Disturbance Attenuation for Constrained Discrete-Time Systems via Receding Horizon Controls

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***Abstract*—In this note, we propose new receding horizon H_∞ control (RHHC) schemes for linear input-constrained discrete time-invariant systems with disturbances. The proposed control schemes are based on the dynamic game problem of a finite-horizon cost function with a fixed finite terminal weighting matrix and a one-horizon cost function with time-varying finite terminal weighting matrices, respectively. We show that the resulting RHHCs guarantee closed-loop stability in the absence of disturbances and H_∞ norm bound for 2-norm bounded disturbances. We also show that the proposed schemes can easily be implemented via linear matrix inequality optimization. We illustrate the effectiveness of the proposed schemes through simulations.**

***Index Terms*—Constrained System, disturbance, H_∞ norm, receding horizon control (RHC), stability.**

I. Introduction

Receding horizon control (RHC) is a closed-loop strategy, where the control is obtained by minimizing the cost function at each sampling time, thus enabling finite horizons to be considered. For this reason, RHC has been widely investigated as one of the easiest ways to handle input/state-constraints [1], [2], disturbances [3], time-varying tracking commands [4], [5], etc.

Many systems have both input/state-constraints and disturbances. These constraints and disturbances often adversely affect performance and stability. Although many methods for handling either constraints or disturbances can be found in the literature, there have been very few stabilizing controls that handle both constraints *and* disturbances. One way to attenuate the effect of disturbance is to minimize an ∞-norm of the transfer function from disturbance to the controlled output, i.e., guarantee a H_∞ norm bound for systems with disturbances. Recently, the paper [6] attempts to systematically guarantee a H_∞ norm bound for systems with constraints and disturbances. However, the proposed algorithm in [6] is very difficult to implement for multiple-input and multiple-output systems since its complicated design parameters must be selected manually. Thus, it will be very interesting to investigate how to easily implement a stabilizing control that guarantees the H_∞ norm bound for systems with constraints and disturbances.

In this note, we propose two control schemes for linear input-constrained discrete time-invariant systems with disturbances, which are based on the dynamic game problem of a finite horizon cost function with a fixed finite terminal weighting matrix and a one-horizon cost function with time-varying finite terminal weighting matrices, respectively. The controls from the proposed schemes are called receding horizon H_∞ controls (RHHC) in this note. We show that the first RHHC under some additional implicit condition and the second RHHC guarantee closed-loop stability in the absence of disturbances and H_∞ norm bound for linear input-constrained systems with disturbances. We also show that the proposed schemes can easily be implemented by using linear matrix inequality (LMI) optimization as in [7]. In implementing the schemes, we also suggest how to obtain a unique saddle-point solution. We illustrate the effectiveness of the proposed schemes through simulations.

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II. RHHC

Consider a linear input-constrained discrete time-invariant system with disturbances

$$x(i+1) = Ax(i) + B_2u(i) + B_1w(i), \qquad x(0) = x_0$$
$$z(i) = \left[(Cx(i))^T, \left(R_2^{\frac{1}{2}} u(i) \right)^T \right]^T \tag{1}$$

subject to $-u_{\lim} \le G_u u(i) \le u_{\lim}$, $i = 0, 1, \ldots, \infty$, where $x(i) \in R^n$ is the state, $u(i) \in R^m$ the control, $w(i) \in R^l$ the disturbance, $z(i) \in R^p$ the output, $R_2 = R_2^T > 0$, and G_u the given constant matrix. From now on, assume that the pair (A, B_2) is stabilizable.

For this system, consider the following dynamic game problem:

$$J^*(i, i+N) = \min_u \max_w J(i, i+N) \tag{2}$$

subject to the input constraint

$$-u_{\lim} \le G_u u(\tau) \le u_{\lim}, \qquad \tau = i, i+1, \ldots, i+N-1 \tag{3}$$

where $J(i, i+N) = \sum_{\tau=i}^{i+N-1} [x^T(\tau)Qx(\tau) + u^T(\tau)R_2u(\tau) - \gamma^2 w^T(\tau)R_1w(\tau)] + x^T(i+N)Q_f x(i+N)$, $Q = C^TC$, $R_1 = R_1^T > 0$, $Q_f = Q_f^T > 0$, $N \ge 1$, and γ is the disturbance attenuation level.

