

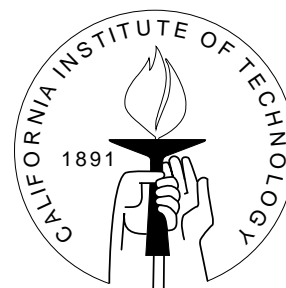
DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

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## ELECTORAL COMPETITION WITH ENTRY

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# Electoral Competition with Entry\*

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## Abstract

By extending the established theoretical models of electoral competition with entry (eg. Palfrey (1984)) to incorporate simultaneous competition for multiple districts I produce a unique two party equilibrium under plurality rule with non-centrist party platforms. This equilibrium also precludes entry of additional parties. This result is used to provide a domain for which Duverger's Law could be expected to apply. I also present new results under the run-off rule for both the single district and multiple district frameworks. In the single district case I find that for the run-off rule the model is more consistent with empirical observation than it is for the plurality rule, but that this performance is reversed when we consider multiple districts. The paper also sheds some light on how the different levels of elections in the U.S. and other systems relate to each other.

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# 1 Introduction

It is an overwhelming, and often cited, empirical fact that plurality rule elections involving single member districts typically produce a two party competition structure, and that these two parties choose non-centrist platforms.<sup>1</sup> Simultaneously, theoretical models of electoral competition under a plurality rule lead to either platform convergence or the entry of more than two parties.<sup>2</sup> It is important to resolve this paradox if we are to understand party behavior in the process of electoral competition.

I propose to extend the theoretical models of electoral competition with entry by assuming that the parties are competing in multiple single member districts simultaneously and that they are constrained to establishing a single party platform for all districts. I then show that as long as the distributions of voters in these districts are not identical an equilibrium under plurality rule which involves only two parties can exist in general conditions and that it is non-centrist. Limits on the dispersion of the voter distribution across districts can be calculated to establish boundaries within which Duverger's Law could be expected to apply (that is, when the equilibrium involves at most two parties). Such a restricted domain for Duverger's Law is appropriate because, as mentioned, empirically the law does not hold everywhere. The performance of the model under the run-off rule will also be established.

## 2 The Basic Model

In this section I will be considering a model of electoral competition with entry in a single district. There will be two incumbent parties who choose their platforms simultaneously. A potential entrant then makes an entry decision, and if he chooses to enter he selects a platform position. The entered parties then engage in the election. This is identical to Palfrey (1984), except that the entrant may be allowed to choose whether to enter the election at all. This structure is actually more general than it seems. It will be seen that one party can never prevent the entry of a second party and so if we allow them to choose sequentially then the first player will act as if there is a second player anyway and choose the same platform as it would have if the parties had chosen simultaneously.<sup>3,4</sup> Also, if the first potential entrant chooses to stay out then we can say that all potential entrants (who similarly consider their entry decision in isolation) would stay out and so

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<sup>1</sup>For support of Duverger's Law see the references in Riker's (1982) survey. In Riker's view, "There are indeed counterexamples [to the law], but not, I believe, definitive ones..." (Riker 1982, p.760). For support of the non-convergence assertion, at least for the case of the U.S., see Alesina and Rosenthal (1995, chapter 2).

<sup>2</sup>For models predicting platform convergence see, for example, Downs (1957), or Feddersen, Sened and Wright (1990). Examples of models involving the entry of more than two parties are Palfrey (1984), and Cox (1987).

<sup>3</sup>This was shown formally in Weber (1992).

<sup>4</sup>In the single district case we would need to retain the assumption that indifferent voters would randomize over the first two parties. In the multiple districts case entry of a second party cannot be deterred, even without this assumption, as long as the distribution of districts is not degenerate.

we have solved for a model involving an arbitrary number of parties. I will denote the two incumbents as  $I_1$  and  $I_2$ , and the entrant as  $E$ .

The issue space is the real line,  $\mathfrak{R}$ . There is a continuum of voters with symmetric, single peaked preferences over the issue space. The voters ideal points are distributed according to a non-degenerate cumulative distribution function,  $F$ , defined on  $\mathfrak{R}$ . The associated pdf is denoted  $f$ .  $F$  and  $f$  have the following properties:

**Assumption 1** If  $F(\alpha) > 0$ , and  $\alpha < 0$ , then  $F$  is strictly increasing on  $(\alpha, -\alpha)$ .

**Assumption 2**  $F$  is continuous and twice differentiable on  $\mathfrak{R}$ .

**Assumption 3**  $F(x) = 1 - F(x) \forall x \in \mathfrak{R}$ .

**Assumption 4**  $f'(x) \geq 0 \forall x \leq 0$ , and  $f'(x) \leq 0 \forall x \geq 0$ .

These assumptions specify that the distribution of ideal points for voters is symmetric about zero, and that the mass at any point is at least as great as at any point further from zero. This requires  $f$  to be quasi-concave. It can be seen that the uniform distribution is one boundary of such distributions. Assumption 1 ensures that there are no gaps in the distribution but without assuming that voter ideal points span all of  $\mathfrak{R}$  (that is, voter ideal points can be contained in a bounded interval, for example  $[-1,1]$ ).

Voters are assumed to be sincere and so vote for the party closest to their ideal point. I will further assume that if a voter is indifferent between the two incumbents then they randomize, but if they are indifferent between an incumbent and the entrant then they vote for the incumbent.<sup>5,6</sup> This assumption prevents entrants from wanting to locate on top of an incumbent. It can be defended simply, by claiming that voters have a preference for established parties if all else is the same. Any ties in the election are then decided randomly. Denote voter  $i$ 's ideal point  $v_i$  and, in an abuse of notation, let  $I_1, I_2$  and  $E$  represent the parties electoral platforms.

**Assumption 5** If  $|v_i - E| < |v_i - I_j|$  for  $j = 1, 2$  then  $vote(i) = E$ . Otherwise,  $vote(i) = I_j$  if  $|v_i - I_j| < |v_i - I_k|$  where  $j, k = 1, 2$  and  $j \neq k$ . If  $|v_i - I_1| = |v_i - I_2|$  then  $prob[vote(i) = I_j] = \frac{1}{2}$  for  $j = 1, 2$ .

It should be noted here that this assumption does not place any restrictions on the voter's utility function other than that utility is decreasing in the distance from his

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<sup>5</sup>This assumption is not needed for any of the plurality results. In fact, it was not made by Palfrey (1984). However, it is crucial to the run-off results, as otherwise entry prevention would not be possible in any district (the entrant could locate on top of either incumbent and obtain a positive probability of victory). I make the assumption for all models in order to facilitate comparisons between the two electoral rules.

<sup>6</sup>Alternatively, we could assume that ties in the overall election between an incumbent and the entrant are decided in favor of the incumbent, and that ties between incumbents are decided randomly.

ideal point. More specifically, a quadratic loss utility function is allowable with this assumption.

Parties are free to locate at any point in the policy space,  $\mathfrak{R}$ . I will assume that parties have lexicographic preferences with probability of victory on the primary dimension and vote share on a second dimension.<sup>7</sup> So if a party has a set of points which maximize its probability of winning then it chooses the point in this set that maximizes its vote share. If there is more than one point that maximizes a party's utility then it is assumed that the party randomizes equally over these points.<sup>8</sup>

A more substantial problem is that there may not exist a vote maximizing choice for the entrant. This technicality arises when  $E$  attempts to maximize his vote share over the set of points that maximizes his probability of winning. The probability of winning for  $E$  can only take on a finite set of values (as we have only three parties and voting is deterministic) and so a set of maximizers over this dimension can always be found. To deal with this existence problem I shall use the limit equilibrium concept introduced by Palfrey (1984). I shall assume that if a maximum doesn't exist then the entrant 'almost' maximizes his vote share when choosing from the set of points which maximize his probability of winning. A perturbed game is defined for each  $\varepsilon$ , where  $\varepsilon$  is how close  $E$  comes to maximizing his vote share. An equilibrium is then defined as any pair of strategies for  $I_1$  and  $I_2$  which are best responses to each other for an infinite sequence of the perturbed games, with the perturbation approaching zero in the limit.

Letting  $W$  denote the winner of the election, the set of points that maximize the entrant's probability of victory is defined as follows.

$$X(I_1, I_2) = \arg \max_{x \in \mathfrak{R}} \{prob(W = E | E = x) | I_1, I_2\}$$

Letting  $V_E$  denote the entrant's vote share, the set of points that  $E$  equally randomizes over, for a given  $\varepsilon$ , is given by  $C_E^\varepsilon$ , where,

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<sup>7</sup>Once again, the plurality results would not change if instead we assumed that parties simply vote maximize. This was the approach of Palfrey (1984). However, when considering the entrant's decision under the run-off rule vote maximization and the maximization of probability of victory do not necessarily coincide. As the probability of victory dictates the entry decision of this party it would then seem natural to assume that this rule also dictates the location decision. As above, the assumption is made for both models in order to facilitate comparison.

<sup>8</sup>We can refer to a party's utility level as even though they have lexicographic preferences their preferences are representable by a utility function. This is because the first dimension of preferences, probability of victory, can take on only a finite number of values  $(0, \frac{1}{3}, \frac{1}{2}, 1)$ , and the second dimension, vote share, can be mapped into the interval  $[0,1]$ . An example of such an utility function is given by,

$$U = \left\{ \begin{array}{ll} V & \text{if } P = 0 \\ 1 + V & \text{if } P = \frac{1}{3} \\ 2 + V & \text{if } P = \frac{1}{2} \\ 3 + V & \text{if } P = 1 \end{array} \right\}$$

$$C_E^\varepsilon(I_1, I_2) = \{E \in X(I_1, I_2) | V_E(I_1, I_2, E) > V_E(I_1, I_2, y) - \varepsilon, \forall y \in X(I_1, I_2)\}$$

Anticipating this entry decision the expected utility for the incumbents, given their own locations, is the expectation over  $C_E^\varepsilon(I_1, I_2)$ . Denote their expected utilities by  $U_1^\varepsilon(I_1, I_2)$  and  $U_2^\varepsilon(I_1, I_2)$ .

**Definition 1** [Palfrey (1984)] A pair of locations,  $\{I_1, I_2\}$ , is a limit equilibrium if,

- (a) for every  $y \neq I_1$ , there is a number  $\varepsilon(y)$ , such that for all  $\varepsilon \in (0, \varepsilon(y))$ ,  $U_1^\varepsilon(y, I_2) < U_1^\varepsilon(I_1, I_2)$ . And,
- (b) for every  $w \neq I_2$ , there is a number  $\varepsilon(w)$ , such that for all  $\varepsilon \in (0, \varepsilon(w))$ ,  $U_2^\varepsilon(I_1, w) < U_2^\varepsilon(I_1, I_2)$ .

It is possible that if  $X(I_1, I_2)$  is not a singleton then  $E$  will randomize over one or several intervals. The entrant will locate, if  $I_1 \leq I_2$ , in some subset of the intervals  $(I_1 - \delta, I_1)$ ,  $(I_2, I_2 + \delta)$  and  $(x - \phi, x + \phi)$ , where  $x \in (I_1, I_2)$  and  $\delta$  and  $\phi$  are determined from  $\varepsilon$ . As  $\varepsilon \rightarrow 0$  these intervals typically get smaller and  $\delta \rightarrow 0$  and  $\phi \rightarrow 0$ . We shall denote these types of intervals, as represented in this example, by  $I_1^-, I_2^+$  and  $x^{-+}$ , respectively.

I will be considering the equilibria under two voting rules, and two different assumptions on entrant behavior. The voting rules will be plurality and run-off. Under plurality the party that gains the most votes, regardless of whether this constitutes a majority, wins the election. Under run-off the party with the smallest number of votes is eliminated from the ballot and the remaining parties compete again with the same platforms (effectively preferences on votes for the eliminated candidate are distributed to the remaining candidates). When we get down to two remaining parties it is the one with a majority that wins the election. This process can be carried out with only one ballot and voters ranking the candidates, or in a series of ballots. In this model of full information and sincere voting the two techniques are equivalent. It will be seen that the two voting rules produce vastly different equilibria.

Under the run-off rule I will assume that from the set of points that maximize a parties probability of winning, the party will choose one of those that (almost) maximizes its primary vote share. The primary vote share for a party is the proportion of voters whose first preference is that party. This objective was chosen as, at least in Australia, the results from run-off elections report primary vote levels and which party is the winner. They do not report which parties survived the rounds. This run-off information can be easily kept out of public view if, like Australia, the single ballot technique in which voters rank the candidates is employed. Thus, as we would expect parties to be aiming for public prominence then maximizing their primary vote share would seem the most natural objective.

The assumptions on entrant behavior revolve around whether the party would enter even if it knew it wasn't going to win. In the first treatment I will assume that one entrant

(and only one entrant) will enter no matter what, even if its probability of winning is zero. This is the assumption used by Palfrey (1984). The second treatment will assume that the potential entrant will enter only if it has a strictly positive probability of victory. This assumption is used by Feddersen, Sened and Wright (1990). It can be justified in many ways, such as through a cost of entry variable. It will be seen that these two alternative assumptions also produce vastly different results.

As the parties have no ideological motivation in the selection of their platforms it is obvious that any equilibrium found will point to another equilibrium in which the two incumbent parties simply switch positions. Any pair of such equilibria will be considered to be the same, and so constitute just one equilibrium.

All proofs have been relegated to the appendix. Here I will just present the results and an intuitive explanation. The equilibria themselves are very intuitive, the complication is in proving that they are unique.

Before I present the equilibria I will present some intuition about the results.

**Run-off** If the incumbents do not locate symmetrically then the entrant will locate just outside the one closest to the center, thereby trapping this party in the middle and eliminating it in the first round. By choosing close enough to this incumbent the entrant will then be closer to the center than the other incumbent and so win the second stage runoff. This is assuming that the gap in the middle isn't too big. If this is the case then the entrant can locate just inside the incumbent party closest to the center and squeeze it on the outside and then win the run off with the incumbent on the other side. So to prevent the entrant winning, the incumbents will locate symmetrically and not too far from the center.<sup>9</sup> So incumbents do not have incentive to move from a symmetric location pair as this will incite the entrant to enter and win.

**Plurality** If the incumbents are on the same side of the center then the entrant can locate in the middle and win the election with a majority. So this can't constitute an equilibrium, and the incumbents must locate on opposite sides of the center (the median voter). They can't locate too far apart either, as then the entrant could locate between them and win the election. Likewise if the incumbents are located too asymmetrically around the center (eg.  $I_1 \approx 0$  and  $I_2 \gg -I_1$ , then an entrant with  $E = I_1^-$  would win the election). If these requirements are not violated then an entrant who locates on the outside of an incumbent steals all of its votes from this incumbent, but the other incumbent still has too many votes to enable its defeat

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<sup>9</sup>For some distributions there may exist unique asymmetric incumbent platforms that preclude entry. However, these will not constitute equilibria as the widest party will always lose and so will have incentive to deviate towards the center. These points require one incumbent to be relatively far from the center. Consequently these points cannot be reached by a single profitable deviation if the incumbents are close enough to the center. Thus, if the incumbents choose symmetric positions close enough to the center they will be in equilibrium. The limit of this dispersion will be seen in the characterization of the equilibria in the next section.

as well. The entrant can't simultaneously punish both incumbents sufficiently. If the entrant has positive cost then he will never enter and so the incumbents have a two party game and their unbridled incentive to move inwards leads to convergence. This intuition is the basis of the convergence result in Feddersen, Sened and Wright (1990). If the entrant will enter no matter what then by moving to the center an incumbent provides more space on its flank and encourages the entrant to enter there, which is bad for that incumbent. Thus we have a countering force to convergence and so, potentially, a non-centrist equilibrium. This is the intuition of the non-centrist equilibrium result of Palfrey (1984).

### 3 Results I

#### 3.1 Enter no matter what

##### 3.1.1 Run-off

- (a) *There exists an infinite number of equilibria,  $\{I_1, I_2\} = \{x, -x\} \forall x \in [W^*, 0]$ , where  $W^*$  solves  $1 - 2F(\frac{W^*}{2}) = F(W^*)$ . If  $x = W^*$  then  $E \in \{I_1^-, I_2^+, 0^{-+}\}$  for all  $F$ .<sup>10</sup> If  $x \neq W^*$  then  $E \in \{I_1^-, I_2^+, \}$ . For all equilibria,  $\text{prob}(W = I_1) = \text{prob}(W = I_2) = \frac{1}{2}, \text{prob}(W = E) = 0$ .*
- (b) *Depending on  $F$ , there may exist additional equilibria which satisfy the following necessary, but not sufficient conditions,  $\{I_1, I_2\} = \{y, -y\}$  where  $y \in [F^{-1}(\frac{1}{4}), W^*)$  and  $F(\frac{y}{2}) \geq \frac{1}{3}$ .  $E \in 0^{-+}$  for all such equilibria.  $\text{prob}(W = I_1) = \text{prob}(W = I_2) = \frac{1}{2}, \text{prob}(W = E) = 0$ .*

\*\*Insert figure 1 here\*\*

Entry will affect each incumbent equally, and so each will still have equal chance of winning the election. The entrant has zero probability of winning the election. Except in the case of  $x = W^*$  the entrant randomizes over entering on the two flanks. The entrant squeezes one of the incumbents out in the first round but is then defeated by the other incumbent in the runoff. If  $x = W^*$  then the entrant will randomize over the flanks and zero for all  $F$  other than the uniform. When  $F$  is uniform the entrant can randomize over the center interval as well as the flanks. As described in the previous section, deviation by the incumbents provides scope for the entrant to win, so neither moves and we have many equilibria. Notice that if  $x < F^{-1}(\frac{1}{4})$  then this violates the 'too far apart' intuition and the entrant could choose  $E = x + \beta$  ( $\beta > 0$ ) and crowd  $I_1$  on the flank and then defeat  $I_2$  in the runoff as it is closer to the center. It should be noted that of the equilibria in (a),  $x = W^*$  is the only one in which the entrant is defeated in the first round.

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<sup>10</sup>If  $F$  is uniform then  $0^{-+} \equiv (I_1, I_2)$ , and so as  $\varepsilon \rightarrow 0$  the interval doesn't collapse.



The second group of equilibria presented above are difficult to characterize as they depend critically on the particular distribution of voters in the electorate. The first group of equilibria are independent of the particular  $F$ , as long as  $F$  satisfies the assumptions of the model. Thus, under the run-off rule we have, at the least, a continuum of equilibria in which the entrant never wins the election. And, independent of  $F$ , all equilibria require the incumbents to be located symmetrically about the middle, and on all but a set of measure zero involve non-centrist platforms.

