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DYNAMIC EFFICIENCY AND VOLUNTARY IMPLEMENTATION IN MARKETS WITH REPEATED PAIRWISE BARGAINING

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Abstract

We examine a simple bargaining setting, where heterogeneous buyers and sellers are repeatedly matched with each other. We begin by characterizing efficiency in such a dynamic setting, and discuss how it differs from efficiency in centralized static setting. We then study the allocations which can result in equilibrium when the matched buyers and sellers bargain through some extensive game form. We take an implementation approach, characterizing the possible allocation rules which result as the extensive game form is varied. We are particularly concerned with the impact of making trade voluntary: imposing individual rationality *on and off* the equilibrium path. No buyer or seller consumates an agreement which leaves them worse off than the discounted expected value of their future rematching in the market. Finally, we compare and contrast the efficient allocations with those that could ever arise as the equlibria of some voluntary negotiation procedure.

Keywords: Implementation, Bargaining, Search, Matching

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Dynamic Efficiency and Voluntary Implementation in Markets with Repeated Pairwise Bargaining*

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1 Introduction

This paper is part of a broad research agenda to understand how markets should be designed, and the extent to which efficient outcomes can be acheived. In particular, this paper is motivated by our interest in understanding the process of trade in markets that are limited to decentralized bilateral trading. To this end, we employ a simple model of matching and search with an infinity of buyers and sellers, where search costs are represented by a delay between random rematchings. Each buyer has a valuation for one (indivisible) unit of a good, and each seller is endowed with one unit and also has a valuation. There is a known distribution of seller and buyer valuations. Trade occurs in discrete periods. In the first period, buyers and sellers are randomly matched into pairs. Each pair then plays a bargaining game that either results in a trade at some price, or no trade. If a buyer-seller match does not result in a trade, then each is randomly rematched with a new potential trading partner in the next period. There is discounting between periods. In this model we characterize the efficient allocations and study the allocations that are achievable via various bargaining procedures.

This paper departs from past work in this area, by approaching the problem from the implementation theory perspective. On the one hand, consistent with much of the previous literature on decentralized bilateral trade, the matching and search technology described above is taken as given. But contrary to past work on decentralized bilateral trade, we do not treat the rules of trade as fixed a priori. That is, our objective is not to study properties of equilibira under some specific game form according to which bilateral trade is governed (say, the Rubinstein bargaining game, or the Nash bargaining solution),

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but rather to ask instead the implementation question: what allocation rules can be implemented as unique equilirium outcome of some finite extensive form bargaining game of perfect information? We first characterize the set of efficient allocation rules in this environment, and then characterize the set of all allocation rules that can be implemented by some bargaining game. Using the conditions for implementability, we show that there exist robust distributions of buyer and seller valuations in which efficient allocations cannot be implemented by any bargaining game, and in fact are not even attainable as an equilibrium of any bargaining game.

The characterization of efficient allocations is nontrivial. Given that trading is constrained to occur only between matched buyers and sellers, and that there are search costs (i.e. time is valuable), there are generally distortions that must occur relative to traditional static efficiency. If one thinks of a simple supply and demand curve (here, representing seller and buyers' reservation values), the static efficient solution would be to have all buyers with values above the competitive equilibrium price trade with all sellers whose values are below that price. In the setting examined here, that may not be possible as these buyers and sellers may not be matched. If a very low valuation seller is matched with a buyer whose valuation is slightly below the equilibrium price should a trade be enacted? Given that there is value to the time before a rematching can occur, and that the rematching might also result in a similar match, the answer is possibly yes. We show that the dynamically efficient allocations are uniquely determined (up to sets of measure zero), and we fully characterize such allocations and establish a number of their properties.

As mentioned above, once we have a characterization of efficient allocation rules, we need to check on the implementability of those rules. To do so, we first need to specify the class of all bargaining mechanisms that we allow. We consider bargaining mechanisms which are finite-length extensive game forms of perfect information. Also, the game form is augmented by appending to each terminal node (except no trade terminal nodes) a signature move for both the buyer and the seller. Both signatures are needed, or the mechanism results in no trade for that match. The role of the signatures is to ensure that trade is *voluntary*, or respects individual rationality constraints. It is assumed that the the buyer and seller in the match have complete information about each others' valuations, so the solution concept we employ is backward induction. If anything, this informational assumption should make it easier to implement the efficient allocations.

We show first that in very special environments (homogeneous seller valuations and no search costs) the efficient allocation rule is implemented by the bargaining game whereby for any buyer–seller match, the seller makes a take-it-or-leave-it offer to the buyer. To establish implementability in more general environments, we first establish general conditions for implementability. Given a simple set of necessary conditions, we provide a robust example showing that if seller valuations are not identical, then efficient allocations are not implementable, even if there is no discounting. We close with a characterization of implementable allocation rules.

2 Relation to the Literature

Because this paper bridges several different areas, we discuss separately how it fits in with previous work in two broad themes: competitive bargaining and implementation. Basically, what we are doing here is layering the implementation question on to a standard model of search and competitive bargaining. Thus, our work relates to both of these areas.

Relation to the Competitive Bargaining Literature

The underlying model that we study involves a combination of matching, bargaining, search and rematching over a sequence of trading periods. As such, it is useful for studying pure exchange economies from a non-cooperative, game-theoretic perspective. Past work in the area [1], has typically assumed both the technological features underlying the matching and search technologies and also has assumed the formal rules according to which bargaining between paired agents is required to follow. It is this latter set of assumptions that marks the first key difference between what we are doing and what has been done before. While the bargaining rules usually are modeled as a specific process of offers and counteroffers such as one based on Rubinstein (1982) and Stahl (1972), we explicitly *do not* assume a particular game form for the bargaining process. Rather, we are trying to identify the set of allocation rules (Walrasian or otherwise) that can be achieved as unique Nash equilibrium outcomes of *some* bargaining mechanism.

The second difference between this paper and earlier work is that we do not focus on the question of the equivalence between Walrasian and competitive bargaining outcomes when market frictions are small. In fact, we are not particularly interested in the case of frictionless markets *per se*, but focus instead on the properties of markets in which frictions exist, despite the large numbers of traders. To this end, we characterize dynamically efficient allocation rules subject to the matching constraints, and show how these differ in systematic and interesting ways from competitive allocations. The study of allocation rules generated by matching processes, combined with noncooperative competitive bargaining is the subject of several recent papers, including Lu and McAfee (1995) and Peters (1995). Related work is also underway by Shimer and Smith (1996), who do not investigate the role of the bargaining game or other implementation questions, but instead address issues efficient sorting subject to the constraints of the matching process, where agreements (when reached) are assumed to be governed by the Nash bargaining solution.

[1] By now the collection of papers in this area is too large to summarize exhaustively. The most closely related papers include Gale (1986ab), Rubinstein and Wolinsky (1985), Binmore and Herrero (1988), and McLennan and Sonnenschein (1991) which follow in the footsteps of the early work on search and matching by Butters (1980), Mortensen (1982), Diamond (1982) and others. The bulk of this work is interested in identifying conditions under which game-theoretic equilibria in these decentralized matching and bilateral bargaining institutions will approximate Walrasian allocations when the frictions (seach costs, discount factors, etc.) become infinitesimal. We lump all these together under the general heading of "competitive bargaining".

Relation to the Implementation Literature

This paper is also related to the extensive implementation literature which studies the classes of allocation rules that can be acheived when self-interested individuals interact through some mechanism. This approach models mechanisms as game forms and obtains general characterizations of what can be acheived by some mechanism in various specified classes of game forms, when behavior is modeled according to a variety of game theoretic solution concepts.

This theory can be useful from at least two perspectives. First, if one takes a designer's normative point of view it helps one identify the set of allocation rules that can be decentralized and provides recipes for that decentralization. Second, even if one takes a more positive point of view where one is not choosing the mechanism, but rather is interested in knowing the properties of allocation rules that result the outcome of some interaction between individuals. The second perspective puts more weight on identifying necessary conditions for implementation, while the first perspective puts additional emphasis on sufficiency conditions.

Although the necessary conditions that come out of this literature must be taken seriously, there is somewhat less consensus about the practicality of the many of the sufficiency results, where very general and abstract mechanisms are constructed in order to demonstrate that a certain class of allocation rules can be implemented. There are two bases on which the canonical mechanisms have been criticized. The first is purely subjective and is simply that the mechanisms seem "unnatural" in the sense that they do not closely resembling commonly used institutions or have seemingly artificial parts of the message space such as the annoucement of preference profiles and the reporting of integers. There is also a basis for criticizing mechanisms on specific properties that they sometimes fail to posess. For instance, applying a theorem by Maskin (1977) tells us that a constrained version of the Walrasian correspondence is Nash implementable when there are three or more agents. But this result places no axiomatic constraints of the nature of the mechanism. One can also ask if the same result will hold if the mechanism is required to have a continuous and balanced outcome function (e.g., Postlewaite and Wettstein (1989)); or if the message space of the mechanism is constrained to be simple in some way (e.g., Dutta, Sen, and Vohra (1993), Saijo, Tatamitani, and Yamato (1993), and Sjöström (1995)). One might also wish to require properties that are necessary for a solution concept to always be well-defined on various parts of the mechanism (the boundedness condition relating to eliminating dominated strategies, and the best response condition relating to Nash equilibrium in Jackson (1992) and Jackson, Palfrey and Srivastava (1994)). A related critique (Jackson (1992) and Abreu and Matsushima (1992)) is that (until recently) research in implementation has ignored mixed strategies.

In this paper, we want to avoid the problems of artificiality as well as the problems inherent in mechanisms for which behavior is not always well-defined relative to the solution concept. In addition, we wish to begin to remedy two other shortcomings to the existing work in implementation theory.

First, we wish to avoid the use of implausible threats, used either to enforce certain actions in equilibrium, or to prevent certain strategy profiles from being "undesirable" equilibria. An extreme example of such a threat (which appears often in sufficiency constructions) is for the planner to destroy the all or part of the social endowment, if a particular out-of-equilibrium message profile is announced. The basic problems with this is that such outcomes may not be credible or enforceable and agents should anticipate this when deciding on strategies. Such mechanisms seem particularly far-fetched in cases where the players have inherent property rights (such as an initial endowment or outside option) that provide a lower bound of the utility the agent can expect in the mechanism, for *all* message profiles. In our model, because the buyer and seller in a match will be rematched in the next period, should they fail to agree to exchange, this places a natural type of "individual rationality" constraint on the process: no buyer or seller should consumate an agreement which leaves them worse off than the discounted expected value of their future rematching in the market. This gives us a natural notion of what we call *voluntary implementation.*

Voluntary implementation is related to problems of renegotiation-proofness and individual rationality, both of which have been considered to a limited extent in implementation theory. Maskin and Moore (1988) investigated "renegotiation-proof implementation", where any possible outcome of the mechanism that specifies a Pareto dominated allocation is replaced by a Pareto efficient allocation rule according to an exogenously specified renegotiation function. Rubinstein and Wolinsky (1992) carried this further, by examining renegotiation-proof implementation in a pairwise bargaining setting where the renegotiation was explicitly modeled, showing that the question depends in important ways on the type of renegotiation process. Ma, Moore and Turnbull (1988) impose an individual rationality constraint, which takes the form as an "opt-out" for each player. In this paper, individual rationality is a bit more. Specifically, voluntary implementation takes the form of an *endogenous* individual rationality constraint, since individual rationality is determined by the value of future rematching, which in turn depends on the bargaining mechanism itself.

A more distantly related (but similarly motivated) problem in implementation theory is "credibility", or the ability of the planner to commit to off-equilibrium-path outcomes that are known to be undesirable, in order to implement desirable outcomes on the equilibrium path. Chakravorti, Corchon, and Wilkie (1992) investigate this, and Baliga, Corchon, and Sjöström (1995) and Baliga and Sjöström (1995) go further, by including the planner as a player in the mechanism. In our work, the problem of credibility is finessed by the voluntary nature of the transaction.

The second issue where we depart from past work in implementation theory is to study dynamic allocation rules. Previously, the implementation problem has almost without exception been cast in a static setting where the set of agents interacting is not changing with time.