For simplicity, throughout the rest of this note, define $\bar{F}$ and $\hat{F}$ as functions mapping F as the following augmented matrices:

$$\bar{F} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ F & \ddots & \ddots & \ddots & \vdots \\ 0 & F & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & F & 0 \end{bmatrix} \quad \hat{F} = \begin{bmatrix} F & 0 & \cdots & 0 \\ 0 & F & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & F \end{bmatrix}. \tag{4}$$

We use $\bar{F}$ and $\hat{F}$ for presentation of $\bar{A}$, $\bar{B}_1$, $\bar{B}_\gamma$, $\bar{B}_2$, $\hat{C}$, $\hat{Q}$, $\hat{R}_1$, $\hat{R}_2$, and $\hat{G}_u$, which consist of A, B_1, B_γ, B_2, C, Q, R_1, R_2, and G_u, respectively, where $B_\gamma = \gamma^{-1}B_1$. Then, if $N \ge 2$, (2) subject to (3) can be converted to the equivalent optimization problem

$$\underset{U(i)}{\text{Minimize}} \ \beta(i) \tag{5}$$

subject to

$$\begin{bmatrix} \beta(i) - V_1(i)U(i) - V_0(i) & \left(V_2^{\frac{1}{2}} U(i) \right)^T \\ V_2^{\frac{1}{2}} U(i) & I \end{bmatrix} \ge 0 \tag{6}$$

$$-\left[u_{\lim}^T, u_{\lim}^T, \cdots, u_{\lim}^T \right]^T \le \hat{G}_u U(i) \le \left[u_{\lim}^T, u_{\lim}^T, \cdots, u_{\lim}^T \right]^T \tag{7}$$

where

$$V_1(i) = 2\left[X_0^T(i)\alpha_1\bar{B}_2 + x^T(i)A^{NT}\alpha_2\bar{B}_2 + X_0^T(i)\alpha_2^T B_{2\phi} + x^T(i)A^{NT}Q_{f\gamma}B_{2\phi} \right]$$
$$U(i) = \left[u^T(i), \ u^T(i+1), \ \cdots, \ u^T(i+N-1) \right]^T$$
$$V_0(i) = X_0^T(i)\alpha_1 X_0(i) + 2x^T(i)A^{NT}\alpha_2 X_0(i) + x^T(i)A^{NT}Q_{f\gamma}A^N x(i)$$
$$V_2 = \hat{R}_2 + \bar{B}_2^T\alpha_1\bar{B}_2 + B_{2\phi}^T\alpha_2\bar{B}_2 + \bar{B}_2^T\alpha_2^T B_{2\phi} + B_{2\phi}^T Q_{f\gamma} B_{2\phi}$$

$$
\begin{aligned}
Q_{f\gamma} &= \left(Q_f^{-1} - B_{\gamma\phi}\left(\hat{R}_1 - \bar{B}_\gamma^T \hat{Q}_{\bar{A}} \bar{B}_\gamma\right)^{-1} B_{\gamma\phi}^T\right)^{-1} \\
X_0(i) &= \left[x^T(i),\ 0,\ \cdots,\ 0\right]^T \\
\alpha_1 &= \hat{Q}_{\bar{A}} + \hat{Q}_{\bar{A}} \bar{B}_\gamma \alpha_3 \bar{B}_\gamma^T \hat{Q}_{\bar{A}} \\
\alpha_2 &= Q_{f\gamma} B_{\gamma\phi}\left(\hat{R}_1 - \bar{B}_\gamma^T \hat{Q}_{\bar{A}} \bar{B}_\gamma\right)^{-1} \bar{B}_\gamma^T \hat{Q}_{\bar{A}} \\
\alpha_3 &= \left(\hat{R}_1 - \bar{B}_\gamma^T \hat{Q}_{\bar{A}} \bar{B}_\gamma - B_{\gamma\phi}^T Q_f B_{\gamma\phi}\right)^{-1} \\
\hat{Q}_{\bar{A}} &= (I - \bar{A})^{-T} \hat{Q} (I - \bar{A})^{-1} \\
B_{2\phi} &= [A^{N-1}B_2,\ A^{N-2}B_2,\ \cdots,\ B_2] \\
B_{\gamma\phi} &= [A^{N-1}B_\gamma,\ A^{N-2}B_\gamma,\ \cdots,\ B_\gamma]. \qquad (8)
\end{aligned}
$$

When $N = 1$, $V_1(i)$, $V_0(i)$, and V_2 are replaced by $V_1(i) = 2x^T(i)A^T Q_{f\gamma} B_2$, $V_0(i) = x^T(i)Qx(i) + x^T(i)A^T Q_{f\gamma} Ax(i)$, and $V_2 = R_2 + B_2^T Q_{f\gamma} B_2$, respectively with $Q_{f\gamma}$ replaced by $(Q_f^{-1} - B_\gamma R_1^{-1} B_\gamma^T)^{-1}$. Appendix shows the formulation of the linear matrix inequality (LMI) form (6).

Now, we establish the existence of a saddle-point solution and investigate closed-loop stability. To this end, we introduce the following conditions:

$$
\begin{aligned}
&\hat{R}_1 > \bar{B}_\gamma^T (I - \bar{A})^{-T} \hat{Q} (I - \bar{A})^{-1} \bar{B}_\gamma + B_{\gamma\phi}^T Q_f B_{\gamma\phi} && (9) \\
&R_1 > B_\gamma^T Q_f B_\gamma && (10) \\
&Q_f \geq Q + H^T R_2 H + (A - B_2 H)^T \left(Q_f^{-1} - B_\gamma R_1^{-1} B_\gamma^T\right)^{-1} \\
&\qquad \times (A - B_2 H) \ \text{ for some } H. && (11)
\end{aligned}
$$

Lemma 1: If (9) when $N \geq 2$ [(10) when $N = 1$] is satisfied, and $V_2 > 0$, there exists an unique saddle-point solution for (2) subject to (3).