The group of candidate equilibria in (b) may not be equilibria as even though in the limit  $E \in I_1^+$  may not lead to  $E$  having a positive probability of winning, there could still exist a point  $E = I_1 + \lambda$ ,  $\lambda > 0$ , such that  $E$  has a strictly positive chance of winning the election. Whether such a point exists will depend on  $F$  and the locations of the incumbents. The incumbent locations in (a) prevent the existence of such points, which can be seen from the definition of  $W^*$ . If a pair of incumbent locations are not in the domain of (a) or (b) then such a point must exist and so they cannot constitute an equilibrium.

### 3.1.2 Plurality

*There exists a unique equilibrium,  $\{I_1, I_2\} = \{y, -y\}$  where  $1 - 2F(\frac{y}{2}) = F(y)$ ,  $E \in \{I_1^-, I_2^+, 0^{+-}\}$  for all  $F$ .  $\text{prob}(W = I_1) = \text{prob}(W = I_2) = \frac{1}{2}$ ,  $\text{prob}(W = E) = 0$ .*

The entrant never wins and the two incumbents each have a  $\frac{1}{2}$  probability of winning the election as the location of the entrant affects both parties equally. The incentive for an incumbent to deviate towards the center is tempered by the resultant added incentive for the entrant to enter on that flank. The equilibrium is the exact point where further deviation inwards will ensure that the entrant locates on the deviating incumbent's flank. This is the same equilibrium found by Palfrey (1984), but on a more general policy space, and with slightly different assumptions on parties objective functions and voter behavior. The non-centrist equilibrium found here relies critically on the assumption that the entrant will enter despite having zero probability of victory.

## 3.2 Enter only if have a positive probability of victory

### 3.2.1 Run-off

(a) *There exists an infinite number of equilibria,  $\{I_1, I_2\} = \{x, -x\} \forall x \in [W^*, 0]$ , where  $W^*$  solves  $1 - 2F(\frac{W^*}{2}) = F(W^*)$ .  $E = \phi$  (doesn't enter). For all equilibria,  $\text{prob}(W = I_1) = \text{prob}(W = I_2) = \frac{1}{2}$ ,  $\text{prob}(W = E) = 0$ .*

(b) *Depending on  $F$ , there may exist additional equilibria which satisfy the following necessary, but not sufficient conditions,  $\{I_1, I_2\} = \{y, -y\}$  where  $y \in [F^{-1}(\frac{1}{4}), W^*)$  and  $F(\frac{y}{2}) \geq \frac{1}{3}$ .  $E = \phi$  (doesn't enter).  $\text{prob}(W = I_1) = \text{prob}(W = I_2) = \frac{1}{2}$ ,  $\text{prob}(W = E) = 0$ .*

The entrant stays out. If an incumbent deviates then, as above, the entrant could win the election. So if the incumbent deviated we would see the reaction by the entrant to enter and win the election. Thus, we have the same incentives as when facing compulsory entry and so the same incumbent equilibria.

### 3.2.2 Plurality

*A pure strategy equilibrium does not exist.*

We can see that for any symmetric location of the incumbent parties an inward deviation can always be found that is small enough so that even though entering on this party's flank will uniquely maximize the entrant's vote share it will still have zero chance of winning the election. This can be seen through the intuition that entry on a flank will only steal votes from one candidate and so give the election to the candidate on the other side of the median. Consequently, under this assumption of entrant behavior such a party will not enter, and so the disincentive for incumbent parties to shift inwards disappears. This then rules out the non-centrist equilibrium of section 3.1.2. Asymmetric equilibria are also eliminated as the incumbent furthest from the center has positive incentive to at least shift to a symmetric position.

Fedderson, Sened and Wright (1990) extract an equilibrium from a framework similar to this by assuming that voters are able to vote strategically. By this they mean that groups of voters are able to coordinate and ensure that a preferred candidate wins the election. Their unique equilibrium is  $\{I_1, I_2\} = \{0, 0\}$ . In the model presented here an entrant could locate at  $E = \sigma$ , where  $\sigma$  is small, and win the election with less than a majority as the incumbents would split the rest of the vote. In this situation the entrant is able to steal votes off both incumbents, and thus punish both sufficiently to win the election. However, if  $\sigma > 0$  then Fedderson et. al. assume that all voters with ideal points less than zero are able to coordinate on one of the incumbents and ensure that the chosen incumbent wins the election (and vice versa for  $\sigma < 0$ ). In anticipation of this ability the potential entrant does not enter. This assumption removes the ability of the entrant to punish both incumbents with its entry and so removes its ability to win the election. Consequently this produces an equilibrium as the incumbent parties are now happy to locate at the median voter.

The various results of the preceding sections are summarized in the following table.

	Enter no matter what	Enter only if have a positive probability of victory
Run-off	Continuum of non-centrist equilibria	Continuum of non-centrist equilibria
Plurality	Unique non-centrist equilibrium	No pure strategy equilibria

## Remark

It is hard to say which assumption of entrant behavior is the more appropriate. In a repeated model we can certainly imagine an entrant who enters despite having zero probability of victory in the current period. They may have aspirations for future electoral success and need to start building a support base at the expense of other parties, or they may simply want to have a voice and feel that the cost of entry is outweighed by the value of the audience that electoral participation brings.<sup>11</sup> Though, we must then ask why there is only one such party, and not many of them? Consequently, in this one shot model that we are presenting here it would seem inappropriate. If we manage to represent a dynamic model then hopefully these considerations could be accounted for.

However, there is one further criticism of the assumption of ‘entry no matter what’ that is far more concerning. The assumption implies that the final outcome should consist of three competing parties (at least), as the entrant will always contest the election. However, the empirical fact that we are attempting to explain is that we only observe two parties. This leads to the conclusion that either the assumption of compulsory entry is misguided, or that the framework of two incumbents facing a potential entrant is inaccurate. Either way the power of the model under this assumption to explain the phenomena at hand is questionable. An additional problem with this assumption is the lack of justification for why there is only one potential entrant, particularly if the probability of victory is of no concern to their entry decision. We are left wondering why, if there exists one, there isn’t more parties poised to enter, and what this would mean for the equilibria.

The run-off results presented here are, to the best of my knowledge, new. In fact, formal modeling of the run-off rule is very sparse indeed. In what appears to be the only formal study of the run-off rule, Osborne and Slivinski (1996) study a model of citizen candidates under both the plurality and run-off rules. The primary difference between the models is that Osborne and Slivinski assume that candidates are policy oriented and thus, most importantly, policy restricted. That is, a candidate is restricted to select as his campaign platform his true ideal point. For certain parameter values for cost of entry and benefit of office they find that two party, non-centrist equilibria occur. A similar model was also used in a study of the plurality rule only by Besley and Coate (1997). A weakness of these models is that they rely too heavily on the definition of a Nash equilibrium. It may be the case that a candidate would prefer a different candidate to run in his stead, and that this alternative candidate would also prefer this option (for example, someone fractionally closer to the center who could guarantee electoral success). However, such deviations are not allowed when determining Nash equilibria

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<sup>11</sup>The assumption that parties immediately receive the support of all voters for whom they are the closest party implies a more long term view is captured by this one shot model as difficulties of party establishment, such as name recognition, are assumed away. That is, assuming voters always vote sincerely implies that if an entrant can’t win in the one shot model it won’t be able to win no matter how many periods we model the competition over (unless, of course, its entry incites additional entry). For a dynamic model to differ from and extend what is presented here we would need to consider additional party competition for characteristics such as name recognition or platform credibility.

as they involve the simultaneous deviations of two players. Thus, these models extend the analysis by adding policy preferences to a candidate's objective function but it would seem that the equilibria produced may not be coalition proof. This extension is the basis of a broad literature in which candidates are policy motivated and, consequently, restricted in the platforms and policies they can select. These restrictions are used to obtain non-centrist platform choices that are significantly different in intuition from the voluntary choices of the purely Downsian candidates modelled here.<sup>12</sup>

In contrast, the results presented here for the plurality rule are known, at least in approximate form. However, the predictions of the model fail to coincide with the empirical phenomena that we are trying to explain. The model either predicts a three party outcome, or the absence of an equilibrium altogether. But as was pointed out earlier, under the plurality rule we most commonly observe two party competition with non-centrist platforms. The only way to extract a two party outcome is if we make the further assumption of Feddersen, Sened and Wright (1990) as to voter sophistication, but even then the competing parties will both locate at the median voter. These shortcomings highlight the absence of a theoretical model that predicts the two party, non-centrist electoral outcomes we observe when the plurality rule is used.

The predictions of the model under the run-off rule, however, do coincide with empirical observation. The unique use of the run-off rule in federal elections has been in Australia, and there we have seen the emergence of an effective two party electoral system with non-centrist platforms.<sup>13</sup> And as we have seen, the model here produces a continuum of two party equilibria which, on all but a set of measure zero, are non-centrist.

Despite this agreement of the model and empirical fact, these predictions stand in conflict with established theoretical predictions for the run-off rule. These arguments are encapsulated in what Riker (1982) refers to as 'Duverger's Hypothesis', which covers the class of electoral rules that were expected to favor multi-partism. This class incorporates the run-off and proportional representation rules as it was believed that they do not encourage parties to maximize their vote count and so the incentive to rationalize into only two parties was absent. However, the results presented here indicate that the ability of two parties to prevent successful competition from additional parties limits the number of competing parties to two, in contrast to the prediction of Duverger.

The results of this section for the plurality rule are concerning. The lack of a reasonable theoretic justification for two party, non-centrist outcomes makes it difficult to conclude that we understand the process of platform selection by competing parties. It is in pursuit of this understanding that we now turn to the extended model.

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<sup>12</sup>See Wittman (1983) and Calvert (1985).

<sup>13</sup>Riker (1992).

## 4 The Extended Model

We are left with the problem of explaining why, under plurality rule, parties choose non-centrist platforms. If we look closer at the empirical phenomena that we are trying to explain we notice that the assumption of a single district is not appropriate. Political parties compete in many districts simultaneously. For example, in the U.S. there are 435 congressional districts that elect a representative simultaneously, and the two main parties compete in most, if not all, of these districts at every election. The candidates in each district are associated with their nominating party and so all candidates from the one party effectively compete with the same platform.

When we expand the framework under consideration in this way by extending the analysis to simultaneous competition for multiple districts we reverse the findings of the previous sections and find that a stable, non-centrist equilibrium exists under plurality rule, and that run-off results in entry and instability. This result serves to fill the hole in our understanding of electoral competition under the plurality rule, and brings the prediction for the run-off rule into line with the prediction of Duverger (and Riker). However, this also means that our prediction is no longer consistent with the electoral situation in Australia. This discrepancy in theoretical prediction and empirical observation for the run-off rule now becomes the open question in this area. However, with only one data point as the basis of this discrepancy its importance should not be overstated.

To incorporate these extensions into the model I will make the following further assumptions.

**Assumption 6** There exists a continuum of districts where district  $i$  has the median voter's ideal point being  $Z_i$ .  $Z_i$  is distributed symmetrically about 0 (the mean of the original district) on the support  $[\underline{Z}, \bar{Z}]$ , where  $\underline{Z} = -\bar{Z}$ . The distribution of  $Z_i$ 's is represented by the cdf  $G$ , where  $G(\underline{Z}) = 0$  and  $G(\bar{Z}) = 1$ . The associated pdf is  $g$ , which is continuous and can be either strictly quasi-concave or quasi-convex.<sup>14</sup> The distribution of voters ideal points in district  $i$  is given by the cdf  $F(x - Z_i)$  for all  $x \in \mathfrak{R}$ .

**Assumption 7** Each of the two incumbent parties must choose a single platform on which they will compete in every electorate.

Without the constraint of assumption 7 the additional districts would not constitute a different approach as the single district results would apply in each district separately. The assumption of a continuum of districts is, of course, not realistic. However, it has been employed as it captures the effect and intuition of the multiple district scenario whilst avoiding the complexity of calculation associated with a lumpy distribution of

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<sup>14</sup>Note that this permits uniform distributions as they are quasi-convex. The restriction to strict quasi-concavity is to rule out particular flat spots in the distribution that may produce multiple weak Nash equilibria.

district median voters. It is in the same spirit as the assumption of a continuum of voters in the single district case.

With an extended structure the objective function of the parties also has to be extended. In the multiple district case I shall once again assume that parties have lexicographic preferences. The primary dimension is the expected share of districts that the party wins in the election, and the second dimension is their total vote share.<sup>15,16</sup> As we have a continuum of districts it is only natural to talk of ‘share of districts’ rather than number of districts. This is analogous to the single district case where parties seek to maximize their vote share over the continuum of voters.

As I am attempting to explain the two party phenomena, from now on I will make the assumption that parties will enter only if they have a strictly positive probability of electoral success. This implies a natural modelling of party entry that if a party enters, though it may enter in many districts simultaneously, it will enter and compete only in those districts in which it has a positive probability of victory.

To incorporate this assumption about entry I will relax the assumption of only one potential entrant and instead assume there are many potential entrants, but that only one will enter in each district, and only if it has a positive probability of victory in that district. This is assumed in order to simplify the analysis. If there are many potential entrants for a single district then the entry decisions of these parties are interrelated and would require a more complicated stage game to be specified. The intention, which is maintained by the assumption, is that if a single party could enter and win a district then the incumbents lose that district. This will not require as many entrants as may be thought. In most instances one entrant, with one platform, will be able to win many districts from the incumbents. Indeed, for the only equilibrium result specifying entry, Proposition 3, only two entrants are required to secure all but an arbitrarily small number of the districts lost by the incumbents.<sup>17</sup> This modeling technique is not as restrictive as it may seem. In fact, if we assumed that the potential entrants were strategic and conscious of further entry in districts they attack then as long as the incumbents are on either side of the median and entry is possible it can be shown that one entrant can secure victory in a district and prevent further entry.<sup>18,19</sup> This framework is rather general and

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<sup>15</sup>As governments can be formed with a minority of seats, or by forming a coalition of parties, a complex model of government formation would need to be incorporated if it were to be assumed that parties were attempting to maximize their probability of winning government. Consequently, the more tractable assumption of seat maximization has been made.

<sup>16</sup>The second dimension is only required in order to rule out potential equilibria in which neither of the incumbent parties win any of the districts and are unable to move their platforms anywhere such that they do. Without the second dimension such locations pairs would constitute an equilibrium even though we may ask why the incumbents themselves would enter given they have a zero probability of winning any districts.

<sup>17</sup>This is achieved by the entrants locating at points arbitrarily close to each incumbent.

<sup>18</sup>For example, if  $|I_1| > |I_2|$  relative to the district median, and successful entry on the right flank is possible, then  $E = I_2 + \delta$ , where  $1 - F(\frac{I_2 + E}{2}) > F(\frac{I_1 + I_2}{2})$  but  $1 - F(E) < F(\frac{I_1 + I_2}{2})$ , secures victory for the entrant but prevents further entry.

<sup>19</sup>The assumption of only one entrant in each district allows me to deal with problematic situations

is consistent with several types of political entrant. It can be seen as covering the entry of multiple independents into the legislature, or the creation of regional or issue based parties which pick off certain sections of the electorate.

## The Literature

There have been several papers which have considered the issue of multiple districts. However, they are infrequent in the political theory literature and so the significance of the extension has not been fully investigated. The first investigation of multiple districts was by Hinich and Ordeshook (1974) in a study of the electoral college. Hinich and Ordeshook were interested in distortionary effects of the electoral college in comparison to a direct vote for the President. They proved the extension of the single district case, that with two candidate competition both candidates would converge to the median of the median district. This result makes two candidate competition in the multiple district case look almost identical to that in the single district case (though maybe with a different convergent point). This question was examined further in Hinich, Mickelsen and Ordeshook (1975) where they attempted to assess the potential magnitude of such distortions through simulations. Further work has been done by Austen-Smith (1981,1984,1986,1987,1989) in a series of papers. The first paper is the most similar to the model presented here as parties are assumed to choose a unique platform which is applicable for candidates in all districts (assumption 7 here). The question of entry and entry deterrence is not considered. Austen-Smith investigates the existence of equilibria when parties compete not only in policy space but in distributive dimensions as well. Parties are assumed to have a fixed campaign budget which has to be allocated to the districts individually. He shows that under certain conditions an equilibrium will exist. He also points out that if parties are asymmetrically endowed then this equilibrium will involve different policy platforms. This result is significantly different from that presented here as it requires an assumption of asymmetric parties and the incorporation of distributive dimensions to get the non-convergence of party platforms, whereas the result of this paper does not. In the second paper Austen-Smith takes an alternative approach. He considers that the final party platform is some function of the individual choices of candidates, who are free to choose their positions, and so studies the optimal choice for individual candidates. This framework is then used in the third and fourth papers to consider bargaining games in the elected legislature and what this means for individual vote choice. His final paper surveys the literature on electing legislatures (which also includes work on proportional representation and multiple member districts).

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in which the incumbents are on the same side of the median, and so wouldn't be expected to win the district, but where one entrant can't prevent subsequent entry.

## 5 Results II

**Run-off** *Suppose  $\underline{Z} \neq \overline{Z}$ . Then if an equilibrium exists it must involve the entry of more than two parties.*

Notice that in order to prevent entry in the central district the incumbents must be located symmetrically and no further from the center than  $[W^*, -W^*]$ , or in a possible asymmetric position. Now consider an arbitrary district,  $r$ , with median voter other than at 0. Let the median of this district be  $Z_r = \varepsilon$ , where  $\varepsilon > 0$ . If the incumbents are located symmetrically in the central district then they can't be located symmetrically in district  $r$ . Thus, entry will be possible. The entrant can locate at  $E = I_2 + \frac{\varepsilon}{4}$ . So the entrant beats  $I_2$ , who is eliminated in the first round. The entrant is then closer to the median voter of  $r$  and so wins the runoff. As  $\varepsilon$  could be arbitrarily small we can see that if the support of median voters across all districts is non-degenerate then entry cannot be prevented by the two incumbents in all districts simultaneously. A similar analysis shows that if the incumbents are located asymmetrically in the central district and are preventing entry, then these locations can't prevent entry in districts with different medians.

So under a run-off rule the two party, entry excluding, equilibria of the single district case are not robust to simultaneous competition in many districts. The positive results for this decision rule which were obtained for the single district fall apart under even the smallest heterogeneity of districts.

