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TRADE RULES FOR UNCLEARED MARKETS

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Abstract

We analyze markets in which the price of a traded commodity is such that the supply and the demand are unequal. Under standard assumptions, the agents then have single peaked preferences on their consumption or production choices. For such markets, we propose a class of *Uniform Trade rules* each of which determines the volume of trade as the median of total demand, total supply, and an exogenous constant. Then these rules allocate this volume "uniformly" on either side of the market. We evaluate these "trade rules" on the basis of some standard axioms in the literature. We show that they uniquely satisfy *Pareto optimality, strategy proofness, no-envy*, and an informational simplicity axiom that we introduce. We also analyze the implications of *anonymity, renegotiation proofness*, and *voluntary trade* on this domain.

JEL classification numbers: D5, D6, D7

Key words: market disequilibrium, trade rule, efficiency, strategy proofness, anonymity, no-envy, renegotiation proofness, voluntary trade

Trade rules for uncleared markets *

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1 Introduction

We analyze markets in which the price of a traded commodity is fixed at a level where the supply and the demand are unequal. This phenomenon is observed in many markets, either because the price adjustment process is slow, such as in the labor market, or because the prices are controlled from outside the market (*e.g.* by the state), such as in health, education, or agricultural markets. These observations are conceptualized in the idea of market disequilibrium which has been particularly central in Keynesian economics after Clower (1965) and Leijonhufvud (1968). For more on this, see Benassy (1982).

For markets in inequilibrium, it is important to understand how trade takes place and how the current practice can be improved upon through the *design* of "good" rules that regulate it. In this paper, we propose such "trade rules" and evaluate them on the basis of some standard properties in the literature.

In our model, a set of producers face demand from a set of consumers (who might be individuals as well as other producers that use the traded commodity as input). We assume that the individuals have convex preferences on consumption bundles. They thus have single-peaked preferences on the boundary of their budget sets, and therefore, on their consumption of the commodity in question. Similarly, we assume that the producers have convex production sets. Their profits are thus single-peaked in their output or input. Due to these observations, our paper is related to earlier studies on single-peaked preferences.[1]

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[1] For a firm s, the preference relation R_s is an ordinal representation of how it compares two production or input-consumption levels in terms of profits. For firms with convex production sets, R_s will be single-peaked under fixed prices.

A trade rule, in our model, takes in the preferences of the buyers and the sellers and in turn, delivers (*i*) the volume of trade (*i.e.* the total trade that will be carried out between the buyers and the sellers) and (*ii*) how the volume of trade will be allocated among the agents on either side of the market. We introduce a class of *Uniform Trade rules* each of which, in step (*i*), determines the volume of trade as the median of total demand, total supply, and an exogenous constant and in step (*ii*), allocates this volume "uniformly" among agents in either side of the market.

There are earlier papers related to either one of the above steps but not both. The second (allocation) step is related to the literature starting with Sprumont (1991) who analyzes the problem of allocating a fixed social endowment of a private commodity among agents with single-peaked preferences. The social endowment in those problems corresponds in our model to the volume of trade which, in the second step is treated as fixed, and is allocated as total supply among the buyers and total demand among the sellers. On Sprumont's domain, an allocation rule called the Uniform Rule turns out to be central. It can be described as follows: if the sum of the agents' peaks is more (respectively, less) than the social endowment, each agent receives the minimum (respectively, the maximum) of his peak and a constant amount. The value of this constant is uniquely determined by the feasibility of the allocation. Sprumont (1991) shows that this rule uniquely satisfies (*i*) *Pareto optimality, strategy proofness*, and *anonymity* as well as (*ii*) *Pareto optimality, strategy proofness*, and *no-envy.* The Uniform rule satisfies many other desirable properties (e.g. see Ching (1992, 1994) and Thomson (1994 a, b)). Thus it is no surprise that in our model, the aforementioned *Uniform Trade rules* employ the Uniform rule to allocate the trade volume among agents on either side of the market.

The first (trade-volume determination) step is intuitively (though not formally) related to Moulin (1980) who analyzes the determination of a one-dimensional policy issue among agents with single-peaked preferences.[2] This relation is particularly apparent (and formal) when there is a single buyer and a single seller. Then the volume of trade is exactly like a public good for these two agents. While this is no more true when there are multiple buyers or seller (who are sharing the trade volume among themselves), the mechanics of determining the trade volume as a function of the total demand and total supply still resemble Moulin's (1980) model. This similarity becomes apparent in our results: parallel to the extended median rules proposed there, strategical considerations lead us to propose the determination of the volume of trade as the median of total demand, total supply and an exogenous constant.

Let us however note that our model is richer than a simple conjunction of the two models mentioned above. This is particularly due to the interaction between the determination of the agents' shares and the determination of the trade volume. For example, the agents can manipulate their allotments also by manipulating (possibly as a group) the volume of trade. Also, single-economy requirements like *Pareto optimality* or "fairness" become much more demanding as what is to be allocated becomes endogenous. Another

[2]Consider, for example, the determination of a tax rate, the budget of a project, or the provision of a public good.

important difference is the existence of two types of agents (buyers and sellers) in our model. This duality limits the implications of requirements like *anonymity* or *no-envy* and, for example in comparison to Moulin (1980), allows for a much larger class of median rules some of which discriminates between the buyers and the sellers.

Our model is also related to those of Barbera and Jackson (1995), Thomson (1995), and Klaus, Peters, and Storcken (1997, 1998). Barbera and Jackson (1995) analyze a pure exchange economy with an arbitrary number of agents and commodities. Each agent has a positive endowment of the commodities and a continuous, strictly convex, and monotonic preference relation on his consumption. The authors look for *strategy-proof* rules to facilitate trade in this exchange economy. With this consideration, they introduce and characterize a class of "fixed-proportion trading rules" where (*i*) trade can only occur in one proportion which is selected from an *a priori* fixed set of proportions satisfying certain restrictions due to which the set of feasible allocations is restricted to be a one-dimensional set on which the agents have single-peaked preferences and (*ii*) given a proportion for trade, the final allocation is chosen by rationing the agents *uniformly*. Thomson (1995) and Klaus, Peters, and Storcken (1997, 1998) alternatively analyze a single-commodity model where they consider the reallocation of an infinitely divisible good among agents with single-peaked preferences and individual endowments.[3] In their models, the agents whose endowments are greater than their peaks (the suppliers) supply to those whose endowments are less then their peaks (the demanders). They show that a set of basic properties characterize a "Uniform reallocation rule".

The relation between these models and ours is quite similar to the one between pure exchange and production economies. In the pure exchange models, whether an agent is a supplier or a demander of the commodity in question depends on the relation between his preferences and his endowment. For example, by changing his preferences, a supplier can turn into a demander of the commodity in question and vice versa. In our production model, however, producers and consumers are exogenously distinct entities. This difference has significant implications on the analysis to be carried out. For example, fairness properties such as anonymity or no-envy compare all agents in the pure exchange version of the model whereas, in the production version, they can only compare agents on the same side of the market.[4] Also, in our model, there are no exogenously set individual endowments. Only after the shares are determined, the production decisions are made.[5] These differences reflect to the results obtained in the two models as well. In the pure exchange model, basic properties imply that the short side of the market always clears whereas this is not the case in our model.[6] We thus interpret the exchange and pro-

[3]Thomson (1995) also allows an "open economy" extension where a transfer from the outside world (aside from the individual endowments) is to be allocated.

[4]Indeed, to consider envy between a producer and a consumer, one would need an environment where each consumer has access to a production technology and maybe even less realistically, each firm can turn itself into a consumer.

[5]Note that this is more than simply setting the endowments in Klaus *et al* to zero since in that case all agents in their model would become demanders of the commodity.

[6]The *short side* of a market is where the aggregate volume of desired transaction is smallest. It is thus the demand side if there is excess supply and the supply side if there is excess demand. The other

duction models (and their findings) as complements of each other in the aforementioned sense.

We look for trade rules that satisfy a set of standard properties such as *Pareto optimality*, *(coalitional) strategy proofness*, and *no-envy*. We also introduce a new property specific to this domain: *separability in total trade* requires the volume of trade only to depend on the total demand and supply but not on their individual components. For example, increasing agent i's demand and decreasing agent $j's$ demand so as to keep total demand unchanged should have no effect on the volume of trade. Note that this change can still effect the shares of these two agents as well as others.