Proof: If (9) when $N \geq 2$ [(10) when $N = 1$] is satisfied, there exists a concave solution $w(\cdot)$ as shown in (26) of Appendix. There also exists a convex solution $u(\cdot)$ since $V_2 > 0$. □

The saddle-point solution is obtained by solving (2) subject to (3), (9)–(11) at each time i. From now on, it is denoted as $u^*(\tau)$ and $w^*(\tau)$ for $\tau \in [i, i + N - 1]$. Then, $u^*(\tau)|_{\tau=i}$ is called the RHHC.

Next, we suggest an important result for closed-loop stability and H_∞ norm bound.

Theorem 1: If Q_f and γ for some H satisfy (10), (11), and

$$
\begin{aligned}
&-u_{\lim} \leq -G_u H x(i + N) \leq u_{\lim} \\
&\qquad \text{with } x(i + N) \text{ as specified in (31) of} \\
&\qquad \text{Appendix 6.2} && (12) \\
&\text{then } J^*(i, i + N) \\
&\qquad \geq x^T(i)Qx(i) + u^{*T}(i) R_2 u^*(i) - \gamma^2 w^T(i) R_1 w(i) \\
&\qquad\quad + J^*(i + 1, i + N + 1) && (13)
\end{aligned}
$$

where $x(i+1) = Ax(i) + B_2 u^*(i) + B_1 w(i)$ and $w(i)$ is the unknown disturbance at i.

Proof: See the Appendix. □

It is almost impossible to handle (12) at the current time i, since it requires the optimal solutions both at i and at $i + 1$ as shown in (31) of Appendix. We can check (12) only at $i + 1$ by converting it to $-u_{\lim} \leq -G_u H \Gamma(i) \leq u_{\lim}$ where $\Gamma(i) = A^N x(i - 1) + B_{2\phi} U^*(i - 1) + A^{N-1} B_1 w(i - 1) + [A^{N-2}B_\gamma, A^{N-3}B_\gamma, \cdots, B_\gamma, 0]\alpha_3[\bar{B}_\gamma^T \hat{Q}_{\bar{A}}(\bar{B}_2 U(i) + X_0(i)) + B_{\gamma\phi}^T Q_f (A^N x(i) + B_{2\phi} U(i))]$ and $U^*(i - 1) = [u^{*T}(i - 1), u^{*T}(i), \cdots, u^{*T}(i + N - 2)]^T$ when $N \geq 2$. If $N = 1$, $\Gamma(i)$ is replaced by $\Gamma(i) = Ax(i - 1) + B_2 u^*(i - 1) + B_1 w(i - 1)$. Note that (12) is very easy to handle in linear quadratic (LQ) problems in which $\gamma = \infty$ and $w(i) = 0$ as shown in [1] and [2].

From this discussion, we can state closed-loop stability of the RHHC with $w(\cdot) = 0$ as follows.

Theorem 2: Assume that the pair (A, C) is detectable, and Q_f and γ satisfy (9)–(11) for some H. If (2) subject to (3) is always feasible and (12) is satisfied when $w(i) = 0$ for all times, then system (1) with the resulting RHHC is bounded and attractive. It is asymptotically stable if C is positive definite.

Proof: For unconstrained systems without (3), $J^*(i, i + N) \geq a\|x(i)\|^2$ for some positive constant a since the pair (A, C) is detectable, (9) for $N \geq 2$ ((10) for $N = 1$) is satisfied, and $Q_f > 0$ [8]. Since $J^*(i, i + N)$ is greater for the constrained system with (3) than for the unconstrained system without (3), we have $J^*(i, i+N) \geq a\|x(i)\|^2$ for all i. From (13), when $w(i) = 0$, we know that $J^*(i, i + N)$ is bounded and monotonically nonincreasing, and $u^*(i) \to 0$ as $i \to \infty$. Therefore, by detectability, $x(i) \to 0$ as $i \to \infty$. Thus, the resulting closed-loop system is attractive. If C is positive definite, then $J^*(i, i + N)$ is a Lyapunov function. □

Theorem 3: For 2-norm bounded disturbances, if (12) is satisfied for all times, then the proposed RHHC scheme guarantees that

$$
\gamma^2 \geq \frac{\|z\|^2}{\|w\|_{R_1}^2} - \frac{J^*(0, N)}{\|w\|_{R_1}^2}, \quad \text{where } \|w\|_{R_1}^2 = \sum_{i=0}^{\infty} w^T(i) R_1 w(i). \qquad (14)
$$