For the case of plurality we will need to define the following condition.

**Condition 1**  $g(\underline{Z}) \geq \frac{2}{3}g(0)$ .<sup>20</sup>

This condition ensures that the weight of districts with median voters at the boundary of the distribution is enough so that the incumbents do not have the incentive to abandon them to entrants by deviating inwards in order to win more districts at the center of the distribution. Recall that  $g$  is assumed to be symmetric, so that  $\underline{Z} = -\overline{Z}$ , and the same condition holds for  $\overline{Z}$ . Define  $M(j)$ ,  $j = I_1, I_2, E$ , to be the share of the districts won by party  $j$ .

**Proposition 1** *Suppose  $0 > \underline{Z} \geq Z^*$ , where  $Z^*$  satisfies  $F(Z^*) = \frac{1}{3}$ .<sup>21</sup> Then if condition 1 is satisfied the unique equilibrium is given by,  $\{I_1, I_2\} = \{2\underline{Z}, -2\underline{Z}\}$ ,  $E_i = \phi \forall i$  (do not enter).  $M(I_1) = M(I_2) = \frac{1}{2}$ ,  $M(E_i) = 0 \forall i$ .*

In the one district case the incumbents had an incentive to deviate towards the center and win the election. This incentive still exists in the multiple district case. However,

<sup>20</sup>Note that this condition only restricts strictly quasi-concave distributions and places no restrictions on quasi-convex distributions of districts.

<sup>21</sup>I should point out that if we were to drop assumption 5 and if  $\underline{Z} = Z^*$ , an entrant would be able to locate at  $E = I_1$  and tie in the district with median at  $\underline{Z}$ , but lose in all other districts. As the district won has a measure zero then this possibility still wouldn't incite entry. Likewise for entry at  $I_2$ .



the incumbents reach a point where further deviation in order to win central districts will allow entry in the districts with the most extreme median voter as in those districts the ‘too asymmetric’ intuition is violated, and condition 1 ensures that the amount of districts lost on the edge by deviating inwards outweighs the amount won in the center.

The incumbents win half of the districts each. They tie in the central district, which is then decided by randomizing. However, as we have a continuum of districts, this central district has a weight of zero and so doesn’t affect the proportion of districts won by each of the incumbents. If we consider the continuum of districts as the limit of a finite distribution of districts, then it is only in the limit that the winner of the central district does not achieve a majority and win government outright.<sup>22</sup>

Only occasionally do we observe legislatures where the seats are evenly divided between the two major parties, or where they are separated by only one seat. It is quite normal for us to observe legislatures where one party holds a significant majority. Consequently, it would be desirable if our theoretical model could produce such uneven seat allocations as an equilibrium. Obviously the equal proportion of seats for the incumbents predicted by this model is a direct consequence of the symmetry of the set up and so we would want to relax the symmetry to produce an asymmetric outcome. This is possible in the multiple district framework presented here as Proposition 1 does not necessarily rely on the symmetry of  $g$  (the statement and use of Condition 1 certainly does, but the logic of the proof does not). Indeed, as long as an analog of Condition 1 holds (Condition 1A below) then the equilibrium depends solely upon the width of the distribution of districts and not on the shape of the distribution (e.g. the mean or the median). This is an interesting result as it is not automatic that asymmetric distributions produce asymmetric outcomes for the parties. For example, if we incorporate asymmetric distributions in models that predict party convergence then the parties still converge to the median (of the median district in the multiple district case), though this may no longer be in the geographic center of the distribution. So we may have a different set of platform choices by the parties but they still receive symmetric outcomes. If we consider the Palfrey model (result 3.1.2 here) then for rather special distributions it is possible to produce asymmetric outcomes for the incumbents. However, we are left to wonder why the losing incumbent would itself enter if it had no chance of victory. To produce such a result we are required to lean even harder on the assumption that parties are willing to enter an election regardless of their chances of victory. Consequently an interpretation of asymmetric outcomes in the single district framework is difficult to develop. However, in the multiple district framework presented here such asymmetric outcomes are easily conceptualized. Even though the minor incumbent party has no chance of winning a majority it still wins some districts contested and thus secures a voice in the legislature. We could also justify entry in this instance as some members of the losing party still gain personally by winning their own district and this may justify the existence of the party.

To characterize the equilibria for an asymmetric distribution of districts we shall

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<sup>22</sup>Of course, that the result of plurality 1 still constitutes an equilibrium with only a finite number of districts remains to be proven.

need to generalize and strengthen condition 1. To maintain tractability I shall continue to assume that  $\underline{Z} = -\overline{Z}$ , though this too could be relaxed.

**Condition 1A**  $g(x) > \frac{2}{3}g(y) \forall x, y \in [\underline{Z}, \overline{Z}]$ .

The tightening of this condition is required to rule out certain flat spots in quasi-convex distributions. Such an additional restriction was not required in the statement of Condition 1 as symmetry ensured that even if such flat spots existed they would not cause a problem. This tightening is overly strong. Consequently, whereas Condition 1 was sufficient and necessary for the result of Proposition 1 to hold, this condition is sufficient for the following result but not necessary.<sup>23</sup> This leads to a generalization of Proposition 1.

**Proposition 1A** *Suppose  $0 > \underline{Z} \geq Z^*$ , where  $Z^*$  satisfies  $F(Z^*) = \frac{1}{3}$ , and relax the assumption that  $g$  is symmetric. Then if condition 1A is satisfied the unique equilibrium is given by,  $\{I_1, I_2\} = \{2\underline{Z}, -2\underline{Z}\}, E_i = \phi \forall i$  (do not enter).  $M(I_1) = \int_{\underline{Z}}^0 g(x)dx, M(I_2) = \int_0^{\overline{Z}} g(x)dx, M(E_i) = 0 \forall i$ .*

We can see immediately that unless  $G(0) = \frac{1}{2}$  then  $M(I_1) \neq M(I_2)$  and one of the incumbent parties will hold an outright majority. Thus we can produce an equilibrium selection of party platforms such that one party is guaranteed of winning a majority of the districts. The existence of such equilibria could be used to explain elections where one party is predicted to win a clear majority and does so, and where the losing party does not seem to have a platform that could win a majority of the seats.<sup>24</sup> This is consistent with a common analyst observation that a party has ‘captured the middle ground.’ A distribution of districts which is skewed to one side would produce such an outcome.

**Proposition 2** *Suppose  $\underline{Z} < Z^*$ . Then if an equilibrium exists it must involve the entry of more than two parties.*

In this situation the dispersion of districts is too broad for the incumbents to compete successfully in all of them. To satisfy the constraint preventing successful entry at a point between the incumbents in the central district, the incumbents must leave open the possibility for successful entry in the extreme districts by violating the ‘too far apart’ intuition. The result here is, in fact, stronger than what is stated; we could say that it is impossible for the two incumbents to prevent entry whether they are, or are not, in equilibrium. As it is equilibria we are interested in the result has been stated in its weaker form.

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<sup>23</sup>This difficulty is a consequence of the dropping of symmetry. To develop a statement that was also necessary would require excessive complication which would only cloud the result. Even with this simple condition we can see that there are many distributions that would produce asymmetric outcomes. An alternative tightening would have been to rule out weakly quasi-convex distributions.

<sup>24</sup>Potentially we could also explain such an outcome if we considered a dynamic model in which the distribution of districts changed from election to election but parties were restricted in changes to their platforms. The purpose of the result here is to show that such an uneven outcome is also possible in a single election model with parties completely free to select their platforms.

**Proposition 3** *Assume that condition 1 is not satisfied ( $g$  must be strictly quasi-concave). Then if an equilibrium exists it is unique and is given by,  $\{I_1, I_2\} = \{2Z^\#, -2Z^\#\}$ , where  $Z^\# < 0$  and satisfies  $g(Z^\#) = \frac{2}{3}g(0)$ . Entrants enter and win districts with median voter's ideal points in the intervals,  $[\underline{Z}, Z^\#)$  and  $(-Z^\#, \bar{Z}]$ . If  $g$  is concave then such an equilibrium always exists.*

With condition 1 violated the incumbents have positive incentive to deviate inwards to win districts in the center off the other incumbent, even though this involves giving up the extreme districts to entrants. We notice that as compared to the equilibrium in Proposition 1 the incumbents still have equal shares of expected district wins, but now neither party will hold a majority. The equilibrium is given by the point where further inward deviation involves more districts lost on the edges than gained in the center. The continuity of the pdf  $g$  ensures that such a point exists. This is very similar to the equilibrium when condition 1 was satisfied. Condition 1 simply ensured that the critical point was reached before any entry occurred. Therefore, condition 1 can be seen as a necessary condition for an equilibrium to involve only two parties. Though the districts abandoned on the edges by the incumbents could be won by a different party entering in each district (effectively independents), we could have as few as two parties entering and winning arbitrarily close to one half of these districts each. To determine the final party structure in this instance we would need to formalize a more extensive model of entry. As I am primarily concerned with two party outcome structures this issue will not be explored any further here.

Just like with Proposition 1, the symmetry of  $g$  could be relaxed here to produce an equilibrium involving entry and asymmetric seat shares for the incumbent parties. Indeed particular  $g$  functions could be found to produce any variety of multiple party equilibria, for example involving entry only on one flank.

This location pair may not constitute an equilibrium for non-concave  $g$  functions if there is too much district share that is lost to entrants. That is, the district share of the incumbents is so small that they each have incentive to deviate from the prospective equilibrium to the outside of the other incumbent as they can win a greater share of the districts on the flank. If  $g$  is concave then the density on the flanks is small enough such that these deviations are not profitable and so we have an equilibrium.

The result Proposition 2 provides another necessary condition for an equilibrium to involve only two parties, that the dispersion of  $Z'_i$ 's isn't too wide. This can be expressed with the following condition.

**Condition 2**  $[\underline{Z}, \overline{Z}] \subseteq [Z^*, -Z^*]$ .

We can see that Conditions 1 and 2 together form a necessary and sufficient condition for an equilibrium to involve at most two parties. As such, these two conditions can be interpreted as the limit of Duverger's Law. If both conditions are satisfied then we would expect electoral competition amongst no more than two parties and the law to be satisfied. If either condition isn't satisfied then there would be entry and the law would not apply. This gives the following theorem.

**Theorem 1** Under assumptions 1-7, Conditions 1 and 2 are necessary and sufficient conditions for Duverger's Law to hold.

As such, the requirement for cumulative density functions  $G$  to satisfy conditions 1 and 2 can be seen as characterizing the domain of Duverger's Law. Outside of this domain the law would not be expected to hold. This is an appropriate result because to explain a law such as Duverger's that doesn't hold universally we would expect, and indeed desire, a theory that predicts a restricted domain of applicability. Hopefully a recourse to empirics will inform us as to whether this is the correct restriction.

## 6 Discussion

One thing that we notice when comparing the two batches of results is the discontinuity in the predictions of the model for both electoral rules. In the single district framework under the run-off rule the model produced a continuum of pure strategy two party equilibria. With probability one we would have an equilibrium with non-centrist party platforms. In contrast under the plurality rule the model failed to have an equilibrium unless we made the somewhat worrisome assumption that the entrant entered even if its chance of electoral success was zero. However, when we expanded the model to incorporate multiple districts the predictions of the model under these rules reversed completely. With even the smallest degree of district heterogeneity the two party equilibria for the run-off rule no longer existed. At the same time, for the plurality rule we not only guarantee an equilibrium exists but produce a unique, non-centrist two party equilibrium. These discontinuities in prediction are quite startling.

The results of the extended model can also provide some insight into the relationship between U.S. Congressional elections and Presidential elections. To maximize performance in the House elections and to preclude entry of a third party each of the two incumbent parties must choose a non-centrist platform. However, the Presidential candidate of each party competes in only one district, the grand district (with mean zero in this symmetric framework), and so would like to move towards this center to maximize his vote in the Presidential race. However, his party is constrained to its non-centrist platform. So to achieve any centripetal movement a Presidential candidate must try and detach himself from his party so that he can move towards the center without disrupting

the equilibrium for the House elections. One obvious way to achieve this objective would be on non-policy issues (as they are party platform constrained on policy issues). This can be seen to lead to the cult of personality phenomena in Presidential races. Personality traits are one way for a candidate to make himself seem more central without dragging his party with him. In fact these incentives for detachment from the party base are applicable to all candidates, including Senators and district candidates, who want to move towards the median in their given district. It is primarily because Presidential and Senatorial candidates are more visible that they can achieve this detachment more effectively than the district candidates. To be pedantic here, the Presidential candidates would not attempt to move completely to the center as they are really the sole candidate in a multiple district election with each district representing each state that the candidate carries. As Hinich and Ordeshook (1974) pointed out, the candidates would attempt to move to the median of the state that contained the median electoral college vote.

This explanation for the cult of personality campaigns so evident in U.S. elections can also be used to explain why such campaigns are not as evident in other single member district elections, such as in Britain and Australia.<sup>25</sup> In those countries the Prime Minister is elected indirectly by voting for his candidate in your local district. Thus leaders of the incumbent parties, the Prime Ministerial candidates, maximize their probability of success by maximizing the number of electorates that their party wins. And this is achieved by sticking firmly to the non-centrist party platform.<sup>26</sup>

We can also use this analysis to consider the phenomena of third party candidates in Presidential elections. For a wide dispersion of median voter points we have the prediction that the two incumbent parties are also widely spaced. If the Presidential candidates cannot achieve detachment from their party platform, or cannot do it very well, then there will exist a large gap between the positions of the two incumbent party Presidential candidates. It is potentially this hole that the third party candidates have tried to exploit. However, the model also predicts that if we have an entry precluding equilibria then the two incumbents are located no further apart than  $[2Z^*, -2Z^*]$ . And we know that this isn't wide enough for an entrant to steal the central district, and so it isn't wide enough for a third candidate to steal the Presidential election. The third candidate will, however, potentially receive a large share of the votes even though they have no chance of victory. This prediction is also consistent with history where third party candidates have received a surprisingly large vote but have never been victorious.<sup>27</sup> This thinking leads to the question of why doesn't one of the incumbent parties enter

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<sup>25</sup>For a discussion and review of this topic, with particular reference to these three countries, see Crewe and King (1994).

<sup>26</sup>Israel is an interesting example of how direct versus indirect election of the leader of the Government can have a significant effect on the political landscape. Electoral changes introduced for the 1996 elections added an additional ballot to the Knesset elections in order to directly elect the Prime Minister. Previously the Prime Minister had been elected indirectly as in other parliamentary systems. This apparently innocuous change has had a dramatic impact on Israeli politics. As the Knesset elections employ proportional representation the results of the model presented here are not directly applicable. For a full account of the effects of this change on Israel see Arian (1998).

<sup>27</sup>Smallwood (1983, p.13).

a stooge near the other incumbent's platform to break up the oppositions vote and so ensure victory for themselves? Staying strictly within the framework presented here it would be hard to answer that they wouldn't. The constraint would be finding a credible independent candidate, and those that exist would be unlikely to stoop to such behavior to aid a party that they, by definition of being an independent, have little affiliation with. Consequently, such behavior has been ruled out as unachievable (not to mention unethical).

Another empirical fact is that district members in the U.S. House express far greater vote independence than do the equivalent members in, for example, Britain's House of Commons.<sup>28</sup> It could be conjectured that this greater independence is reflective of the ability of the individual members to detach themselves to some degree from the party platform. Of course, if complete detachment was possible we would have the centrist equilibrium result for each district that we had in the single district analysis. Maybe this extra ability allows the U.S. incumbent parties to support a wider dispersion of median voter points whilst still precluding entry. It could be that the greater dispersion of median voters in the U.S. necessitates such flexibility if entry is to be precluded. That is, maybe U.S. dispersion is beyond the bounds specified in our equilibrium and increased flexibility for candidates is what is needed to preclude entry of a third party. Unfortunately, these are only conjectures, and would need further study for us to be able to comment on them confidently.

A further point which the model predicts that is consistent with the data is that the dispersion of median points produces some districts that are safely in the hands of one party, others safely in the hands of the other party, and some districts that are fought for fiercely. This is a direct consequence of the constraint that the parties are constrained to choose one platform which they must use in every district regardless of its particular distribution of voters. This result can be seen as a formalization and explanation of what Robertson (1977) categorized as marginal and safe seats.

## 7 Empirical Prediction

We have seen that the predictions of the model under the plurality rule are consistent with the two fundamental empirical phenomena: Duverger's Law and non-centrist platforms. In addition the model requires certain conditions to hold and makes further predictions as to the actual platform choice of the parties. To test that the structure of the model presented here is in fact what is underlying the main empirical phenomena these additional requirements and predictions should be investigated. The model places restrictions on both the support and the distribution itself of the district median voters. These restrictions are conditions 2 and 1, respectively. The model also predicts what the equilibrium party platforms will be for a given distribution of median voters. These conditions and predictions should, in principle, be empirically testable. We could test

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<sup>28</sup>Cain, Ferejohn, and Fiorina (1987, p.43).

the null hypothesis that the dispersion of median voter points across districts is strictly contained in the interval bounded by the two incumbent party platforms, and that the distribution of median voters has sufficient density at the edges.

## 8 Conclusion

By extending the theoretical model of electoral competition with entry to incorporate simultaneous competition for multiple districts I produce a unique two party equilibrium under plurality rule with non-centrist party platforms. This equilibrium also precludes entry of additional parties. This result is used to provide a domain for which Duverger's Law could be expected to apply. I also investigated the equilibrium characteristics of the model under the run-off rule and the plurality rule in both the single and multiple district frameworks. The paper has also shed some light on how the different levels of elections in the U.S. and other systems relate to each other.

## 9 Appendix

Let tilde (eg.  $\tilde{I}_1$ ) denote a deviation by an incumbent.

We note that  $E$  will never locate at the same point as either incumbent. At such a point  $V_E = 0$  and  $P(W = E) = 0$  by assumption 4. As  $F$  is non-degenerate and strictly increasing such a point is strictly dominated.

For simplicity some arguments of functions have been omitted. This occurs when they are  $I_1, I_2$ , or  $E$  (these typically represent party positions prior to any deviations).

WOLOG assume that if  $I_1 \neq I_2$  then  $I_1 < I_2$ .

### 9.1 Single District

#### 9.1.1 Plurality: enter no matter what

**Case 1**  $I_1, I_2 \leq 0$ .

- $I_1 = I_2 < 0$ .