The importance of intertemporal tradeoffs is critical since many problems that economists are interested in, such as bargaining, investment, growth, repeated interactions, and so forth, are dynamic. Unfortunately, implementation theory has little to contribute to questions of mechanism design in this large arena. Extensive form games have been examined, but only in the context of using them to implement static allocations.[2] Finally, we wish to emphasize that the implementation in this paper is somewhat stronger than just implementation by subgame perfect equilbrium. Our implementation results are for mechanisms that are constructed as games of perfect information, so our concept of equilibrium is actually "backward induction" (Herrero and Srivastava (1992)), which is a stronger notion of implementation than subgame perfection.

Summarizing our contributions relative to the implementation literature: using a competitive bargaining model with rematching, we are able to characterize implementability in a dynamic environment, without imposing implausible threats, and without employing artificial or suspicious features to the implementing mechanism. Thus, we obtain a characterization of what is implementable is this class of dynamic allocation problems, without resorting to the usually cumbersome methods of proof in implementation theory.

The remainder of the paper is organized as follows. The model and definitions are presented in Section 3. Dynamically efficient allocation rules are characterized in Section 4. Section 5 establishes two simple necessary conditions for voluntary implementation of a dyanamic allocation rule. Section 6 presents a robust example that demonstrates the general non-implementability of dynamically efficient allocation rules. Section 7 presents a full characterization of dynamic allocation rules that can be voluntarily implemented. Section 8 contains some concluding remarks.

3 Definitions

The Economy

There are two goods. One good is indivisible and the other is divisible. *Sellers* are endowed with one unit of the indivisible good, and *buyers* are endowed with one unit of the divisible (numeraire) good.

Preferences

Agents' preferences are are characterized by a reservation value of the indivisible good, $v \in [0,1]$. There are a finite number of dates, $t \in \{1, \dots, T\}$, at which trade can take place, and a common discount parameter $\delta \in [0,1]$. A seller with reservation value s who sells her indivisible good for p units of the numeraire good at time t receives utility $\delta^t(p-s)$, and a buyer with reservation value b who buys a unit of the indivisible good for p units of the numeraire good at time t receives utility $\delta^t(b-p)$. An agent who never trades receives utility 0.

[2]Two recent exceptions are Kalai and Ledyard (1995) and Brusco and Jackson (1996).

Distributions of Values

Initially, there is a continuum buyers and of sellers. Information concerning the reservation values of the agents remaining in the economy at the beginning of a time $t \in \{1, \ldots, T\}$ is summarized by the following functions.

$B_t(b)$ – the mass of buyers at time t with value no more than b.

$S_t(s)$ – the mass of sellers at time t with value no more than s.

These are not cumulative distribution functions, since, for instance, it may be that $S_t(1) \neq 1$. The corresponding distribution functions (for $S_t(1) > 0$ and $B_t(1) > 0$) are $\frac{S_t(v)}{S_t(1)}$ and $\frac{B_t(v)}{B_t(1)}$. The initial mass of buyers and sellers is the same, $B_1(1) = S_1(1)$, so it will always be true that $B_t(1) = S_t(1)$, for all t. This is without loss of generality, since we can model other cases by adding buyers or sellers who should never trade.[3]

We assume that at least one of the two distributions is atomless. Specifically, we will assume that the initial distribution of buyers, B_1, is continuous and increasing at all $b > 0$. This rules out masses of buyers with identical valuations and assures that there are buyers with values in an any open subinterval of [0,1]. This assumption simplifies the analysis in that we do not have to worry about rationing agents with the same valuation, or randomizing. The one exception is that we allow for the possibility of a mass of buyers at $b = 0$.

Pairwise Matching

At the beginning of each period, the remaining buyers and sellers who have not yet traded are pairwise matched with each other. The matching is described by a probability measure μ_t on $[0,1]^2$ where for any measurable $A_t \subset [0,1]^2$

$$\mu_t(A_t) = \int_s \left(\int_{b:(s,b)\in A_t} \frac{dB_t(b)}{B_t(1)} \right) \frac{dS_t(s)}{S_t(1)}.$$

The distribution over values that any specific seller with valuation s will be matched with at time t is $\frac{dB_t(b)}{B_t(1)}$. Similarly, the distribution over values that any specific buyer with valuation b will be matched with at time t is $\frac{dS_t(s)}{S_t(1)}$.

Remarks on the Matching.

Note that there is generally a measurability problem associated with a law of large numbers over a continuum of i.i.d. random variables (see Judd (1985) and Feldman and Gilles (1985)). Our random variables (the buyer matched to a given seller) are naturally

[3]For instance, $B_1(1) > S_1(1)$, is handled by adding sellers with $s = 1$.

not i.i.d., as no two sellers are ever matched to the same buyer; however, there still remain problems with finding a foundation for the matching process described above.[4]

For the case where the distributions are step functions (i.e., buyers' and sellers' valuations only take on a finite number of values) we can describe a matching process with the above specified properties, as follows. Consider a given valuation of the sellers and suppose that there is a mass m_t^s of sellers with this valuation. Select a subset of the buyers with total mass m_t^s so that the distribution of buyer valuations in this subset is $\frac{B_t(v)}{B_t(1)}$. Next, arrange these buyers around the edge of a wheel (in a measurable way) and the sellers around the outside of the wheel. Finally, "spin" the wheel and match up the sellers and buyers who are lined up when the wheel stops.

The difficulty with trying to extend the above described process to a situation where, for instance, the buyers values span a continuum is now easily seen. For any v it must be that there is a measurable subset of buyers with mass $S_t(v)$ and with distribution of valuations $\frac{B_t(\cdot)}{B_t(1)}$. Starting with a set with mass $B_t(1) > S_t(v)$, it is not always possible to find a measurable subset (with mass $S_t(v)$) which preserves the original distribution of valuations. This can be shown along the same lines as Proposition 1 in Feldman and Gilles (1985). To get the rough intuition behind this, one can think of a measurable function (the one indicating which buyers have been matched) as being 'almost' continuous. It is not possible to be almost continuous and still select a given proportion from *every* interval of buyers.

There are ways around the difficulty discussed above that would suit our purposes. One is to notice that we can find a matching process that comes arbitrarily close to fitting the above description. Alternatively, we could work with step function distributions where one can exactly satisfy the representation and arbitrarily approximate the limiting distribution. We choose to work directly at the limit distributions and to note that we can come arbitrarily close to finding a matching process that formally justifies the assumed one. (See Al-Najjar (1996) for more discussion of this.) The reason for doing this, rather than working with step functions, is that it slightly simplifies the analysis of efficient allocations.

Allocation Rules

To describe allocations, we need to describe which buyers and sellers will trade at each time, and what price will be paid (i.e., what transfer is made). We restrict our attention to allocation rules which depend only on the time and on the buyers' and sellers' valuations (but not their names). This restriction reflects our interest in anonymous processes and simplifies our notation without any real loss of generality under our assumptions.[5]

[4]See McClennan and Sonnenschein (1991) for some additional discussion in the context of a model that has the same kind of measurability problem.

[5]The trades which are made at a time t could be allowed to depend on the history of trades made up until that point, and a buyer or sellers' name rather than just their value. However, given the continuum

A *trading rule* is a collection, $A = (A_1, \ldots, A_T)$, of measurable subsets A_t of $[0,1] \times [0,1]$. A pair $(s, b) \in A_t$ indicates that any seller with valuation s and buyer with valuation b who are matched at time t should trade.

A *price rule* is a collection of measurable functions $p = (p_1, \ldots, p_T)$, where $p_t : A_t \to [0,1]$. A price rule indicates that if a buyer and seller trade then the buyer transfers $p_t(s, b)$ units of the divisible good to the seller.

An *allocation rule* consists of a trading rule and a price rule.

Cutoff Rules

One type of trading rule that will play an important role in our results is a cutoff rule. This is a rule such that the set of buyers who trade with any given seller form an upper interval of the set of buyer types, and the set of sellers who trade with a given buyer form a lower interval of the set of seller types. More formally, A is a *cutoff rule* if for all t and s,[6]

(i) either $\{b|(s,b) \in A_t\} = \{b \in [0,1]|b \geq b'\}$ or $\{b|(s,b) \in A_t\} = \{b \in [0,1]|b > b'\}$ for some $b' \in [0,1]$, and

(ii) $\{b|(s,b) \in A_t\} \subset \{b|(s',b) \in A_t\}$ whenever $s > s'$.

In many cases it will not matter whether the inequalities in (i) are weak or strict (see the definition of equivalence below), and we represent a cutoff rule by functions $\beta_t(s)$ (corresponding to b' in (i)).

Evolution of Distributions over Values

Any trading rule A and initial distributions S_1 and B_1 induce $S_2, \ldots, S_T$ and $B_2, \ldots, B_T$, according to the matching process. The resulting distributions are defined recursively by:

$$S_{t+1}(v) = S_t(v) - \int_{s \leq v} \left(\int_{b:(s,b) \in A_t} \frac{dB_t(b)}{B_t(1)} \right) dS_t(s) \tag{1}$$

and

$$B_{t+1}(v) = B_t(v) - \int_{b \leq v} \left(\int_{s:(s,b) \in A_t} \frac{dS_t(s)}{S_t(1)} \right) dB_t(b). \tag{2}$$

model and the properties of the matching process, the history will be known as a function of t (up to sets of measure 0). Also, there will not be any positive mass of matched agents with the same pair of valuations, so knowing the agents' names is unnecessary.

[6]The definition can equivalently be stated from the buyer's perspective.

Equivalence of Trading and Allocation Rules

Given the continuum of agents, we define an equivalence over allocation rules that differ only on sets of measure 0.

The trading rules A and $\widehat{A}$ are *equivalent* if for each t, $\mu_t(A_t \cap \widehat{A}_t) = \mu_t(A_t \cup \widehat{A}_t)$, where μ_t is the measure defined in (0) induced by A according to (1) and (2).[7]

The allocation rules (A, p) and $(\widehat{A}, \widehat{p})$ are *equivalent* if A and $\widehat{A}$ are equivalent and for each t, $\mu_t(\{(s,b) \in A_t | p(s,b) \neq \widehat{p}(s,b)\}) = 0$.

Expected Utility

The expected utility $u_t^s(s; A, p)$ of a seller with valuation s under an allocation rule (A, p) at time t conditional on not having traded yet is given by

$$u_t^s(s; A, p) = \sum_{\tau=t}^{T} \delta^{\tau-t} \left(\Pi_{i=t}^{\tau-1} \left[1 - \int_{b:(s,b)\in A_i} \frac{dB_i(b)}{B_i(1)} \right] \right) \left(\int_{b:(s,b)\in A_\tau} (p_\tau(s,b) - s) \frac{dB_\tau(b)}{B_\tau(1)} \right).$$

Similarly, the expression for the expected utility $u^b(b, A, p)$ of a buyer with valuation b under an allocation rule (A, p) is given by

$$u^b(b, A, p) = \sum_{\tau=t}^{T} \delta^{\tau-t} \left(\Pi_{i=t}^{\tau-1} \left[1 - \int_{s:(s,b)\in A_i} \frac{dS_i(s)}{S_i(1)} \right] \right) \left(\int_{v_s:(s,b)\in A_\tau} (b - p_\tau(s,b)) \frac{dS_\tau(s)}{S_\tau(1)} \right).$$

Reservation Prices

It will often be useful to work with the *reservation prices*, $\overline{p}_t^s(s; A, p)$ and $\overline{p}_t^b(b; A, p)$, induced by an allocation rule. The reservation price at time t is simply the price at which an indivdual would be indifferent between trading and not trading at time t. These follow immediately from above:

$$\overline{p}_t^s(s; A, p) - s = \delta u_{t+1}^s(s; A, p), t = 1, \cdots, T-1$$

$$b - \overline{p}_t^b(b; A, p) = \delta u_{t+1}^b(b; A, p), t = 1, \cdots, T-1$$

$$\overline{p}_T^s(s; A, p) = s$$

$$\overline{p}_T^b(b; A, p) = b.$$

When an allocation rule is fixed, we may simply write $u_t^s(s)$ and $\overline{p}_t^s(s)$.

[7]Notice that in this case that the measure $\widehat{\mu}_t$ induced by $\widehat{A}_t$ will coincide with μ_t.

Dynamic Efficiency

We say that a trading rule A is *dynamically efficient* if there exists a price rule p such that (A, p) maximizes the total expected surplus:

$$\int_s u^s(s; A, p) dS_1(s) + \int_b u^b(b; A, p) dB_1(b).$$

Notice that dynamic efficiency is a property of trading rules, and thus is independent of the choice of p.

