PHYSICAL REVIEW D

PARTICLES AND FIELDS

THIRD SERIES, VOL. 6, NO. 8 15 October 1972

Scalar Debye Potentials for Electromagnetic Fields in Spherical Gravity and Spherical Media. I*†

Tse Chin Mo and C. H. Papas

Electrical Engineering Department, California Institute of Technology, Pasadena, California 91109

(Received 19 May 1972)

Modified scalar Debye potentials for electromagnetic (EM) waves in spherical gravity and spherical media are found. These potentials decompose the EM waves into two completely independent electric and magnetic radial modes and achieve scalarization and boundary fitting. Their equations, being different in a vacuum medium from $\Phi^{;\mu}{}_{;\mu}=0$ of a scalar field, are reduced to one-dimensional Helmholtz equations under a separability condition, and can have their gravity effect "nullified" by a particular medium. Also the reflection coefficient R_l for an l spherical wave satisfies a Ricatti equation, and the phase shifts δ_l and scattering cross sections are related to R_l. For an incident plane EM wave, the nonforward differential scattering cross section is expressed in terms of the R_l for the case where the medium and/or gravity tapers off slower than $(\text{radius})^{-1}$ and R_l, δ_l themselves diverge.

I. INTRODUCTION AND SUMMARY

To investigate and solve any electromagnetic (EM) field problem, one must decouple the field equations and the boundary conditions (BC) into independent scalar functions.[1] The possibility and the method of achieving this depend on the choice of an observational frame, the geometry of the BC's, and the properties of the gravity and media.[2] In a flat vacuum with spherical BC's, the well-known Debye potentials serve the purpose.[3] In this paper we investigate such a decomposition and the results it produces for a space-time with spherically inhomogeneous and time-changing gravity and medium.

Starting from the physical Maxwell's equations we derive in Sec. II the modified Debye potentials, their differential equations, and their BC's. Also we point out the advantages of these potentials over the 4-vector potential A^μ, and emphasize how they differ from the scalar wave Φ. In Sec. III A we separate variables under certain conditions and obtain one-dimensional Helmholtz equations for the l spherical partial waves of the Debye potentials. Then a particular medium that "nullifies" gravity is constructed, and other examples are given. In Sec. III B, nonseparable cases are discussed. Section IV presents the Ricatti equation for the reflection coefficients R_l of l waves, and the relations of the phase shifts δ_l as well as scattering cross sections σ_{sc} and σ_{tot} with R_l. Finally for an incident plane EM wave we find an expression of the differential scattering cross section $d\sigma_{sc}/d\Omega$ in the nonforward directions in terms of R_l and Q_l (for magnetic wave), for the case where the medium and/or gravity tapers off at radial infinity slower than $(\text{radius})^{-1}$ and δ_l, R_l themselves diverge. This expression for the EM wave is analogous to a well-known expression for the scattering of scalar waves.

II. EM FIELDS AND MODIFIED DEBYE POTENTIALS

A. Physical EM Fields

In a spherical and time-changing gravity and medium, naturally a "spherical" coordinate frame should be used both to fit BC's and to separate variables. Now, any spherical gravity can be described by a spherical coordinate system $\{t, b, \theta, \phi\}$ with[4]

$$ds^2 = g_{00}dt^2 + g_{11}db^2 - \lambda^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where g_{00}, $-g_{11}$, and λ^2 are positive functions of (t, b) and $-\infty < t < \infty$, $0 < b < \infty$.[5]

For observers $\{O\}$ fixed in this $\{t, b, \theta, \phi\}$ frame with their spatial positions $(b, \theta, \phi) \equiv$ fixed, the physical Maxwell equations are obtained by projecting the tensor equations $G^{\mu\nu}{}_{;\nu} = -J^\mu$ and $*F^{\mu\nu}{}_{;\nu} = 0$ on the co-moving tetrads carried by $\{O\}$.[6] A straightforward algebraic calculation gives[7]

$$\vec{\nabla}_g \times [(g_{00})^{1/2}\vec{H}] = (g_{00})^{1/2}\vec{J} + \frac{\partial \vec{D}}{\partial t} + \vec{D}\frac{\partial}{\partial t}\ln[\lambda(-g_{11})^{1/2}] + \vec{D}\cdot\vec{e}_{(b)}\vec{e}_{(b)}\frac{\partial}{\partial t}\ln\left(\frac{\lambda}{(-g_{11})^{1/2}}\right), \tag{2}$$