Then  $C_E^\varepsilon(I_1, I_2) = I_2^+$  and  $P(W = E | E \in C_E^\varepsilon(I_1, I_2)) = 1$ .

For  $I_1, P(W = I_1) = 0$  and as  $\varepsilon \rightarrow 0, V_{I_1} \rightarrow \frac{1}{2}F(I_1)$ .

If  $F(I_1) \neq 0$  then as  $F$  is atomless,  $\delta$  small enough can be found s.t.  $\tilde{I}_1 = I_1 - \delta \implies V_{I_1}(\tilde{I}_1) = F(I_1 - \frac{\delta}{2}) > \frac{1}{2}F(I_1)$  and so  $I_1$  is better off.

If  $F(I_1) = 0$  then as  $F$  is non-degenerate a  $\gamma$  small enough can be found s.t.  $0 < F(I_2 + \gamma) < \frac{1}{2}$ . Then  $\tilde{I}_2 = I_2 + \gamma \implies C_E^\varepsilon(I_1, I_2) = I_2^+$  and  $V_{I_2}(\tilde{I}_2) = F(\tilde{I}_2) - F(\frac{I_1 + \tilde{I}_2}{2}) > 0$ , as  $F$  is strictly increasing once  $F > 0$ . No equilibrium.

- $I_1 \neq I_2$ .

Then  $C_E^\varepsilon(I_1, I_2) = I_2^+$  and  $P(W = E | E \in C_E^\varepsilon(I_1, I_2)) = 1$ .

For  $I_1, P(W = I_1) = 0, V_{I_1} = F(\frac{I_1 + I_2}{2})$ .

If  $F(I_2) \neq 0$  then as  $F$  is atomless there exists an  $\alpha$  small enough such that  $\tilde{I}_1 = I_2 - \alpha \implies V_{I_1}(\tilde{I}_1, I_2, E) > V_{I_1}$ . If  $F(I_2) = 0$  then consider the same deviation as in the subcase above.

- $I_1 = I_2 = 0$ .

Then  $C_E^\varepsilon(I_1, I_2) = \{I_1^-, I_2^+\}$  and  $P(W = E | E \in C_E^\varepsilon(I_1, I_2)) = 1$ .

For  $I_1, P(W = I_1) = 0$  and  $V_{I_1} \rightarrow \frac{1}{4}$  as  $\varepsilon \rightarrow 0$ .

$\tilde{I}_1 = I_1 - \alpha, \alpha > 0 \implies V_{I_1}(\tilde{I}_1) = F(\frac{-\alpha}{2}) > \frac{1}{4}$  for  $\alpha$  small enough. No equilibrium.

There is no equilibrium when  $I_1, I_2 \leq 0$ , and so by symmetry when  $I_1, I_2 \geq 0$ . Therefore any equilibrium must involve the two incumbents locating on opposite sides of the median voter.

Define  $y$  where  $1 - 2F(\frac{y}{2}) = F(y)$  and  $Y = (y, -y)$ .

**Case 2**  $I_1, I_2 \notin Y \cup \{y, -y\}$ . (So  $I_1 < y, I_2 > -y$ )

- $|I_1| \neq |I_2|$ .

Let  $|I_1| < |I_2|$ . i.e.  $I_1$  is closer to the center.

If  $E = 0, V_E = F(\frac{I_2}{2}) - F(\frac{I_1}{2}) > F(\frac{-y}{2}) - F(\frac{y}{2}) = 1 - 2F(\frac{y}{2})$

If  $E = I_2^+, V_E \rightarrow 1 - F(I_2) < 1 - F(-y) = F(y) \implies P(W = E | E \in C_E^\varepsilon(I_1, I_2)) = 0$ .

If  $E = I_1^-, V_E \rightarrow F(I_1) < F(y) \implies P(W = E | E \in C_E^\varepsilon(I_1, I_2)) = 0$  as  $F(y) < \frac{1}{3}$ .

So for  $\varepsilon$  small enough,  $I_1^-, I_2^+ \notin C_E^\varepsilon$ . i.e.  $E$  locates in the center.

To optimize  $E$  will choose a point  $x$  s.t.  $V_{I_1} = V_{I_2}$ . If such a point does not exist then  $E \in I_1^+$  with  $E \rightarrow I_1$  as  $\varepsilon \rightarrow 0$ . This is optimal as if  $V_{I_1} > V_{I_2}$  then  $V_E$  will increase if the entrant deviates to  $\tilde{E} = E - \delta$ , where  $\delta > 0$  and  $V_{I_1}(I_1, I_2, \tilde{E}) > V_{I_2}(I_1, I_2, \tilde{E})$ . So  $E$  continues to deviate until  $V_{I_1} = V_{I_2}$ , or  $E \rightarrow I_1^+$ .



- Assume  $E = x$  s.t.  $V_{I_1} = V_{I_2}$ .

Then consider  $\tilde{I}_2 = -I_1, C_E^\varepsilon = 0^{-+}$ , as  $I_1, \tilde{I}_2 \notin Y \cup \{y, -y\}$ , and  $V_{I_1} = V_{\tilde{I}_2}$ . As  $[I_1, \tilde{I}_2] \subset [I_1, I_2] \implies V_E(I_1, \tilde{I}_2) < V_E(I_1, I_2)$  and so  $V_{I_2}(\tilde{I}_2) > V_{I_2}$ . As no other candidate's vote share increased relative to that of  $I_2$ 's then  $P(W = I_2 | I_1, \tilde{I}_2) \geq P(W = I_2)$  and  $I_2$  will deviate. No equilibrium.

- Assume  $E \in I_1^+$  and  $V_{I_1} > V_{I_2}$ , so  $P(W = I_2) = 0$ .

Consider  $\tilde{I}_2 = -I_1$ . Then again  $C_E^\varepsilon = 0^{-+}, V_E(I_1, \tilde{I}_2) < V_E(I_1, I_2)$  and  $V_{I_1} = V_{I_2}(\tilde{I}_2)$ . Thus,  $V_{I_2}(\tilde{I}_2) > V_{I_2}$ . So  $I_2$  is better off. No equilibrium.

- $|I_1| = |I_2|$ .

Then  $C_E^\varepsilon = 0^{-+}$  and  $V_{I_1} = V_{I_2}$ .

Consider,  $\tilde{I}_1 = I_1 + \delta$ , where  $\delta$  is small such that  $\tilde{I}_1 < y$ , i.e.  $\delta < |\frac{I_1}{2}|$ .

Then  $E$  will still choose a point such that  $V_{I_1}(\tilde{I}_1) = V_{I_2}$ .

As  $[\tilde{I}_1, I_2] \subset [I_1, I_2], V_E(\tilde{I}_1, I_2) < V_E(I_1, I_2)$ .

And so  $V_{I_1}(\tilde{I}_1) > V_{I_1}$  and  $P(W = I_1 | \tilde{I}_1, I_2) \geq P(W = I_1 | I_1, I_2)$ .

So  $I_1$  will deviate. No equilibrium.

**Case 3**  $I_1, I_2 \in Y$ .

If  $E \in (I_1, I_2)$  then  $V_E < 1 - 2F(\frac{y}{2}) \implies P(W = E) = 0$ .

If  $E = I_2^+$  then  $V_E > 1 - F(-y) = F(y)$ .

If  $E = I_1^-$  then  $V_E > F(y)$ . So  $E$  will locate on a flank.

We note that if  $|I_1| = |I_2|$  then  $C_E^\varepsilon = \{I_1^-, I_2^+\}$ ,  $P(W = E) = 0$ , and  $P(W = I_1) = P(W = I_2) = \frac{1}{2}$ . So if for  $I_1, I_2, P(W = E) > 0$  then either  $P(W = I_1) < \frac{1}{2}$  or  $P(W = I_2) < \frac{1}{2}$ . If  $P(W = I_1) < \frac{1}{2}$  then  $\tilde{I}_1 = -I_2 \implies P(W = I_1 | \tilde{I}_1, I_2) = \frac{1}{2}$ . So this can't be an equilibrium. Likewise for  $P(W = I_2) < \frac{1}{2}$ .

So in an equilibrium  $E$  must be solely vote maximizing (as it can't win the election).

- $|I_1| < |I_2|$ .

For small enough  $\varepsilon, C_E^\varepsilon = I_1^-. V_E > F(y) \implies V_{I_1} < 1 - 2F(\frac{y}{2}) < V_E$ . And as we only need consider  $P(W = E) = 0$  then  $P(W = I_2) = 1, P(W = I_1) = 0$ .

So  $\tilde{I}_1 = -I_2$  and  $P(W = I_1 | \tilde{I}_1, I_2) = \frac{1}{2} \implies I_1$  is strictly better off. No equilibrium.

- $|I_1| = |I_2|$ .

$C_E^\varepsilon = \{I_1^-, I_2^+\}$ . Consider  $\tilde{I}_1 = I_1 - \delta$ , where  $\delta < 2I_1, \tilde{I}_1 \geq y$ .

Then  $C_E^\varepsilon = I_2^+ \implies V_{I_1}(\tilde{I}_1, I_2, E) > V_E(\tilde{I}_1, I_2, E)$  and  $V_{I_1}(\tilde{I}_1, I_2, E) > V_{I_2}(\tilde{I}_1, I_2, E)$ .  
So  $P(W = I_1 | \tilde{I}_1, I_2, E) = 1$ . No equilibrium.

**Case 4**  $I_1 \in Y, I_2 \notin Y$ . (So  $I_1 \in (y, 0), I_2 > -y$ ).

$E = I_2^+$  is dominated by  $E = I_1^-$ . So  $I_2^+ \notin C_E^\varepsilon$ .

$V_{I_2}$  is bounded by  $F(\frac{I_1+I_2}{2})$  (when  $E = I_1^-$ ).

Consider  $\tilde{I}_2 = -I_1 + \alpha$ , where  $\alpha$  is small ( $0 > \frac{-\alpha}{2} > I_1$  such that  $\tilde{I}_2 \in Y$ ). For small enough  $\varepsilon, C_E^\varepsilon = I_1^-$ . So  $V_E(I_1, \tilde{I}_2, E) = F(I_1) > V_{I_1}(I_1, \tilde{I}_2, E)$ .

$V_{I_2}(I_1, \tilde{I}_2, E) = 1 - F(\frac{\alpha}{2}) > F(I_1)$ . So  $P(W = I_2 | I_1, \tilde{I}_2, E) = 1$  and vote share has increased.

Therefore, regardless of  $P(W = I_2 | I_1, I_2, E)$ ,  $I_2$  has incentive to deviate. No equilibrium.

**Case 5**  $I_1 = y$ . Need to show that  $I_2 = -y$  is a strict best response.

If  $I_2 = -y \implies P(W = I_2) = \frac{1}{2}$ .

Consider  $\tilde{I}_2 < -y$ . For  $\tilde{I}_2 \leq 0$  we know that  $P(W = I_2 | I_1, \tilde{I}_2, E) = 0$ .  $I_2$  is strictly worse off. So consider  $\tilde{I}_2 \in (0, -y)$ . Then  $E = \tilde{I}_2^+$  and  $P(W = I_2 | I_1, \tilde{I}_2, E) = 0$ .  $I_2$  is strictly worse off.

Consider  $\tilde{I}_2 > -y$ . For small enough  $\varepsilon, I_1^-, \tilde{I}_2^+ \notin C_E^\varepsilon$ . E optimizes with a point s.t.

$$- V_{I_1}(I_1, \tilde{I}_2, E) = V_{I_2}(I_1, \tilde{I}_2, E).$$

In this case we must have  $P(W = I_2 | I_1, \tilde{I}_2, E) \leq \frac{1}{2}$ . As  $[I_1, I_2] \subset [I_1, \tilde{I}_2]$  then  $V_E(I_1, \tilde{I}_2, E) > V_E(I_1, I_2, E)$ , so  $V_{I_2}(I_1, \tilde{I}_2, E) < V_{I_2}(I_1, I_2, E)$  and  $P(W = I_2 | I_1, I_2, E) \geq P(W = I_2 | I_1, \tilde{I}_2, E)$ .

Therefore  $I_2$  is strictly worse off.

$$- V_{I_1} (I_1, \tilde{I}_2, E) > V_{I_2} (I_1, \tilde{I}_2, E).$$

$P(W = I_2 \mid I_1, \tilde{I}_2, E) = 0$ .  $I_2$  is strictly worse off.

So  $I_2 = -y$  is  $I_2$ 's strict best response to  $I_1 = y$ .

Thus  $\{y, -y\}$  is an equilibrium, and the only equilibrium involving  $I_1 = y$  or  $I_2 = -y$ .

### 9.1.2 Plurality: Enter only if have a positive probability of victory

**Case 1**  $I_1, I_2 \leq 0$ .

The entrant always wins in this case. So the proof is the same as for the ‘enter no matter what’ result.

Define  $\omega$ , where  $1 - 2F(\frac{\omega}{2}) = F(\frac{\omega}{2})$ . That is  $F(\frac{\omega}{2}) = \frac{1}{3}$ . And set  $\Omega = (\omega, -\omega)$  with  $\hat{\Omega}$  being the closure of  $\Omega$ .

**Case 2**  $I_1, I_2 \in \Omega / \{0\}$ .

Note that if  $\tilde{I}_2 = -I_1$  then  $E = \phi$  and  $P(W = I_1 \mid I_1, \tilde{I}_2) = P(W = I_2 \mid I_1, \tilde{I}_2) = \frac{1}{2}$ .

So in equilibrium  $P(W = I_j) \geq \frac{1}{2}$  for  $j = 1, 2$ . Therefore  $P(W = E \mid I_1, I_2) = 0$  and so  $E = \phi$ .

- $|I_1| > |I_2|$ .

Then  $P(W = I_2) = 1$  for  $E = \phi$ . Can't be an equilibrium.

- $I_1 = -I_2$ .

Consider  $\tilde{I}_1 = I_1 + \delta$ , where  $\delta \in (0, \frac{2}{3}|I_1|)$ .  $E \in (\tilde{I}_1, I_2)$  still gives  $P(W = E \mid \tilde{I}_1, I_2) = 0$ . So  $E \notin (\tilde{I}_1, I_2)$ .

For  $E = \tilde{I}_1^-$ ,  $V_E(\tilde{I}_1, I_2, E)$  is bounded by  $F(\tilde{I}_1)$ . And  $F(\tilde{I}_1) < 1 - F(\frac{\tilde{I}_1 + I_2}{2}) = V_{I_2}(\tilde{I}_1, I_2, E)$ . So  $E \notin \tilde{I}_1^-$ . Likewise for  $E = I_2^+$ . Therefore  $E = \phi$ . So we must have  $V_{I_1}(\tilde{I}_1, I_2, E) > V_{I_2}(\tilde{I}_1, I_2, E)$  and  $P(W = I_1 \mid \tilde{I}_1, I_2, E) = 1$ .  $I_1$  is strictly better off. No equilibrium.

**Case 3**  $I_1, I_2 \notin \hat{\Omega}$ .

The entrant always wins in this case (e.g. by locating at 0). So the proof is the same as the analogous case under the ‘enter no matter what’ assumption.

**Case 4**  $I_1 \in \hat{\Omega}/\{0\}, I_2 \notin \hat{\Omega}$ .

If  $E = \phi$ , then  $P(W = I_1) = 1$ . And  $\tilde{I}_2 = -I_1 \implies P(W = I_1 | I_1, \tilde{I}_2) = P(W = I_2 | I_1, \tilde{I}_2) = \frac{1}{2}$ , making  $I_2$  better off.

So for this to be an equilibrium  $E$  must enter. Therefore  $P(W = E) > 0$  and  $P(W = I_2) \geq \frac{1}{2}$  must hold in equilibrium.

This requires  $P(W = I_1) = 0$ . So we must have  $V_E = V_{I_2} > V_{I_1}$ .

- Assume  $E \in (I_1, I_2)$ .

Let  $E = t$  be the point at which  $V_E = V_{I_2} > V_{I_1}$  holds. By the continuity of  $F$  this point is unique. Then, also by the continuity of  $F$  and that  $V_{I_2} > V_{I_1}$ , there exists a small enough  $\delta$  such that  $\tilde{E} = t + \delta \implies V_{\tilde{E}} > V_{I_2} > V_{E_1}$  and so  $P(W = \tilde{E} | I_1, I_2, \tilde{E}) = 1$ . Therefore  $\tilde{E}$  is strictly better off and so can't be optimizing by playing  $E = t$ . No equilibrium.

- Assume  $E \in C_E^\varepsilon = I_1^-$ .

If for  $\varepsilon'$ ,  $V_E = V_{I_2}$  then by the continuity of  $F$ , there must exist a point  $r$ , where  $r < I_1$ , such that  $E = r \implies V_E = V_{I_2}$ . For small enough  $\varepsilon$   $I_1^- \subset (r, I_1)$  and so  $V_E > V_{I_2} \implies P(W = I_2) = 0$ . So for  $E$  in this interval there cannot be a tie between  $E$  and  $I_2$ . No equilibrium.

- Assume  $E \in C_E^\varepsilon = I_2^+$ .

Then  $V_{I_1} > \frac{1}{2}$  and so  $P(W = I_1) = 1$ . No equilibrium.

**Case 5**  $I_1 = \omega, I_2 \in \hat{\Omega}$ .

- Assume  $I_2 = -\omega$ .

Then  $E = 0$  and  $P(W = I_1 | I_1, I_2, E) = P(W = I_2 | I_1, I_2, E) = P(W = E | I_1, I_2, E) = \frac{1}{3}$ .

Consider  $\tilde{I}_1 = \omega + \delta$ , where  $\delta \in (0, \frac{2}{3}|w|)$ .  $E \in (\tilde{I}_1, I_2)$  gives  $P(W = E | \tilde{I}_1, I_2, E) = 0$ . So  $E \notin (\tilde{I}_1, I_2)$ .

For  $E = \tilde{I}_1^-$ ,  $V_E(\tilde{I}_1, I_2, E)$  is bounded by  $F(\tilde{I}_1)$ . And  $F(\tilde{I}_1) < 1 - F(\frac{\tilde{I}_1 + I_2}{2}) = V_{I_2}(\tilde{I}_1, I_2, E)$ . So  $E \notin \tilde{I}_1^-$ . Likewise for  $E = I_2^+$ . Therefore  $E = \phi$ . And so we must have  $V_{I_1}(\tilde{I}_1, I_2, E) > V_{I_2}(\tilde{I}_1, I_2, E)$  and  $P(W = I_1 | \tilde{I}_1, I_2, E) = 1$ .  $I_1$  is strictly better off. No equilibrium.