Proof: From Theorem 1, $J^*(0, N) \geq \sum_{i=0}^{k} x^T(i)Qx(i) + u^{*T}(i)R_2 u^*(i) - \gamma^2 w^T(i) R_1 w(i) + J^*(k + 1, k + N + 1)$. As $k \to \infty$, $w(k) \to 0$ and thus, $J^*(k + 1, k + N + 1) \to 0$ as shown in Theorem 2. Therefore, we have (14). □

For implementation, we convert conditions (9)–(11) to the equivalent LMI forms

$$
\begin{aligned}
&\begin{bmatrix} \hat{R}_1 - \bar{B}_\gamma^T (I - \bar{A})^{-T} \hat{Q} (I - \bar{A})^{-1} \bar{B}_\gamma & B_{\gamma\phi}^T \\ B_{\gamma\phi} & S \end{bmatrix} > 0 \\
&\begin{bmatrix} R_1 & B_\gamma^T \\ B_\gamma & S \end{bmatrix} > 0 && (15) \\
&\begin{bmatrix} S & (AS - B_2 Y)^T & (CS)^T & \left(R_2^{\frac{1}{2}} Y\right)^T \\ AS - B_2 Y & S - B_\gamma R_1^{-1} B_\gamma^T & 0 & 0 \\ CS & 0 & I & 0 \\ R_2^{\frac{1}{2}} Y & 0 & 0 & I \end{bmatrix} \geq 0 && (16)
\end{aligned}
$$

where $S = Q_f^{-1}$ and $Y = HS$.

The RHHC can then be implemented as follows.

a) At the initial time, obtain Q_f and γ satisfying $V_2 > 0$ in (8), (15) and (16) for some H.
b) With the obtained Q_f, solve the problem (5) subject to (6) and (7).
c) Implement the RHHC $u^*(i)$.
d) At the next time i, repeat procedure b) and c).

Since we cannot consider (12) explicitly in the proposed scheme, closed-loop stability is not guaranteed by the proposed RHHC. However, our simulations show that the proposed RHHC satisfies (13) and (14) in many cases.

In Section III, we propose another RHHC scheme, which handles (12) directly and thereby guarantees closed-loop stability and feasibility.

III. Stabilizing One-Horizon RHHC

Consider the following dynamic game problem:

$$J^*(i) = \min_{u(i),Q_f(i)} \max_{w(i)} \beta_1(i) + \beta_2(i) \tag{17}$$

subject to

$$\beta_1(i) \geq x^T(i)Qx(i) + u^T(i)R_2u(i) \tag{18}$$

$$\beta_2(i) \geq -\gamma^2 w^T(i)R_1w(i) + x^T(i+1)Q_f(i)x(i+1) \tag{19}$$

$$-u_{\lim} \leq G_u u(i) \leq u_{\lim} \tag{20}$$

and (21) and (22), as shown at the bottom of the page, where $S(i) = \beta_2(i)Q_f^{-1}(i)$ and $Y(i) = H(i)S(i)$. The resulting saddle-point solution $u^*(i)$ at each time i is called the one-horizon RHHC. Here, we would like to mention that (21) and (22) are another equivalent LMI forms of (10) and (11), respectively. In order to investigate closed-loop stability, we introduce a well-known lemma.

Lemma 2: [9] Assume that there exist $Q_f(i)$, γ, $H(i)$, and $\beta_2(i)$ satisfying

$$\begin{bmatrix} E(i) & G_uY(i) \\ Y^T(i)G_u^T(i) & S(i) \end{bmatrix} \geq 0 \quad E_{j,j}(i) \leq u_{lim,j}^2 \tag{23}$$

where $u_{lim,j}$ and $E_{j,j}(i)$ are the jth and (j,j) elements of $u_{\lim}$ and $E(i)$, respectively. If $x(i+1) \in \mathcal{E}_{Q_f(i)}$ where $\mathcal{E}_{Q_f(i)} = \{\xi | \xi^T Q_f(i)\xi \leq \beta_2(i)\}$, then $u(i+1) = -H(i)x(i+1)$ satisfies the input constraint (3).

Based on Lemma 2, we suggest the following important theorem for closed-loop stability.

Theorem 4: If (17) subject to (18)–(23) is feasible at $i = 0$ and $w(i) = 0$ at each time i, then it is feasible for all times.