Dynamic efficiency and Pareto efficiency coincide if ex-ante transfers of the divisible good can be made among the buyers, and among the sellers. Without such transfers, dynamic efficiency as we have defined it is utilitarian. To see the differences in this setting, consider a situation where some sellers are forced to trade with any buyer they meet in the first period whose valuation falls *below* a certain level, even if the buyer's value is less than the seller's. Such trades can be part of a Pareto efficient allocation if there are no transfers possible, since these sellers are taking low valued buyers out of the market. This benefits the other sellers since the remaining pool of buyers has higher average valuations. This is clearly inefficient if transfers can be made among the sellers.

Also, notice that the definition of efficiency takes the set of agents in the system as given. One cannot tell agents to leave without trading. If one admits the ability to control the agents in the system, then a stronger notion of efficiency emerges where one could tell all the agents other than the ones who trade in the competitive allocations not to take part in the matching at all. The efficient allocation under such a definition would coincide with the competitive one. One could implement such a rule by making the competitive price the only one available, so that indeed the 'inefficient' agents would have no incentive to take part in the matching. One has to take our model of agents' valuations as a simplification of a more complicated model where the value to each agent of an agreement to trade depends on the specific pair of matched agents, and our reservation values are just expected values over a more complicated function. For instance, sellers with high expected valuations could still have some small probability that they would meet someone with whom they would have substantial gains from trade. Rather than complicate the model in this way to make it more realistic, we will work with the simpler model and take as given the presence of the agents.

Negotiation and Game Forms

The formal, or informal, negotiation process which goes on between a matched buyer and seller is represented by an extensive game form. We restrict attention to finite stage extensive game forms of perfect information. (The results extend to infinite stage game forms, but finite ones are all that are needed.) The same extensive form is played by each matched pair. The procedure can depend on the time, and thus on the measures of agents remaining. However, the procedure cannot depend on the specific history of any measure 0 set of agents.

This last assumption is important. If one permits full contingency of the mechanism on the history, the implementation problem can become trivial. The future stages of the mechanism could then be chosen to enforce no trade if *any* agent deviates from prespecified actions. This would defeat the idea of individual rationality as capturing voluntary trade with an endogenous outside option, as the outside option could be controlled as a function of any single agent's actions.

One can argue that we should use the stronger assumption that the mechanism be the same in each period. That is, the form of negotiation available at any time should be the same if it is representing some primitive set of available actions. While we agree with this in certain contexts (for instance in a richer model where there were inflows of agents too), allowing for the larger set of mechanisms will strengthen our impossibility results.

Individual Rationality

The heart of our analysis is the assumption that no agreement becomes binding until it is signed by each of the two agents. After negotiations have led to a suggested trade and price, the trade does not take place unless both agents "sign" the agreement. This is captured as follows. Consider, γ_t, an extensive game form with perfect recall to be played between an arbitrary buyer and seller at some time t, such that each terminal node suggests either a trade and price, or no trade. Given γ_t, let us define a dynamic version, $\Gamma(\gamma_t)$, as follows. First, replace any terminal node of γ_t which recommends a trade and price, with a node that has a binary choice node (yes, no) for the buyer. Let "no" lead to a terminal node with no trade as the outcome. Let "yes" lead to a binary choice node (Yes, No) for the seller. Let "No" lead to a terminal node with no trade as the outcome, and "Yes" lead to a terminal node with the originally prescribed trade and price. We have simply augmented γ_t by additional moves which require both the buyer and seller's "signature" before completing the trade.

Now at any time t, each matched buyer and seller play the augmented version of γ_t. If the outcome of $\Gamma(\gamma_t)$ is trade, then the trade is consumated and the buyer and seller are removed from the matching process. If the outcome is no trade, then the buyer and seller are returned to their respective pools to be rematched in the next period.

Equilibrium

We can now define an equilibrium. Consider a specification of a pure behavioral strategy for each agent for each $\Gamma(\gamma_t)$ as a function of the agent's own value and role (i.e., . buyer or seller) as well as the value of the other agent he is matched with. We restrict attention to strategies that are measurable across agents' valuations. Such a collection of strategies induces an allocation rule (A, p).

An *equilibrium* of the augmented sequence of mechanisms is a specification of strategies (as defined above) where for each t and pair of agents' values:

(i) the strategies form a subgame perfect equilibrium of $\Gamma(\gamma_t)$, given the anticipated values of the no-trade option under the induced allocation rule (A,p),[8] and

(ii) at any node where an agent's actions may lead either to current trade at some price or to rematching, the agent chooses an action leading to rematching only if it offers an expected utility higher than any of the other available actions.

Part (i) of the definition of equilibrium imposes sequential rationality in the form of subgame perfect equilibrium. Part (ii) of the definition of equilibrium is a tie-breaking rule when an agent is indifferent between trading today or waiting an being rematched. The particular form of the tie-breaking rule is not important: we could have defined it to have agents always favoring delay in such situations. One can think of this as being equivalent to a lexicographic preference assumption that eliminates indifference.[9] This simplifies the analysis, as we will have a unique prediction of an outcome of a given extensive game form as a function of agent's reservation prices. As we shall see, however, the implementation problem is still non-trivial, as reservation prices are endogenous.

Voluntary Implementation

An allocation rule (A,p) is *voluntarily attainable* if there exist $(\gamma_1,\ldots,\gamma_T)$ such that at least one equilibrium of the augmented sequence of mechanisms results in an allocation rule that is equivalent to (A,p).

The essential difference between attainability and implementability is uniqueness. Attainability does not require uniqueness, and hence is a very weak form of implementation, roughly equivalent to what has been known in the literature as 'truthful' implementation (Dasgupta, Hammond, and Maskin 1979). However, in this setting there is no revelation principle and we do not employ direct mechanisms, so we have defined attainability directly. More generally, one may be interested in knowing all the equilibria of a mechanism, which motivates the definition below.

An allocation rule (A,p) is *voluntarily implemented* if there exist $(\gamma_1,\ldots,\gamma_T)$ such that each equilibrium of the augmented sequence of mechanisms results in an allocation rule that is equivalent to (A,p).

[8]We could have simply referred to a subgame perfect equilibrium of the overall game form with the continuum of players and T periods. In doing so we would have to specify what information was known to agents concerning matches and other players' behavior at each point in time. In order for subgame perfection to have any bite in a specific match and time, extreme informational assumptions would be necessary to avoid information sets (such as knowing the previous behavior of all agents and the full outcome of the matching). The definition we employ applies subgame perfection directly to each time and match and thus avoids such assumptions.

[9]We did not model it that way since it would preclude a utility representation.

More generally, we may simply be concerned that an efficient trading rule be implemented (or attainable) and not concerned with the particular prices that are realized. We say that a trading rule A is *voluntarily implemented* if there exists a sequence γ_t such that for each equilibrium there exists a price rule p such that the equilibrium results in an allocation rule equivalent to (A, p). A trading rule A is *voluntarily attainable* if there exists a sequence γ_t such that there exists some equilibrium and price rule p such that the equilibrium results in an allocation rule equivalent to (A, p).

4 Dynamic Efficiency

Homogeneous Sellers

We begin with the case where the distribution over seller values has a simple form. In particular, they either have a reservation value of 0 or 1. Thus, $S_1(v) = S_1(0)$ for all $v < 1$. We call this the case of homogenous sellers since all the sellers with whom trade is ever efficient have $s = 0$. Notice that this is essentially equivalent to having 'fewer' sellers than buyers, as being matched with a seller who has $s = 1$ is the same as not being matched at all.

Example 1.

Let $T = 2$ so that there are two periods of matching and potential trade, and consider the case where $\delta = 1$. Buyers' valuations are uniformly distributed across [0,1] with a total mass of 1. A mass $0 < m < 1$ of sellers have valuation 0 and the remaining mass, $1 - m$, have valuation 1. This is represented by $B_1(b) = b$ for all b and $S_1(s) = m$ for all $s < 1$.

If there were a centralized market, or one had control over the matching procedure, then all Pareto efficient allocations would involve the assets going to the buyers with value at least $1-m$. The competitive allocations are an obvious choice, where sellers sell to the buyers with values above the competitive price, $p = 1 - m$.

In our model, trade is constrained through the matching process, and the characterization of an efficient allocation becomes more complicated. Some of the higher value buyers might never be matched to a seller with whom they can trade, and it is sometimes better to clear a trade with a low-valued buyer than to wait for a buyer with a higher expected value.

Let us describe the dynamically efficient allocation rule for this example. It is clear that in the second (last) period, all positive value trades should be cleared, since there will be no further matching. It is also clear that a dynamically efficient trading rule will be a cutoff rule. It suffices to specify the minimum value of a buyer that a 0 value seller should trade with. (These and other claims in this example are proved in Theorem 1.) For any value c set as a cutoff today, the remaining distribution tomorrow will be $B_2(b) = b$ for $v \leq c$, and $B_2(b) = (1-m)(b-c) + c$ for $b > c$. The gain from clearing a trade today with a buyer of value b, is simply b. Sellers who do not trade today are rematched in the second period. The expected value of the buyer that they will trade with in the second period is simply the expected value of b under the distribution $B_2(v)/B_2(1)$ which is

$$\frac{1 - m(1-c)(1+c)}{2(1-m(1+c)}$$

We can find a dynamically efficient trading rule by setting the cutoff value equal to the expected value of rematching. That is, on the margin, a trade should be cleared today if (and only if) it offers at least as much total value as could be expected by waiting and clearing the trade tomorrow. Thus efficient trading rules are characterized by any $c*$ satisfying,

$$c* = \frac{1 - m(1-c*)(1+c*)}{2(1-m(1+c*))}$$

or

$$c* = \frac{\sqrt{1-m} - (1-m)}{m}.$$

The cutoff rule is decreasing in m. As the mass of sellers m increases, the current cutoff has less of a reduction effect on tomorrow's expected trading value. Also notice that the cutoff value is always lower than the competitive price $(1-m)$, which makes sense since some of the competitive trades can never occur because of the matching process.

Dynamic Efficiency with Homogenous Sellers and No Discounting.

We first offer a characterization of dynamic efficiency for the case where sellers have either values of 0 or 1, and where $\delta = 1$. As we shall see later, these assumptions offer considerable simplifications. All proofs are relegated to the appendix.

Theorem 1 *Suppose that $\delta = 1$ and $S_1(v) = S_1(0)$ for all $v < 1$. There exists a unique (up to sets of measure 0) dynamically efficient trading rule. It can be represented by a cutoff rule, with associated functions $\beta_1, \ldots \beta_T$. The cutoff rules are the unique (up to sets of measure 0) solutions to the following equations:*

$$\beta_t(1) = 1 \ \forall t$$

$$\beta_T(0) = 0$$

and

$$\beta_t(0) = \int_0^1 \max[b, \beta_{t+1}(0)] \frac{dB_{t+1}(b)}{B_{t+1}(1)} \tag{3}$$

for each $t < T$, where B_{t+1} is defined recursively by (2).

Notice that one cannot simply calculate $\beta_t(0)$ by evaluating the right hand side of (3), as B_{t+1} depends on $\beta_t(0)$. This means that (3) is a characterization of the efficient rule rather than a formula for it. It offers intuitive insight into the efficient solution, stating that it is the fixed point of a certain relationship. (Once one has specified a specific initial distribution, one can use (3) to obtain a closed form for the cutoff rule, as illustrated in the previous example.) The uniqueness of the solution to (3) in the above theorem is thus very important.

Given the characterization of efficiency in (3), we can deduce certain properties of efficient rules.

Corollary 1 *(i) The sequence of cutoff values is strictly decreasing:* $\beta_1(0) > \cdots > \beta_T(0)$.

(ii) The static (centralized) competitive price is larger than $\beta_1(0)$.

Analogous results hold for the case of homogeneous buyers and heteregeneous sellers. When there is discounting, the results extend in a natural way, but the argument used in proving Theorem 1 becomes messy. So, for the rest of the paper we restrict attention to two periods, $T = 2$.

Dynamic Efficiency with Heterogeneous Sellers and/or Discounting:

We now offer a characterization of dynamic effiency for more general distributions of sellers' values, with discounting and $T = 2$.