- Assume  $I_2 \in (0, -\omega)$ .

Note that  $\tilde{I}_1 = -I_2 \implies E = \phi$  and  $P(W = I_1 | \tilde{I}_1, I_2, E) = P(W = I_2 | \tilde{I}_1, I_2, E) = \frac{1}{2}$ . And  $\tilde{I}_2 = -I_1 \implies E = 0$  and  $P(W = I_1 | I_1, \tilde{I}_2, E) = P(W = I_2 | I_1, \tilde{I}_2, E) = P(W = E | I_1, I_2, E) = \frac{1}{3}$ . So in equilibrium  $P(W = I_1) \geq \frac{1}{2}$  and  $P(W = I_2) \geq \frac{1}{3}$ . So  $E = \phi$ . But then  $P(W = I_2) = 1$  and  $P(W = I_1) = 0$ . No equilibrium.

### 9.1.3 Run-Off: enter no matter what

Firstly I shall prove a lemma which is valid for both assumptions on entry. Vote share refers to primary vote share.

**Lemma 1** *If  $|I_1| < |I_2|$  then if  $E$  is (almost) maximizing  $P(W = I_2) = 0$ . That is, the incumbent furthest from the center never wins.*

**Proof.** If  $E = \phi$  then we have two candidate plurality and as  $|I_1| < |I_2|$  then  $P(W = I_1) = 1$  and so  $P(W = I_2) = 0$ . So consider  $E \neq \phi$ . If  $E \in I_2^+$  then  $V_{I_1} > \frac{1}{2}$  and so  $P(W = I_1) = 1$ . But  $V_E(E \in I_1^-) > V_E(E \in I_2^+)$  and so  $E \notin I_2^+$ . So if  $P(W = E) = 0$  then to maximize vote share  $E \in \{I_1^-, (I_1, I_2)\}$ , in which case  $|E| < |I_2|$  which implies  $P(W = I_2) = 0$  (as no matter who  $I_2$  faces in the run-off it will lose). So if  $P(W = I_2) > 0$  we must have  $P(W = E) > 0$ . For both of these conditions to hold we must have  $|E| = |I_2|$  and  $V_{I_1} \leq V_{I_2}, V_E$ . So  $E = -I_2$ . But if such a point exists then by the continuity of  $F$  there exists another point that makes  $E$  strictly better off and so this original point can't constitute an optimizer for  $E$ . Consider  $\tilde{E} = \frac{E+I_1}{2}$ , then  $V_E(I_1, I_2, \tilde{E}) > V_{I_1}(I_1, I_2, \tilde{E})$  and  $|\tilde{E}| < |I_2|$  so  $P(W = E|\tilde{E}) = 1$ . So there doesn't exist an almost maximizing location in which  $P(W = I_2) > 0$  and  $P(W = E) > 0$ . And as  $P(W = E) = 0 \implies P(W = I_2) = 0$  then if  $E$  is almost optimizing  $P(W = I_2) = 0$ . ■

Define  $W' = [W^*, -W^*]$ , where  $W^*$  solves  $F(W^*) = 1 - 2F(\frac{W^*}{2})$ .

**Case 1**  $I_1, I_2 \in W'$ .

Consider symmetric locations,  $|I_1| = |I_2|$ . If  $E \in (I_1, I_2)$  then  $V_E < 1 - 2F(\frac{W^*}{2})$  and  $V_{I_1}, V_{I_2} > F(W^*)$ , which imply that  $E$  loses in the first round. If  $E \in I_1^-, I_2^+$  then  $|E| < |I_1|, |I_2|$  and so  $E$  will never win the run-off. So  $P(W = E) = 0$ . If  $x \neq W^*$  then  $C_E^\varepsilon = \{I_1^-, I_2^+\}$ . If  $x = W^*$  and  $F$  isn't uniform then  $C_E^\varepsilon = \{I_1^-, I_2^+, 0^{--}\}$ . If  $x = W^*$  and  $F$  is uniform then  $C_E^\varepsilon = \{I_1^-, I_2^+, (I_1, I_2)\}$ . In all circumstances the incumbents are affected equally and we have  $P(W = I_1) = P(W = I_2) = \frac{1}{2}$ .

So any equilibrium in this range must satisfy  $P(W = I_1) \geq \frac{1}{2}, P(W = I_2) \geq \frac{1}{2}$ , as either incumbent could deviate to symmetry.

Consider deviations from symmetry. If an incumbent deviates outwards then by Lemma 1 their probability of victory is zero and they are strictly worse off. Now consider a deviation inwards, let  $\tilde{I}_1 \in (I_1, 0]$ .  $E = \tilde{I}_1^-$  then implies  $V_{\tilde{I}_1} < 1 - 2F(\frac{W^*}{2})$  as  $E, I_2 \in W'$ . As  $V_E = F(\tilde{I}_1) > F(W^*)$  and  $V_{I_2} = 1 - F(\frac{\tilde{I}_1+I_2}{2}) > F(W^*)$  then  $V_E, V_{I_2} > V_{\tilde{I}_1}$  and so  $P(W = \tilde{I}_1) = 0$  and  $I_1$  is strictly worse off (we also note that as for small enough  $\varepsilon$ ,  $|E| < |I_2|$  then  $P(W = E|\tilde{I}_1, I_2, E) = 1$ ). Thus,  $\{I_1, I_2\} = \{y, -y\}$  where  $y \in W'$  is a strict Nash equilibrium.

Now consider asymmetric positions, say  $|I_1| < |I_2|$ . From the lemma if  $E$  is almost maximizing  $P(W = I_2) = 0$ . So this can't be an equilibrium as  $\tilde{I}_2 = -I_1 \implies P(W = I_1) = \frac{1}{2}$ . No asymmetric equilibria.

**Case 2**  $I_1 \in W', I_2 \notin W'$ .

This implies that  $|I_2| > |I_1|$ , and so  $P(W = I_2) = 0$ . If  $\tilde{I}_2 = -I_1$  then  $I_1, \tilde{I}_2 \in W'$  and so, by Case 1,  $P(W = I_2) = \frac{1}{2}$ . So  $I_2$  is strictly better off. No equilibrium.

Define  $\overline{W} = [F^{-1}(\frac{1}{4}), F^{-1}(\frac{3}{4})]$ .

**Case 3**  $I_1, I_2 \notin \overline{W}$ .

This case provides examples of situations where an entrant does not maximize its utility by maximizing its primary vote share.

- If  $I_1, I_2 < F^{-1}(\frac{1}{4})$ .

We notice that  $E \in I_2^+ \implies P(W = E) = 1$ . The proof that this cannot constitute an equilibrium uses the same deviations and analysis as for Case 1 from ‘Plurality: enter no matter what’ (of course we only need consider the first two subcases from that proof).

- $I_1 < F^{-1}(\frac{1}{4}), I_2 > F^{-1}(\frac{3}{4})$ .

$E$  can win the election.  $E = F^{-1}(\frac{1}{4}) \implies V_{I_1} = F[\frac{I_1 + F^{-1}(\frac{1}{4})}{2}] < \frac{1}{4}$  and  $V_E = F[\frac{I_2 + F^{-1}(\frac{1}{4})}{2}] - F[\frac{I_1 + F^{-1}(\frac{1}{4})}{2}] > \frac{1}{4}$ . And so  $E$  isn’t eliminated in the first round. Then as  $|E| < |I_1|, |I_2|$   $E$  will win the run-off against whoever survives. So  $P(W = E) = 1$ .

There exists many points whereby  $E$  wins. We have to establish which point (approximately) maximizes its primary vote share, and so where it will locate.

Let  $|I_1| \leq |I_2|$ . Using the arguments of Case 2 from ‘Plurality: enter no matter what’ we can say the  $E$  will maximize its vote share by approaching  $I_1$  until  $V_{I_1} = V_{I_2}$ , if such a point exists, else  $E \in I_1^+$  and  $V_{I_1} > V_{I_2}$ . In the latter case  $V_{I_1} < \frac{1}{4}$  and  $V_E > \frac{1}{4}$ , so  $P(W = E) = 1$ . And we have  $C_E^\varepsilon = I_1^+$ . If  $E$  is chosen such that  $V_{I_1} = V_{I_2}$ , and  $V_{I_1} = V_{I_2} < \frac{1}{3}$  then  $P(W = E) = 1$  and so this is optimal for  $E$ . If  $V_{I_1} = V_{I_2} \geq \frac{1}{3}$  then  $P(W = E) \leq \frac{1}{3}$  and so this isn’t optimal for  $E$ .  $E$  must move towards an incumbent. Recall that if  $E$  locates at any point between the incumbents then it wins an interval of voters of constant length. Its choice of location is effectively a choice of where this interval should be placed on  $(I_1, I_2)$ . As  $E$  is moving towards an incumbent he is losing vote share and so wants to move as little as possible. Noting that at  $E = F^{-1}(\frac{1}{4})$ ,  $V_E > V_{I_1}$ , then by the continuity of  $F$  a point, call it  $K$ , will exist such that  $E = K \implies V_{I_1} = V_E$ . This corresponds to a selection of location for the interval of voters that  $E$  wins. If  $E$  were to reflect this interval about zero then by the symmetry of  $f$  we would have  $V_{I_2} = V_E$ . Let the location of  $E$  that corresponds to this location of its interval be denoted  $\overline{K}$ , and by the symmetry of  $f$  such a point exists. Note that for  $|I_1| \neq |I_2|$  we will have  $K \neq -\overline{K}$ . As for  $E \in K^-, \overline{K}^+$  we have  $|E| < |I_1|, |I_2|$  then  $C_E^\varepsilon = \{K^-, \overline{K}^+\}$ .

Now we know how  $E$  will react we can consider possible deviations for the incumbents. It suffices to consider only vote share with these deviations as prior to deviation  $P(W = I_1) = P(W = I_2) = 0$ .

- Assume  $|I_1| < |I_2|$

Consider  $\tilde{I}_2 = -I_1$ . After this deviation  $E$  will maximize vote share by locating at zero. With this deviation in mind consider the original subcases,

- Previously  $E$  was chosen such that  $V_{I_1} > V_{I_2}$  (i.e.  $E \in I_1^+$ ).

If  $E = 0 \implies V_{I_1} = V_{I_2} < \frac{1}{3}$  then  $C_E^\epsilon = 0^{-+}$ . As  $(I_1, \tilde{I}_2) \subset (I_1, I_2)$  then  $V_E(I_1, \tilde{I}_2) < V_E$ , and so  $V_{I_2}(I_1, \tilde{I}_2) > V_{I_2}$ .  $I_2$  is strictly better off. No equilibrium.

If  $E = 0 \implies V_{I_1} = V_{I_2} \geq \frac{1}{3}$  then  $E$  will move towards an incumbent, and its vote share will be even less than if it had located at zero (though now it will win the election). So  $V_{I_1}(I_1, \tilde{I}_2) + V_{I_2}(I_1, \tilde{I}_2) > V_{I_1} + V_{I_2}$ , and as the entrant will attack both incumbents equally we have  $E[V_{I_1}(I_1, \tilde{I}_2)] = E[V_{I_2}(I_1, \tilde{I}_2)]$ , which implies that  $E[V_{I_2}(I_1, \tilde{I}_2)] > V_{I_2}$  (where the expectation is because  $E$  is randomizing over two distinct intervals).  $I_2$  is strictly better off. No equilibrium.

- Previously  $C_E^\epsilon = x^{-+}$  and  $V_{I_1} = V_{I_2}$  at  $E = x$  (not just equal in expectation).

If  $E = 0 \implies V_{I_1} = V_{I_2} < \frac{1}{3}$  then  $C_E^\epsilon = 0^{-+}$ . As above,  $I_2$  is then strictly better off. No equilibrium.

If  $E = 0 \implies V_{I_1} = V_{I_2} \geq \frac{1}{3}$  then  $E$  will move towards an incumbent and, as above,  $I_2$  will be strictly better off. No equilibrium.

- Previously  $C_E^\epsilon = \{K^-, \overline{K}^+\}$  so that  $V_{I_1} = V_{I_2}$  only in expectation (i.e.  $E$  attacked one incumbent).

Thus  $E = 0 \implies V_{I_1} = V_{I_2} \geq \frac{1}{3}$  and so  $C_E^\epsilon(I_1, \tilde{I}_2) = \{K_*^-, \overline{K}_*^+\}$ . As  $(I_1, \tilde{I}_2) \subset (I_1, I_2)$  then  $V_E(I_1, \tilde{I}_2) < V_E$ . As in expectation we still have  $V_{I_1}(I_1, \tilde{I}_2) = V_{I_2}(I_1, \tilde{I}_2)$  then  $E[V_{I_2}(I_1, \tilde{I}_2)] > E[V_{I_2}]$ . So  $I_2$  is strictly better off. No equilibrium.

- Assume  $|I_1| = |I_2|$

Now it must be the case that  $V_{I_1} = V_{I_2}$ . This can result from  $C_E^\epsilon = 0^{-+}$  or  $C_E^\epsilon = \{K^-, \overline{K}^+\}$ , but not  $C_E^\epsilon = I_1^-$  or  $C_E^\epsilon = I_2^+$ .

- $C_E^\epsilon = 0^{-+}$  implies that  $V_{I_1} = V_{I_2} < \frac{1}{3}$ .

Then there exists a  $\gamma$  small enough such that  $\tilde{I}_2 = I_2 - \gamma$  whereby  $C_E^\epsilon = (\frac{2}{3})^{-+}$  and  $V_{I_1}(\tilde{I}_1, I_2) = V_{I_2}(\tilde{I}_1, I_2) < \frac{1}{3}$ . And so this is the optimal choice for  $E$ . As  $(I_1, \tilde{I}_2) \subset (I_1, I_2)$  then  $V_E(I_1, \tilde{I}_2) < V_E$ . This implies that  $V_{I_2}(\tilde{I}_1, I_2) > V_{I_2}$ . And so  $I_2$  is strictly better off. No equilibrium.

-  $C_E^\varepsilon = \{K^-, \overline{K}^+\}$ . This implies that  $E[V_{I_1}] = E[V_{I_2}] \geq \frac{1}{3}$ .

If  $\tilde{I}_2 = I_2 - \delta$  such that  $\tilde{I}_2 \notin \overline{W}$  then as  $(I_1, \tilde{I}_2) \subset (I_1, I_2)$  we have for small enough  $\delta$  that  $E = \frac{\delta}{2} \implies V_{I_1}(\tilde{I}_1, I_2) = V_{I_2}(\tilde{I}_1, I_2) \geq \frac{1}{3}$ . So  $E$  will again optimize by moving towards one of the incumbents. Also because  $(I_1, \tilde{I}_2) \subset (I_1, I_2)$  we have  $V_E(\tilde{I}_1, I_2) < V_E$  and so as  $E[V_{I_1}(\tilde{I}_1, I_2)] = E[V_{I_2}(\tilde{I}_1, I_2)]$  then  $E[V_{I_2}(\tilde{I}_1, I_2)] > E[V_{I_2}]$ . Thus  $I_2$  is strictly better off. No equilibrium.

**Case 4**  $I_1, I_2 \in \overline{W}/W'$ .

Note that the region  $\overline{W}/W'$  may be empty. For example if  $f$  is uniform on  $[-\frac{1}{2}, \frac{1}{2}]$ , this implies that  $\overline{W} = W'$ .

I need to show that if the condition on  $y$  isn't satisfied, or the incumbents aren't located symmetrically then they can't be in equilibrium.

- Firstly consider  $I_1, I_2 < W^*$ .

The proof that this cannot constitute an equilibrium uses the same deviations and analysis as for Case 1 from 'Plurality: enter no matter what'. No equilibrium.

- Secondly consider when the incumbents are symmetric but the condition isn't satisfied. That is  $I_2 = -I_1$  and  $F(\frac{I_1}{2}) < \frac{1}{3}$ .

Then  $E = 0 \implies V_{I_1}, V_{I_2} < \frac{1}{3}, V_E > \frac{1}{3}$ , and as  $|E| < |I_1|, |I_2|$   $E$  wins the run-off.  $P(W = E) = 1$ . Consider  $\tilde{I}_1 = I_1 + \gamma$  such that  $F(\frac{\tilde{I}_1}{2}) < \frac{1}{3}$ . Then  $E = -\frac{\gamma}{2} \implies P(W = E) = 1$  but  $V_E(\tilde{I}_1, I_2) < V_E$  and so  $V_{I_1}(\tilde{I}_1, I_2) > V_{I_1}$ . Thus  $I_1$  is strictly better off. No equilibrium.

- Now consider asymmetric locations where the condition isn't satisfied.

For  $I_1 < W^*, I_2 > -W^*$  let  $|I_1| = |I_2| - \alpha$ , where  $\alpha > 0$ . From Lemma 1 we have in equilibrium  $P(W = I_2) = 0$ . To maximize vote  $E \in (I_1, I_2)$ . If  $\tilde{I}_2 = -I_1$  then  $E$  maximizes his vote and wins the election at  $E \in 0^{-+}$ . But as  $(I_1, \tilde{I}_2) \subset (I_1, I_2)$  then  $V_E(I_1, \tilde{I}_2) < V_E$ . This implies that  $V_{I_2}(\tilde{I}_1, I_2) > V_{I_2}$ . And so  $I_2$  is strictly better off. No equilibrium.

- Finally, consider asymmetric locations when the condition is satisfied.



Once again let  $|I_1| = |I_2| - \alpha$ , where  $\alpha > 0$ , and so  $P(W = I_2) = 0$ . Let  $\tilde{I}_2 = -I_1$ . For  $E$  to win  $C_E^\varepsilon = \{K^-, \overline{K}^+\}$ , where  $K^-$  and  $\overline{K}^+$  are as defined above. This is because if  $E$  locates on the flanks then  $|E| > |I_1|, |I_2|$  so  $E$  couldn't win the run-off. If such points don't exist then  $E$  maximizes its vote at  $E = 0$ , by definition of  $W^*$ , and this implies that  $P(W = I_1|I_1, \tilde{I}_2) = P(W = I_2|I_1, \tilde{I}_2) = \frac{1}{2}$ . So  $I_2$  is strictly better off. No equilibrium.