We observe that the above properties are logically independent and in Theorem 1, we show that they are uniquely satisfied by a class of *Uniform Trade rules*. As noted above, these rules do not necessarily clear the short side of the market. Such practice might seem unrealistic at first glance. However, real life examples to it are in fact more common than one would initially expect, especially in markets with strong welfare implications for the society. In health or education sectors for example, it is not uncommon to observe excess demand due to price regulations and an overutilization of services (such as overfilled schools or hospitals). Similarly, there are many countries (such as that of the authors) where in response to an excess supply of labor, governments tend to over-employ in the public sector. Even in the private sector, since most labor contracts include restrictions on when and how the contract can be terminated, firms regularly experience periods in which they overemploy. Finally let us note that, especially when several interconnected markets are concerned, clearing the short side in every one of these markets might be problematic. Benassy (1982, pages 11-12) presents the example of a firm that buys from an input market in excess demand and sells to an output market in excess demand. If the short side clears in the input market, the firm cannot produce at its profit maximizing level even though it faces excess demand. Thus in this example, application of the short side rule in the input market has efficiency implications on the output market.

We later analyze the implications of a stronger separability property. In Proposition 2, we show that any *Pareto optimal* and *strategy proof* trade rule that satisfies *strong separability in total trade* has to determine the volume of trade by an extended median rule that is constant across different societies. Adding *no-envy* (in allocations) and *anonymity* (in determination of the trade volume) restricts the admissible class of rules to *Uniform Trade rules* (i) that are constant across societies and (ii) that do not discriminate between buyers and sellers.

We observe that among *Uniform Trade rules*, *renegotiation proof* ones are those that clear exactly one side of the market in economies where there are less agents on the short side of the market than there is on the long side. Interestingly enough, *renegotiation proofness* has no implications for societies with an equal number of buyers and sellers.

We also observe that only the *Uniform Trade rule* that clears the short side of the

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side is called the *long side*.

market satisfies a *voluntary trade* requirement that gives each agent the right to choose zero trade for himself (the term is introduced by Benassy (1982), Chapter 6). For this, we show in Proposition 3 that any *Pareto optimal* and *strategy proof* trade rule that satisfies *voluntary trade* has to clear the short side of the market. Note that, in examples such as health services for infants or compulsory education for children, consumers (*i.e.* the parents) do not have the right to choose zero consumption. For such markets therefore, *voluntary trade* is not a desirable property. On the other hand, with the exception of certain epidemics, adults have *voluntary trade* power in determining their consumption of health services.

The paper is organized as follows. In Section 2, we introduce the model and in Section 3, we introduce and discuss Uniform trade rules. Section 4 contains the main results. We conclude in Section 5.

2 The Model

There is a universal set $\mathcal{B}$ of potential buyers and a universal set $\mathcal{S}$ of potential sellers. Let $|\mathcal{B}| = |\mathcal{S}|$. There is a perfectly divisible commodity that each seller produces and each buyer consumes. Let $\mathbb{R}_+$ be the consumption/production space for each agent. Each $i \in \mathcal{B} \cup \mathcal{S}$ is endowed with a continuous preference relation R_i over $\mathbb{R}_+$. Let P_i denote the strict preference relation associated with R_i. The preference relation R_i is **single-peaked** if there is $p(R_i) \in \mathbb{R}_+$, called the **peak** of R_i, such that for all x_i, y_i in $\mathbb{R}_+$, $x_i < y_i \leq p(R_i)$ or $x_i > y_i \geq p(R_i)$ implies $y_i P_i x_i$. Let $\mathcal{R}$ denote the set of all continuous and single-peaked preference relations on $\mathbb{R}_+$.

Given a finite set $B \subset \mathcal{B}$ of buyers and a finite set $S \subset \mathcal{S}$ of sellers, let $N = B \cup S$ be a **society**. Let $\mathcal{N} = \mathcal{B} \cup \mathcal{S}$ be the set of all societies. A preference profile R_N for a society N is a list $(R_i)_{i \in N}$ such that for each $i \in N$, $R_i \in \mathcal{R}$. Let $\mathcal{R}^N$ denote the set of all profiles for the society N. Given $R_N \in \mathcal{R}^N$, let $p(R_N) = (p(R_i))_{i \in N}$. Given $N' \subset N$ and $R_N \in \mathcal{R}^N$, let $R_{N'} = (R_i)_{i \in N'}$ denote the restriction of R_N to N'. A **market for society** $B \cup S$ is a profile of preferences for buyers and seller $(R_B, R_S) \in \mathcal{R}^{B \cup S}$. Let

$$\mathcal{M} = \bigcup_{(B \cup S) \in \mathcal{N}} \mathcal{R}^{B \cup S}$$

be the set of all markets.

A **(feasible) trade** for $(R_B, R_S) \in \mathcal{M}$ is a vector $z \in \mathbb{R}_+^{B \cup S}$ such that $\sum_B z_i = \sum_S z_i$. For each buyer (seller) i, z_i denotes how much he buys (sells). Let $Z(B \cup S)$ denote the set of all trades for (R_B, R_S). A *trade* $z \in Z(B \cup S)$ is **Pareto optimal with respect to** (R_B, R_S) if there is no $z' \in Z(B \cup S)$ such that for all $i \in B \cup S$, $z'_i R_i z_i$ and for some $j \in B \cup S$, $z'_i P_i z_i$. In our framework, Pareto optimal trades posses the following property.

Lemma 1 *For each $(R_B, R_S) \in \mathcal{M}$, the trade $z \in Z(B \cup S)$ is Pareto optimal with respect to (R_B, R_S) if and only if for $K \in \{B, S\}$, $\sum_K p(R_k) \leq \sum_{N \setminus K} p(R_k)$ implies* *(i)* $p(R_k) \leq z_k$ *for each* $k \in K$, *(ii)* $z_j \leq p(R_j)$ *for each* $j \in N \setminus K$, *and thus (iii)* $\sum_K p(R_k) \leq \sum_K z_k \leq \sum_{N \setminus K} p(R_k)$.

Proof: Let $(R_B, R_S) \in \mathcal{M}$ be such that $\sum_K p(R_k) \leq \sum_{N \setminus K} p(R_k)$.

Assume that $z \in Z(B \cup S)$ is Pareto optimal. First note that if there is $i \in K$ such that $z_i < p(R_i)$ and there is $j \in N \setminus K$ such that $z_j < p(R_j)$, then there is $\varepsilon > 0$ such that $z' \in Z(B \cup S)$ defined as for all $k \notin \{i, j\}$, $z'_k = z_k$, $z'_i = z_i + \varepsilon$, and $z'_j = z_j + \varepsilon$ Pareto dominates z. Similarly, if there is $i \in K$ such that $z_i > p(R_i)$ and there is $j \in N \setminus K$ such that $z_j > p(R_j)$, we obtain a similar contradiction.

Now note that if $\sum_K z_k < \sum_K p(R_k) \leq \sum_{N \setminus K} p(R_k)$, then there is $i \in K$ such that $z_i < p(R_i)$ and there is $j \in N \setminus K$ such that $z_j < p(R_j)$. Similarly, if $\sum_K p(R_k) \leq \sum_{N \setminus K} p(R_k) < \sum_K z_k$, then there is $i \in K$ such that $z_i > p(R_i)$ and there is $j \in N \setminus K$ such that $z_j > p(R_j)$. Thus $\sum_K p(R_k) \leq \sum_K z_k \leq \sum_{N \setminus K} p(R_k)$.

Finally, if there is $i, j \in K$ such that $z_i < p(R_i)$ and $z_j > p(R_j)$, there is $\varepsilon > 0$ such that $z'_i = z_i + \varepsilon$, $z'_j = z_j - \varepsilon$, and for all $k \in K \setminus \{i, j\}$, $z'_k = z_k$ is a Pareto improvement over z. This and $\sum_K p(R_k) \leq \sum_K z_k$ implies that for each $i, j \in K$, $z_i \geq p(R_i)$ and $z_j \geq p(R_j)$. A similar argument proves that for each $i, j \in N \setminus K$, $z_i \leq p(R_i)$ and $z_j \leq p(R_j)$.

For the converse, assume $p(R_k) \leq z_k$ for each $k \in K$ and $z_l \leq p(R_l)$ for each $l \in N \setminus K$. Let $z' \in Z(B \cup S)$ be such that for some $i \in K$, $z'_i P_i z_i$. Then $z'_i < z_i$. This implies that either there is $j \in K$ such that $z'_j > z_j \geq p(R_j)$ or there is $l \in N \setminus K$ such that $z'_l < z_l \leq p(R_l)$. Thus z' does not Pareto dominate z. A similar argument follows if there is $i \in N \setminus K$ such that $z'_i P_i z_i$. Thus z is Pareto optimal. ∎

A **trade rule** $F : \mathcal{M} \rightarrow \bigcup_{N \in \mathcal{N}} Z(N)$ associates each market (R_B, R_S) with a trade $z \in Z(B \cup S)$. Let $\Omega_F(.) = \sum_{i \in B} F_i(.)$ be the associated rule that determines the **volume of trade**. In what follows, we introduce properties that are related to the four main titles in axiomatic analysis: efficiency, nonmanipulability, fairness, and stability.

