}, \quad i = 1, \cdots, m.$$

Suppose that for each z in a neighborhood of z_e, $\ker d\tilde{\mathcal{C}}(z) = \mathfrak{g} \cdot z$. Then for any choice of positive constants k_i, the feedback $u_i = -c_i F_i(z) - k_i \dot{F}_i(z)$ asymptotically stabilizes the relative equilibrium, i.e., asymptotically stabilizes the corresponding equilibrium $\zeta_e \in J^{-1}(\mu)/G_\mu$ of the reduced system.

Proof: Set $u_i = -c_i F_i + v_i$. This gives us the following reduced system on $(T^*Q)_\mu$:

$$\dot{\zeta} = X_{\tilde{H}_\mu}(\zeta) - X_{F_{1\mu}}(\zeta) v_1 - \cdots - X_{Fm\mu}(\zeta) v_m.$$

Now, A_μ is positive definite and $C := \operatorname{diag}\{c_1, \cdots, c_m\}$ has been chosen such that $B_\mu + K^T C K$ is positive definite. Thus $\delta^2 \tilde{H}_\mu(\zeta_e)$ is positive definite. Now $d\tilde{\mathcal{C}}(z)$ is of maximal dimension for all z on an open neighborhood of z_e, and thus $d\tilde{\mathcal{C}}_\mu(\zeta)$ is also of maximal dimension on an open neighborhood of ζ_e. The functions $F_{i\mu}$ Poisson commute, and thus, by Theorem 2.1 and Lemma 2.2, the feedback $v_i = -k_i \dot{F}_{i\mu}(\zeta) = -k_i \dot{F}_i(z)$ asymptotically stabilizes the equilibrium ζ_e. Thus the feedback for the original system is $u_i = -c_i F_i(z) - k_i \dot{F}_i(z)$. ∎

E. Asymptotic Stabilization for the "Cowboy" Solution of the Double Spherical Pendulum

We will now illustrate the use of Theorem 3.3 by applying it to get a feedback that asymptotically stabilizes the cowboy solution of the double spherical pendulum. Here we will give a description of the steps involved, but the actual calculations are too involved to reproduce here. (The computation of the relative equilibria of the DSP and their stability analysis is described in [6]; see also [4].)

Recall that q_1, q_2 are the positions of the bobs of the first and second pendula, respectively, relative to their points of suspension. This system has a symmetry group $G = S^1$, which is abelian. Note that the configuration of the system is specified by $q_1^\perp$, $q_2^\perp$, which are defined to be horizontal projections of q_1, q_2. The implicit assumption here is that both the pendula point downwards—i.e., for each pendulum, the height of the bob is lower than the height of the point of suspension. There do exist relative equilibria with one or both pendula pointing upwards, but we will not need to discuss those equilibria here.

Let (r_1, θ_1) and (r_2, θ_2) be polar coordinates for $q_1^\perp$ and $q_2^\perp$. The configuration space Q for this system is thus four dimensional and is parameterized by $(r_1, \theta_1, r_2, \theta_2)$. The symmetry group for this system, $G = S^1$, acts as follows. If $\psi \in S^1$, then

$$\psi \cdot (r_1, \theta_1, r_2, \theta_2) = (r_1, \theta_1 + \psi, r_2, \theta_2 + \psi).$$

If we define $\varphi: \;= \theta_2 - \theta_1$, then $(\theta_1, r_1, r_2, \varphi)$ is another set of coordinates on the configuration space. With respect to these new coordinates, the action of the group S^1 is as follows: $\psi \cdot (\theta_1, r_1, r_2, \varphi) = (\theta_1 + \psi, r_1, r_2, \varphi)$. Thus (r_1, r_2, φ) is a set of coordinates on the shape space $S = Q/G$, and Q is regarded as a principle S^1-bundle over Q/G.