So assume that such points exist. At  $E \in K^-$  we have  $V_E(I_1, \tilde{I}_2) > V_{I_1}(I_1, \tilde{I}_2)$ . Now consider the original decision of  $E$  before any deviations (being careful to distinguish between variables in terms of  $I_2$  and  $\tilde{I}_2$ ). If  $E \in I_1^-$  then  $V_E < V_{I_1}(I_1, \tilde{I}_2)$ . And if  $E \in K^-$  then  $V_E > V_E(I_1, \tilde{I}_2)$ . This implies that  $V_E(E = K^-) > V_E(E = I_1^-)$ . And as  $E \in K^- \implies P(W = E) = 1$ , then  $I_1^- \notin C_E^\varepsilon$ . So both before and after  $I_2$  deviates inwards the entrant locates between the incumbents and attacks one of them in the first round. As  $(I_1, \tilde{I}_2) \subset (I_1, I_2)$  then  $V_E(I_1, \tilde{I}_2) < V_E$ . This implies that  $E[V_{I_2}(\tilde{I}_1, I_2)] > E[V_{I_2}]$ . And so  $I_2$  is strictly better off. No equilibrium.

**Case 5**  $I_1 \notin \overline{W}, I_2 \in \overline{W}/W'$ .

- If  $I_1, I_2 < 0$  then  $E = I_2^+ \implies V_E > \frac{1}{2}$  and  $P(W = E) = 1$ . As  $f$  is atomless there exists a  $\tau$  small enough such that  $\tilde{I}_1 = I_2 - \tau$  implies  $V_{I_1}(\tilde{I}_1, I_2) > V_{I_1}$ . So  $I_1$  is strictly better off. No equilibrium.
- Let  $I_1 < F^{-1}(\frac{1}{4}), I_2 \in (W^*, F^{-1}(\frac{3}{4})]$ . So  $P(W = I_1) = 0$ . Consider  $\tilde{I}_1 = -I_2$  and repeat the analysis of case 4 above. No equilibrium.

#### 9.1.4 Run-Off: Enter only if have a positive probability of victory

**Case 1**  $I_1, I_2 \in W'$ .

Consider symmetric locations. Then  $P(W = E) = 0$  which implies  $E = \phi$  and  $P(W = I_1) = P(W = I_2) = \frac{1}{2}$ . So to have an equilibrium in this domain we must have  $P(W = I_1), P(W = I_2) \geq \frac{1}{2}$ .

For inwards deviations, say  $\tilde{I}_1 = I_1 + \alpha, \alpha > 0$ , then from 'Run-off: enter no matter what' Case 1 we know that this implies  $P(W = E|\tilde{I}_1, I_2, E) = 1 \implies P(W = \tilde{I}_1|\tilde{I}_1, I_2, E) = 0$  and so  $I_1$  is strictly worse off and wouldn't deviate. Considering deviations outwards then from lemma 1 we have  $P(W = \tilde{I}_1|\tilde{I}_1, I_2, E) = 0$  and so  $I_1$  is strictly worse off. Thus,  $\{I_1, I_2\} = \{y, -y\}$  where  $y \in W'$  is a strict Nash equilibrium.

Consider asymmetric locations,  $|I_1| < |I_2|$ . By lemma 1  $P(W = I_2) = 0$ . So  $I_2$  could deviate to  $\tilde{I}_2 = -I_1$  and be strictly better off. No equilibrium.

**Case 2**  $I_1 \in W', I_2 \notin W'$ .

By lemma 1  $P(W = I_2) = 0$ . So  $I_2$  could deviate to  $\tilde{I}_2 = -I_1$  and be strictly better off. No equilibrium.

**Case 3**  $I_1, I_2 \notin \overline{W}$ .

In Case 3 of ‘Run-off: enter no matter what’  $E \neq \phi$  both before and after the deviations considered. As such, the proof of Case 3 here is identical to that above. No equilibrium.

**Case 4**  $I_1, I_2 \in \overline{W}/W'$ .

Repeat Case 4 from ‘Run-off: enter no matter what’ with the following additions. If the deviations mentioned cause the entrant to alter its strategy to  $E = \phi$  (which wasn’t allowed in ‘enter no matter what’) then the vote share of the incumbents must be as least as great as it was when the entrant had to enter the market. As the deviating incumbent’s vote share went up in that case, then it must also go up when the entrant chooses to not enter. As for all deviators considered in the previous proof the probability of victory was originally zero then the increased vote share alone implies that the deviator is strictly better off. No equilibrium.

So consider when  $E = \phi$  before any deviation. If the condition  $F(\frac{y}{2}) \geq \frac{1}{3}$  isn’t satisfied then  $E = 0 \implies P(W = E) = 1$ , so  $E \neq \phi$ , a contradiction. So we need only consider asymmetric incumbent locations when the condition of the equilibrium is satisfied (remember symmetric locations may in fact constitute an equilibrium). Let  $|I_1| < |I_2|$ . Then  $P(W = I_2) = 0$ , from lemma 1. This implies that  $E \in (I_1, I_2) \implies V_E < V_{I_1}, V_{I_2}$  (as otherwise  $E$  would enter). If  $\tilde{I}_2 = -I_1$  then  $E$  still can’t win by locating in the center, and as the incumbent’s are symmetric  $E$  can’t win on the flanks, so again  $E = \phi$ . But now  $P(W = I_2 | I_1, \tilde{I}_2) = \frac{1}{2}$ . So  $I_2$  is strictly better off. No equilibrium.

**Case 5**  $I_1 \notin \overline{W}, I_2 \in \overline{W}/W'$ .

Proceed as in Case 5 from ‘run-off: enter no matter what’, with the same additional remarks as in Case 4 of this proof. No equilibrium.

## 9.2 Multiple Districts

### 9.2.1 Proposition 1

First I shall prove a lemma. Define  $Z' = [2Z^*, -2Z^*]$ .

**Lemma 2** For  $I_1, I_2 \in Z'$  and  $I_1 \leq I_2$  the districts won by both incumbents combined are  $D = [\frac{3I_1+I_2}{4}, \frac{I_1+3I_2}{4}]$ .

**Proof.** Note that all of these districts may not actually exist for a given  $G$  (i.e. existence requires  $g(\cdot) > 0$ ). This lemma is proven by showing that successful entry is not possible in these districts, and only these districts.

Consider entry between the incumbents. In a given district  $E$  can secure an interval of voters of length  $(\frac{I_2-I_1}{2})$  for itself. Ignoring for the moment the location of  $I_1$  and  $I_2$ , to maximize vote share with a given length interval on a symmetric, single peaked distribution, such as  $F$ , it is weakly optimal to center the interval about the peak, the median voter.

As  $I_1, I_2 \in Z'$  then  $I_2 - I_1 \leq 4Z^* \implies (\frac{I_2-I_1}{2}) \leq 2Z^*$ . From the definition of  $Z^*$  we have  $V_E \leq \frac{1}{3}$ . As for a fixed  $E$ ,  $V_{I_1}$  and  $V_{I_2}$  are strictly monotone functions in  $z_r$  (the district median voter; as  $I_1, I_2 \in Z'$ ) then only on a set of measure zero (that is, at one point) can  $V_{I_1} = V_{I_2} = V_E = \frac{1}{3}$  and  $P(W = E) \neq 0$ . Thus  $E$  can't win in any measurable set of districts by locating between  $I_1$  and  $I_2$ .

Consider now entry on a flank. Let district  $l$  have median  $z_l = \frac{3I_1+I_2}{4} - \delta$ , where  $\delta \in [\frac{I_1-I_2}{4}, \infty)$ .

Let  $E = I_1^-$ , then  $V_E(l)$  is bounded by  $F(I_1 - z_l)$  as  $\varepsilon \rightarrow 0$  (where  $V_E(l)$  is  $E$ 's vote share in district  $l$ ). This is the case as for the given range of  $\delta$   $E = I_2^+$  is dominated by  $E = I_1^-$  (as  $I_1$  is closer to the median in these districts).

$V_E(l) < F(I_1 - \frac{3I_1+I_2}{4} + \delta) = F(\frac{I_1-I_2}{4} + \delta)$ .  $V_{I_2}(l) = 1 - F(\frac{I_2-I_1}{4} + \delta)$ . As  $F$  is symmetric  $F(\frac{I_1-I_2}{4}) = 1 - F(\frac{I_2-I_1}{4})$ .

As  $F$  is a cdf then for  $\delta \leq 0, z_l \in D, F(\frac{I_1-I_2}{4} + \delta) \leq 1 - F(\frac{I_2-I_1}{4} + \delta) \implies V_E(l) < V_{I_2}(l)$  and so  $P(W = E) = 0$ .

For  $\delta > 0, z_l \notin D, F(\frac{I_1-I_2}{4} + \delta) > 1 - F(\frac{I_2-I_1}{4} + \delta) \implies V_E(l) > V_{I_2}(l)$ . Also, as  $\varepsilon \rightarrow 0, V_{I_1}(l) \rightarrow F(\frac{I_2-I_1}{4} + \delta) - F(\frac{I_1-I_2}{4} + \delta) < F(\frac{I_2-I_1}{4}) - F(\frac{I_1-I_2}{4}) = 1 - F(\frac{I_1-I_2}{4}) - F(\frac{I_1-I_2}{4}) \leq \frac{1}{3}$ . So for small enough  $\varepsilon$   $V_{I_1}(l) < \frac{1}{3}$  and then  $P(W = I_1) = 0$ . Therefore for  $\delta > 0$   $P(W = E) = 1$ .

So the incumbents win districts for  $\delta \leq 0$  and lose districts when  $\delta > 0$ . Thus, as  $F$  is symmetrical they win only districts in  $D$ . ■

It is obvious that  $I_1$  wins  $[\frac{3I_1+I_2}{4}, \frac{I_1+I_2}{2}]$  and  $I_2$  wins  $[\frac{I_1+I_2}{2}, \frac{3I_1+I_2}{4}]$ , with a tie in the middle district. Call these intervals  $D(I_1)$  and  $D(I_2)$ , respectively.

Define  $\overline{D}(I_1) = D(I_1) \cap [\underline{Z}, \overline{Z}]$ , and likewise for  $\overline{D}(I_2)$ . These are the districts won by each incumbent that actually exist.

Define  $M(I_1) = \int_{D(I_1)} g(z)dz$ , and likewise for  $M(I_2)$ . These are the shares of the districts won by each incumbent.

Define  $H = (2\underline{Z}, 2\overline{Z})$ , and  $\hat{H} = \text{closure}(H)$ .

**Case 1**  $I_1, I_2 \in H$ .

If  $\overline{D}(I_1) = \phi$  then,

If  $I_2 \neq 0$  set  $\tilde{I}_1 = -I_2$ . This implies  $\overline{D}(I_1) = [\frac{3I_1+I_2}{2}, 0]$  and so  $M(\tilde{I}_1) > 0$ . No equilibrium. Likewise for  $\overline{D}(I_2)$ .

If  $I_2 = 0$  set  $\tilde{I}_1 = \underline{Z}$ . This implies  $\overline{D}(I_1) = [\frac{3\underline{Z}}{4}, \frac{\underline{Z}}{2}]$  and so  $M(\tilde{I}_1) > 0$ . No equilibrium.

So in equilibrium both  $M(I_1), M(I_2) > 0$ .

Therefore we must have  $\frac{I_1+I_2}{2} \in (\underline{Z}, \overline{Z})$ .

If  $\frac{3I_1+I_2}{4} < \underline{Z}$  then set  $\tilde{I}_1 = I_1 + \delta$ , where  $\delta$  is s.t.  $\frac{3\tilde{I}_1+I_2}{4} = \underline{Z}$ .

We have  $D(I_1) = [\frac{3I_1+I_2}{4}, \frac{I_1+I_2}{2}] \implies \overline{D}(I_1) = [\underline{Z}, \frac{I_1+I_2}{2}]$ . Now  $D(\tilde{I}_1) = [\frac{3\tilde{I}_1+I_2}{4}, \frac{\tilde{I}_1+I_2}{2}] \implies \overline{D}(I_1) = [\underline{Z}, \frac{\tilde{I}_1+I_2}{2}] = [\underline{Z}, \frac{I_1+I_2}{2} + \frac{\delta}{2}]$ .

And so  $\overline{D}(I_1) \subset \overline{D}(\tilde{I}_1)$ , making  $I_1$  strictly better off. No equilibrium. Likewise for  $\frac{I_1+3I_2}{4} > \overline{Z}$ .

So for an equilibrium we must have that  $[\frac{3I_1+I_2}{4}, \frac{I_1+3I_2}{4}] = D \subseteq [\underline{Z}, \overline{Z}]$ . Though  $\frac{3I_1+I_2}{4} \neq \underline{Z}$  as this implies that  $I_1 \leq 2\underline{Z}$ , but then  $I_1 \notin H$ . Likewise for  $\frac{I_1+3I_2}{4} \neq \overline{Z}$ . So  $D \subset [\underline{Z}, \overline{Z}]$ .

Now, assuming  $|I_1| \geq |I_2|$ , consider a deviation  $\tilde{I}_1 = I_1 - \gamma$ , where  $\gamma > 0$  and such that  $\tilde{I}_1 \in H$ .

$D(I_1) = \overline{D}(I_1) = [\frac{3I_1+I_2}{4}, \frac{I_1+I_2}{2}]$ . And  $D(\tilde{I}_1) = \overline{D}(\tilde{I}_1) = [\frac{3\tilde{I}_1+I_2}{4}, \frac{\tilde{I}_1+I_2}{2}] = [\frac{3I_1+I_2}{4} - \frac{3\gamma}{4}, \frac{I_1+I_2}{2} - \frac{\gamma}{2}]$ .

And so we get the relationship,  $\overline{D}(\tilde{I}_1) = \overline{D}(I_1) - [\frac{I_1+I_2}{2} - \frac{\gamma}{2}, \frac{I_1+I_2}{2}] + [\frac{3I_1+I_2}{4} - \frac{3\gamma}{4}, \frac{3I_1+I_2}{4}]$ .

If  $g$  is strictly quasi-concave then by condition 1 we see that  $M(\tilde{I}_1) > M(I_1)$ .

If  $g$  is quasi-convex then as for small enough  $\gamma$ ,  $\frac{3I_1+I_2}{4} < \frac{I_1+I_2}{2} - \frac{\gamma}{2} < 0$ , then  $M(\tilde{I}_1) > M(I_1)$  (as there is more density at the extremes). No equilibrium.

**Case 2**  $I_1, I_2 \in Z'/\hat{H}$ . Note that for  $\underline{Z} = Z^*$  this set is empty.

- $I_1, I_2 < 2\underline{Z}$ . Then  $E = I_2^+$  wins all districts,  $M(I_1) = M(I_2) = 0$ .

Consider  $\tilde{I}_2 = -I_1$ .  $D(I_2) = [0, \frac{\tilde{I}_2}{2}] \implies \overline{D}(I_2) = [0, \overline{Z}]$  as  $I_1 < 2\underline{Z}$ . This implies  $M(I_2) > 0$ .

$I_2$  is strictly better off. No equilibrium.

- $I_1 < 2\underline{Z}, I_2 > -2\underline{Z}$ . Let  $|I_1| \geq |I_2|$ .

$D(I_1) = [\frac{3I_1+I_2}{4}, \frac{I_1+I_2}{2}] \implies \overline{D}(I_1) = [\underline{Z}, \frac{I_1+I_2}{2}]$ . Consider a deviation,  $\tilde{I}_1 = I_1 + \delta$ , where  $\delta$  is s.t.  $\tilde{I}_1 < 2\underline{Z}$ .

$D(\tilde{I}_1) = [\frac{3\tilde{I}_1+I_2}{4}, \frac{\tilde{I}_1+I_2}{2}] = [\frac{3I_1+I_2}{4} + \frac{3\delta}{4}, \frac{I_1+I_2}{2} + \frac{\delta}{2}] \implies \overline{D}(\tilde{I}_1) = [\underline{Z}, \frac{I_1+I_2}{2} + \frac{\delta}{2}]$ . As  $[\frac{I_1+I_2}{2}, \frac{I_1+I_2}{2} + \frac{\delta}{2}]$  is measurable we have  $M(\tilde{I}_1) > M(I_1)$ . No equilibrium.

**Case 3**  $I_1 \in Z', I_2 \notin Z'$ .

I will use the  $D$  notation from lemma 2, though calling it  $D'$  here because  $I_2 \notin Z'$  means the lemma may not be applicable. What can be seen is that some of the arguments from the lemma can be preserved. Entry on the flank is still precluded but there may be entry in the center. So we see that  $D(I_1) \subseteq D'(I_1)$  and  $D(I_2) \subseteq D'(I_2)$ . That is, the set of districts that would be won (if they existed) is a subset of  $D'$ .

WOLOG let  $I_1 \leq 0$ . If  $I_2 < 2Z^*$ , then  $D'(I_2) = [\frac{3I_2+I_1}{4}, \frac{I_2+I_1}{2}]$ . As  $\frac{I_2+I_1}{2} < Z^*$ ,  $\overline{D}(I_2) = 0$ .

If  $I_1 < 0$ , the deviation of  $\tilde{I}_2 = -I_1 \implies D(\tilde{I}_2) = [0, \frac{-I_1}{2}]$  (as now  $\tilde{I}_2 \in Z'$ )  $\implies M(\tilde{I}_2) > 0$  as  $I_1 \neq 0$ .

If  $I_1 = 0$ , the deviation of  $\tilde{I}_2 = \overline{Z} \implies D(\tilde{I}_2) = [\frac{\overline{Z}}{2}, \frac{3\overline{Z}}{4}]$  (as now  $\tilde{I}_2 \in Z'$ )  $\implies M(\tilde{I}_2) > 0$  as  $\overline{Z} \neq 0$ .

So for  $I_1 \leq 0$  consider  $I_2 > -2Z^*$ .  $D(I_2) \subseteq D'(I_2) = [\frac{I_1+I_2}{2}, \frac{I_1+3I_2}{4}] \implies \overline{D}(I_2) \subseteq \overline{D}'(I_2) = [\frac{I_1+I_2}{2}, \overline{Z}]$ . If this set is empty then consider the deviations above.

Now consider  $\overline{D}'(I_2) \neq \phi$  (so  $\frac{I_1+I_2}{2} < \overline{Z}$ ) and let  $\tilde{I}_2 = -2Z^*$ .  $D(\tilde{I}_2) = [\frac{I_1+\tilde{I}_2}{2}, \frac{I_1+3\tilde{I}_2}{4}] \implies \overline{D}(I_2) = [\frac{I_1+\tilde{I}_2}{2}, \overline{Z}]$  as  $I_1 \geq 2Z^*$  (the lemma is now applicable).