We start with efficiency. A trade rule F is **Pareto optimal** if for each $(R_B, R_S) \in \mathcal{M}$, the trade $F(R_B, R_S)$ is *Pareto optimal with respect to* (R_B, R_S).

We present two properties on nonmanipulability. A trade rule F is **strategy proof** if for each $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$, $F_i(R_i, R_{N \setminus i}) R_i F_i(R'_i, R_{N \setminus i})$. That is, regardless of the others' preferences, an agent is best-off with the trade associated with her true preferences. Strategy proof rules do not give the agents incentive for individual manipulation. They however are not immune to manipulation by groups. For this, a stronger property is necessary: a trade rule F is **coalitional strategy proof** if for each $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, $M \subset N$, and $R'_M \in \mathcal{R}^M$, if there is $i \in M$ such that $F_i(R'_M, R_{N\setminus M}) P_i F_i(R_M, R_{N\setminus M})$ then, there is $j \in N'$ such that $F_j(R_M, R_{N\setminus M}) P_j F_j(R'_M, R_{N\setminus M})$.[7]

Our first fairness property is after Foley (1967). Since in our model the agents on different sides of the market are exogenously differentiated, our version of the property only compares agents on the same side of the market. A trade rule F is **envy free** (equivalently, satisfies **no-envy**) if for each $(R_B, R_S) \in \mathcal{M}$, $K \in \{B, S\}$, and $i, j \in K$, $F_i(R_B, R_S) R_i F_j(R_B, R_S)$.

Note that **no-envy** does not have any implications on the determination of the volume of trade. The next two fairness properties deal with this issue. A bijection $\pi : \mathcal{N} \to \mathcal{N}$ which satisfies $\pi(i) \in \mathcal{B}$ ($\pi(i) \in \mathcal{S}$) if and only if $i \in \mathcal{B}$ ($i \in \mathcal{S}$) is called an *in-group-permutation*. Let Π be the set of all in-group-permutations and let $R^{\pi}_{\pi(i)} = R_i$ for each $\pi \in \Pi$ and $i \in \mathcal{N}$. A trade rule F satisfies **in-group anonymity in total trade** if for each $(R_B, R_S) \in \mathcal{M}$ and each $\pi \in \Pi$, $\Omega_F(R_B, R_S) = \Omega_F(R^{\pi}_{\pi(B)}, R^{\pi}_{\pi(S)})$. That is the volume of trade should not depend on the identity of the agents in a group. A bijection $\phi : \mathcal{N} \to \mathcal{N}$ which satisfies $\phi(i) \in \mathcal{B}$ ($\phi(i) \in \mathcal{S}$) if and only if $i \in \mathcal{S}$ ($i \in \mathcal{B}$) is called a *between-group-permutation*. Let Φ be the set of all between-group-permutations and let $R^{\phi}_{\phi(i)} = R_i$ for each $\phi \in \Phi$ and $i \in \mathcal{N}$. A trade rule F satisfies **between-group anonymity in total trade** if for each $(R_B, R_S) \in \mathcal{M}$ and each $\phi \in \Phi$, $\Omega_F(R_B, R_S) = \Omega_F(R^{\phi}_{\phi(S)}, R^{\phi}_{\phi(B)})$. This property requires that the volume of trade should not depend on the identity of the groups. A trade rule F satisfies **anonymity in total trade** if it satisfies both of these anonymity properties.

If the universal sets of buyers and sellers are allowed to be of infinite size, *between-group anonymity in total trade* implies *in-group anonymity in total trade*. The simple proof, for any given $\pi \in \Pi$, constructs a pair ϕ, $\phi' \in \Phi$ such that $\pi = \phi' \circ \phi$. Thus, a buyer $b_i \in \mathcal{B}$ who is mapped by π to another buyer $\pi(b_i)$ can alternatively be mapped first to a seller $\phi(b_i) \in \mathcal{S}$ and then this seller can be mapped to the buyer $\pi(b_i) = \phi'(\phi(b_i))$. With finite universal sets, however, the two properties are independent. To see this, let $|\mathcal{B}| = |\mathcal{S}| = n$. Let $\pi \in \Pi$ be such that for each $b_i \in \mathcal{B}$, $\pi(b_i) = b_i$ and for each $s_i \in \mathcal{S}$, $\pi(s_i) = s_{i+1}$ (with $\pi(s_n) = s_1$). Suppose there are $\phi, \phi' \in \Phi$ such that $\phi' \circ \phi = \pi$. Then $\phi'(\phi(b_i)) = b_i$ implies $\phi = \phi'$ and $\phi'(\phi(s_i)) = s_{i+1}$ implies $\phi \neq \phi'$, a contradiction.

Our fourth title is stability. We introduce two properties related to it. The first property is for markets where a buyer-seller pair can renegotiate a deal among themselves. A trade rule F is **renegotiation proof** if for each $(R_B, R_S) \in \mathcal{M}$ there is no $i \in S$ and $j \in B$ such that for some $r \in \mathbb{R}_+$, $r P_i F_i(R_B, R_S)$ and $r P_j F_j(R_B, R_S)$. This is a weak no-blocking property.[8] Our final stability property is for markets where each agent is entitled to leaving the market, that is, buying or selling zero units. A trade rule F satisfies **voluntary trade** if for each $(R_B, R_S) \in \mathcal{M}$ and $i \in B \cup S$, $F_i(R_B, R_S) R_i 0$.

[7]Note that ours is the stronger formulation of the property. A weaker version considers only coalitional manipulations that make all agents in the coalition strictly better-off.

[8]We will later note that requiring a stronger version of the property that allows any coalition to form does not affect our results. Allowing some agents in a blocking-coalition to remain indifferent, on the other hand, has strong implications.

Lastly, we introduce the following informational simplicity property. It requires the volume of trade only to depend on the total demand and supply but not on their individual components. A trade rule F satisfies **separability (in total trade)** if for each $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S), (R'_B, R'_S) \in \mathcal{R}^{B \cup S}$, $\sum_{i \in B} p(R_i) = \sum_{i \in B} p(R'_i)$ and $\sum_{i \in S} p(R_i) = \sum_{i \in S} p(R'_i)$ implies $\Omega_F(R_B, R_S) = \Omega_F(R'_B, R'_S)$. Note that this property is not logically related to *anonymity in total trade* since it does not make the determination of trade volume independent of the agents' identities. It merely relates two problems with the same set of agents. A stronger separability property would totally disregard the agents' identities: a trade rule F satisfies **strong separability (in total trade)** if for each $(B \cup S)$, $(B' \cup S') \in \mathcal{N}$, $(R_B, R_S) \in \mathcal{R}^{B \cup S}$, and $(R'_{B'}, R'_{S'}) \in \mathcal{R}^{B' \cup S'}$, $\sum_{i \in B} p(R_i) = \sum_{i \in B'} p(R'_i)$ and $\sum_{i \in S} p(R_i) = \sum_{i \in S'} p(R'_i)$ implies $\Omega_F(R_B, R_S) = \Omega_F(R'_{B'}, R'_{S'})$. Note that since the sets B and B' (as well as S and S') are allowed to be of different cardinality, this property is stronger than a conjunction of *separability* and *in-group anonymity in total trade*.

3 Uniform Trade Rules

Let $\beta : \mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$ and $\sigma : \mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$ be two functions such that for each $B \cup S \in \mathcal{N}$, $B = \emptyset$ or $S = \emptyset$ implies $\beta(B \cup S) = \sigma(B \cup S) = 0$. The **Uniform Trade rule with respect to** β **and** σ, $\mathbf{UT}^{\beta\sigma}$, is then defined as follows. We first determine the volume of trade: given $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S) \in \mathcal{R}^{B \cup S}$, let

$$\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \begin{cases} median\{\beta(B \cup S), \sum_B p(R_i), \sum_S p(R_i)\} \text{ if } \sum_B p(R_i) \leq \sum_S p(R_i)\}, \\ \\ median\{\sigma(B \cup S), \sum_B p(R_i), \sum_S p(R_i)\} \text{ if } \sum_B p(R_i) \geq \sum_S p(R_i)\}. \end{cases}$$

That is, a median rule with the exogenous reference-point $\beta(B \cup S)$ is used when the buyers are the short side of the market. If, on the other hand, the sellers are the short side, then the reference point $\sigma(B \cup S)$ is used to calculate the median.