Proof: We have only to show that if (17) subject to (18)–(23) is feasible at any i, it is feasible at $i+1$. Since $J^*(u^*(i), w^*(i)) \geq J(u^*(i), w(i))$, we have $J^*(u^*(i), w^*(i)) - J(u^*(i), w(i) = 0) = -\gamma^2 w^{*T}(i)w^*(i) + x^T(i+1|i)Q_f(i)x(i+1|i) - x^T(i+1)Q_f(i)x(i+1) \geq 0$ where $x(i+1|i) = Ax(i) + B_2u^*(i) + B_1w^*(i)$ and $x(i+1) = Ax(i) + B_2u^*(i)$. Hence, if (19) is satisfied, then $\beta_2(i) \geq x^T(i+1)Q_f(i)x(i+1)$ and $u(i+1) = -H(i)x(i+1)$ satisfies (20) from Lemma 2. Thus, $u(i+1) = -H(i)x(i+1)$ and $w^*(i+1) = \gamma^{-1}R_1^{-1}B_\gamma^T(Q_f^{-1}(i) - B_\gamma R_1^{-1}B_\gamma^T)^{-1}(A - B_2H(i))x(i+1)$ can be a solution to the dynamic game problem at $i+1$, since (22) shows that $-\gamma^2 w^{*T}(\sigma)R_1w^*(\sigma) + x^T(\sigma+1|\sigma)Q_f(i)x(\sigma+1|\sigma) = x^T(\sigma)(A - B_2H(i))^T(Q_f^{-1}(i) - B_\gamma R^{-1}B_\gamma^T)^{-1}(A - B_2H(i))x(\sigma) \leq x^T(\sigma)Q_f(i)x(\sigma) \leq \beta_2(i)$, where $\sigma = i+1$. Therefore, the problem is feasible at $i+1$ when $Q_f(i+1)$ and $\beta_2(i+1)$ are replaced by $Q_f(i)$ and $\beta_2(i)$, respectively. □

Now, we are ready to state closed-loop stability of the one-horizon RHHC.

Theorem 5: Assume that the pair (A, C) is detectable. If (17) subject to (18)–(23) is feasible at $i = 0$, then system (1) with the one-horizon RHHC is bounded and attractive. It is asymptotically stable if C is positive definite.

Proof: From the proof of Theorem 4, we know that (12) with $w(i) = 0$ is satisfied at each time i where $x(i+N)$ in (12) is replaced by $x(i+1) = Ax(i) + B_2u^*(i)$ in the one-horizon RHHC scheme. Theorems 1 and 4 show that $J^*(i) \geq x^T(i)Qx(i) + u^{*T}(i)R_2u^*(i) + J^*(i+1)|_{Q_f(i)}$ where $J^*(i+1)|_{Q_f(i)} = \min_{u(i)} \max_{w(i)} x^T(i+1)Qx(i+1) + u^T(i+1)R_2u(i+1) - \gamma^2 w^T(i+1)R_1w(i+1) + x^T(i+2)Q_f(i)x(i+2)$ with the solution $Q_f(i)$ for $J^*(i)$. Since $J^*(i+1)|_{Q_f(i)} \geq J^*(i+1)$ by optimality, $J^*(i) \geq x^T(i)Qx(i) + u^{*T}(i)R_2u^*(i) + J^*(i+1)$. The remaining proof follows that of Theorem 2. □

It is straightforward that the one-horizon RHHC scheme also guarantees the H_∞ norm bound (14) for 2-norm bounded disturbances, if (12) is satisfied with $x(i+N)$ replaced by $x(i+1) = Ax(i) + B_2u^*(i) + B_1w(i)$. For implementation, we convert (18) and (19) to the equivalent LMI forms

$$\begin{bmatrix} \beta_1(i) - x^T(i)Qx(i) & \left(R_2^{\frac{1}{2}}u(i)\right)^T \\ R_2^{\frac{1}{2}}u(i) & I \end{bmatrix} \geq 0$$

$$\begin{bmatrix} 1 & (Ax(i) + B_2u(i))^T \\ Ax(i) + B_2u(i) & S(i) - \beta_2(i)B_\gamma R_1^{-1}B_\gamma^T \end{bmatrix} \geq 0. \tag{24}$$

Thus, we can summarize the proposed one-horizon RHHC as follows.

a) At $i = 0$, set γ with which (17) subject to (20)–(22), and (24) is feasible.

b) Solve (17) subject to (20)–(24). If infeasible, solve (17) subject to (20)–(22), and (24).

c) Implement the one-horizon RHHC $u^*(i)$.

d) At the next time i, repeat procedure b) and c).

Although the second part of b) is not used when $w(i) = 0$ and initial states are small, it allows us to get the one-horizon RHHC even in the presence of large disturbances and initial states. Our simulations show that the resulting one-horizon RHHC satisfies the cost monotonicity (13) and H_∞ norm bound (14) in many cases, even though (23) is sometimes not satisfied. Note that the proposed one-horizon RHHC scheme cannot consider more than one horizon; however, it can deal with (12) directly and thereby guarantees closed-loop stability and feasibility.