As  $\tilde{I}_2 < I_2$ , and  $[\frac{I_1+\tilde{I}_2}{2}, \frac{I_1+I_2}{2}]$  is measurable then  $\overline{D}'(I_2) \subset \overline{D}(\tilde{I}_2)$  and so it must be that  $M(\tilde{I}_2) > M(I_2)$ .  $I_2$  is strictly better off. No equilibrium.

**Case 4**  $I_1 = 2\underline{Z}$ . Show that  $I_2 = -2\underline{Z}$  is a strict best response for  $I_2$ .

- $I_2 = -2\underline{Z}$ . We have  $D(I_1) = [\underline{Z}, 0], D(I_2) = [0, -\underline{Z}] \implies M(I_2) = \frac{1}{2}$ .
- $I_2 \leq \underline{Z}$ .  $E = \{\max[I_1, I_2]\}^+$ . If  $I_1 \leq I_2$  then  $D(I_2) \subset [I_1, I_2]$ , or if  $I_1 > I_2$  then  $D(I_2) \subset [I_2, I_1]$ . Either way this implies that  $\overline{D}(I_2) = 0$ . And so  $M(I_2) = 0$ .
- $\underline{Z} < I_2 < -2\underline{Z}$ . We have  $M(I_2) \geq 0$ . For  $M(I_2) = 0$  we are done. For  $M(I_2) > 0$  consider  $\tilde{I}_2 = I_2 + \alpha$ , where  $\alpha > 0$ , and such that  $\tilde{I}_2 \leq 2\overline{Z}$ . Then  $I_2$  will win districts with measure  $\frac{3\alpha}{4}$  but lose districts with, at most, measure  $\frac{\alpha}{2}$ . If  $g$  is strictly quasi-concave then by Condition 1,  $M(\tilde{I}_2) > M(I_2)$ . So as  $\tilde{I}_2 \rightarrow 2\overline{Z}$ ,  $M(\tilde{I}_2)$  is increasing and approaching  $M(I_2 = 2\overline{Z}) = \frac{1}{2}$ . So for  $I_2 < 2\overline{Z}$  we have  $M(I_2) < \frac{1}{2}$ .

This is not necessarily true for  $g$  strictly quasi-convex. If  $I_2$  was to win districts in an interval of length  $|\underline{Z}|$  on  $g$ , then it can be seen that the optimal location of this interval is  $[\underline{Z}, 0]$ , or  $[0, \overline{Z}]$ , in which case  $M(I_2) = \frac{1}{2}$ . For  $I_2 < 2\overline{Z}$  the measure of  $\overline{D}(I_2)$  is less than  $\frac{1}{2}$ . As  $G$  is strictly increasing once  $G(\cdot) > 0$ , we must have  $M(I_2) < \frac{1}{2}$  for  $I_2 < 2\overline{Z}$ .

- $I_2 > -2\underline{Z}$ .  $D'(I_2) = [\frac{2\underline{Z}+I_2}{2}, \frac{2\underline{Z}+3I_2}{4}] \implies \overline{D}(I_2) = [\frac{2\underline{Z}+I_2}{2}, -\underline{Z}]$ .

And as  $2\underline{Z} + I_2 > 0$  we have  $M(I_2) < \frac{1}{2}$ .

So  $I_2 = -2\underline{Z}$  is a strict best response for  $I_2$ . Therefore  $\{I_1, I_2\} = \{2\underline{Z}, -2\underline{Z}\}$  is a strict equilibrium, consequently it is the only equilibrium involving  $I_1 = 2\underline{Z}$  or  $I_2 = -2\underline{Z}$ .

### Case 5 $I_1, I_2 \notin Z'$ .

There is entry in every district. For the case  $I_1 < 2Z^*, I_2 > -2Z^*$  the entrants locate between the incumbents. This is because  $\frac{|I_1 - I_2|}{2} > 2Z^*$ , and using the arguments of lemma 2 an entrant centering its interval of voters won at the median in a district will win that district. When  $I_1, I_2 < 2Z^*$  entry at  $E = [\max\{I_1, I_2\}]^+$  wins every district. So we have that  $M(I_1) = M(I_2) = 0$ . Recall that it was assumed that if successful entry was possible then only one new party would enter and win the district.

- $I_1, I_2 < 2Z^*$ .

- Let  $I_1 < I_2$ .

For a district with median  $z_l \in [\underline{Z}, \overline{Z}]$ ,  $I_2$ 's vote share is given by  $F(I_2 - z_l) - F(\frac{I_2 + I_1}{2} - z_l)$ . If  $\tilde{I}_2 = 2Z^*$  then  $E = \tilde{I}_2^+$  and now  $V_{I_2}(l|I_1, \tilde{I}_2) = F(2Z^* - z_l) - F(\frac{2Z^* + I_1}{2} - z_l) > F(I_2 - z_l) - F(\frac{I_2 + I_1}{2} - z_l)$  as  $I_2, \tilde{I}_2 < \underline{Z}$  and  $f(2Z^* - z_l) > 0$ . Consequently  $I_2$  is strictly better off. No equilibrium.

- Let  $I_1 = I_2$ .

Then, for a district with median  $z_l$ ,  $I_1$ 's vote share approaches  $\frac{1}{2}F(I_1 - z_l)$  as  $\varepsilon \rightarrow 0$ .

If  $F(I_1 - z_l) \neq 0$  for any  $z_l$  then as  $F$  is strictly increasing once  $F(\cdot) > 0$ ,  $\delta$  small enough can be found s.t.  $\tilde{I}_1 = I_1 - \delta \implies V_{I_1}(l|\tilde{I}_1, I_2) = F(I_1 - \frac{\delta}{2} - z_l) > \frac{1}{2}F(I_1 - z_l)$  and so  $I_1$  is weakly better off in every district and strictly better off in some.

If  $F(I_1) = 0$  for all  $z_l$  then consider  $\tilde{I}_2 = 2Z^*$ . As above this implies  $E = \tilde{I}_1^+$  and  $V_{I_1}(l|\tilde{I}_1, I_2) > 0$  for all  $z_l$ . And so  $I_1$ 's vote share increases in every district. No equilibrium.

- $I_1 < 2Z^*, I_2 > -2Z^*$ .

Let  $|I_1| \leq |I_2|$ . Earlier it was shown that when an entrant locates between incumbents it optimizes by choosing  $E$  such that  $V_{I_1} = V_{I_2}$  if it can, or by approaching the incumbent closest to the median voter.

So we can see immediately that  $V_{I_1}(z_l \leq 0) \geq V_{I_2}(z_l \leq 0)$ . For districts where  $z_l > 0$ , it may be the case that  $I_2$  is closer to the median. I wish to show that even in these districts  $V_{I_1}(z_l > 0) \geq V_{I_2}(z_l > 0)$ . Consider district  $l$  where  $z_l > 0$ . For this result to not hold we require that  $f(\frac{I_1+I_2}{2} - z_l) < f(I_2 - \bar{Z})$  so that there doesn't exist an  $E$  such that  $V_{I_1} = V_{I_2}$ . But as  $|I_1| \leq |I_2|$  we get that  $\frac{I_1+I_2}{2} \geq 0 \implies \frac{I_1+I_2}{2} - z_l \geq \underline{Z}$ . Also  $I_2 > 2Z^*$  and so  $I_2 - z_l > \bar{Z}$ . Therefore  $f(\frac{I_1+I_2}{2} - z_l) > f(I_2 - \bar{Z})$  and so  $V_{I_1}(l) \geq V_{I_2}(l)$  for all  $z_l \in [\underline{Z}, \bar{Z}]$ .

Now consider a deviation by  $I_2$  to  $\tilde{I}_2 = -2Z^*$ . As  $|I_1| > |\tilde{I}_2|$ ,  $V_{I_1}(l|I_1, \tilde{I}_2, E) \leq V_{I_2}(l|I_1, \tilde{I}_2, E)$  for all districts. In every district the entrant will still locate between the incumbents. As  $|\frac{\tilde{I}_2 - I_1}{2}| < |\frac{I_2 - I_1}{2}|$  the vote share for the entrant in each district must decline. And so  $V_{I_1}(l|I_1, \tilde{I}_2, E) + V_{I_2}(l|I_1, \tilde{I}_2, E) > V_{I_1}(l) + V_{I_2}(l)$  for every  $l$ . As  $V_{I_1}(l) \geq V_{I_2}(l)$  and  $V_{I_1}(l|I_1, I_2, E) \leq V_{I_2}(l|I_1, \tilde{I}_2, E)$  we must have that  $V_{I_2}(l|I_1, \tilde{I}_2, E) > V_{I_2}(l)$  for every  $l$ . And so  $I_2$  is strictly better off. No equilibrium.

**Case 6**  $I_1 \in H, I_2 \in Z'/\hat{H}$ .

If  $I_1, I_2 \leq 0$  then  $I_1 \in (2\underline{Z}, 0], I_2 \in [2Z^*, 2\underline{Z}]$ .  $D(I_2) = [\frac{I_1+3I_2}{4}, \frac{I_1+I_2}{2}] \implies \bar{D}(I_2) = \phi$  as  $\frac{I_1+I_2}{2} < \underline{Z}$ . Thus  $M(I_2) = 0$ . If  $I_1 < 0$  then consider the deviation  $\tilde{I}_2 = -I_1$ . If  $I_1 = 0$  then consider the deviation  $\tilde{I}_2 = \bar{Z}$ . Both deviations imply, as shown previously, that  $M(\tilde{I}_2) > 0$ . No equilibrium. Likewise for  $I_1, I_2 \geq 0$ .

So consider  $I_1 < 0$  and  $I_2 \in (2\bar{Z}, 2Z^*]$ . Thus  $D(I_2) = [\frac{I_1+I_2}{2}, \frac{I_1+3I_2}{4}] \implies \bar{D}(I_2) = [\frac{I_1+I_2}{2}, \bar{Z}]$ , as  $\frac{I_1+3I_2}{4} > \bar{Z}$ . Now consider the deviation,  $\tilde{I}_2 = I_2 - \alpha, \alpha > 0$ , such that  $\frac{I_1+3\tilde{I}_2}{4} = \bar{Z}$ . Then  $D(\tilde{I}_2) = [\frac{I_1+\tilde{I}_2}{2}, \frac{I_1+3\tilde{I}_2}{4}] \implies \bar{D}(\tilde{I}_2) = [\frac{I_1+I_2-\alpha}{2}, \bar{Z}]$ . And so  $D(I_2) \subset D(\tilde{I}_2)$  and  $I_2$  is strictly better off. No equilibrium.

### 9.2.2 Proposition 1A

The proof is identical to that for Proposition 1, with Condition 1A substituted for Condition 1. The extra restriction in Condition 1A ensures that the arguments of Proposition 1 can also be used to prove Proposition 1A.

### 9.2.3 Proposition 2

Using the notation of the previous section, we can see that for  $I_1 \leq I_2$  we have  $D' = [\frac{3I_1+I_2}{4}, \frac{I_1+3I_2}{4}]$ . As  $D \subseteq D'$  to preclude entry we require  $\frac{3I_1+I_2}{4} \leq \underline{Z}$  and  $\frac{I_1+3I_2}{4} \geq \bar{Z}$ . Solving these two requirements simultaneously and recalling that  $\underline{Z} = -\bar{Z}$ , we have

$I_1 \leq 2\underline{Z}$  and  $I_2 \geq -2\underline{Z}$ . As  $\underline{Z} < Z^*$  this implies that  $I_1 < 2Z^*$  and  $I_2 > -2Z^*$ , but then we will have entry between the incumbents in every district (see Proposition 1, case 5). So there does not exist a pair of locations for the incumbents which are able to preclude entry in every district. Thus, there does not exist an equilibrium which precludes entry in all districts.

### 9.2.4 Proposition 3

This proof is very similar to that of Proposition 1, in many instances the only difference being a change in the domain of a case.

Define  $H^\# = (2Z^\#, -2Z^\#)$ , and  $\hat{H}^\# = \text{closure}(H^\#)$ .

**Case 1**  $I_1 I_2 \in H^\#$ .

Proceed as with Case 1 in Proposition 1. We can ignore quasi-convexity requirement. No equilibrium.

**Case 2**  $I_1, I_2 \in Z'/\hat{H}^\#$ .

- Consider firstly  $2Z^\# \leq \underline{Z}$ .

(a)  $I_1, I_2 < 2Z^\#$

Then  $E = I_2^+$  wins all districts,  $M(I_1) = M(I_2) = 0$ . Consider  $\tilde{I}_2 = -I_1$ .  $D(I_2) = [0, \frac{\tilde{I}_2}{2}] \implies M(I_2) > 0$ .  $I_2$  is strictly better off. No equilibrium.

(b)  $I_1 < 2Z^\#, I_2 > -2Z^\#$ .

Let  $|I_1| - \gamma = |I_2|$ , where  $\gamma > 0$ . Now  $D(I_1) = [\frac{-I_2}{2} - \frac{3\gamma}{4}, \frac{-\gamma}{2}]$  and  $D(I_2) = [\frac{-\gamma}{2}, \frac{I_2}{2} - \frac{\gamma}{4}]$ . If  $\frac{3I_1 + I_2}{4} < \underline{Z}$  then the deviation  $\tilde{I}_1 = I_1 + \delta$ , such that  $\frac{3\tilde{I}_1 + I_2}{4} = \underline{Z}$ , makes  $I_1$  strictly better off (see Proposition 1, case 2). For  $\frac{3I_1 + I_2}{4} \geq \underline{Z}$  we shall consider two deviations. Consider  $\tilde{I}_1 = -I_2$ , and  $\tilde{I}_2 = -I_1$ . When  $I_1$  deviates with  $\tilde{I}_1 = -I_2$  we find  $D(\tilde{I}_1) = [\frac{-I_2}{2}, 0] \implies M(\tilde{I}_1) = M(I_1) + [G(0) - G(\frac{-\gamma}{2})] - [G(\frac{-I_2}{2}) - G(\frac{-I_2}{2} - \frac{3\gamma}{4})]$ . If  $[G(0) - G(\frac{-\gamma}{2})] > [G(\frac{-I_2}{2}) - G(\frac{-I_2}{2} - \frac{3\gamma}{4})]$  then  $I_1$  will deviate. No equilibrium. Otherwise consider the deviation by  $I_2$ . In this case we find  $D(\tilde{I}_2) = [0, \frac{I_2 + \gamma}{2}] \implies M(\tilde{I}_2) = M(I_2) + [G(\frac{I_2 + \gamma}{2}) - G(\frac{I_2}{2} - \frac{\gamma}{4})] - [G(0) - G(\frac{-\gamma}{2})]$ . By the symmetry of  $g$ ,  $G(\frac{I_2 + \gamma}{2}) - G(\frac{I_2}{2} - \frac{\gamma}{4}) > G(\frac{-I_2}{2}) - G(\frac{-I_2}{2} - \frac{3\gamma}{4})$ . And so if it isn't profitable for  $I_1$  to deviate then we have  $[G(\frac{I_2 + \gamma}{2}) - G(\frac{I_2}{2} - \frac{\gamma}{4})] > [G(0) - G(\frac{-\gamma}{2})] \implies M(\tilde{I}_2) > M(I_2)$ , and so it is then profitable for  $I_2$  to deviate. So no equilibrium.

Now for  $I_1 < 2Z^\#, I_2 > -2Z^\#$ , let  $|I_1| = |I_2|$ . And so  $D(I_1) = [\frac{I_1}{2}, 0]$ . If  $I_1$  deviates inwards,  $\tilde{I}_1 = I_1 + \alpha$ , and so  $D(\tilde{I}_1) = [\frac{I_1}{2} + \frac{3\alpha}{4}, \frac{\alpha}{2}]$ . As we know that  $\frac{I_1}{2} < Z^\#$  then we have  $g(\frac{I_1}{2}) < \frac{2}{3}g(0)$ . Because  $g$  is continuous there exists a  $\alpha$  small enough such that  $\forall \alpha' < \alpha$ ,  $g(\frac{I_1}{2} + \frac{3\alpha'}{4}) < \frac{2}{3}g(\frac{\alpha'}{2})$ , which implies that  $M(\tilde{I}_1) > M(I_1)$  as then  $I_1$  gains more districts in the center than it loses on the fringe. No equilibrium.



- Now consider  $\underline{Z} < 2Z^\#$ .

For  $I_1 < 0, I_2 > 0$  this is the same as for  $\underline{Z} \geq 2Z^\#$ . So from now assume that  $I_1, I_2 < 2Z^\#$ . We recall that by setting  $\tilde{I}_1 = -I_2$ ,  $M(I_1) > 0$ , and likewise for  $I_2$ , so in equilibrium we require that  $M(I_1), M(I_2) > 0$ . Considering positions such that  $M(I_1), M(I_2) > 0$  and noting that  $I_1 = I_2 \implies M(I_1) = M(I_2) = 0$  we need only consider  $I_1 < I_2 < 2Z^\#$ . Consider now the deviation  $\tilde{I}_2 = I_2 + \delta$ , such that  $\tilde{I}_2 < 2Z^\#$ . Then  $D(\tilde{I}_2) = D(I_2) + [\frac{I_1+3I_2}{4}, \frac{I_1+3I_2}{4} + \frac{3\delta}{4}] - [\frac{I_1+I_2}{2}, \frac{I_1+I_2}{2} + \frac{\delta}{2}]$ . As  $\tilde{I}_2 < 0$  then as  $M(I_2) > 0$  and  $g$  is strictly increasing on this domain it implies that  $M(\tilde{I}_2) > M(I_2)$ . No equilibrium.

**Case 3**  $I_1 \in Z', I_2 \notin Z'$ .

Proceed as with Case 3 in Proposition 1. No equilibrium.

**Case 4**  $I_1 = 2Z^\#$ . Show that  $I_2 = -2Z^\#$  is a strict best response for  $I_2$ , given that  $g$  is concave, and show that when  $g$  is not concave  $I_2 \neq -2Z^\#$  cannot constitute an equilibrium.

- $I_2 = -2Z^\#$ . We have  $D(I_1) = [Z^\#, 0], D(I_2) = [0, -Z^\#] \implies M(I_2) = \frac{1}{2} - G(Z^\#)$ .