Next, we allocate the volume of trade among the agents: for $K \in \{B, S\}$, let

$$UT_K^{\beta\sigma}(R_B, R_S) = \begin{cases} (\min\{\lambda, p(R_i)\})_{i \in K} \text{ if } \sum_K p(R_i) \geq \Omega_{UT^{\beta\sigma}}(R_B, R_S), \\ \\ (\max\{\lambda, p(R_i)\})_{i \in K} \text{ if } \sum_K p(R_i) \leq \Omega_{UT^{\beta\sigma}}(R_B, R_S). \end{cases} \quad (1)$$

where $\lambda \in \mathbb{R}_+$ satisfies

$$\sum_K \min\{\lambda, p(R_i)\} = \Omega_{UT^{\beta\sigma}}(R_B, R_S) \text{ if } \sum_K p(R_i) \geq \Omega_{UT^{\beta\sigma}}(R_B, R_S)$$

and

$$\sum_K \max\{\lambda, p(R_i)\} = \Omega_{UT^{\beta\sigma}}(R_B, R_S) \text{ if } \sum_K p(R_i) < \Omega_{UT^{\beta\sigma}}(R_B, R_S).$$

The class of *Uniform Trade rules* is very rich. It contains rules that for example always favor the buyers ($\beta = 0$ and $\sigma = \infty$), rules that always favor the short side of the market ($\beta = \sigma = 0$), or rules that guarantee a fixed volume of trade unless both sides of the market wish to deviate from it ($\beta = \sigma = c \in \mathbb{R}_+$), as well as rules that mix between these and many other arbitration methods based on the identities of the agents and who constitutes the short side of the market. The following proposition analyzes the properties that *Uniform Trade rules* satisfy.

Proposition 2 *All* Uniform Trade rules satisfy *Pareto optimality,* coalitional strategy proofness, *no-envy,* and *separability in total trade.*

Proof: *Separability in total trade* follows from the median definition of $\Omega_{UT^{\beta\sigma}}$. To show that $UT^{\beta\sigma}$ satisfies *Pareto optimality,* note that by the median definition of $\Omega_{UT^{\beta\sigma}}$, we have

$$\sum_K p(R_i) \leq \Omega_{UT^{\beta\sigma}}(R_B, R_S) \leq \sum_{N\backslash K} p(R_i)$$

for $K \in \{B, S\}$. Thus there is $\rho, \lambda \in \mathbb{R}_+$ such that

$$\sum_K \max\{\rho, p(R_i)\} = \Omega_{UT^{\beta\sigma}}(R_B, R_S) = \sum_{N\backslash K} \min\{\lambda, p(R_i)\}.$$

Thus for each $i \in K$, $UT_i^{\beta\sigma}(R_B, R_S) \geq p(R_i)$ and for each $i \in N \setminus K$, $UT_i^{\beta\sigma}(R_B, R_S) \leq p(R_i)$. This, by Lemma 1 implies the desired conclusion.

To show that $UT^{\beta\sigma}$ satisfies *no envy,* let $R_{B\cup S} \in \mathcal{M}$ and $i \in K \in \{B, S\}$. No envy trivially holds if $UT_i^{\beta\sigma}(R_N) = p(R_i)$. Alternatively $UT_i^{\beta\sigma}(R_N) < p(R_i)$ implies $UT_j^{\beta\sigma}(R_N) \leq UT_i^{\beta\sigma}(R_N)$ for each $j \in K$. Similarly $UT_i^{\beta\sigma}(R_N) > p(R_i)$ implies $UT_j^{\beta\sigma}(R_N) \geq UT_i^{\beta\sigma}(R_N)$ for each $j \in K$. Therefore, $UT_i^{\beta\sigma}(R_N) R_i UT_j^{\beta\sigma}(R_N)$ for each $j \in K$.

To show that $UT^{\beta\sigma}$ satisfies *coalitional strategy proofness,* take an arbitrary market $R_N = (R_B, R_S) \in \mathcal{M}$. Let $z = UT^{\beta\sigma}(R_N)$, $\omega = \Omega_{UT^{\beta\sigma}}(R_N)$, $M \subset N$, and $R'_M \in \mathcal{R}^M$. Let $R'_N = \left(R'_M, R_{N\backslash M}\right)$, $z' = UT^{\beta\sigma}(R'_N)$ and $\omega' = \Omega_{UT^{\beta\sigma}}(R'_N)$. Suppose there is $i \in M$ such that $z'_i P_i z_i$. This implies $z_i \neq p(R_i)$. Without loss of generality, let $i \in S$. Then, $\sum_S p(R_k) \neq \omega$. Without loss of generality, let $\sum_S p(R_k) > \omega$. Then, by the definition of $UT^{\beta\sigma}$, there is $\lambda \in \mathbb{R}_+$ such that $z_i = \lambda = \min\{\lambda, p(R_i)\} < z'_i$.

Case 1: $\omega' \leq \omega$ and $\sum_S p(R'_k) \geq \omega'$

By the definition of $UT^{\beta\sigma}$, there is $\lambda' \in \mathbb{R}_+$ such that $z'_i = \min\{\lambda', p(R'_i)\} \leq \lambda'$. This implies $\lambda' > \lambda$. Since

$$\sum_S z'_k = \omega' \leq \omega = \sum_S z_k$$

there is $j \in S$ such that $z'_j < z_j$ which implies $z_j P_j z'_j$. Moreover, $j \in M$. To see this suppose $j \notin M$. Then, $R'_j = R_j$. This implies $z'_j = \min\{\lambda', p(R_j)\} \geq \min\{\lambda, p(R_j)\} = z_j$, a contradiction.

Case 2: $\omega' \leq \omega$ and $\sum_S p(R'_k) < \omega'$

Then there is $\theta \in \mathbb{R}_+$ such that $z'_i = \max\{\theta, p(R'_i)\} > z_i = \min\{\lambda, p(R_i)\}$. Since $\omega' \leq \omega$, there is $j \in S$ such that $z'_j < z_j$ which implies $z_j P_j z'_j$. We claim that $j \in M$. To see this suppose $j \notin M$. Then, $z'_j = \max\{\theta, p(R_j)\} \geq p(R_j)$ and $z_j = \min\{\lambda, p(R_j)\} \leq p(R_j)$. This implies $z'_j \geq z_j$, a contradiction.

Case 3: $\omega' > \omega$

Then, $\sum_B p(R'_k) \geq \omega'$. To see this, suppose $\sum_B p(R'_k) < \omega'$. But $\beta(B \cup S) \leq \omega < \omega'$ then contradicts

$$\omega' = median\{\beta(B \cup S), \sum_B p(R'_k), \sum_S p(R'_k)\}.$$

By the definition of $UT^{\beta\sigma}$, there are $\rho, \rho' \in \mathbb{R}_+$ such that $z_k = \max\{\rho, p(R_k)\}$ and $z'_k = \min\{\rho', p(R'_k)\}$ for each $k \in B$. Since $\omega = \sum_B z_k < \omega' = \sum_B z'_k$, there is $j \in B$ such that $z_j < z'_j$. Then $p(R_j) \leq z_j < z'_j$ which implies $z_j P_j z'_j$. We claim that $j \in M$. Suppose this is not the case. Then $R_j = R'_j$. So, $z_j = \max\{\rho, p(R_j)\} < z'_j = \min\{\rho', p(R_j)\}$, a contradiction. ∎

All Uniform Trade rules satisfy a core-like property which requires that no coalition of agents can make all its members better-off by reallocating the shares (assigned by a trade rule) of its members among themselves. On the other hand, properties such as *anonymity in total trade, strong separability, renegotiation proofness,* and *voluntary trade* are not satisfied by all *Uniform Trade rules.* In the next section, this is discussed in further detail.

4 Results

The following two lemmas are extensions of standard results by Ching (1994) on Sprumont's (1991) domain. They prove to be useful for our purposes too.

Lemma 3 *Let the trade rule F satisfy* Pareto optimality *and* strategy proofness. *Then for each $N \in \mathcal{N}$, $i \in N$, and $(R_i, R_{-i}), (R'_i, R_{-i}) \in \mathcal{R}^N$, if $p(R_i) \leq p(R'_i)$, then $F_i(R_i, R_{-i}) \leq F_i(R'_i, R_{-i})$.*

Proof: Suppose $F_i(R'_i, R_{-i}) < F_i(R_i, R_{-i})$. Then there are three possible cases. If

$$F_i(R'_i, R_{-i}) < F_i(R_i, R_{-i}) \leq p(R_i) \leq p(R'_i)$$

then with preferences R'_i, agent i has an incentive to declare R_i. If $p(R_i) \leq p(R'_i) \leq F_i(R_i, R_{-i})$, then let $K \in \{B, S\}$ be such that $i \in K$ and note that $p(R'_i) + \sum_{K \setminus \{i\}} p(R_k) \leq$

$\Omega_F(R_{N\setminus K}, R_K) \leq \sum_{N\setminus K} p(R_k)$. Thus by *Pareto optimality*, $p(R_i') \leq F_i(R_i', R_{-i})$ and we have

$$p(R_i) \leq p(R_i') \leq F_i(R_i', R_{-i}) < F_i(R_i, R_{-i})$$

and then with preferences R_i, agent i has an incentive to declare R_i'. Finally if $p(R_i) \leq F_i(R_i, R_{-i}) \leq p(R_i')$, then with preferences R_i', agent i has an incentive to declare R_i. Since in all cases, *strategy proofness* is violated, the supposition is false. ∎

It follows from Lemma 2 that if $(R_i, R_{-i}), (R_i', R_{-i}) \in \mathcal{R}^N$ is such that $p(R_i) = p(R_i')$, then $F_i(R_i, R_{-i}) = F_i(R_i', R_{-i})$.