The fact that the symmetry group is abelian leads to some simplification in the conditions the statement of Theorem 3.3, as we shall proceed to show. Let $z_e = \alpha_\mu(q_e) \in T^*Q$ be a relative equilibrium. Now, since the group is abelian, we know that $\mathfrak{g}_\mu = \mathfrak{g}$, and $\mathfrak{g}_\mu^\perp = \{0\}$. Thus

$$\mathcal{V}_{\text{RIG}} = \mathfrak{g}_\mu^\perp \cdot q_e = \{0\}$$

and

$$\mathcal{V}_{\text{INT}} = \mathcal{V} = (\mathfrak{g}_\mu \cdot q_e)^\perp = (\mathfrak{g} \cdot q_e)^\perp.$$

Thus $\mathcal{V}_{\text{INT}}$ is the orthogonal complement of the vertical space.

We will assume that our generalized force is of the form $\tau = \sum_{i=1}^m dF_i u_i$, where the F_i are G-invariant functions on Q, and the u_i are control inputs.

Following the general theory, let $\tilde{V}_\mu$ be defined by

$$\tilde{V}_\mu(q) = V_\mu(q) + \tfrac{1}{2} \sum_{i=1}^m c_i F_i^2(q)$$

where the c_i are constants. We need to find values for c_i such that $\delta^2 \tilde{V}_\mu(q_e)$ is positive definite on $\mathcal{V} = \mathcal{V}_{\text{INT}}$. Now, since G is abelian, the amended potential V_μ, defined by $V_\mu(q) = V(q) + \langle \mu, \mathbb{I}_q^{-1} \mu \rangle$, is G-invariant. Thus $\tilde{V}_\mu$ is also G-invariant. Hence it is enough to show that $\delta^2 \tilde{V}_\mu(q_e)$ is positive definite on *any* complement of the vertical space $\mathfrak{g} \cdot q_e$. The functions r_1, r_2, φ form a coordinate chart on the shape space, so they can also be regarded as G-invariant functions on the configuration space. It will be convenient to choose

$$W := \operatorname{span}\{\partial/\partial r_1, \partial/\partial r_2, \partial/\partial \varphi\} \subset T_{q_e} Q$$

as the space on which the positive-definiteness of $\delta^2 \tilde{V}_\mu(q_e)$ is to be verified.

The matrix of $\delta^2 V_\mu(q_e)|W$ with respect to the above basis of W is

$$B_\mu = \begin{bmatrix} \dfrac{\partial^2 V_\mu}{\partial r_1^2}(q_e) & \dfrac{\partial^2 V_\mu}{\partial r_1 \partial r_2}(q_e) & \dfrac{\partial^2 V_\mu}{\partial r_1 \partial \varphi}(q_e) \\ \dfrac{\partial^2 V_\mu}{\partial r_2 \partial r_1}(q_e) & \dfrac{\partial^2 V_\mu}{\partial r_2^2}(q_e) & \dfrac{\partial^2 V_\mu}{\partial r_2 \partial \varphi}(q_e) \\ \dfrac{\partial^2 V_\mu}{\partial \varphi \partial r_1}(q_e) & \dfrac{\partial^2 V_\mu}{\partial \varphi \partial r_2}(q_e) & \dfrac{\partial^2 V_\mu}{\partial \varphi^2}(q_e) \end{bmatrix}.$$

For this system (with the values of the system parameters as specified below), upon diagonalizing the symmetric bilinear form B_μ, we get only one positive entry on the diagonal. Since we need B_μ to be positive definite on the kernel of $dF(q_e)$: $W \to \mathbb{R}^m$, $\ker dF(q_e)$ can have dimension at most one, and thus F needs to have at least two components. Roughly speaking, to use this technique, we must have actuation in all directions along which the second derivative of the Hamiltonian is not positive definite. It interesting to compare this with Bloch *et al.* [1], [2]. They change the *kinetic energy* (rather than the potential energy) to get a new mechanical system, and they do not necessarily require actuation along all the unstable directions.