In the next section, we illustrate the proposed schemes via simulation examples.

$$\begin{bmatrix} \beta_2(i)R_1 & \beta_2(i)B_\gamma^T \\ \beta_2(i)B_\gamma & S(i) \end{bmatrix} > 0 \tag{21}$$

$$\begin{bmatrix} S(i) & (AS(i) - B_2Y(i))^T & (CS(i))^T & \left(R_2^{\frac{1}{2}}Y(i)\right)^T \\ AS(i) - B_2Y(i) & S(i) - \beta_2(i)B_\gamma R_1^{-1}B_\gamma^T & 0 & 0 \\ CS(i) & 0 & \beta_2(i)I & 0 \\ R_2^{\frac{1}{2}}Y(i) & 0 & 0 & \beta_2(i)I \end{bmatrix} \geq 0 \tag{22}$$

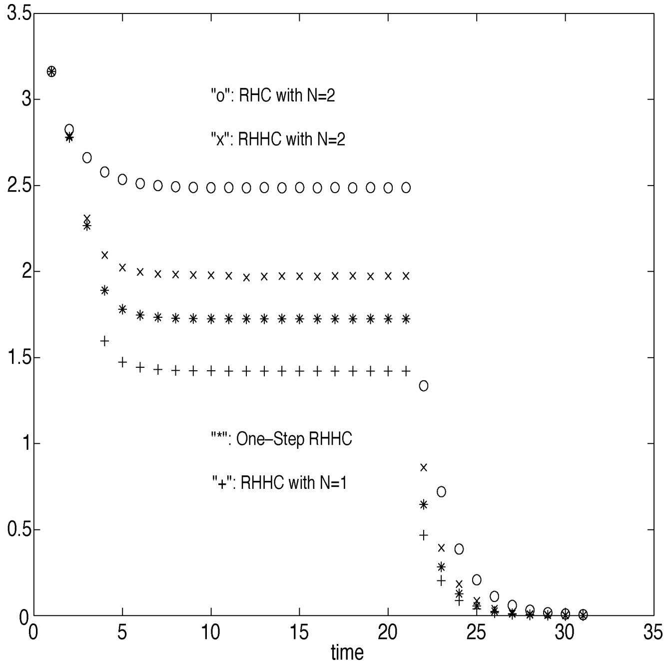

Fig. 1. Norm of state at each time: Case I.

IV. Simulation Examples

Consider the following:

$$A = \begin{bmatrix} 1.2 & 0.2 \\ 0.3 & 0.5 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}$$
$$C = [1 \quad 0]$$
$$x_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad u_{\lim} = 3 \quad R_1 = R_2 = 1.$$

For the RHHC scheme, we obtain Q_f and H by using the "feasp()" function in the LMI Toolbox [7].

To illustrate the proposed schemes, we put three types of disturbances into the system during two thirds of the simulation time: the first is chosen from a uniform distribution on the interval $(-1.0, 1.0)$; the second is set to a constant value 2; and the third is the sum of the first disturbance and $(-1)^i 2$. Under this situation, the proposed RHHC scheme has unique saddle-point solutions if $\gamma \geq 1$ when $N = 1$, if $\gamma \geq 1.5$ when $N = 2$, and if $\gamma \geq 2.2$ when $N = 3$. The proposed one-horizon RHHC scheme has a unique saddle-point solution when $\gamma \geq 1.2$.

For comparison, we consider the receding horizon linear quadratic control (RHC), which is obtained from the RHHC scheme with $\gamma = \infty$. Although page limit restrictions only allow us to show the result for the first type of disturbance in Fig. 1, the simulation results show that the RHHC when $N = 1$ and $\gamma = 1$ outperforms the RHHC when $N = 2$ and $\gamma = 1.5$, the one-horizon RHHC when $\gamma = 1.2$, and the RHC with $\gamma = \infty$. In these examples, all the resulting RHHC's satisfy (13) and (14) and perform better with a smaller γ. Condition (23) does not affect the performance. Note that the RHHC in [6] is very difficult to design even for this simple model.

For the scalar system and disturbance $(w(i) = (-1)^i 0.4)$ used in [6], the proposed RHHC when $N = 1$ and $\gamma = 100$ has nearly the same performance as the RHHC when $\gamma^2 = 1.9$ in [6].

Finally, note that for many cases, the proposed schemes seem to satisfy the saddle-point value monotonicity (13) and the H_∞ norm bound (14) even in the presence of large disturbances or initial states.

V. Conclusion

In this note, we proposed new RHHC schemes, which guarantee closed-loop stability in the absence of disturbances and H_∞ norm bound for linear input-constrained discrete time-invariant systems with disturbances. The control schemes are based on the dynamic game problem of a finite-horizon cost function with a fixed finite terminal weighting matrix and a one-horizon cost function with time-varying finite terminal weighting matrices, respectively. We show that the proposed schemes can easily be implemented by using LMI optimization. In implementing the schemes, we suggest how to obtain an unique saddle-point solution. The effectiveness of the proposed schemes is illustrated by simulations.

The proposed RHHC and one-horizon RHHC schemes are simple practical methods to implement stabilizing H_∞ controls for input-constrained systems with disturbances. The proposed schemes can be extended to various constrained H_∞ problems.