Lemma 4 *Let the trade rule F satisfy* Pareto optimality *and* strategy proofness. *Let $N \in \mathcal{N}$, $i \in N$, and $(R_i, R_{-i}), (R_i', R_{-i}) \in \mathcal{R}^N$. If $p(R_i) < F_i(R_i, R_{-i})$ and $p(R_i') \leq F_i(R_i, R_{-i})$, then $F_i(R_i', R_{-i}) = F_i(R_i, R_{-i})$. Similarly if $p(R_i) > F_i(R_i, R_{-i})$ and $p(R_i') \geq F_i(R_i, R_{-i})$, then $F_i(R_i', R_{-i}) = F_i(R_i, R_{-i})$.*

Proof: To prove the first statement, suppose $p(R_i) < F_i(R_i, R_{-i})$, $p(R_i') \leq F_i(R_i, R_{-i})$, and $F_i(R_i', R_{-i}) \neq F_i(R_i, R_{-i})$. There are two possible cases. If $p(R_i) \leq p(R_i')$ then by Lemma 2, $F_i(R_i, R_{-i}) < F_i(R_i', R_{-i})$ and with preferences R_i', agent i has an incentive to declare R_i. Alternatively if $p(R_i') < p(R_i)$ then by Lemma 2, $F_i(R_i', R_{-i}) < F_i(R_i, R_{-i})$. Let $R_i'' \in \mathcal{R}$ be such that $p(R_i'') = p(R_i)$ and $0 P_i'' F_i(R_i, R_{-i})$. By Lemma 2, $F_i(R_i'', R_{-i}) = F_i(R_i, R_{-i})$. Thus $F_i(R_i', R_{-i}) < F_i(R_i'', R_{-i})$ and with preferences R_i'', agent i has an incentive to declare R_i'. Since in all cases, *strategy proofness* is violated, the supposition is false. The proof of the second statement is similar. ∎

Our main result characterizes *Uniform Trade rules*.

Theorem 5 *A trade rule F satisfies* Pareto optimality, strategy proofness, no-envy, *and* separability in total trade *if and only if it is a* Uniform Trade rule.

Proof: We already showed that the *Uniform Trade rules* satisfy these properties. Conversely, let F be a trade rule satisfying all properties. Let $N = B \cup S \in \mathcal{N}$.

Step 1. For each $K \in \{B, S\}$, $(R_{N\setminus K}, R_K), (R_{N\setminus K}, R_K') \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) < \sum_K p(R_k)$ and $\Omega_F(R_{N\setminus K}, R_K) < \sum_K p(R_k')$ implies $\Omega_F(R_{N\setminus K}, R_K') = \Omega_F(R_{N\setminus K}, R_K)$. Similarly, for each $K \in \{B, S\}$, $(R_{N\setminus K}, R_K), (R_{N\setminus K}, R_K') \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) > \sum_K p(R_k)$ and $\Omega_F(R_{N\setminus K}, R_K) > \sum_K p(R_k')$ implies $\Omega_F(R_{N\setminus K}, R_K') = \Omega_F(R_{N\setminus K}, R_K)$.

To prove the first statement, let $K \in \{B, S\}$, $(R_{N\setminus K}, R_K), (R_{N\setminus K}, R_K') \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) < \sum_K p(R_k)$ and $\Omega_F(R_{N\setminus K}, R_K) < \sum_K p(R_k')$.

Let $R^* \in \mathcal{R}$ be such that $p(R^*) = \frac{\sum_K p(R_k)}{|K|}$. By *separability in total trade*, $\Omega_F(R_{N\setminus K}, R_K^*) = \Omega_F(R_{N\setminus K}, R_K)$. By *Pareto optimality* and *no-envy*, for each $k \in K$, $F_k(R_{N\setminus K}, R_K^*) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$. Note that $\frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|} < p(R^*)$.

Now let $R^{**} \in \mathcal{R}$ be such that $p(R^{**}) = \frac{\sum_K p(R'_k)}{|K|}$ and $p(R^*)P^{**}\frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$. Since $\Omega_F(R_{N\setminus K}, R_K) < \sum_K p(R'_k)$, we have $\frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|} < p(R^{**})$.

Let $K = \{1, ..., n\}$. Now for each $i \in K$, we claim

$$F_K(R_{N\setminus K}, R^*_{\{i,...,n\}}, R^{**}_{\{1,...,i-1\}}) = F_K(R_{N\setminus K}, R^*_{\{i+1,...,n\}}, R^{**}_{\{1,...,i\}}).$$ [9]

To prove, note that $F_K(R_{N\setminus K}, R^*_K) = (\frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|})_{k\in K}$ and for $i \geq 2$, assume that the statement holds up to agent i. Thus for each $k \in K$,

$$F_k(R_{N\setminus K}, R^*_{\{i,...,n\}}, R^{**}_{\{1,...,i-1\}}) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|} < \min\{p(R^*), p(R^{**})\}.$$

Then by Lemma 3, $F_i(R_{N\setminus K}, R^*_{\{i+1,...,n\}}, R^{**}_{\{1,...,i\}}) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$. Let $j \in K \setminus \{i\}$. If $R_j = R^{**}$, then[10] by *no-envy* $F_j(R_{N\setminus K}, R^*_{\{i+1,...,n\}}, R^{**}_{\{1,...,i\}}) = F_i(R_{N\setminus K}, R^*_{\{i+1,...,n\}}, R^{**}_{\{1,...,i\}}) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$. Alternatively assume $R_j = R^*$. If $F_j(R_{N\setminus K}, R^*_{\{i+1,...,n\}}, R^{**}_{\{1,...,i\}}) < \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$, then j *envies* i and if $\frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|} < F_j(R_{N\setminus K}, R^*_{\{i+1,...,n\}}, R^{**}_{\{1,...,i\}})$, since by *Pareto optimality*, $F_j(R_{N\setminus K}, R^*_{\{i+1,...,n\}}, R^{**}_{\{1,...,i\}}) \leq p(R^*)$, we have $F_j(R_{N\setminus K}, R^*_{\{i+1,...,n\}}, R^{**}_{\{1,...,i\}})P^{**}\frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$, that is, i *envies* j. Thus

$$F_j(R_{N\setminus K}, R^*_{\{i+1,...,n\}}, R^{**}_{\{1,...,i\}}) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}.$$

By this claim we have, for each $i \in K$,

$$\Omega_F(R_{N\setminus K}, R^*_{\{i,...,n\}}, R^{**}_{\{1,...,i-1\}}) = \Omega_F(R_{N\setminus K}, R^*_{\{i+1,...,n\}}, R^{**}_{\{1,...,i\}}).$$

This implies $\Omega_F(R_{N\setminus K}, R^{**}_K) = \Omega_F(R_{N\setminus K}, R_K)$. Finally note that $\sum_K p(R'_k) = |K|\, p(R^{**})$. This, by *separability in total trade*, implies that $\Omega_F(R_{N\setminus K}, R'_K) = \Omega_F(R_{N\setminus K}, R_K)$.

The proof of the second statement of this step is similar.

Step 2. For each $(R_{N\setminus K}, R_K)$, $(R_{N\setminus K}, R'_K) \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) \leq \sum_K p(R_k)$ and $\sum_{N\setminus K} p(R_k) \leq \sum_K p(R'_k) \leq \Omega_F(R_{N\setminus K}, R_K)$ implies $\Omega_F(R_{N\setminus K}, R'_K) = \sum_K p(R'_k)$. Similarly, for each $(R_{N\setminus K}, R_K)$, $(R_{N\setminus K}, R'_K) \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) \geq \sum_K p(R_k)$ and $\sum_{N\setminus K} p(R_k) \geq \sum_K p(R'_k) \geq \Omega_F(R_{N\setminus K}, R_K)$ implies $\Omega_F(R_{N\setminus K}, R'_K) = \sum_K p(R'_k)$.

To prove the first statement, let $(R_{N\setminus K}, R_K)$, $(R_{N\setminus K}, R'_K) \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) \leq \sum_K p(R_k)$ and $\sum_{N\setminus K} p(R_k) \leq \sum_K p(R'_k) \leq \Omega_F(R_{N\setminus K}, R_K)$. Note that by *Pareto optimality* $\Omega_F(R_{N\setminus K}, R'_K) \leq \sum_K p(R'_k)$. Suppose $\Omega_F(R_{N\setminus K}, R'_K) < \sum_K p(R'_k)$. Then by Step 1, $\Omega_F(R_{N\setminus K}, R'_K) = \Omega_F(R_{N\setminus K}, R_K)$, a contradiction.