Theorem 2 *There exists a unique (up to sets of measure 0) dynamically efficient trading rule. It is a described by cutoff rules, with associated functions* $\beta_1(s)$ *and* $\beta_2(s)$*. These cutoff values uniquely satisfy the following equations:*

$$\beta_2(s) = s \qquad \forall s \in [0,1]$$

and

$$\beta_1(s) - s \;=\; \delta \int_0^1 \max[b' - s, 0]\frac{dB_2(b')}{B_2(1)} \;+\; \delta \int_0^1 \max[\beta_1(s) - s', 0]\frac{dS_2(s')}{S_2(1)} \; - \; \delta \int_0^1 \left(\int_0^1 \max[b' - s', 0]\frac{dB_2(b')}{B_2(1)} \right) \frac{dS_2(s')}{S_2(1)}, \tag{4}$$

if this is feasible with $\beta_1(s) < 1$*, and* $\beta_1(s) = 1$ *otherwise; where* S_2 *and* B_2 *are determined by (1) and (2), respectively. Furthermore,* $\beta_1(s)$ *is continuous and is strictly increasing at values of* s *such that* $\beta_1(s) < 1$.

Let us examine the intuition behind (4) as a characterization of efficiency. Consider a planner deciding on whether to clear a currently matched pair. Let these be the ones on the left hand side. If this trade is cleared, then the left hand side represents the marginal value of consummating that trade today. If this trade is not cleared, then the right hand side gives the marginal expected value from throwing both players back in the pool to be rematched tomorrow. The first two expressions are the discounted expected values of the trades that these agents will have from being rematched in the next period. The last expression accounts for the fact that rematching these two agents uses up two agents from the next period. Thus, in addition to counting the expected value from the trades from rematching these two agents next period, we also have to subtract the expected value of the one match equivalent that is displaced due to the rematching. This is precisely the calculation in (4).

In the special case where the sellers have reserve value of either s or 1 (the homogenous case), (4) reduces to

$$\beta_1(s) - s \;=\; \delta \int_0^1 \max[b' - s, 0] \frac{dB_2(b')}{B_2(1)},$$

which is simply a generalization of (3) to include discounting.

5 Necessary Conditions for Voluntary Implementation and Attainability

Next, we turn to the issue of voluntary implementation, and consider the case of arbitrary (finite) T. A characterization of voluntary implementation will provide us with the complete collection of allocation rules which could ever be the unique equilibrium outcome of such a dynamic interaction - under any negotiation process (which is representable by a finite extensive game form of perfect information). With such a characterization in hand, we will return to check whether it is compatible with the characterization of dynamic efficiency.

Given the individual rationality that is at the heart of our definition of voluntary implementation, it is clear that the trades suggested under an implementable (or attainable) allocation rule must be individually rational. By individually rational, it means that the price p of any trade consummated between s and b in period t must lie between the corresponding seller and buyer reservation values.

Individual Rationality An allocation rule (A, p) satisfies *individual rationality* if for any t and almost every $(s, b) \in A_t$

$$\overline{p}_t^s(s; A, p) \leq p_t(s, b) \leq \overline{p}_t^b(b; A, p),$$

A second necessary condition for voluntary implementation (or attainability) is individual irrationality, which states that there is no price which is traded at by some pair of agents at time t which is simultaneously individually rational for a pair of agents who do not trade.

Individual Irrationality An allocation rule (A, p) satisfies *individual irrationality* if for each t, $(s', b') \notin A_t$ and $(s, b) \in A_t$: either

$$p_t(s,b) > \overline{p}_t^b(b'; A, p)$$

or

$$p_t(s,b)) < \overline{p}_t^s(s'; A, p).$$

A third condition that is necessary for voluntary implementation (or attainability) is that prices be non-decreasing in reservation prices.

Non-decreasing Prices An allocation rule (A, p) has *non-decreasing* prices as a function of reservation prices, if for each t, (s, b) and (s', b') in A_t:

$$p_t(s,b) \geq p_t(s', b')$$

whenever $\overline{p}_t^s(s; A, p) \geq \overline{p}_t^s(s'; A, p)$ and $\overline{p}_t^b(b; A, p) \geq \overline{p}_t^b(b'; A, p)$.

Rubinstein and Wolinsky (1991) state a related necessary condition for implementation without the possibility of rematching. Here we must state the condition in terms of reservation prices rather than agents' valuations, as reservation prices reflect the relevant valuations in the dynamic context. This distinction is important since reservation prices may not always be non-decreasing in an agent's value, as reservation prices depend on future prospects for trade under an allocation rule.

Notice that an implication of the above condition is that the price rule can only vary with the reservation prices of the agents.

Theorem 3 *Consider a trading rule, A, that is voluntarily attainable (or implementable), and $(\widehat{A}, \widehat{p})$, an allocation rule corresponding to one of the equilibria of an implementing mechanism, where $\widehat{A}$ is equivalent to A. Then $(\widehat{A}, \widehat{p})$ satisfies individual rationality, individual irrationality, and non-decreasing prices.*

Let us make two remarks on Theorem 3. First, these conditions are necessary simply when one considers attainability. In other words, these conditions are needed simply to ensure that $(\widehat{A}, \widehat{p})$ can arise as an equilibrium of any mechanism. The conditions are not arising from multiple equilibrium considerations. Second, these conditions are also necessary when one admits infinite stage mechanisms. Details on this are given in a footnote to the proof.

These conditions play a central role in the full characterization. We defer the full characterization of dynamic implementation to Section 7. The full characterization tackles difficulties associated with possible discontinuities in the implemented price function, as well as the usual implementation challenge of ruling out equilibria which do not result in an allocation rule equivalent to (A, p). For now, let us show that these conditions are also sufficient in some cases of interest.

Theorem 4 *Suppose that B_1 and S_1 are continuous and increasing. Consider an allocation rule, (A, p), such that A_t is a continuous cutoff rule,[10] and p_t is continuous on A_t for all t. (A, p) is voluntarily attainable if and only if it satisfies individual rationality, individual irrationality, and has non-decreasing prices.*

6 Implementing Efficient Rules

For the case with homogenous sellers and no discounting, the voluntary implementation of the dynamically efficient trading rule problem is easily handled. The characterization of dynamic efficiency from Theorem 1 has one comparing the value of a match today with the expected value of buyers' values tomorrow. This exactly matches the sellers' individual rationality constraint if the seller sees all of the gains from trade. The dynamically efficient trading rule is voluntarily implemented by the mechanism which has sellers making take it or leave it offers in each period. (A proof of this claim appears in the appendix.)

When there are heterogeneous sellers, however, implementing the dynamically efficient trading rule becomes more difficult. In particular there is a rich set of individual rationality and individual irrationality conditions to be satisfied simultaneously. The following proposition establishes that this can be impossible, even with very well behaved initial distributions of buyers and sellers.

Proposition 1 *There exists a rich set of continuous and increasing distributions of buyer and seller valuations for which the dynamically efficient trading rule is not voluntarily attainable (and hence not voluntarily implementable).*

The formal proof of Proposition 1 appears in the appendix. It shows, by means of a robust example, that it is not generally possible to assure individual rationality and individual irrationality are simultaneously satisfied.[11]

The intuition is the following. When a buyer and seller trade in the first period, this creates (at the margin) an externality on the other traders, in that the trade has a marginal effect on the distribution of traders who are rematched in the next period,

[10]The cutoff values from both the buyer and seller's perspectives are continuous functions.

[11]In fact, the only nontrivial example we have where the dynamically efficient allocation is voluntarily attainable is in the case of homogenous sellers.

which affects the expected gains from period two trade for the other pairs who do not trade in the first period. This is reflected in the last term of equation (4). Thus in the optimal solution, it is possible that some "good" trading pairs should be left in the market to offset this externailty. This can be true even though the expected surplus from that transaction in the first period exceeds the sum of the expected surpluses of the two transacting parties were they to search one more period. If this is the case, then for any game that tries to implement the efficient solution, some of these trading pairs end up being tempted to trade in the first period, which will prevent the efficient solution from being an equilibrium outcome.

7 The Characterization of Voluntary Implementation

We now offer a complete characterization of voluntarily implementable allocation rules.

Given $P \subset [0,1]$, denote

$$\mathrm{IR}(P) = \{(q,r) \in [0,1]^2 \mid \exists p \in P \; q \leq p \leq r\}.$$

An allocation rule (A, p) satisfies *condition* $*$ if for each t there exists $P_t \subset [0,1]$ and $\widehat{p}_t : [0,1]^2 \to P_t$ such that:

(*i) [Reservation Price Measurability] for every $s, b \in A_t$, $p_t(s,b) = \widehat{p}_t(\overline{p}_t^s(s), \overline{p}_t^b(b))$

(*ii) [Individual Rationality] $(\overline{p}_t^s(s), \overline{p}_t^b(b)) \in \mathrm{IR}(P_t)$, for every $s, b \in A_t$, and $\overline{p}^s \leq \widehat{p}_t(\overline{p}^s, \overline{p}^b) \leq \overline{p}^b$, for every $(\overline{p}^s, \overline{p}^b) \in \mathrm{IR}(P_t)$,

(*iii) [Individual Irrationality] $(\overline{p}_t^s(s), \overline{p}_t^b(b)) \notin \mathrm{IR}(P_t)$, for every $s, b \notin A_t$

(*iv) [Non-Decreasing Prices] $\widehat{p}_t$ is non-decreasing over the domain $\mathrm{IR}(P_t)$,

(*v) [Separating Prices] for every $(\overline{p}^{s'}, \overline{p}^b) \in \mathrm{IR}(P_t)$ and $\overline{p}^s$ such that $(\overline{p}^s, \overline{p}^b) \in \mathrm{IR}(P_t)$, if $\widehat{p}_t(\overline{p}^{s'}, \overline{p}^b) < \widehat{p}_t(\overline{p}^s, \overline{p}^b)$ then there exists $p \in P_t$ such that $\overline{p}^{s'} \leq p < \overline{p}^s$. Similarly, for every $\overline{p}^{b'}$ if $\widehat{p}_t(\overline{p}^s, \overline{p}^{b'}) > \widehat{p}_t(\overline{p}^s, \overline{p}^b)$ then there exists $p \in P_t$ such that $\overline{p}^{b'} \geq p > \overline{p}^b$.

Let us discuss some of the differences in the above condition from the conditions stated in Section 4. The conditions stated in Section 4 are weaker versions of the above, but are still strong enough to reach the negative results in Section 5.

The conditions (*ii)–(*iv) are more complicated than the statements appearing in Section 4. The set P_t corresponds to the set of prices that are reachable by the implementing mechanism. Sometimes it is necessary for this to be larger than the set of prices which are supposed to be traded at in equilibrium, as off equilibrium behavior will be important in determining equilibrium behavior. (See the example below.) Then, for instance, the individual irrationality condition must be satisfied relative to all of the prices in P_t. If some price in P_t is individually rational, then an equilibrium which results in trade will exist. So (*iii) must hold relative to all of P_t.

Condition (*i) is new relative to what we presented in Section 4. The function $\widehat{p}_t$ has as domain reservation prices, as these are what matter in determining equilibrium actions. It is necessary then that the implemented price function be measurable with respect to reservation prices, which is condition (*i).

The last condition (*v) is also new relative to what we presented in Section 4. It states that the implemented price function can only be increasing in places where we can distinguish the reservation prices of the agent in question. If for instance $\overline{p}^{s\prime} < \overline{p}^s$, but there are no prices from P_t in between $\overline{p}^{s\prime}$ and $\overline{p}^s$, then these two types would have exactly the same preferences over trades in P_t (the only ones possible from the implementing mechanism). In such a case, the equilibrium actions of these two types would be the same.

Condition $*$ is necessary for implementation (and attainability).

Theorem 5 *If an allocation rule (A,p) is voluntarily implementable (or simply attainable), then there exists $(\widetilde{A},\widetilde{p})$ which is equivalent to (A,p) and satisfies condition $*$.*

Let us now turn to the full implementation problem. First let us consider an example that illustrates the above condition, and shows that there is a multiple equilibrium problem.

Example 2 Consider the dynamically efficient trading rule defined in Example 1, when $m = 1/2$.

Consider a fixed price of $c = \frac{2}{\sqrt{2}} - 1$ in the first period, which corresponds to $c*$. So 0 valued sellers trade with all buyers with values above c in the first period at a price of c. In the second period let 0 valued sellers trade with all buyers, and trade at a price equal to the buyer's valuation.

Let us check that condition $*$ is satisfied relative to this A, p. Let $P_1 = \{c\}$ and $P_2 = [0,1]$. Let $\widehat{p}_1(\overline{p}^s, \overline{p}^b) = c$ and $\widehat{p}_2(\overline{p}^s, \overline{p}^b) = \overline{p}^b$. It is then simple to verify (i)–(v). Thus, we should be able to find a mechanism which has A, p as an equilibrium.