Appendix

A. Derivation of (6) and Proof of Lemma 1

Using the variables defined in (8), the cost function (2) can be represented by

$$\begin{aligned} J(i, i+N) &= X^T(i)\hat{Q}X(i) + U^T(i)\hat{R}_2U(i) \\ &\quad - \gamma^2 W^T(i)\hat{R}_1W(i) \\ &\quad + \left[A^N x(i) + B_{2\phi}U(i) + B_{1\phi}W(i)\right]^T Q_f \\ &\quad \times \left[A^N x(i) + B_{2\phi}U(i) + B_{1\phi}W(i)\right] \end{aligned} \tag{25}$$

where $X(i) = [x^T(i) \ \ x^T(i+1) \ \ \ldots \ \ x^T(i+N-1)]^T$ and $W(i) = [w^T(i) \ \ w^T(i+1) \ \ \ldots \ \ w^T(i+N-1)]^T$. Since $X(i) = \bar{A}X(i) + \bar{B}_2U(i) + \bar{B}_1W(i) + X_0(i) = (I - \bar{A})^{-1}[\bar{B}_2U(i) + \bar{B}_1W(i) + X_0(i)]$, we can convert the cost function $J(i, i+N)$ in (2) to

$$\begin{aligned} J(i, i+N) &= U^T(i)\hat{R}_2U(i) + \left(\bar{B}_2U(i) + X_0(i)\right)^T \\ &\quad \times \hat{Q}_{\bar{A}}\left(\bar{B}_2U(i) + X_0(i)\right) \\ &\quad + W^T(i)\bar{B}_1^T\hat{Q}_{\bar{A}}\left(\bar{B}_2U(i) + X_0(i)\right) \\ &\quad + \left(\bar{B}_2U(i) + X_0(i)\right)^T\hat{Q}_{\bar{A}}\bar{B}_1W(i) \\ &\quad + W^T(i)\bar{B}_1^T\hat{Q}_{\bar{A}}\bar{B}_1W(i) - \gamma^2W^T(i)\hat{R}_1W(i) \\ &\quad + \left[A^N x(i) + B_{2\phi}U(i) + B_{1\phi}W(i)\right]^T Q_f \\ &\quad \times \left[A^N x(i) + B_{2\phi}U(i) + B_{1\phi}W(i)\right]. \end{aligned} \tag{26}$$

From this, it is easy to see that there exists $W(i)$ maximizing $J(i, i+N)$ if (9) for $N \geq 2$ [(10) for $N = 1$] is satisfied. Then, if $N \geq 2$, the resulting $W(i)$ is given by

$$\begin{aligned} W(i) &= \left(\gamma^2\hat{R}_1 - \bar{B}_1^T\hat{Q}_{\bar{A}}\bar{B}_1 - B_{1\phi}^TQ_fB_{1\phi}\right)^{-1} \\ &\quad \times \left[\bar{B}_1^T\hat{Q}_{\bar{A}}\left(\bar{B}_2U(i) + X_0(i)\right)\right. \\ &\quad \left. + B_{1\phi}^TQ_f\left(A^N x(i) + B_{2\phi}U(i)\right)\right] \\ &= \gamma^{-1}\alpha_3\left[\bar{B}_\gamma^T\hat{Q}_{\bar{A}}\left(\bar{B}_2U(i) + X_0(i)\right)\right. \\ &\quad \left. + B_{\gamma\phi}^TQ_f\left(A^N x(i) + B_{2\phi}U(i)\right)\right]. \end{aligned} \tag{27}$$

If $N = 1$, then $w(i)$ is given by $w(i) = \gamma^{-1}(R_1 - B_\gamma^TQ_fB_\gamma)^{-1}B_\gamma^TQ_f(Ax(i) + B_2u(i))$.

Next, we introduce the following equations to simplify the computation of complex matrix equations:

$$\left(A_1 - A_2^T A_3 A_2\right)^{-1} A_2^T A_3 = A_1^{-1} A_2^T \left(A_3^{-1} - A_2 A_1^{-1} A_2^T\right)^{-1} \tag{28}$$

$$A_1 \left(I + A_2 \left(A_1^{-1} - A_2\right)^{-1}\right) = \left(A_1^{-1} - A_2\right)^{-1}. \tag{29}$$

By using (26)–(29), we can convert (2) to

$$\min_u U^T(i) V_2 U(i) + V_1(i) U(i) + V_0(i) \leq \beta(i). \tag{30}$$

Using the Schur complement, we can convert (30) to (6).

B. Proof of Theorem 1

Let the pairs $(u_1(\tau), w_1(\tau))$ and $(u_2(\tau), w_2(\tau))$ be the saddle-point solutions at $i+1$ and i for problem (2) subject to (6) and (7), where Q_f satisfies (9)–(12). Then, let $x_1(\tau)$ and $x_2(\tau)$ be the state trajectories determined by these pairs, respectively, where $x_1(i+1) = Ax_2(i) + B_2 u_2(i) + B_1 w(i)$. Define $\sigma = i+N$ and $\Delta J^*(i) = J^*(i+1, \sigma+1) - J^*(i, \sigma)$. Then, since $J^*(u^*(\cdot), w^*(\cdot)) \geq J(u^*(\cdot), w(\cdot))$, replacing $w_2(i)$ with $w(i)$, and $w_2(\tau)$ with $w_1(\tau)$ from $i+1$ to $\sigma - 1$ leads to