[9] With an abuse of notation, for $i = 1$, let $\{1, ..., i-1\} = \emptyset$ and for $i = n$, let $\{i+1, ..., n\} = \emptyset$.

[10] Note that we use R_j to denote the "generic" preference relation of agent j. On the other hand, R^{**} denotes a particular preference relation defined above.

The proof of the second statement of this step is similar.

Step 3. Determining the functions β and σ.

For $c \in \mathbb{R}_+$, let $R^c \in \mathcal{R}$ be such that $p(R^c) = c$ and let $R^c_{N'} = (R^c)_{i \in N'}$. Now for $d \in \mathbb{R}_+$, consider $(R^0_B, R^d_S) \in \mathcal{R}^{B \cup S}$ and

1. if there is $d^* \in \mathbb{R}_+$ such that $d^* |S| > \Omega_F(R^0_B, R^{d^*}_S)$, let $\beta(B \cup S) = \Omega_F(R^0_B, R^{d^*}_S)$,
2. if for each $d \in \mathbb{R}_+$, $d|S| = \Omega_F(R^0_B, R^d_S)$, let $\beta(B \cup S) = \infty$.

Similarly obtain $\sigma(B \cup S)$ by using the profiles $(R^{c^*}_B, R^0_S) \in \mathcal{R}^{B \cup S}$ for $c^* \in \mathbb{R}_+$. If no such c^* exists, set $\sigma(B \cup S) = \infty$.

Step 4. If $(R_B, R_S) \in \mathcal{R}^{B \cup S}$ satisfies $\sum_B p(R_k) \leq \sum_S p(R_k)$, then

$$\Omega_F(R_B, R_S) = median\{\beta(B \cup S), \sum_B p(R_k), \sum_S p(R_k)\}.$$

If $\sum_B p(R_k) = \sum_S p(R_k)$, the statement trivially holds. So let $\sum_B p(R_k) < \sum_S p(R_k)$.

First assume there is $d^* \in \mathbb{R}_+$ such that $d^* |S| > \Omega_F(R^0_B, R^{d^*}_S)$. Then by Step 3, $\beta(B \cup S) = \Omega_F(R^0_B, R^{d^*}_S)$.

There are three possible cases.

Case 1. $\sum_B p(R_k) < \beta(B \cup S) < \sum_S p(R_k)$.

Then since $0|B| < \beta(B \cup S) = \Omega_F(R^0_B, R^{d^*}_S) < d^* |S|$, applying Step 1 twice, we get $\Omega_F(R^0_B, R^{d^*}_S) = \Omega_F(R_B, R^{d^*}_S) = \Omega_F(R_B, R_S)$.

Case 2. $\beta(B \cup S) \leq \sum_B p(R_k) < \sum_S p(R_k)$.

Then since $0|B| \leq \beta(B \cup S) = \Omega_F(R^0_B, R^{d^*}_S) < d^* |S|$, applying Step 1 to S, we get $\Omega_F(R^0_B, R^{d^*}_S) = \Omega_F(R^0_B, R_S)$ and applying Step 2 to B, we get $\Omega_F(R_B, R_S) = \sum_B p(R_k)$.

Case 3. $\sum_B p(R_k) < \sum_S p(R_k) \leq \beta(B \cup S)$.

Then since $0|B| < \beta(B \cup S) = \Omega_F(R^0_B, R^{d^*}_S) < d^* |S|$, applying Step 1 to B, we get $\Omega_F(R^0_B, R^{d^*}_S) = \Omega_F(R_B, R^{d^*}_S)$ and applying Step 2 to S, we get $\Omega_F(R_B, R_S) = \sum_S p(R_k)$.

Next assume that for each $d \in \mathbb{R}_+$, $d|S| = \Omega_F(R^0_B, R^d_S)$. Then by Step 3, $\beta(B \cup S) = \infty$. Let $d > 0$ be such that $d|S| = \sum_S p(R_k)$. Then $\Omega_F(R^0_B, R^d_S) = \sum_S p(R_k) > 0$. Thus by Step 1, $\Omega_F(R_B, R^d_S) = \sum_S p(R_k)$. Finally by Step 2 $\Omega_F(R_B, R_S) = \sum_S p(R_k)$.

Since in all cases $\Omega_F(R_B, R_S) = median\{\beta(B \cup S), \sum_B p(R_k), \sum_S p(R_k)\}$, the proof is complete.

Step 5. If $(R_B, R_S) \in \mathcal{R}^{B\cup S}$ satisfies $\sum_B p(R_k) \geq \sum_S p(R_k)$, then

$$\Omega_F(R_B, R_S) = median\{\sigma(B \cup S), \sum_B p(R_k), \sum_S p(R_k)\}.$$

The proof is similar to that of Step 4.

Step 6. $F = UT^{\beta\sigma}$

Suppose $F_K(R_N) \neq UT^{\beta\sigma}_K(R_N)$ for some $R_N \in \mathcal{R}^N$ and $K \in \{B, S\}$. By steps 4 and 5, $\Omega_F(R_N) = \Omega_{UT^{\beta\sigma}}(R_N)$ and by our supposition, $\sum_K p(R_k) \neq \Omega_F(R_N)$.

First assume that $\sum_K p(R_k) > \Omega_F(R_N)$. Since $F_K(R_N) \neq UT^{\beta\sigma}_K(R_N)$, there is $i \in K$ such that

$$F_i(R_N) < UT^{\beta\sigma}_i(R_N) \leq p(R_i).$$

Let $R'_i \in \mathcal{R}$ be such that $p(R'_i) = p(R_i)$ and for each $x > F_i(R_B, R_S)$, $xP'_iF_i(R_B, R_S)$. By Lemma 2,

$$F_i(R'_i, R_{-i}) < UT^{\beta\sigma}_i(R'_i, R_{-i}) \leq p(R'_i).$$

Now since $\sum_K F_k(R_N) = \sum_K UT^{\beta\sigma}_k(R_N)$, there is $j \in K$ such that $UT^{\beta\sigma}_j(R'_i, R_{-i}) < F_j(R'_i, R_{-i})$. Thus $UT^{\beta\sigma}_j(R'_i, R_{-i}) < p(R_j)$ and by definition of $UT^{\beta\sigma}$, $UT^{\beta\sigma}_i(R'_i, R_{-i}) \leq UT^{\beta\sigma}_j(R'_i, R_{-i})$. Then $F_i(R'_i, R_{-i}) < F_j(R'_i, R_{-i})$ and with preferences R'_i, agent i envies agent j, a contradiction.

The proof of the second case where $\sum_K p(R_k) < \Omega_F(R_N)$ is similar. ∎

Note that the properties of Theorem 1 are logically independent. First, the simple rule which always chooses zero trade satisfies all properties but *Pareto optimality.* Second, the rule which always clears the short side of the market and rations the long side proportionally (that is, each agent gets a constant proportion of his peak) satisfies all properties but *strategy proofness.* Third, the rule which always clears the short side of the market and rations the long side by a priority order (according to which agents are served sequentially until the volume of trade is exhausted) satisfies all properties but *no-envy.* Finally, the following is an example of a rule that satisfies all properties but *separability in total trade.*[11] Let $N = \{1, 2, 3\}$ and $K = \{1, 2\}$. Let

$$\Omega_F(R_1, R_2, R_3) = median\,\{p(R_3), 2p(R_1), 2p(R_2)\}\,.$$

That is, given a market where agents 1 and 2 are on one side and Agent 3 is on the other side, the volume of trade is determined as a median of the three quantities above. Then let F determine the shares of agents 1 and 2 similar to the *Uniform trade rules* (see Equation 1). Finally, let F coincide with an arbitrary *Uniform Trade rule* for every $(B \cup S) \in \mathcal{N}$ with $|B \cup S| \neq 3$.

Next we analyze the implications of *in-group* and *between-group anonymity in total trade* on *Uniform Trade rules.*

[11]This rule is in fact *coalitional strategy proof.*

Corollary 6 *Let F be a trade rule that satisfies* Pareto optimality, strategy proofness, no-envy, *and* separability in total trade. *Then*
(i) F satisfies in-group anonymity in total trade *if and only if it is a* Uniform Trade rule $UT^{\beta\sigma}$ *where for each* $(B \cup S)$, $(B' \cup S') \in \mathcal{N}$ *such that* $|B| = |B'|$ *and* $|S| = |S'|$, $\beta(B \cup S) = \beta(B' \cup S')$ *and* $\sigma(B \cup S) = \sigma(B' \cup S')$,
(ii) F satisfies between-group anonymity in total trade *if and only if it is a* Uniform Trade rule $UT^{\beta\sigma}$ *where for each* $(B \cup S)$, $(B' \cup S') \in \mathcal{N}$ *such that* $|B| = |S'|$ *and* $|S| = |B'|$, $\beta(B \cup S) = \sigma(B' \cup S')$ *and* $\sigma(B \cup S) = \beta(B' \cup S')$.