The rest of the locations for  $I_2$  are shown to produce  $M(I_2) < \frac{1}{2} - G(Z^\#)$  with the same techniques as in Proposition 1 and thus can't constitute equilibria. The only addition to the proof is for  $I_1 < I_2$  when  $2Z^\# > \underline{Z}$  (when there are districts to the left of  $I_1$ ).

For  $g$  concave we need to make sure that there isn't more density on the flanks that would make  $I_2$  better off. We have that  $g(Z^\#) < \frac{2}{3}g(0)$ , and so by the concavity of  $g$ , we also have  $g(2Z^\#) < \frac{1}{3}g(0)$ , and  $g(3Z^\#) = 0$ . So for  $I_2 < I_1$ ,  $\overline{D}(I_2) \subset [I_2, I_1]$  and its measure is bounded by  $\frac{|Z^\#|}{4}$ . And so  $M(I_2)$  is bounded by  $G(Z^\#) - G(Z^\# - \frac{|Z^\#|}{4})$  which by the concavity of  $g$  implies  $M(I_2) < \frac{1}{2} - G(Z^\#)$ . So  $I_2 = -2Z^\#$  is a strict best response for  $I_2$  when  $g$  is concave. So  $\{I_1, I_2\} = \{2Z^\#, -2Z^\#\}$  is a strict equilibrium, and therefore is the only equilibrium involving  $I_1 = 2Z^\#$  or  $I_2 = -2Z^\#$ .

If  $g$  is not concave then  $I_2$  may wish to locate at some point such that  $I_2 < I_1$ . In this case we need to show that  $I_1$  would have incentive to deviate, thus precluding an equilibrium. Consider  $\tilde{I}_1 = I_1 + \delta$ , then  $\overline{D}(\tilde{I}_1) = [\frac{\tilde{I}_1+I_2}{2}, \frac{3\tilde{I}_1+I_2}{4}] = \overline{D}(I) + [\frac{3I_1+I_2}{4}, \frac{3I_1+I_2}{4} + \frac{3}{4}\delta] - [\frac{I_1+I_2}{2}, \frac{I_1+I_2}{2} + \frac{1}{2}\delta]$ . As, for small enough  $\delta$ ,  $g$  is strictly increasing over this range then  $I_1$  is strictly better off. So for quasi-concave  $g$  functions  $I_2 \neq -2Z^\#$  can't be an equilibrium. Thus if an equilibrium exists in this domain then it must be  $\{I_1, I_2\} = \{2Z^\#, -2Z^\#\}$ .

**Case 5**  $I_1, I_2 \notin Z'$ .

Proceed as with Case 5 in Proposition 1. No equilibrium.

**Case 6**  $I_1 \in H^\#, I_2 \in Z'/\hat{H}^\#$ .

If  $I_1, I_2 \leq 0$  and  $\frac{I_1+I_2}{2} \leq \underline{Z}$  then we proceed as in Proposition 1. Likewise for  $I_1, I_2 \geq 0$ . No equilibrium. So consider  $Z^\# > \frac{I_1+I_2}{2} > \underline{Z}$ , where  $D(I_1) = \overline{D}(I_1) = [\frac{I_1+I_2}{2}, \frac{I_2+3I_1}{4}]$ . For  $I_1 < 0$  let  $\tilde{I}_1 = \frac{I_1}{2}$ . Then  $D(\tilde{I}_1) = \overline{D}(\tilde{I}_1) = [\frac{\tilde{I}_1+I_2}{2}, \frac{I_2+3\tilde{I}_1}{4}] = [\frac{I_1+I_2}{2} - \frac{I_1}{4}, \frac{I_2+3I_1}{4} - \frac{3I_1}{8}]$ . As  $\frac{I_1}{2} < 0$  then the strict quasi-concavity and symmetry of  $g$  imply that  $M(I_1) < M(\tilde{I}_1)$ , and so  $I_1$  is strictly better off. No equilibrium. Consider now  $I_1 = 0$ . Let  $\tilde{I}_1 = \alpha$ , where  $\alpha > 0$  is such that  $|\frac{I_2+3\alpha}{4}| < 0$ . And then  $D(\tilde{I}_1) = \overline{D}(\tilde{I}_1) = [\frac{\tilde{I}_1+I_2}{2}, \frac{I_2+3\tilde{I}_1}{4}] = [\frac{I_1+I_2}{2} + \frac{\alpha}{2}, \frac{I_2+3I_1}{4} + \frac{3\alpha}{4}]$ , and once again by the strict quasi-concavity and symmetry of  $g$ ,  $M(I_1) < M(\tilde{I}_1)$ , and so  $I_1$  is strictly better off. No equilibrium.

So consider  $I_1 \in (2Z^\#, 0)$  and  $I_2 \in (-2Z^\#, -2Z^*)$ . As  $|I_1| < |I_2|$  we have that  $\frac{I_1+I_2}{2} > 0$ . If  $\frac{I_1+3I_2}{4} > \underline{Z}$  then set  $\tilde{I}_2 = I_2 - \rho$ , such that  $\frac{I_1+3\tilde{I}_2}{4} = \underline{Z}$  (as in Proposition 1). And so  $\overline{D}(I_1) \subset \overline{D}(\tilde{I}_1)$  and  $I_1$  is strictly better off. No equilibrium. So consider where  $\frac{I_1+3I_2}{4} \leq \underline{Z}$ . Now analyze dual deviations by the incumbents as done in Case 2 of this result. At least one incumbent has incentive to deviate. No equilibrium.

### 9.2.5 Run-Off

I will proceed by showing that the two incumbents cannot choose platforms such that entry is prevented in all districts. Thus, in any potential equilibrium there must be entry of third parties.

Consider the central district with the incumbents located asymmetrically. Let  $|I_1| = |I_2| - \alpha, \alpha > 0$ . Now, if  $I_1, I_2 < 0$  then  $E = 0 \implies P(W = E) = 1$  and so entry occurs: we are done. Therefore consider  $I_1 < 0, I_2 > 0$ .

If  $E \in I_1^+$  then  $V_E \rightarrow F(\frac{I_1+I_2}{2}) - F(I_1) = F(\frac{\alpha}{2}) - F(I_1), V_{I_1} \rightarrow F(I_1), V_{I_2} \rightarrow 1 - F(\frac{\alpha}{2})$ . As  $|E| < |I_1|, |I_2|$  to prevent entry we require at least that  $1 - F(\frac{\alpha}{2}) \geq F(\frac{\alpha}{2}) - F(I_1) \implies F(\frac{\alpha}{2}) \leq \frac{1}{2}(1 + F(I_1))$ , and  $F(I_1) \geq F(\frac{\alpha}{2}) - F(I_1) \implies F(\frac{\alpha}{2}) \leq 2F(I_1)$ .

If  $E \in I_1^-$  then  $V_E \rightarrow F(I_1), V_{I_1} \rightarrow F(\frac{\alpha}{2}) - F(I_1), V_{I_2} = 1 - F(\frac{\alpha}{2})$ . As  $|E| > |I_1|$  to prevent entry we may require  $V_{I_1} > V_E \implies F(\frac{\alpha}{2}) - F(I_1) \geq F(I_1) \implies F(\frac{\alpha}{2}) \geq 2F(I_1)$ . Combined with the first necessary conditions this implies that  $F(\frac{\alpha}{2}) = 2F(I_1)$ . That this requirement must hold with equality means that we must also ensure that if  $E \in I_1^+$  then  $E$  doesn't beat or tie with  $I_1$  for strictly positive values of  $\varepsilon$  (that is  $E$  loses in the limit but wins at points on the convergent path). A sufficient condition for this to happen is that there exists a  $\rho$  such that  $f(\frac{\alpha}{2} + \rho) \geq 2f(I_1 + \rho)$ . If this is true then  $E = I_1 + 2\rho \implies V_E = F(\frac{\alpha}{2} + \rho) - F(I_1 + \rho) \geq F(\frac{\alpha}{2}) - F(I_1) + F(I_1 + \rho) - F(I_1) = F(I_1 + \rho) = V_{I_1}$ . Which implies  $P(W = E) > 0$  and so entry occurs.

Alternatively, if  $V_{I_1} \leq V_E$  then we require  $V_{I_1} > V_{I_2}$  which implies  $F(\frac{\alpha}{2}) \geq \frac{1}{2}(1 + F(I_1))$ . Combining this with the above conditions implies  $F(\frac{\alpha}{2}) = \frac{1}{2}(1 + F(I_1))$ .

So one or both of these two identities must hold for the central district.

- Assume both conditions hold. That is,  $F(\frac{\alpha}{2}) = 2F(I_1)$  and  $F(\frac{\alpha}{2}) = \frac{1}{2}(1 + F(I_1))$ .

This implies that  $F(I_1) = \frac{1}{3}$  and  $F(\frac{\alpha}{2}) = \frac{2}{3}$ . Therefore  $\frac{\alpha}{2} = -I_1$ . Consider  $E = I_1 + \Delta$ . Then  $V_E = F(\frac{\alpha}{2} + \frac{\Delta}{2}) - F(I_1 + \frac{\Delta}{2}) < \frac{1}{3} < V_{I_1}$  and  $V_{I_2} = 1 - F(\frac{\alpha}{2} + \frac{\Delta}{2})$ .

$$V_E - V_{I_2} = 2F(\frac{\alpha}{2} + \frac{\Delta}{2}) - F(I_1 + \frac{\Delta}{2}) - 1.$$

$$\lim_{\Delta \rightarrow 0} (V_E - V_{I_2}) = 2F(\frac{\alpha}{2}) - F(I_1) - 1 = 2 \cdot \frac{2}{3} - \frac{1}{3} - 1 = 0.$$

$$\frac{d(V_E - V_{I_2})}{d\Delta} = f(\frac{\alpha}{2} + \frac{\Delta}{2}) - \frac{1}{2}f(I_1 + \frac{\Delta}{2}).$$

By the continuity and symmetry of  $f$  (recall  $-I_1 = \frac{\alpha}{2}$ ) then  $\exists \Delta^*$  such that  $\forall \Delta \in (0, \Delta^*)$ ,  $f(\frac{\alpha}{2} + \frac{\Delta}{2}) > \frac{1}{2}f(I_1 + \frac{\Delta}{2})$ . Thus  $\exists \Delta$  such that  $|E| < |I_1|, |I_2|$  and  $V_E > V_{I_1} \implies P(W = E) = 1$  and so entry must still occur.

Thus if both conditions are satisfied for the central district then entry must still occur and we are done. So I shall now consider when each condition is satisfied alone. If one of these conditions doesn't hold for the central district then by the continuity of  $F$  there exists an interval,  $[\Delta, 0)$ , of district medians in which this condition can't hold either. Thus when we consider these districts in the following cases the sole condition that held for the central district is the only one that can hold in these districts as well. Thus if I can show that the condition that held in the central district can't simultaneously hold in these other districts and prevent entry, then I will have shown that third party entry will occur. This is the method of the following cases.

- Considering  $F(\frac{\alpha}{2}) = 2F(I_1)$  firstly.

Consider districts  $[\Gamma, 0]$  where  $|\Gamma| < |I_1|$ . This implies that  $\frac{\alpha}{2} - \Gamma > 0$  (as  $\Gamma < 0$ ). To preclude entry in each district we need  $F(\frac{\alpha}{2} - \gamma) = 2F(I_1 - \gamma) \forall \gamma \in [\Gamma, 0]$ . Therefore as  $\gamma$  changes  $\frac{\partial}{\partial \gamma} F(\frac{\alpha}{2} - \gamma) = \frac{\partial}{\partial \gamma} 2F(I_1 - \gamma) \implies -f(\frac{\alpha}{2} - \gamma) = -2f(I_1 - \gamma)$ . As  $I_1 - \gamma < 0$  and  $\frac{\alpha}{2} - \gamma > 0$  this can only be true if  $f(I_1 - \gamma) = \frac{1}{2}f(\frac{\alpha}{2} - \gamma) = \text{constant}$ , for all  $\gamma \in [\Gamma, 0]$ , otherwise the condition is violated and entry occurs. If  $f(I_1 - \gamma) = \frac{1}{2}f(\frac{\alpha}{2} - \gamma) = \text{constant}$ , for all  $\gamma \in [\Gamma, 0]$ , then the additional sufficient condition for entry above (where  $E$  wins on the convergent path) holds and  $E = I_1 + 2\gamma$  for any  $\gamma \in [\Gamma, 0]$ , implies  $V_E \geq V_{I_1}$  and  $E$  enters.

- Consider  $F(\frac{\alpha}{2}) = \frac{1}{2}(1 + F(I_1))$ .

Similarly to the case above, by considering districts  $[\Delta, 0]$  where  $|\Delta| < |I_1|$  we can establish that if entry is to be prevented  $f(I_1 - \delta) = 2f(\frac{\alpha}{2} - \delta) \forall \delta \in [\Delta, 0]$ . But as  $V_{I_1} > V_{I_2}$  this leads to (as  $\Delta < 0$ )  $E = I_1 - \Delta \implies V_E = F(\frac{\alpha}{2} - \frac{\Delta}{2}) - F(I_1 - \frac{\Delta}{2}) = F(\frac{\alpha}{2}) - F(I_1) - [F(\frac{\alpha}{2} - \frac{\Delta}{2}) - F(\frac{\alpha}{2})] = 1 - F(\frac{\alpha}{2} - \frac{\Delta}{2}) = V_{I_2}$ , by the condition on  $f$  and the identity  $F(\frac{\alpha}{2}) = \frac{1}{2}(1 + F(I_1))$ . And so  $V_E = V_{I_2}$ , combined with the fact that  $|E| < |I_1|, |I_2|$  this implies that  $P(W = E) > 0$  and so  $E$  will enter.

Thus, for  $0 \neq |I_1| < |I_2|$  the incumbents are unable to preclude entry in all districts. So consider where  $|I_1| = |I_2|$ . For any  $\beta \in (0, \overline{Z})$  the incumbents are asymmetric in that district and so the analysis from above holds for any  $\beta$  if we consider districts  $\overline{\beta} \in [\frac{\beta}{2}, \overline{Z}]$ . Also consider  $|I_1| = 0 < |I_2|$ . Then  $\forall z_r \in [\underline{Z}, 0)$  which is measurable as  $\underline{Z} \neq 0$ , we have  $I_1 - z_r, I_2 - z_r > 0$  and so  $E = I_1^- \implies P(W = E) = 1$ . Therefore there is successful entry in these districts.

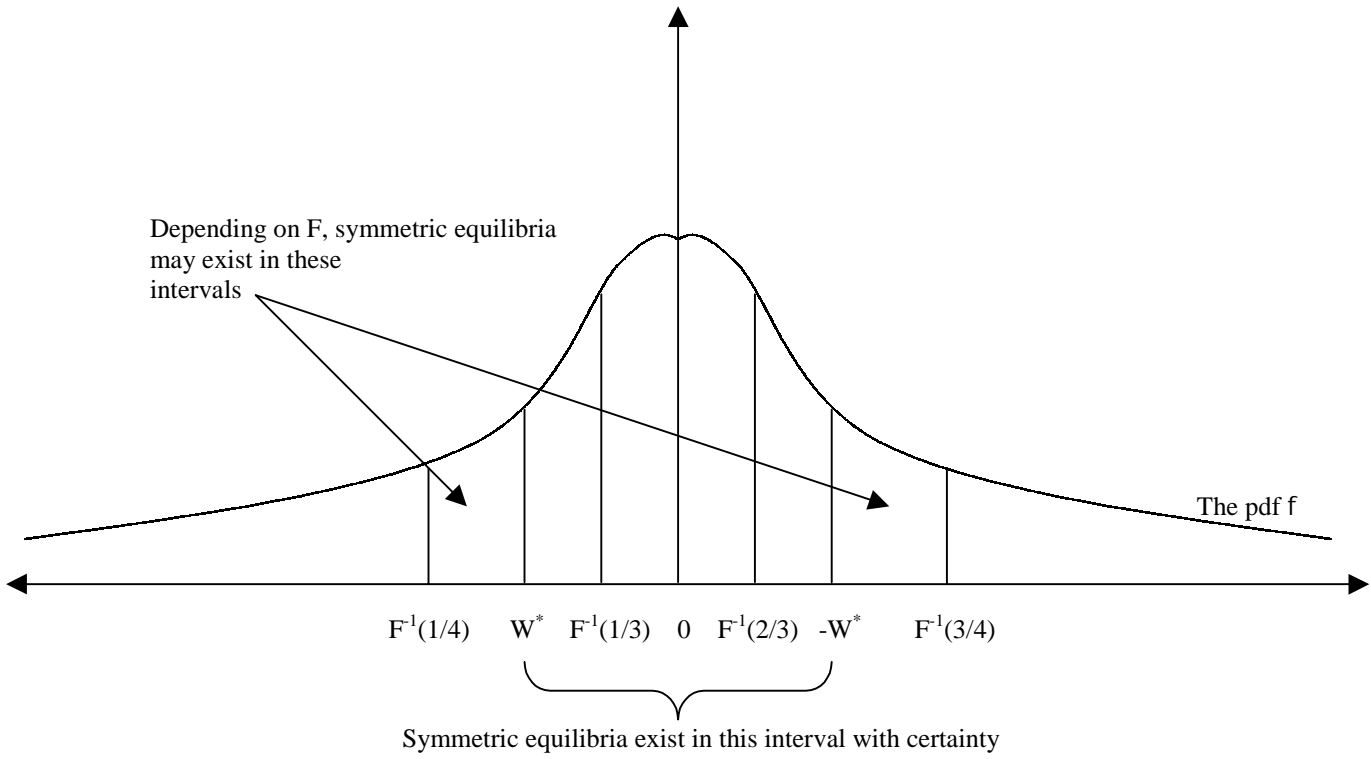
### 9.2.6 Theorem 1

This result is merely a combination of the previous results. It is stated in order to give a clear representation of what has been established. Its proof simply refers to the previous results.

Sufficient  $\implies$  then we have the conditions for Proposition 1. No entry happens. Final outcome involves two parties. Duverger's Law holds.

Necessary  $\implies$  Without Condition 2 we satisfy the requirements for Proposition 2. There is entry in every district. Final outcome involves three or more parties. Duverger's Law fails. Without Condition 1 we satisfy the conditions for Proposition 3. There is entry in intervals of districts on the edges of the distribution of district median voters. Final outcome involves three or more parties. Duverger's Law fails.

Figure 1



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