$$\begin{aligned}
\Delta J^*(i) = &\sum_{\tau=i+1}^{\sigma} \Big[x_1^T(\tau) Q x_1(\tau) + u_1^T(\tau) R_2 u_1(\tau) \\
&\qquad - \gamma^2 w_1^T(\tau) R_1 w_1(\tau)\Big] \\
&+ x_1^T(\sigma+1) Q_f x_1(\sigma+1) \\
&- \sum_{\tau=i}^{\sigma-1} \Big[x_2^T(\tau) Q x_2(\tau) + u_2^T(\tau) R_2 u_2(\tau) \\
&\qquad - \gamma^2 w_2^T(\tau) R_1 w_2(\tau)\Big] - x_2^T(\sigma) Q_f x_2(\sigma) \\
\leq &\sum_{\tau=i+1}^{\sigma} \Big[x_1^T(\tau) Q x_1(\tau) + u_1^T(\tau) R_2 u_1(\tau) \\
&\qquad - \gamma^2 w_1^T(\tau) R_1 w_1(\tau)\Big] \\
&+ x_1^T(\sigma+1) Q_f x_1(\sigma+1) - x_2^T(i) Q x_2(i) \\
&- u_2^T(i) R_2 u_2(i) + \gamma^2 w^T(i) R_1 w(i) \\
&- \sum_{\tau=i+1}^{\sigma-1} \Big[x_3^T(\tau) Q x_3(\tau) + u_2^T(\tau) R_2 u_2(\tau) \\
&\qquad - \gamma^2 w_1^T(\tau) R_1 w_1(\tau)\Big] - x_3^T(\sigma) Q_f x_3(\sigma)
\end{aligned}$$

where $x_3(\cdot)$ is the state trajectory determined by $x_3(i+1) = x_1(i+1)$, $u_2(\tau)$, and $w_1(\tau)$ for $\tau \in [i+1, \sigma-1]$. Since $J(u(\cdot), w^*(\cdot)) \geq J^*(u^*(\cdot), w^*(\cdot))$, replacing $u_1(\tau)$ with $u_2(\tau)$ from $i+1$ to $\sigma - 1$ and $u_1(\sigma)$ with $u(\sigma) = -Hx(\sigma)(x(\sigma) = x_3(\sigma))$ leads to

$$\begin{aligned}
\Delta J^*(i) \leq &\max_{w(\sigma)} J(\sigma, \sigma+1) - x^T(\sigma) Q_f x(\sigma) \\
&- \Big[x_2^T(i) Q x_2(i) + u_2^T(i) R_2 u_2(i) - \gamma^2 w^T(i) R_1 w(i)\Big]
\end{aligned}$$

where $J(\sigma, \sigma+1) = x^T(\sigma) Q x(\sigma) + u^T(\sigma) R_2 u(\sigma) - \gamma^2 w^T(\sigma) R_1 w(\sigma) + x^T(\sigma+1) Q_f x(\sigma+1)$

$$\begin{aligned}
x(\sigma) = &A^N x_2(i) + B_{2\phi} U_2(i) + A^{N-1} B_1 w(i) \\
&+ \left[A^{N-2} B_\gamma, A^{N-3} B_\gamma, \cdots, B_\gamma, 0\right] \alpha_3 \\
&\times \Big[\bar{B}_\gamma^T \hat{Q}_{\bar{A}} \left(\bar{B}_2 U_1(i+1) + X_{01}(i+1)\right) \\
&\quad + B_{\gamma\phi}^T Q_f \left(A^N x_1(i+1) + B_{2\phi} U_1(i+1)\right)\Big]
\end{aligned}$$

$$\begin{aligned}
U_2(i) &= \left[u_2^T(i), u_2^T(i+1), \ldots, u_2^T(\sigma-1)\right]^T \\
X_{01}(i+1) &= \left[x_1^T(i+1), 0, \ldots, 0\right]^T \\
U_1(i+1) &= \left[u_1^T(i+1), u_1^T(i+2), \ldots, u_1^T(\sigma)\right]^T.
\end{aligned} \tag{31}$$

$B_{2\phi}$, $B_{\gamma\phi}$, and α_3 are defined in (8). If $N = 1$, $x(\sigma)$ of (31) is given by $x(\sigma) = Ax_2(i) + B_2 u_2(i) + B_1 w(i)$. Note that $u(\sigma) = -Hx(\sigma)$ satisfies the input constraint under assumption (12).

Then, if $R_1 - B_\gamma^T Q_f B_\gamma > 0$, $w(\sigma)$ maximizing $J(\sigma, \sigma+1)$ is given by

$$w(\sigma) = \gamma^{-1} R_1^{-1} B_\gamma^T \left(Q_f^{-1} - B_\gamma R_1^{-1} B_\gamma^T\right)^{-1} (A - B_2 H) x(\sigma) \tag{32}$$

where (32) can be derived by (28). With $u(\sigma) = -Hx(\sigma)$ and (32), by (29), we have (13).

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