We thus let $m = 2$, and choose $F_1 = r_1$ and $F_2 = \varphi$. F_2 is not continuous everywhere on the shape space—but since we are interested in local behavior in the neighborhood of a relative equilibrium, this is not an issue. In order to be able to use Theorem 3.3, we should, strictly speaking, replace F_2 with a function that is smooth on the whole of the shape space, but coincides with F_2 on a neighborhood of the relative equilibrium. Note that the dimension of the shape space is three, whereas we have only two control inputs. Thus our system is *partially internally actuated*. The matrix of $dF(q_e)\colon W \to \mathbb{R}^2$ is

$$K = \begin{bmatrix} \frac{\partial F_1}{\partial r_1}(q_e) & \frac{\partial F_1}{\partial r_2}(q_e) & \frac{\partial F_1}{\partial \varphi}(q_e) \\ \frac{\partial F_2}{\partial r_1}(q_e) & \frac{\partial F_2}{\partial r_2}(q_e) & \frac{\partial F_2}{\partial \varphi}(q_e) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now the matrix of $\delta^2 \tilde{V}_\mu(q_e)|W$ is $B_\mu + K^T C K$, where $C = \mathrm{diag}\{c_1, c_2\}$. Hence, by Theorem 2.3, if B_μ is positive definite on

$$\ker K = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

it is possible to find c_1, c_2 such that $\delta^2 \tilde{V}_\mu(q_e)|W$ is positive definite. Thus what we need to check is whether $\partial^2 V_\mu / \partial r_2^2(q_e)$ is positive.

It has been shown in [6] that at a relative equilibrium, both the pendula have to lie in the same vertical plane through the origin. Thus the value of the coordinate φ is either zero or π. Thus the internal configuration of the system at a relative equilibrium is determined by the parameter α, defined by the relation $q_2^\perp = \alpha q_1^\perp$. Note that if $\alpha > 0$, we get a straight-stretched-out solution, with $\varphi = 0$; whereas if $\alpha < 0$, we get a cowboy solution, with $\varphi = \pi$. Given a value for α, we can determine q_e and μ using the formulas in [4]. (The value of q_e is not unique: the *internal* configuration is uniquely determined by α, but the value of the group variable θ_1 can be arbitrary.) The value for α cannot be arbitrarily chosen; [4] gives the conditions (involving system parameters like length of the rods and ratio of the masses of the bobs) that α has to satisfy.

Feedback Law for a Specific Choice of System Parameters and μ: If we assume that in our DSP system that both the rods are of unit length, and both the bobs are of unit mass, and if we choose $\alpha = -3/2$, then this value of α does satisfy the required conditions. Note that since $\alpha < 0$, the relative equilibrium corresponds to a "cowboy" solution. We can then find μ and a value of q_e. After determining μ, we can calculate the amended potential $V_\mu(q) = V(q) + \langle \mu, \mathbb{I}_q^{-1}\mu \rangle$. The next step is to check that $\partial^2 V_\mu / \partial r_2^2(q_e)$ is positive. For our system, it indeed is. This ensures (see Theorem 2.3) that we can find constants c_1, c_2 such that $\delta^2 \tilde{V}_\mu(q_e)|W$ is positive definite. For the system parameters we have chosen, it can be checked that the choice $c_1 = 300$ and $c_2 = 20$ will work. Finally, it can be verified that the rank condition (3.11) is satisfied for our system. Thus one can conclude that the feedback $u_1 = 300 r_1 - k_1 \dot{r}_1$ and $u_2 = 20\varphi - k_2 \dot{\varphi}$ will make the cowboy solution an asymptotically stable relative equilibrium for any choice of positive constants k_1, k_2.

IV. Concluding Remarks

In this paper, we have derived feedback laws that asymptotically stabilize relative equilibria of a mechanical system with symmetry and are of the proportional-derivative (PD) form: $u_i = -c_i F_i - k_i \dot{F}_i$, $i = 1, \cdots, m$. The functions F_i depend only on the internal configuration of the system. The proportional term $(-c_i F_i)$ modifies the potential and converts the equilibrium to a minimum of the (modified) reduced Hamiltonian, thereby stabilizing the equilibrium in the sense of Lyapunov. The derivative term $-k_i \dot{F}_i = -k_i dF_i(q) \cdot \dot{q}$, which is a linear function of the velocities $\dot{q}$, is used to introduce dissipation in the system and thereby make the system asymptotically stable.