Proof: By Theorem 1, F is a *Uniform Trade rule* $UT^{\beta\sigma}$. The proof of the first statement is trivial and omitted. For the second statement, first assume that $UT^{\beta\sigma}$ is between-group anonymous in total trade. Let $(B \cup S)$, $(B' \cup S') \in \mathcal{N}$ be such that $|B| = |S'|$ and $|S| = |B'|$. Let $(R_B, R_S) \in \mathcal{R}^{B \cup S}$ and $(R'_{B'}, R'_{S'}) \in \mathcal{R}^{B' \cup S'}$ be such that $R_B = R'_{S'}$, $R_S = R'_{B'}$, and $\sum_B p(R_k) < \beta(B \cup S) < \sum_S p(R_k)$. Then $\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \beta(B \cup S)$ and $\Omega_{UT^{\beta\sigma}}(R'_{B'}, R'_{S'}) = \sigma(B' \cup S')$. By *between-group anonymity in total trade* $\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \Omega_{UT^{\beta\sigma}}(R'_{B'}, R'_{S'})$. Thus $\beta(B \cup S) = \sigma(B' \cup S')$. One similarly obtains $\sigma(B \cup S) = \beta(B' \cup S')$.

Now assume that $UT^{\beta\sigma}$ satisfies the given property. Let $(R_B, R_S) \in \mathcal{R}^{B \cup S}$ and $\phi \in \Phi$. Without loss of generality assume $\sum_B p(R_k) \leq \sum_S p(R_k)$. Then,

$$\Omega_{UT^{\beta\sigma}}(R_B, R_S) = median\{\beta(B \cup S), \sum_B p(R_k), \sum_S p(R_k)\}.$$

By the given property $\beta(B \cup S) = \sigma(\phi(S) \cup \phi(B))$. Also, $\sum_B p(R_k) = \sum_{\phi(B)} p(R_k^\phi)$ and $\sum_S p(R_k) = \sum_{\phi(S)} p(R_k^\phi)$. Thus $\sum_{\phi(B)} p(R_k^\phi) \leq \sum_{\phi(S)} p(R_k^\phi)$ and

$$\begin{aligned} \Omega_{UT^{\beta\sigma}}(R^\phi_{\phi(S)}, R^\phi_{\phi(B)}) &= median\{\sigma(\phi(S) \cup \phi(B)), \sum_{\phi(S)} p(R_k^\phi), \sum_{\phi(B)} p(R_k^\phi)\} \\ &= \Omega_{UT^{\beta\sigma}}(R_B, R_S). \end{aligned}$$

∎

That is, *in-group anonymity in total trade* essentially makes β and σ only dependent on the number of buyers and sellers whereas *between-group anonymity in total trade* requires the treatment of buyers in a $k-$buyer, $l-$seller problem to be the same as the treatment of sellers in an $l-$buyer, $k-$seller problem.

Next we analyze the implications of *strong separability in total trade.*

Proposition 7 *If a trade rule F satisfies* Pareto optimality, strategy proofness, *and* strong separability in total trade, *then there is* $c_\beta, c_\sigma \in \mathbb{R}_+ \cup \{\infty\}$ *such that for each* $(R_B, R_S) \in \mathcal{M}$

$$\Omega_F(R_B, R_S) = \begin{cases} median\{c_\beta, \sum_B p(R_i), \sum_S p(R_i)\} & \text{if} \quad \sum_B p(R_i) \leq \sum_S p(R_i)\}, \\ median\{c_\sigma, \sum_B p(R_i), \sum_S p(R_i)\} & \text{if} \quad \sum_B p(R_i) \geq \sum_S p(R_i)\}. \end{cases}$$

Proof: For $x \in \mathbb{R}_+$, let $R^x \in \mathcal{R}$ be such that $p(R^x) = x$. Fix $b \in \mathcal{B}$ and $s \in \mathcal{S}$.

If there is $x^* \in \mathbb{R}_+$ such that $\Omega_F(R_b^0, R_s^{x^*}) < x^*$, let $c_\beta = \Omega_F(R_b^0, R_s^{x^*})$. Otherwise, let $c_\beta = \infty$. Similarly if there is $y^* \in \mathbb{R}_+$ such that $\Omega_F(R_b^{y^*}, R_s^0) < y^*$, let $c_\sigma = \Omega_F(R_b^{y^*}, R_s^0)$; otherwise, let $c_\sigma = \infty$.

Step 1. For each $(R_b, R_s) \in \mathcal{R}^{\{b,s\}}$, if $p(R_b) \leq p(R_s)$, then $\Omega_F(R_b, R_s) = median\{c_\beta, p(R_b), p(R_s)\}$ and if $p(R_s) \leq p(R_b)$, then $\Omega_F(R_b, R_s) = median\{c_\sigma, p(R_b), p(R_s)\}$.

Let $(R_b, R_s) \in \mathcal{R}^{\{b,s\}}$ and assume that $p(R_b) \leq p(R_s)$ (the proof for the alternative case is similar). Note that $\Omega_F(R_b, R_s) = F_b(R_b, R_s) = F_s(R_b, R_s)$.

Claim 1: If $p(R_b) < c_\beta < p(R_s)$, then $\Omega_F(R_b, R_s) = c_\beta = median\{c_\beta, p(R_b), p(R_s)\}$.

Too see this note that $p(R_b^0) < c_\beta = F_b(R_b^0, R_s^{x^*}) = F_s(R_b^0, R_s^{x^*}) < p(R_s^{x^*})$. Thus by Lemma 3, $c_\beta = F_b(R_b^0, R_s^{x^*}) = F_b(R_b, R_s^{x^*})$ and $F_s(R_b, R_s^{x^*}) = F_s(R_b, R_s) = c_\beta$. This implies $\Omega_F(R_b, R_s) = c_\beta$.

Claim 2: If $c_\beta \leq p(R_b) \leq p(R_s)$, then $\Omega_F(R_b, R_s) = p(R_b) = median\{c_\beta, p(R_b), p(R_s)\}$.

If $p(R_b) = p(R_s)$, the statement trivially holds. So let $p(R_b) < p(R_s)$. Note that by Lemma 3, $\Omega_F(R_b^0, R_s^{x^*}) = \Omega_F(R_b^0, R_s) = c_\beta$. Suppose $\Omega_F(R_b, R_s) > p(R_b)$. Note that $\Omega_F(R_b, R_s) = F_b(R_b, R_s)$. Let $R'_b \in \mathcal{R}$ be such that $p(R'_b) = p(R_b)$ and $c_\beta P'_b F_b(R_b, R_s)$. By Lemma 2, $F_b(R'_b, R_s) = F_b(R_b, R_s)$. Thus $c_\beta = F(R_b^0, R_s) P'_b F_b(R'_b, R_s)$ violates *strategy proofness*. This implies $\Omega_F(R_b, R_s) = p(R_b)$.

Claim 3: If $p(R_b) \leq p(R_s) \leq c_\beta$, then $\Omega_F(R_b, R_s) = p(R_s) = median\{c_\beta, p(R_b), p(R_s)\}$.

The proof Claim 3 is similar to that of Claim 2.

Step 2. For each $B \cup S \in \mathcal{N}$ and $(R_B, R_S) \in \mathcal{R}^N$, if $\sum_B p(R_k) \leq \sum_S p(R_k)$, then $\Omega_F(R_b, R_s) = median\{c_\beta, \sum_B p(R_k), \sum_S p(R_k)\}$ and if $p(R_s) \leq p(R_b)$, then

$$\Omega_F(R_b, R_s) = median\{c_\sigma, \sum_B p(R_k), \sum_B p(R_k)\}.$$

Assume that $\sum_B p(R_k) \leq \sum_S p(R_k)$ (the proof for the alternative case is similar). Let $R_b^*, R_s^* \in \mathcal{R}$ be such that $p(R_b^*) = \sum_B p(R_k)$ and $p(R_s^*) = \sum_S p(R_k)$. By *strong separability in total trade*, $\Omega_F(R_B, R_S) = \Omega_F(R_b^*, R_s^*)$ and by Step 1, $\Omega_F(R_b^*, R_s^*) = median\{c_\beta, p(R_b^*), p(R_s^*)\}$. Combining the two statements gives the desired conclusion.

∎

The following remark trivially follows from Proposition 2 and Theorem 1.