Consider the following mechanism. The first period is the degenerate mechanism that simply has trade at price c. The second period is a mechanism where a seller makes take it or leave it offers to the buyer, where the seller can name any price in [0,1].

Let us check that there is an equilibrium that results in (A, p). It is the obvious one. Buyers approve trade in the first period if and only if $b \geq c$, and the 0-valued sellers approve trade in the first period. Notice that from the characterization of dynamic efficiency (and from Example 1), we know that a 0-valued seller's reservation price is exactly c in the first period. In the second period, 0 valued sellers make the offer of b to the buyer they are matched with, and it is approved.

But there is another equilibrium relative to the above mechanism! It involves all of the sellers rejecting the first period price. The second period is as before. This is an equilibrium, since if all the sellers reject in the first period, then the full mass of buyers is still there in the second period. The average value of the buyers is then $1/2$ in the second period. Since this is larger than c (see Example 1), the sellers are indeed acting optimally.

There are two equilibria of this mechanism. If sellers believe that 0-valued sellers will all trade in the first period when matched with buyers who have values above c, then they are willing to do so, since they expect only a value of c in the second period. However, if sellers believe that a significant (mass > 0) portion of the 0-valued sellers will not be trading in the first period when matched with buyers who have values above c, then they realize that the expected value tomorrow will be above c, and so they will wait too.

In fact, the efficient equilibrium is fragile: even a small variation in the expectations makes it better for the seller to wait.

However, the efficient allocation rule is fully implementable. We simply have to alter the above mechanism. Consider the following change: In the first period the buyer makes a take it or leave it offer to the seller from the set of prices $[c, 1]$. Any buyer with a value above c would always rather trade in the first period, since they expect to have their full value extracted in the second period. High valued buyers can offer sellers enough to get them to trade in the first period, even if the sellers expect a value above c in the second period. This means that the trades will occur in the first period that should. Given that they occur, the buyers will be able to offer c and get it.

The mechanism needs to have a possible range of prices in the first period that is larger than just c in order to implement the efficient rule. This illustrates the role of P_t in condition $*$. It also gives us insight to the full characterization. When we consider this $P_1 = [c, 1]$, condition $*$ is not satisfied relative to the undesired allocation rule where all of the agents wait until the second period to trade. In particular, (iii), individual irrationality is violated in this example. We are now ready to state the full characterization.

We restrict attention to implementation via mechanisms for which there exists a subgame perfect equilibrium of the augmented mechanism for each t relative to every set of reservation prices $\overline{p}^s, \overline{p}^b$.

Notice that first, just by definition of implementation, there must exist a subgame perfect equilibrium of the augmented mechanism for each t relative to the reservation prices $\overline{p}^s, \overline{p}^b$ which are generated by the implemented allocation rule.

So why should we care about existence relative to reservation prices $\overline{p}^s, \overline{p}^b$ which could only be generated by other allocation rules? The reason is that otherwise one could handle the multiple equilibrium problem by simply having subgame perfect equilibrium not exist for some agents relative to the expectations generated by some undesired allocation rules. This would be problematic because in the absence of existence we would not have a well defined prediction of behavior. In such cases it would be hard to say that we have ruled out undesired behavior, for the agents could still act in ways consistent with the undesired allocation rules. In contrast, if there does exist subgame perfect equilibrium relative to the expectations, and it does not correspond to the allocation rule, then we can say that we do have a prediction of how agents will play, and it does not correspond to the undesired allocation rule.

Theorem 6 *If an allocation rule (A, p) is voluntarily implementable by a mechanism satisfying the above existence requirement then*

(1) there exists $(\widetilde{A}, \widetilde{p})$ which is equivalent to (A, p) and satisfies condition $*$, and

(2) for each (A', p') not equivalent to (A, p), (A', p') fails to satisfy condition $*$ relative to the same $\widehat{p}$ and P_t as $(\widetilde{A}, \widetilde{p})$.

Conversely, if (1) and (2) hold and P_t is closed for each t then (A, p) is voluntarily implementable by a mechanism satisfying the above existence requirement.

We know that it is not necessary that P_t be closed. It is an open question whether (1) and (2) are sufficient in the absence of this condition, or whether there are additional necessary conditions.

The implementing mechanism is quite simple. It involves a sequential announcement of both reservation values by both agents. The mechanism γ_t at time t is described as follows:

Stage 1. The seller announces p^s. proceed to stage 2.

Stage 2. The buyer announces $p^{s\prime}, p^b$. Proceed to stage 3.

Stage 3. The seller announces $p^{b\prime}$.

The Outcome:

If $p^{b'} \leq p^b$ and $p^{s'} \geq p^s$ and $(p^{s'}, p^{b'}) \in \mathrm{IR}(P_t)$, then the outcome is $\widehat{p}_t(p^{s'}, p^{b'})$;

If $p^{b'} \leq p^b$ and $p^{s'} < p^s$ and $(p^{s'}, p^b) \in \mathrm{IR}(P_t)$, then the outcome is $\inf\{p \in P_t | p \geq p^{s'}\}$.

If $p^{b'} > p^b$ and $(p^s, p^{b'}) \in \mathrm{IR}(P_t)$, then the outcome is $\sup\{p \in P_t | p \leq p^{b'}\}$.

Otherwise, the outcome is no-trade.

In the above mechanism, the announcement of $p^{s'}$ allows the buyer to challenge the seller's announcement if, for instance, the seller announces $p^s > \overline{p}^s$. The announcement of $p^{b'}$ by the seller serves a similar purpose. If the seller honestly reveals $\overline{p}^s$, then a buyer has no incentive to challenge.

8 Concluding Remarks

There are three main contributions in this paper.

First, we provided a characterization of dynamic efficiency in a setting with random matching and search. In situations where markets are truly decentralized, standard notions of efficiency are inappropriate since goods may not be transferable from one arbitrary agent to another. The matching process imposes constraints on the set of feasible allocations, and introduces search externalities across agents. The constraints and externalities are at the heart of the characterization of dynamic efficiency.

Second, we obtain a clean characterization of implementation in situations where mechanisms cannot impose trade on agents. The characterization is intuitive in terms of the individual rationality conditions which naturally arise from the voluntary choice of agents to accept the outcome of the mechanism, or reject it and search for a new trading partner in the next period. The implementation can be achieved by simple mechanisms using alternating move games with perfect information, with a structure similar to standard bargaining games.

Third, we show that it is often the case that dynamically efficient allocations are inconsistent with voluntary decentralized trade under *any* bargaining game. Even with atomless agents, the externalities cannot be overcome, regardless of the mechanism by which agents negotiate and trade. Thus, in spite of the fact that trading pairs share complete information about each others' valuations, the strong necessary conditions imposed by voluntary trade are incompatible with overcoming the externalities and achieving efficient allocations.

The strength of the first two[12] results we obtain is, of course, tempered by the fact that we have worked in a specific setting. The specific nature of the preferences of the

[12]The last result (inefficiency) naturally still holds in more general settings.

agents (i.e., the "bargaining" structure), the way in which agents may accept or reject the suggestion of the mechanism, and the particular matching technology are important in terms of the clean and intuitive characterizations we obtain.

Relative to the implementation literature, this suggests exploring how the nonimposition restriction behaves in more general environments, especially those where one admits the possibility of some choices in matching, such as those offered by a centralized exchange. Relative to the competitive bargaining problem, it would be interesting to examine how the analysis extends an infinite horizon, and to situations where there are inflows of agents.

Appendix

Proof of Theorem 1: Consider time T with distributions of buyers and sellers, B_T and S_T. It is clear that any trade between a buyer with $0 < b < 1$ and a seller with $s = 0$ should be cleared, but with $s = 1$ should not be cleared. Thus, A_T can be represented as a cutoff rule. It is also clear that this is unique up to sets of measure 0.

We now proceed by induction. Suppose that given t, $(A_1, \ldots, A_t)$, and $(B_1 \ldots, B_t)$, the conclusions of the proposition hold for each $\tau > t$. That is, for each $\tau > t$, $(A_\tau, \ldots, A_T)$, are uniquely determined (up to sets of measure 0) cutoff rules satisfying (3), where $B_{\tau+1}(v)$ is determined by (2). We show that this implies the same is true for t.

Since A is dynamically efficient, it follows that for any price rule, p, the allocation rule $(A_t, \ldots, A_T, p_t, \ldots, p_T)$ maximizes the $T-t$ horizon problem with initial distributions S_t and B_t:

$$W(A_t, \ldots, A_T, p_t, \ldots, p_T) =$$
$$\int_s u_t^s(s, A_t, \ldots, A_T; p_t, \ldots, p_T) dS_t(s) + \int_b u_t^b(b, A_t, \ldots, A_T; p_t, \ldots, p_T) dB_t(b).$$

As the above expression is independent of the choice of the price functions, consider the prices given by $\widehat{p}_t(s,b) = b$. Then $\int_b u_t^b(b; A, \widehat{p}) dB_t(b) = 0$, so the above expression reduces to

$$W(A_t, \ldots, A_T) =$$
$$u_t^s(0, A_t, \ldots, A_T; \widehat{p}_t, \ldots, \widehat{p}_T) S_t(0) + u_t^s(1, A_t, \ldots, A_T; \widehat{p}_t, \ldots, \widehat{p}_T) S_t(1)$$

Now given the induction, we can write

$$W(A_t, \ldots, A_T) =$$
$$S_t(0) \int_{b:(0,b)\in A_t} b \tfrac{dB_t(b)}{B_t(1)} + S_t(1) \int_{b:(1,b)\in A_t} (b-1) \tfrac{dB_t(b)}{B_t(1)} + S_{t+1}(0) \int_0^1 \max\{b, \beta_{t+1}\} \tfrac{dB_{t+1}(b)}{B_{t+1}(1)}$$

Let us rewrite this as

$$W(\pi_t, \ldots, \pi_T) =$$
$$S_t(0) \int_0^1 \pi_t(0,s) b \tfrac{dB_t(b)}{B_t(1)} + S_t(1) \int_0^1 \pi_t(1,s)(b-1) \tfrac{dB_t(b)}{B_t(1)} + \tfrac{S_{t+1}(0)}{B_{t+1}(1)} \int_0^1 \max\{b, \beta_{t+1}\} dB_{t+1}(b)$$

where $(\pi_1, \ldots, \pi_T)$ is a collection of functions $\pi_t(s,b) = 1$ if $(s,b) \in A_t$ and $\pi_t(s,b) = 0$ if $(s,b) \notin A_t$.

We can maximize $W(\pi)$ with respect to all measurable π's and show that the unique solution is a bang-bang function and that it corresponds to the claimed cutoff function in Theorem 1. We employ the Maximum Principle.

First let us consider $\pi(0,b)$. Differentiating $W(\pi)$ with respect to $\pi(0,b)$ provides

$$\frac{d[W(\pi)]}{d[\pi_t(0,b)]} = b dB_t(b) \frac{S_t(0)}{B_t(1)} + \frac{d(S_{t+1}(0)/B_{t+1}(1))}{d\pi_t(0,b)} \int_0^1 \max[v, \beta_{t+1}] dB_{t+1}(v) + \frac{S_{t+1}(0)}{B_{t+1}(1)} \frac{dG}{d\pi_t(0,b)},$$

where

$$G = \int_0^1 \max[v, \beta_{t+1}] dB_{t+1}(v)$$

Notice that (1) and (2) are equivalent to:

$$S_{t+1}(v) = S_t(v) - \int_{s \le v} \left(\int_0^1 \pi_t(s, b) \frac{dB_t(b)}{B_t(1)} \right) dS_t(s)$$

$$B_{t+1}(v) = B_t(v) - \int_{b \le v} \left(\int_0^1 \pi_t(s, b) \frac{dS_t(s)}{S_t(1)} \right) dB_t(b).$$

Therefore

$$dB_{t+1}(v) = \left(1 - \pi_t(0, v) \frac{dS_t(0)}{S_t(1)} \right) dB_t(v).$$

so

$$\frac{dS_{t+1}(0)}{d\pi_t(0, b)} = -dB_t(b) \frac{S_t(0)}{B_t(1)},$$

$$\frac{dB_{t+1}(v)}{d\pi_t(0, b)} = -dB_t(b) \frac{S_t(0)}{B_t(1)},$$

for any $v \ge b$ and 0 otherwise. And

$$\frac{d(dB_{t+1}(v))}{d\pi_t(0, b)} = -dB_t(b) \frac{S_t(0)}{B_t(1)},$$

if $v = b$ and 0 otherwise.