To make the relative equilibrium a minimum of the reduced Hamiltonian, we require that $\delta^2 V_\mu(q_e)$ is positive definite on $\ker dF(q_e)$, i.e., on the space on which we have no control authority. An intuitive but imprecise way of saying this is that we need actuation along all the directions along which the second derivative of the potential is not positive definite.

Assuming that we can make the equilibrium a minimum of the reduced Hamiltonian, a condition that assures that we can asymptotically stabilize the relative equilibrium using the derivative terms is that $d\tilde{\mathcal{C}}_\mu$ equals the whole cotangent space of $(T^*Q)_\mu$ on a neighborhood of ζ_e, where $\tilde{\mathcal{C}}_\mu$ is as defined in (3.10). Another way of writing this condition is as follows:

$$\mathrm{span}\left\{ dF_{i\mu}(\zeta_e),\, d\left\{\tilde{H}_\mu,\, F_{i\mu}\right\}(\zeta_e),\, d\left\{\tilde{H}_\mu,\, \left\{\tilde{H}_\mu,\, F_{i\mu}\right\}\right\}(\zeta_e),\, \cdots, \right\} = T^*_{\zeta_e}((T^*q)_\mu). \tag{4.1}$$

Now using the fact that $B(dL, \cdot) = i_{dL} B = X_L$, where L is any function on $(T^*Q)_\mu$, and B is the Poisson tensor, it is easy to conclude that the above condition is satisfied iff

$$\mathrm{span}\left\{ X_{F_i}(\zeta_e),\, X_{\{\tilde{H}_\mu,\, F_{i\mu}\}}(\zeta_e),\, X_{\{\tilde{H}_\mu,\, \{\tilde{H}_\mu,\, F_{i\mu}\}\}}(\zeta_e),\, \cdots \right\} = T_{\zeta_e}((T^*Q)_\mu) \tag{4.2}$$

which is equivalent to

$$\mathrm{span}\left\{ X_{F_i}(\zeta_e),\, \left[X_{\tilde{H}_\mu},\, X_{F_{i\mu}}\right](\zeta_e),\, \left[X_{\tilde{H}_\mu},\, \left[X_{\tilde{H}_\mu},\, X_{F_{i\mu}}\right]\right](\zeta_e),\, \cdots \right\} = T_{\zeta_e}((T^*Q)_\mu). \tag{4.3}$$

This condition is reminiscent of the condition for local accessibility of a control system. Comparing the above condition with the condition for local accessibility, we see that the above condition is more stringent than local accessibility. It is easy to check that this condition is equivalent to controllability of the linearization of the reduced system at ζ_e (see [8]).

Amongst the space of pairs of matrices A, B of appropriate dimension, the pairs that are controllable form an open dense subset. This suggests that the condition that we need for asymptotic stabilizability is not a very stringent one. Indeed, van der Schaft [10] notes that in general, just one dissipation term $-k_i \dot{F}_i$ is enough to assure asymptotic stability.

This PD feedback scheme has several well-known advantages, some of which we briefly list here. First, it is easy to implement: if we are able to measure the m shape space variables $F_1, \cdots, F_m$ and their rates of change, the computational burden is quite minimal. Second, this scheme *decentralized* in the sense that the$i$th input depends only on F_i and not on any of the other measurements. A third advantage of this scheme is robustness, by which we mean that a controller designed to stabilize the relative equilibrium of a system will work even for a new system obtained by perturbing this system, assuming that the perturbation preserves the relative equilibrium. This is easily verified by noting that the positivity condition on the second derivative of the amended potential, and the condition on the rank of $d\tilde{\mathcal{C}}$ in the statement of Theorem 3.3 will continue to hold if the system data is perturbed by small amounts.

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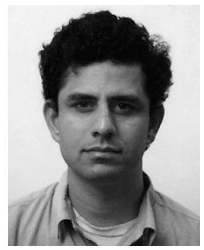

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