Remark 1 A trade rule F satisfies *Pareto optimality, strategy proofness, no-envy,* and *strong separability in total trade* if and only if it is a Uniform Trade rule $UT^{\beta\sigma}$ where there is $c_\beta, c_\sigma \in \mathbb{R}_+ \cup \{\infty\}$ such that for all $(B \cup S) \in N$, $\beta(B \cup S) = c_\beta$ and $\sigma(B \cup S) = c_\sigma$.

Strongly separable Uniform Trade rules treat the buyers (respectively, the sellers) the same way in every problem. Note that thus *strong separability* not only implies *separability* but also *in-group anonymity in total trade. Strongly separable* rules that satisfy *between-group anonymity in total trade* treat all problems the same way and make no difference between buyers and sellers. This observation trivially follows from Corollary 1 and Proposition 2.

Remark 2 A trade rule F satisfies *Pareto optimality, strategy proofness, no-envy, strong separability in total trade,* and *between-group anonymity in total trade* if and only if it is a Uniform Trade rule $UT^{\beta\sigma}$ such that for some $c \in \mathbb{R}_+ \cup \{\infty\}$, $\beta(B \cup S) = \sigma(B \cup S) = c$ for all $(B \cup S) \in N$.

Next, we analyze the implications of *renegotiation proofness.*

Corollary 8 *A trade rule F satisfies* Pareto optimality, strategy proofness, no-envy, separability in total trade, *and* renegotiation proofness *if and only if it is a* Uniform Trade rule $UT^{\beta\sigma}$ *where for each* $(B \cup S) \in \mathcal{N}$, $|B| < |S|$ *implies* $\beta(B \cup S) \in \{0, \infty\}$ *and* $|S| < |B|$ *implies* $\sigma(B \cup S) \in \{0, \infty\}$.

Proof: By Theorem 1, F is a *Uniform Trade rule* $UT^{\beta\sigma}$. For the only if part suppose there is $(B \cup S) \in \mathcal{N}$ such that $|B| < |S|$ and $\beta(B \cup S) \in (0, \infty)$. Let $R^c \in \mathcal{R}$ be such that $p(R^c) = c \in (\frac{\beta(B \cup S)}{|S|}, \frac{\beta(B \cup S)}{|B|})$. Let $(R_B, R_S) \in \mathcal{R}^{B \cup S}$ be such that for each $i \in B \cup S$, $R_i = R^c$. Then,

$$\Omega_{UT^{\beta\sigma}}(R_B, R_S) = median\{\beta(B \cup S), c\,|B|, c\,|S|\} = \beta(B \cup S).$$

By *no-envy*, for each $i \in B$, $UT_i^{\beta\sigma}(R_B, R_S) = \frac{\beta(B \cup S)}{|B|}$ and for each $j \in S$, $UT_j^{m}(R_B, R_S) = \frac{\beta(B \cup S)}{|S|}$. This implies, there is $i \in B$ and $j \in S$ such that $cP_iUT_i^{\beta\sigma}(R_B, R_S)$ and $cP_jUT_j^{\beta\sigma}(R_B, R_S)$ and therefore that $UT^{\beta\sigma}$ is not renegotiation proof. Thus, $\beta(B \cup S) = 0$ or $\beta(B \cup S) = \infty$. A similar argument applies for the case $|S| < |B|$ and $\sigma(B \cup S)$.

The if part is as follows. If $(B \cup S) \in \mathcal{N}$ is such that $\beta(B \cup S), \sigma(B \cup S) \in \{0, \infty\}$, then for each $(R_B, R_S) \in \mathcal{R}^{B \cup S}$, there is $K \in \{B, S\}$ such that $\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \sum_{i \in K} p(R_i)$ and thus, $UT_i^{\beta\sigma}(R_B, R_S) = p(R_i)$ for each $i \in K$. In this case, no member of K is better-off by joining a blocking pair and therefore, renegotiation is not possible.

Next let $(B \cup S) \in \mathcal{N}$ be such that $|B| \geq |S|$ and $\beta(B \cup S) \in (0, \infty)$. Let $(R_B, R_S) \in \mathcal{R}^{B \cup S}$ be such that $\sum_{i \in B} p(R_i) < \beta(B \cup S) < \sum_{i \in S} p(R_i)$ (otherwise, one group gets its peak and has no incentive to renegotiate). Then, $\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \beta(B \cup S)$ and for each $i \in B$, $UT_i^{\beta\sigma}(R_B, R_S) = \max\{\rho, p(R_i)\}$ where $\rho \in \mathbb{R}_+$ satisfies $\sum_B \max\{\rho, p(R_k)\} = \beta(B \cup S)$. Similarly for each $j \in S$, $UT_j^{\beta\sigma}(R_B, R_S) = \min\{\lambda, p(R_j)\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum_S \min\{\lambda, p(R_k)\} = \beta(B \cup S)$. This implies $\lambda \geq \frac{\beta(B\cup S)}{|S|}$, $\rho \leq \frac{\beta(B\cup S)}{|B|}$ and thus, $\rho \leq \lambda$. Now suppose there is a blocking pair $(i, j) \in B \times S$. Since neither i nor j can get his peak,

$$p(R_i) < UT_i^{\beta\sigma}(R_B, R_S) = \rho \leq \lambda = UT_j^{\beta\sigma}(R_B, R_S) < p(R_j).$$

For both agents to be strictly better off at some $r \in \mathbb{R}_+$, we must have $r < UT_i^{\beta\sigma}(R_B, R_S)$ and $r > UT_j^{\beta\sigma}(R_B, R_S)$. This implies $r < UT_i^{\beta\sigma}(R_B, R_S) \leq UT_j^{\beta\sigma}(R_B, R_S) < r$, a contradiction. Thus $UT^{\beta\sigma}$ is renegotiation proof. ∎

It is interesting to observe that *renegotiation proofness* has no implications on problems with an equal number of buyers and sellers while its implications on the remaining problems are quite strong. Let us also note that a stronger version of *renegotiation proofness* which allows blocking pairs where one agent is indifferent (while, of course the other is strictly better-off) is violated by all *Uniform Trade rules.* On the other hand, strenghtening *renegotiation proofness* by allowing larger (than two-agent) coalitions to form has no effect on the conclusion of Corollary 2.[12]

We next analyze the implications of *voluntary trade.*

Proposition 9 *If a trade rule F satisfies* Pareto optimality, strategy proofness, *and* voluntary trade*, then for each $(R_B, R_S) \in \mathcal{M}$, $\Omega_F(R_B, R_S) = \min\{\sum_B p(R_k), \sum_S p(R_k)\}$.*

Proof: Let $(R_B, R_S) \in \mathcal{M}$ and without loss of generality assume that $\sum_B p(R_k) \leq \sum_S p(R_k)$. By *Pareto optimality,* $\sum_B p(R_k) \leq \Omega_F(R_B, R_S) \leq \sum_S p(R_k)$. Suppose $\sum_B p(R_k) < \Omega_F(R_B, R_S)$. Then there is $i \in B$ such that $p(R_i) < F_i(R_B, R_S)$. Let $R_i' \in \mathcal{R}$ be such that $p(R_i') = p(R_i)$ and $0P_i'F_i(R_B, R_S)$. By Lemma 2, $F_i(R_{B\setminus i}, R_i', R_S) = F_i(R_B, R_S)$ and thus $0P_i'F_i(R_{B\setminus i}, R_i', R_S)$, violating *voluntary trade.* ∎

The following remark follows trivially from Proposition 3 and Theorem 1.

Remark 3 A trade rule F satisfies *Pareto optimality, strategy proofness, no-envy,* and *voluntary trade* if and only if it is a *Uniform Trade rule* $UT^{\beta\sigma}$such that $\beta(B \cup S) = \sigma(B \cup S) = 0$ for all $(B \cup S) \in \mathcal{N}$.

[12]Formally, all *renegotiation proof* Uniform trade rules satisfy the following property: a trade rule F is *strong renegotiation proof* if for each $(R_B, R_S) \in \mathcal{M}$ there is no $S' \subset S$, $B' \subset B$, and $z \in Z(B', S')$ such that $z_i P_i F_i(R_B, R_S)$ for each $i \in B' \cup S'$.

5 Conclusions

In this section, we list some open questions. First, our model is motivated by a production economy. We pick a market there that is in inequilibrium, isolate it from other related markets, and then produce a trade vector for it. In doing this, our considerations are at the micro level. That is, our properties focus on a trade rule's performance at that particular market and not on its implications on say, related markets or on the overall competitiveness of the affected firms. In short, we do not analyze the implications of a trade rule on the overall economy. Such an analysis seems to be an important follow-up to our work. Second, in this paper we do not consider population changes. Implications of properties such as consistency or population monotonicity (and in fact, good formulations of these ideas on this domain) remains an open question. Third, we analyze rules that are separable in total trade. We believe *separability* to be an intuitively desirable property and we obtain a very large class of rules that satisfy it. Nevertheless, there might be other interesting rules that violate this property.

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