Thus,

$$\frac{d(S_{t+1}(0)/B_{t+1}(1))}{d\pi_t(0, b)} = -dB_t(b) \frac{S_t(0)}{B_t(1)} \frac{(B_{t+1}(1) - S_{t+1}(0))}{(B_{t+1}(1))^2}$$

and

$$\frac{dG}{d\pi(0, b)} = -\max[b, \beta_{t+1}] dB_t(b) \frac{S_t(0)}{B_t(1)},$$

Substituting these above expressions back into $\frac{d[W(\pi)]}{d[\pi_t(0,b)]}$ gives

$$\frac{d[W(\pi)]}{d[\pi_t(0, b)]} =$$

$$dB_t(b) \frac{S_t(0)}{B_t(1)} \left(b - \frac{(B_{t+1}(1) - S_{t+1}(0))}{B_{t+1}(1)} \int_0^1 \max[v, \beta_{t+1}] \frac{dB_{t+1}(v)}{B_{t+1}(1)} - \max[b, \beta_{t+1}] \frac{S_{t+1}(0)}{B_{t+1}(1)} \right)$$

In order for this to equal 0, it must be that (3) holds. (Notice that it is impossible to set the expression equal to 0 with $b < \beta_{t+1}$, so $\max[b, \beta_{t+1}] = b$.) Furthermore, as b increases, this expression gets larger, which implies that the solution is a cutoff rule and there is a unique b for which the right hand side equals 0 (except for the set of measure 0 b's for which $dB_t(b) = 0$ which are irrelevant in the expected utility). Performing the same exercise with respect to $s = 1$, provides $\frac{d[W(\pi)]}{d[\pi_t(1,b)]} < 0$ for all $b < 1$ which implies that the optimal cutoff for $s = 1$ equals 1. ▮

Proof of Corollary 1:

(i) The proof is inductive. From above, $\beta_T = 0$ and $\beta_{T-1} = \delta \int_0^1 \max\{b, \beta_T\} \frac{dB_t(b)}{B_t(1)} = \delta \int_0^1 b \frac{dB_t(b)}{B_t(1)}$. By construction and by the assumptions that $\delta > 0$ and $dB_1(b) > 0 \; \forall b \in [0,1]$, $\delta \int_0^1 b \frac{dB_t(b)}{B_t(1)} > 0$, so $\beta_{T-1} > 0 = \beta_T$. Hence the statement is true for $T-1$. The induction hypothesis is that $\beta_s > \beta_{s+1}$ for all $s = t, t+1, \ldots, T$. Then for :

$$\beta_t = \delta \int_0^1 \max\{b, \beta_{t+1}\} \frac{dB_{t+1}(b)}{B_{t+1}(1)} < \delta \int_0^1 \max\{b, \beta_t\} \frac{dB_{t+1}(b)}{B_{t+1}(1)}$$

since $\beta_t > \beta_{t+1} \geq 0$. Since B_{t+1} is obtained from B_t by removing mass from the distribution of buyers with valuations above β_t, it follows that $\frac{dB_{t+1}(b)}{B_{t+1}(1)} \leq \frac{dB_t(b)}{B_t(1)}$ for all $b \geq \beta_t$, so by first order stochastic dominance

$$\int_0^1 \max\{b, \beta_t\} \frac{dB_{t+1}(b)}{B_{t+1}(1)} \leq \int_0^1 \max\{b, \beta_t\} \frac{dB_t(b)}{B_t(1)} = \beta_{t-1}.$$

Thus, $\beta_{t-1} > \beta_t$.

(ii) Notice that from the characterization in Theorem 1 it follows that

$$u_1^s(0) = \int_0^1 \max\{b, \beta_1\} dB_1(b).$$

Suppose that to the contrary of (ii), $\beta_1(0) \geq p^c$ where $p^c = B_1^{-1}(1 - S_1(0))$ is the static (centralized) competitive price. Then

$$u_1^s(0) \geq \int_0^1 \max\{b, p^c\} dB_1(b).$$

But

$$\int_0^1 \max\{b, p^c\} dB_1(b) = \int_{p^c}^1 b dB_1(b) \; + \; p^c B_1(p^c).$$

Thus

$$u_1^s(0) > \int_{p^c}^1 b dB_1(b).$$

Notice that this second expression is the total welfare that one could get from a problem where one was not constrained by matching. That is, the total welfare from the static (centralized) problem. This is a contradiction since the total welfare (the value of the objective function) in the constrained problem cannot exceed the total welfare in the unconstrained problem. ▮

Proof of Theorem 2: First, it is clear that an efficient trading rule should have cutoff rules for second period trades with $\beta_2(s) = s$, so $A_2 = \{(s, b) : b > s\}$.

Assuming the form for β_2, and using $\widehat{p}_t(s,b) = b$ (as in the proof of Theorem 1), we can write

$$W(A_1, A_2) =$$
$$\int_0^1 \left(\int_{b\in A_1(s)}(b-s)dB_1(b)\right) dS_1(s) + \delta \int_0^1 \int_s^1 (b-s)\frac{dB_2(b)}{B_2(1)} dS_2(s)$$

where $A_1(s) = \{b : (s,b) \in A_1\}$, and B_2 and S_2 are determined by (1) and (2).

Let us rewrite this as

$$W(\pi, A_2) =$$
$$\int_0^1 \left(\int_0^1 \pi(b,s)(b-s)dB_1(b)\right) dS_1(s) + \delta \int_0^1 \int_s^1 (b-s)\frac{dB_2(b)}{B_2(1)} dS_2(s),$$

where $\pi(s,b) = 1$ if $(s,b) \in A_1$ and $\pi(s,b) = 0$ if $(s,b) \notin A_1$.

As in the proof of Theorem 1, we can maximize $W(\pi)$ with respect to all measurable π's and show that the unique solution corresponds to the cutoff function given by (4).

Recall that

$$S_2(v) = S_1(v) - \int_{s\leq v} \left(\int_0^1 \pi(s,b)dB_1(b)\right) dS_1(s)$$

and

$$B_2(v) = B_1(v) - \int_{b\leq v} \left(\int_0^1 \pi(s,b)dS_1(s)\right) dB_1(b).$$

Differentiate $W(\pi)$ with respect to $\pi(s,b)$ for any (s,b), which leads to:

$$\frac{d[W(\pi)]}{d[\pi(s,b)]} = (b-s)dB_1(b)dS_1(s) + \frac{-\delta}{B_2(1)}\frac{d[B_2(1)]}{d[\pi(s,b)]}\int_0^1 \left(\int_{s'}^1 (b'-s')\frac{dB_2(b')}{B_2(1)}\right) dS_2(s') +$$
$$\delta\frac{d[dS_2(s)]}{d[\pi(s,b)]}\int_s^1 (b'-s)\frac{dB_2(b')}{B_2(1)} + \delta\frac{d[dB_2(b))]}{d[\pi(s,b)]}\int_0^b (b-s')\frac{dS_2(s')}{B_2(1)} \qquad (A1)$$

Next, observe that

$$\frac{d[dS_2(s)]}{d[\pi(s,b)]} = -dB_1(b)dS_1(s),$$
$$\frac{d[dB_2(b)]}{d[\pi(s,b)]} = -dB_1(b)dS_1(s),$$

and

$$\frac{d[B_2(1)]}{d[\pi(s,b)]} = -dB_1(b)dS_1(s).$$

Substituting these expressions into (A1) provides

$$\frac{d[W(\pi)]}{d[\pi(s,b)]} = dB_1(b)dS_1(s)$$

$$\left[(b-s) + \delta \int_0^1 \left(\int_{s'}^1 (b'-s)\frac{dB_2(b')}{B_2(1)}\right)\frac{dS_2(s')}{B_2(1)} - \delta \int_s^1 (b-s)\frac{dB_2(b')}{B_2(1)} - \delta \int_0^b (b-s')\frac{dS_2(s')}{B_2(1)}\right].$$

The solution should have $\pi = 1$ when $\frac{d[W(\pi)]}{d[\pi(s,b)]} > 0$ and $\pi = 0$ when $\frac{d[W(\pi)]}{d[\pi(s,b)]} < 0$. To see that the solution should be a cutoff rule, fix s and notice that the part inside the brackets on the right hand side of the expression for $\frac{d[W(\pi)]}{d[\pi(s,b)]}$ is increasing in b. Setting $\frac{d[W(\pi)]}{d[\pi(s,b)]} = 0$ implies (4). ∎

We next present the proofs of theorems 3–6. We do this in the order: Theorem 6, Theorem 5, Theorem 3, Theorem 4. This is different from the order in the body of the paper, but it is the natural order to present the proofs, since the more general results in Theorem 6 are used to prove Theorem 5, and ultimately Theorems 3 and 4. The claim in Section 6 and Proposition 1 are proved at the end.

Proof of Theorem 6:

We begin by demonstrating the necessity of the conditions. Suppose that (A, p) is implemented by $(\gamma_1, \ldots, \gamma_T)$. Let P_t be the set of prices that correspond to some terminal node of γ_t.

Lemma: *For any t, and for any (s, b) pair, there is a unique subgame perfect equilibrium outcome of $\Gamma(\gamma_t)$ satisfying (ii) in the definition of equilibrium, as a function of $(\overline{p}^s, \overline{p}^b)$. It is trade at some price if and only if $(\overline{p}^s, \overline{p}^b) \in \mathrm{IR}(P_t)$.*

By part (ii) of the definition of equilibrium, an agent's choice from a set of outcomes is uniquely determined. The subgame perfect equilibrium outcomes can be found by backward induction, which results in a unique outcome.

Only If: By the veto power that each agent has under the augmented mechanism, the unique equilibrium outcome must be no trade if $(\overline{p}^s, \overline{p}^b) \notin \mathrm{IR}(P_t)$.

If: We now show that if $(\overline{p}^s, \overline{p}^b) \in \mathrm{IR}(P_t)$, then the unique equilibrium outcome must be trade at some price. Suppose the contrary, so that for some $(\overline{p}^s, \overline{p}^b) \in \mathrm{IR}(P_t)$, the equilibrium outcome is no trade. Consider a pair of equilibrium strategies for $(\overline{p}^s, \overline{p}^b)$ when they are matched at time t and denote these σ. These lead to no trade at time t. Consider also some strategies which lead to the outcome of p at time t and denote these σ'. Alter σ at each node on the play path of σ' to match the action under σ' at that node, and leave the actions at other nodes under σ unchanged. Call this new strategy σ''. Since σ'' results in trade at p, it must not be an equilibrium for $\overline{p}^s$ and $\overline{p}^b$.[13] Find the last node along the play path of σ'' such that there is an improving deviation for the agent choosing at that node. Find a best response for that agent at that node.[14] The new play path must lead to trade at some price since it is improving for the agent and both agents weakly prefer p to no trade. The new strategy combination is now a Nash

[13] Notice that infinite stage mechanisms can be admitted and this proof still works, since σ' (and thus σ'') must result in trade after some finite number of stages.

[14] We know that there exists a best response at that node, since the other actions at that node yield the same outcomes that they would under σ, and there is a best response there under σ.

equilibrium in all subgames from this node on (and all subgames off the current play path). Iterate this logic up the nodes of the play path. This results in a subgame perfect equilibrium which has an outcome of trade at some price, which is a contradiction. ∎

With the lemma in hand, we can conclude the proof of necessity in the theorem.

Define $\widehat{p}_t$ to be the equilibrium price of $\Gamma(\gamma_t)$ as a function of $(\overline{p}^s, \overline{p}^b)$. By the lemma, this is a well defined function on the domain $\mathrm{IR}(P_t)$.

Let $(\widetilde{A}, \widetilde{p})$ denote an allocation rule corresponding to some equilibrium. By the definition of implementability, it is equivalent to (A, p). Define $\overline{p}_t$ relative the equilibrium strategies leading to $(\widetilde{A}, \widetilde{p})$.

We first verify (1). We show that condition $*$ holds relative to $(\widetilde{A}, \widetilde{p})$, for the $\widehat{p}_t$ and P_t defined above.

(*i) and (*iii) follow directly from the lemma.

(*ii) follows from the lemma and the fact that agents will never accept a price that is not individually rational in an equilibrium of the augmented mechanism.

(*iv) Given the uniqueness from the lemma above, this follows directly from the proof of Rubinstein and Wolinsky (1991) [Appendix II of the working paper version].

(*v) By (*iv) we know that $\overline{p}^{s\prime} < \overline{p}^s$ Suppose the contrary of (*v). Then for all $p \in P_t$, either $p \geq \overline{p}^s$ and $p \geq \overline{p}^{s\prime}$, or $p < \overline{p}^s$ and $p < \overline{p}^{s\prime}$. This implies that the set of equilibria is exactly the same for the two agents when either is matched with $\overline{p}^b$. This implies nonuniqueness of the equilibrium outcome, a contradiction.

Next, let us verify (2).

Consider an (A', p') which is not equivalent to (A, p). Consider the $\widehat{p}_t$ and P_t defined for each t as above. Notice that (*iv) and (*v) are satisfied, as they are independent of the allocation rule. We must show that one of (*i), (*ii), and (*iii) fail for (A', p') relative to the $\widehat{p}_t$ and P_t defined above.

By the lemma, for each t and (s, b) there is a unique subgame perfect equilibrium outcome of $\Gamma(\gamma_t)$ relative to the reservation values $(\overline{p}_t^s(s; A', p'), \overline{p}_t^t(b; A', p'))$. Select a subgame perfect equilibrium pair of strategies for each t and (s, b). By the implementation of (A, p), these strategies cannot result in (A', p'). Thus, there exists t and (s, b) such that either

Case 1: $(s, b) \notin A_t'$ and the outcome is trade at some price p, or

Case 2: $(s, b) \in A_t'$ and the outcome is trade at some price $p \neq p'(s, b)$, or

Case 3: $(s, b) \in A'_t$ and the outcome is no trade.

In case 1, it follows from lemma that $(\overline{p}^s_t(s; A', p'), \overline{p}^b_t(b; A', p')) \in \mathrm{IR}(P_t)$, which means that (*iii) fails.

In case 2, it follows from the definition of $\widehat{p}_t$ that $p'_t(s, b) \neq \widehat{p}_t(\overline{p}^s_t(s; A', p'), \overline{p}^b_t(b; A', p'))$, which means that (*i) fails.

In case 3, it follows from lemma that $(\overline{p}^s_t(s; A', p'), \overline{p}^b_t(b; A', p')) \notin \mathrm{IR}(P_t)$, which means that (*ii) fails.

We now prove sufficiency. Assume that (1) and (2) hold. Consider the implementing mechanism, $(\gamma_1, \ldots, \gamma_T)$, described above. The remainder of the proof consists of verifying three claims.

Claim 1: Consider t and a subgame perfect equilibrium of the augmented version of the mechanism described above under part (ii) of the definition of equilibrium, when reservation values are $(\overline{p}^s, \overline{p}^b)$. The outcome is unique, and:

(a) if $(\overline{p}^s, \overline{p}^b) \in \mathrm{IR}(P_t)$ then the outcome is trade at $\widehat{p}_t(\overline{p}^s, \overline{p}^b)$.

(b) if $(\overline{p}^s, \overline{p}^b) \notin \mathrm{IR}(P_t)$ then the outcome is no trade.

Proof of Claim 1: The set of possible outcomes from the above mechanism is P_t. Thus (b) follows by the same logic as lemma, noting that in this case a subgame perfect equilibrium exists because no price is ever approved by both agents. Similarly, if $(\overline{p}^s, \overline{p}^b) \in \mathrm{IR}(P_t)$ then the unique subgame perfect equilibrium outcome is trade at some price, provided an equilibrium exists. We need to show that a subgame perfect equilibrium exists and it is trade at $\widehat{p}_t(\overline{p}^b, \overline{p}^s)$.

Consider the following strategies which result in $\widehat{p}_t(\overline{p}^b, \overline{p}^s)$. It is easily checked that given these expectations, these form a subgame perfect equilibrium.

On the equilibrium path behavior: The seller announces $p^s = \overline{p}^s$. The buyer announces $(p^{s\prime}, p^b) = (\overline{p}^s, \overline{p}^b)$. The seller announces $p^{b\prime} = \overline{p}^b$. Both approve this.

Off the equilibrium path behavior:

Each player approves any price that is indivdually rational for them, and vetos others.

If the seller announces $p^s < \overline{p}^s$, then the buyer announces $(p^{s\prime}, p^b) = (p^s, \overline{p}^b)$.

If the seller announces $p^s > \overline{p}^s$, then the buyer announces $(p^{s\prime}, p^b) = (\overline{p}^s, \overline{p}^b)$.

If the buyer announces $p^b < \overline{p}^b$, then the seller announces $p^{b\prime} = \overline{p}^b$.

To see that this is an equilibrium, notice first that if the seller announces a price in excess of his reservation price, the buyer can correct the announcement and win all the surplus, and (by (*5)) end up paying a lower price. If the seller announces a price below his reservation price, he is directly conceding some surplus to the buyer. If the seller tells the truth, then the buyer cannot claim the seller has a lower reservation price, or this will lead to no trade. Similarly, the buyer cannot gain from understating his reservation price, since the seller could then correct this announcement to the true buyer reservation value, and win all the surplus.

Claim 2: There exists an equilibrium which results in $\widetilde{A}_t, \widetilde{p}_t$.

Proof of Claim 2: If we fix the reservation prices of the buyers and sellers, then there is a unique subgame perfect equilibrium outcome for any matched pair for a specific stage. So fix the reservation prices at those generated by the allocation rule, $\widetilde{A}_t, \widetilde{p}_t$. We will verify that the subgame perfect equilibrium outcome in this case results in $\widetilde{A}_t, \widetilde{p}_t$.

If $(s,b) \notin \widetilde{A}_t$, then no trade is the only subgame perfect outcome of the augmented mechanism. This follows from (*iii) and Claim 1.

If $(s,b) \in \widetilde{A}_t$, then from Claim 1 and (*i) and (*ii) it follows that the outcome is trade at $\widehat{p}_t(\overline{p}^s_t(s;A,p), \overline{p}^b_t(b;A,p))$.

Claim 3: If (A',p') not equivalent to (A,p), then (A',p') is not the result of any equilibrium of the mechanism.

Proof of Claim 3: Suppose to the contrary that there is an equilibrium that results in (A',p'). . .

Consider $(s,b) \notin A'_t$. It must be that the outcome of $\Gamma(\gamma_t)$ is no trade. ¿From Claim 1, it then follows that (*iii) of condition $*$ holds relative to $\widehat{p}_t$ and P_t.

Consider $(s,b) \in A'_t$. It must be that the outcome of $\Gamma(\gamma_t)$ is trade at $p'_t(s,b)$. It then follows from Claim 1 that (*i) and (*ii) of condition $*$ hold relative to $\widehat{p}_t$ and P_t.

This contradicts (2), which implies that (A',p') fails to satisfy (*i), (*ii), or (*iii) relative to $\widehat{p}_t$ and P_t. ▮

Proof of Theorem 5: This is the same as the above proof of the necessity of (1), except that lemma is only stated for $(\overline{p}^s, \overline{p}^b)$ relative to which equilibrium exists. Then one needs to extend $\widehat{p}_t$ to satisfy (*ii), (*iv), and (*v), for $(\overline{p}^s, \overline{p}^b) \in \mathrm{IR}(P_t)$ relative to which there does not exist an equilibrium. For such a $(\overline{p}^s, \overline{p}^b)$, define $\widehat{p}_t(\overline{p}^s, \overline{p}^b)$ by setting it equal to the max of $\overline{p}^s$ and the sup of $\widehat{p}_t$ over $(\overline{p}^{s\prime}, \overline{p}^{b\prime}) \in \mathrm{IR}(P_t)$ such that $\overline{p}^{s\prime} \leq \overline{p}^s$, $\overline{p}^{b\prime} \leq \overline{p}^b$, and for which there exists an equilibrium. This construction clearly satisfies (*ii), (*iv), and (*v). ▮

Proof of Theorem 3: This follows directly from Theorem 5. ∎

Proof of Theorem 4: It follows from Theorem 6 that the conditions are necessary. To see that they are sufficient we show that condition $(*)$ of Theorem 6 is satisfied relative to (A, p). Then the result follows from Claims 1 and 2 in the proof of Theorem 6.

Under the assumptions of Theorem 4, reservation prices are continuous and non-decreasing functions of s and b. Given the continuity of p and A it follows that individual rationality must hold with exact equality for cutoff pairs.[15] Thus, if $b = \beta_t(s)$ (where β_t is the cutoff defined by the cutoff rule A_t), then $\overline{p}_t(b) = \overline{p}_t(s) = p_t(s, b)$. To see this, consider a cutoff pair s, b. By individual rationality $\overline{p}_t(b) \geq p_t(s, b) \geq \overline{p}_t(b)$. For any $b' < b$ we know that either $\overline{p}_t(b) < p_t(s, b)$ or $\overline{p}_t(s) > p_t(s, b)$. We know that the second cannot hold, so it must be that $\overline{p}_t(b) < p_t(s, b)$. Then by continuity, $\overline{p}_t(b) = p_t(s, b)$. Similar reasoning establishes $\overline{p}_t(s) = p_t(s, b)$.

Let $\underline{s}^t$ be the $\min\{s|(s, b) \in A_t$ for some $b\}$, and $\underline{s}^t$ be the $\max\{s|(s, b) \in A_t$ for some $b\}$. Similarly define $\underline{b}^t$ and $\overline{b}^t$. Next, notice that $\overline{p}_t(\underline{s}^t) = \overline{p}_t(\underline{b}^t) = p_t(\underline{s}^t, \underline{b}^t)$ and similarly, $\overline{p}_t(\overline{s}^t) = \overline{p}_t(\overline{b}^t) = p_t(\overline{s}^t, \overline{b}^t)$. Given the assumptions on A_t and p_t, the range of p_t over pairs in A_t is $[p_t(\underline{s}^t, \underline{b}^t), p_t(\overline{s}^t, \overline{b}^t)]$, since $(\underline{s}^t, \underline{b}^t) \in A_t$ and $(\overline{s}^t, \overline{b}^t) \in A_t$ given that A_t is a cutoff rule and cutoffs are non-decreasing in value.

So, let $P_t = [p_t(\underline{s}^t, \underline{b}^t), p_t(\overline{s}^t, \overline{b}^t)]$. For $\overline{p}^s, \overline{p}^b \in IR(P_t)$ define $\widehat{p}_t(\overline{p}^s, \overline{p}^b)$ through $p_t(s, b)$ by setting

$$\widehat{p}_t(\overline{p}^s, \overline{p}^b) = p_t(s', b')$$

where $s' = \min\ \{s|\overline{p}_t^s(s) \geq max\{\overline{p}^s, \overline{p}_t^s(\underline{s}^t)\}\}$ and $b' = \max\ \{b|\overline{p}_t^s(b) \geq min\{\overline{p}^b, \overline{p}_t^b(\overline{b}^t)\}\}$. Using this, we verify that condition $(*)$ is satisfied relative to (A, p). $(*i)$ holds since if $s, b \in A_t$, then $s' = s$ and $b' = b$ in our definition above. $(*ii)$ holds by the construction of $\widehat{p}_t$ and the individual rationality assumed in Theorem 4. $(*iii)$ holds by the individual irrationality assumed in Theorem 4. $(*iv)$ holds by the construction of $\widehat{p}_t$. To see $(*v)$, notice that for $\widehat{p}_t(\overline{p}^{s\prime}, \overline{p}^b) < \widehat{p}_t(\overline{p}^s, \overline{p}^b)$, it must be that $\min\ \{s|\overline{p}_t^s(s) \geq max\{\overline{p}^{s\prime}, \overline{p}_t^s(\underline{s}^t)\}\}$ $< \min\ \{s|\overline{p}_t^s(s) \geq max\{\overline{p}^s, \overline{p}_t^s(\underline{s}^t)\}\}$. Thus, from the definition of $\widehat{p}$, there exists $s' < s$ and b with $\widehat{p}_t(\overline{p}^{s\prime}, \overline{p}^b) = p_t(s', b)$ and $\widehat{p}_t(\overline{p}^s, \overline{p}^b) = p_t(s, b)$. Given the individual rationality in Theorem 4, we know that $(s'b) \in A_t$ and $(sb) \in A_t$. By continuity of p, we can find $s'', b \in A_t$ with $p_t(s', b) < p_t(s'', b) < p_t(s, b)$. Then $(*v)$ is satisfied with $p = p_t(s'', b)$. The same can be done for the other part of the condition concerning buyer values. ∎

At the beginning of Section 6 we made the following claim which we now prove.

Claim: *Given homogenous sellers and no discounting, the dynamically efficient trading rule is voluntarily implemented by the mechanism which has sellers making take it or leave it offers in each period.*

[15] Given the continuity, all claims that were "almost every," no longer have that qualifier.

Proof: First, we show that the dynamically efficient solution is the outcome of an equilibrium of the augmented mechanism. Second, we show that the any equilibrium of the augmented mechanism results in the dynamically efficient solution.

The following strategies form an equilibrium resulting in the efficient solution: Buyers accept any offer at any time that is no more than their valuation. At any time t with some remaining buyer distribution B_t, a seller with $s = 0$ matched with a buyer who has a reservation value b at time t, makes the offer b if $b \geq \beta_t$, and 1 otherwise, where β_t is the dynamically efficient solution. Sellers with $s = 1$ always offer 1. Both approve any trade at a price at least as high as their reservation prices under the continuation. To see that this is a subgame perfect equilibrium of $\Gamma(\gamma_t)$ (given the expectations of the continuation) notice that buyers' choices are best responses, since they will never see an offer lower than their b. Now let us consider sellers' choices. The choices of a seller with $s = 1$ are obvious, so consider a seller with $s = 0$. Given that they make an offer that they want accepted, it is a best response to set the offer at b. Thus we need to verify that their best response is to trade with b (at a price of b) if only if $b \geq \beta_t$. Since $\beta_T = 0$ in the last period, this is clear. Given the analysis of the last period, we know that in the second to last period, sellers either trade with the buyer to whom they are matched, or wait and see an expected value of $\int_0^1 v \, dB_T/B_T(1)$. Thus, a seller in the second to last period is better off trading today if and only if $b \geq \int_0^1 v \, dB_T(v)/B_T(1) = \beta_{T-1}$. By induction, a seller is better off trading today if and only if $b \geq \beta_t = \int_{\beta_{t+1}}^1 v \, dB_{t+1}(v)/B_{t+1}(1) + \beta_{t+1}B_{t+1}(\beta_{t+1})/B_{t+1}(1)$.

Now let us verify that any equilibrium must result in the dynamically efficient solution.

We proceed by analyzing the last period first. Any subgame perfect equilibrium in the last period must have sellers with $s = 0$ making offers equal to the buyers' values and then buyers accepting in the last period. This follows from a standard argument: In a subgame perfect equilibrium, buyers must accept any offer which is less than their value and reject any offer higher than their value in the last period. Thus, it cannot be an equilibrium for the seller to offer the buyer less than their value, since the seller could do better by raising the offer slightly. Similarly, it cannot be an equilibrium for the seller to offer the buyer at a price higher than the buyer's value, since she could gain by making an offer below the buyer's value. Thus, the only possible equilibrium offer must be at the buyer's value. For this to be an equilibrium the buyer must accept.

Given the analysis of the last period, buyers must expect no surplus in the last period, and so any trade that occurs in equilibrium in the second to last period must occur at the buyer's value b. By induction, the price on any trade in any period must occur at b. Now let us examine seller's choices. By the same argument as previously given on sellers' incentives, sellers with $s = 0$ should trade with a buyer of value b in the second to last period if and only if $b \geq \int_0^1 v \, dB_T(v)/B_T(1)$, where B_T is the distribution induced under the equilibrium. This follows the unique efficient solution starting at time $T - 1$ with B_{T-1}. By induction, sellers with $s = 0$ should trade with a buyer of value b in period t if and only if $b \geq \beta_t = \int_{\beta_{t+1}}^1 v \, dB_{t+1}(v)/B_{t+1}(1) + \beta_{t+1}B_{t+1}(\beta_{t+1})/B_{t+1}(1)$, where B_{t+1} (and B_t) are the distributions induced under the equilibrium. Given the uniqueness of the efficient solution at any time, it follows that the equilibrium must result in the dynamically efficient solution from time 1. ∎

Proof of Proposition 1: Consider $0 < \epsilon < 1/2$ and initial distributions of buyers and sellers represented by

$B_1(0) = \frac{1}{2}(1-\epsilon)$,

$B_1(b) = \frac{1}{2}(1-\epsilon) + b\frac{\epsilon}{1-\epsilon}$ for $0 < b < 1-\epsilon$, and

$B_1(b) = \frac{1}{2}(1+\epsilon) + \frac{b-(1-\epsilon)}{\epsilon}\frac{1}{2}(1-\epsilon)$, for $1-\epsilon \leq b$, and

$S_1(s) = \frac{s}{2}(\frac{1}{\epsilon} - 1)$, for $s \leq \epsilon$

$S_1(s) = .5(1-\epsilon) + \frac{s-\epsilon}{1-\epsilon}\epsilon$ for $\epsilon < s < 1$, and

$S_1(1) = 1$.

These distributions have mass $.5(1-\epsilon)$ of sellers who have values uniformly distributed over $(0,\epsilon)$, then mass ϵ with values uniformly distributed over $(\epsilon, 1)$, and the remaining mass of $.5(1-\epsilon)$ with value 1. Buyers are symmetric to this so that there is a mass $.5(1-\epsilon)$ of buyers who have values uniformly distributed over $(1-\epsilon, 1)$, then mass ϵ with values uniformly distributed over $(0, 1-\epsilon)$, and the remaining mass of $.5(1-\epsilon)$ with values $= 0$.

From Theorem 2, the cutoff rule for the efficient allocation satisfies

$$\beta_1(0) = E_2(b) + E_2(\max[\beta_1(0) - s, 0]) - E_2(\max[b - s, 0])$$

As ϵ becomes small, the period 2 distribution of buyers and sellers is almost all on values near 0 and 1. This means that $E_2(b)$ converges to $(1-B_2(.5))/B_2(1)$, $E_2(\max[\beta_1(0)-s,0])$ converges to $\beta_1(0)S_2(.5)/S_2(1)$ and $E_2(\max[b-s,0])$ converges to $(1-B_2(.5))S_2(.5)/(S_2(1)$ $B_2(1))$. So as ϵ becomes small,

$$\beta_1(0) = (1 - B_2(.5))/B_2(1) + \beta_1(0)S_2(.5)/S_2(1) - (1 - B_2(.5))S_2(.5)/(S_2(1)B_2(1))$$

Solving for $\beta_1(0)$

$$\beta_1(0) = (1 - B_2(.5))/B_2(1).$$

This means that $\beta_1(0)$ is at most 1/2, since none of the $b = 1$'s are cleared in the first period. So $B_2(.5) = 1/2$ and $B_2(1) = 3/4$. Thus, $\beta_1(0)$ converges to 1/3 as ϵ goes to zero. Similarly, $\sigma_1(1)$ converges to 2/3 as ϵ goes to zero. Also note that β_1 is continuous as b increases above 0, and similarly for σ_1.

We now show that the necessary conditions for voluntary attainability cannot be satsified. Suppose to the contrary that they are. Pick some small $\gamma > 0$, and apply individual irrationality to $b' = \beta_1(0) - \gamma$ and $s' = 0$. Since we know that individual rationality is satisfied for s close to 0 and some b close to $\beta_1(0)$, it follows that that for almost every γ [16]

$$\beta_1(0) - \gamma - p_1(0, \beta_1(0)) < u_2^b(\beta_1(0) - \gamma, A_2, p_2)$$

We also know that

$$u_2^b(\beta_1(0) - \gamma, A_2, p_2) < \frac{S_2(.5)}{S_2(1)}(\beta_1(0) - \gamma - p_2(\gamma, \beta_1(0) - \gamma)) + 2\epsilon$$

To see this notice that there is a probability of at most $\frac{S_2(.5)}{S_2(1)}$ that the buyer is matched with someone with a value between γ and .5, and the best price they can get is then $p_2(\gamma, \beta_1(0) - \gamma))$. For γ small enough, there is at most 2ϵ chance that they are matched with a seller with value smaller than γ.

Thus, from the two inequalities above,

$$\beta_1(0) - \gamma - p_1(0, \beta_1(0)) < \frac{S_2(.5)}{S_2(1)}(\beta_1(0) - \gamma - p_2(\gamma, \beta_1(0) - \gamma)) + 2\epsilon.$$

A similar argument for $s' = \gamma$ and $b' = \beta_1(0)$ leads to

$$p_1(0, \beta_1(0)) - \gamma < \frac{1 - B_2(.5)}{B_2(1)}(p_2(\gamma, 1 - \gamma) - \gamma) + 2\epsilon.$$

Given the symmetry of the distributions and thus the efficient solution, $\frac{S_2(.5)}{S_2(1)} = \frac{1-B_2(.5)}{B_2(1)}$. So summing the two previous inequalities we find that

$$\beta_1(0) - 2\gamma < \frac{S_2(.5)}{S_2(1)}(\beta_1(0) - \gamma - p_2(\gamma, \beta_1(0) - \gamma) + p_2(\gamma, 1 - \gamma) - \gamma) + 4\epsilon$$

Simplifying

$$(\beta_1(0) - 2\gamma)(1 - \frac{S_2(.5)}{S_2(1)}) < \frac{S_2(.5)}{S_2(1)}(p_2(\gamma, 1 - \gamma) - p_2(\gamma, \beta_1(0) - \gamma)) + 4\epsilon \qquad (l)$$

We can follow the same arguments for around the buyer with value 1 and the cutoff seller $\sigma_1(1)$ to find that

$$(1 - \sigma_1(1) - 2\gamma)(1 - \frac{S_2(.5)}{S_2(1)}) < \frac{S_2(.5)}{S_2(1)}(p_2(\sigma_1(1) + \gamma, 1 - \gamma) - p_2(\gamma, 1 - \gamma))4\epsilon \qquad (k)$$

[16] We will proceed as if this satisfied for $s = 0$ and $b = \beta_1(0)$, while this can be redefined to be some agents arbitrarily close to these.

Summing (l) and (k) we find

$$(\beta_1(0)+1-\sigma_1(1)-4\gamma)(1-\frac{S_2(.5)}{S_2(1)}) < \frac{S_2(.5)}{S_2(1)}(p_2(\gamma,1-\gamma)-p_2(\gamma,\beta_1(0)-\gamma)$$

$$+p_2(\sigma_1(1)+\gamma,1-\gamma)-p_2(\gamma,1-\gamma))+8\epsilon \qquad (m)$$

(m) simplifies to

$$(\beta_1(0)+1-\sigma_1(1)-4\gamma)(1-\frac{S_2(.5)}{S_2(1)}) < \frac{S_2(.5)}{S_2(1)}(p_2(\sigma_1(1)+\gamma,1-\gamma)-p_2(\gamma,\beta_1(0)-\gamma))+8\epsilon$$

As ϵ and γ go to zero, this approaches

$$(\beta_1(0)+1-\sigma_1(1))(1-\frac{S_2(.5)}{S_2(1)}) < \frac{S_2(.5)}{S_2(1)}(p_2(\sigma_1(1)+\gamma,1-\gamma)-p_2(\gamma,\beta_1(0)-\gamma))$$

or approximately

$$4/9 < 1/3(p_2(\sigma_1(1)+\gamma,1-\gamma)-p_2(\gamma,\beta_1(0)-\gamma)).$$

This is impossible to satisfy.

The example is robust in that the above argument will hold for all initial distributions that are close to the above ones since the resulting efficient allocation rules would be close by, as would the reservation prices. ▮

The intuition behind the proof is as follows: In the second period, there will be close to a population with proportion 1/3 high value buyers, 2/3 "dummy" buyers, and similarly 1/3 low value sellers, 2/3 "dummy" sellers. This means that any agent has a chance of only 1/3 of meeting a good match tomorrow. Somebody must get no more than 1/2 of the surplus from tomorrow. This means that either the low valued seller or the high valued buyer has an expected value of no more than 1/6 from trading tomorrow. Say it is the low valued seller. From Theorem 3 (from which the combination of individual rationality and individual irrationality almost imply that individual rationality is exactly binding for $\beta_1(0)$), this implies that the low valued seller can get no more than 1/6 from her cutoff trade today. Since the cutoff is approximately 1/3, the buyer with value 1/3 must get at least 1/6 from the trade today. However this buyer can expect at most 1/3(1/3) from waiting and is thus more than happy with the trade today. Then some other buyer with a lower value, but close to 1/3, who should not trade will also turn out to be happy to trade, contradicting individual irrationality.

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