$$\vec{\nabla}_g \cdot \vec{D} = \rho,$$

$$\vec{\nabla}_g \times [(g_{00})^{1/2}\vec{E}] = -\frac{\partial \vec{B}}{\partial t} - \vec{B}\frac{\partial}{\partial t}\ln[(-g_{11})^{1/2}] + \vec{B}\cdot\vec{e}_{(b)}\vec{e}_{(b)}\frac{\partial}{\partial t}\ln\left(\frac{\lambda}{(-g_{11})^{1/2}}\right), \tag{3}$$

$$\vec{\nabla}_g \cdot \vec{B} = 0.$$

Here the $\vec{\nabla}_g$, $\times$, $\cdot$ operators are defined in accord with the usual 3-vector analysis for 3-space $\{b, \theta, \phi\}$ with metric $d\sigma^2 \equiv -g_{11}db^2 + \lambda^2 d\Omega^2$. Also the $(\vec{E}, \vec{B})$ and $(\vec{D}, \vec{H})$ are the usual macroscopic physical EM fields as observed by $\{O\}$. Similarly $(\rho, \vec{J})$ is the physical observable charge current to $\{O\}$.

Now for the spherical case of present interest, let the space be filled with an isotropic and angularly homogeneous simple medium. Relative to $\{O\}$ this medium is electromagnetically characterized by the two local constitutive parameters $\epsilon(t, b)$ and $\mu(t, b)$ such that $\vec{D} = \epsilon\vec{E}$, $\vec{H} = (1/\mu)\vec{B}$.[8] Notice that this spherical material medium's contribution to gravity through Einstein's equation is included in (1), but the space-time gravity effect of the EM field under investigation is neglected.

B. Radial EM Modes and Debye Potentials

Now consider an EM field in the sourceless region as described in Sec. II A. Since everything is spherically symmetric, we might expect to decompose the field into decoupled scalar representations just as the Debye potentials do in a flat vacuum.[9] By examining the detailed components structure of (2) and (3), and after very lengthy mathematical manipulations, we find that this is exactly the case, but with the modified decompositions

$$\vec{E} \equiv \vec{E}_{(\text{elec})} + \vec{E}_{(\text{mag})} = \frac{1}{\epsilon}\vec{\nabla}_g \times \vec{\nabla}_g \times (\vec{\lambda}V) - \frac{1}{\lambda(g_{00})^{1/2}}\frac{\partial}{\partial t}[\lambda\vec{\nabla}_g \times (\vec{\lambda}U)], \tag{4}$$

$$\vec{B} \equiv \vec{B}_{(\text{elec})} + \vec{B}_{(\text{mag})} = \frac{\mu}{\lambda(g_{00})^{1/2}}\frac{\partial}{\partial t}[\lambda\vec{\nabla}_g \times (\vec{\lambda}V)] + \vec{\nabla}_g \times \vec{\nabla}_g \times (\vec{\lambda}U). \tag{5}$$

Here $\vec{\lambda} \equiv \lambda\vec{e}_{(b)}$ is a radial "position vector" of $\{O\}$. Also the subscripts denote the radial electric mode for V wave and radial magnetic mode for U wave which has respectively vanishing radial magnetic and electric field components. Notice that, of course, the fields relative to other observers moving with respect to $\{O\}$ are obtained just by locally Lorentz transforming (4) and (5) which can be rewritten into tensor expressions by reversing the physical projecting process.[10]

These modified Debye waves, V and U, are scalar functions independent of each other. They obey the decoupled equations

$$\begin{pmatrix}\epsilon\\ \mu\end{pmatrix}\frac{\partial}{\partial b}\left[\left(\frac{g_{00}}{-g_{11}}\right)^{1/2}\begin{pmatrix}(1/\epsilon)(\partial/\partial b)\lambda V\\ (1/\mu)(\partial/\partial b)\lambda U\end{pmatrix}\right] + \frac{(-g_{00}g_{11})^{1/2}}{\lambda\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right)\begin{pmatrix}V\\ U\end{pmatrix} - \begin{pmatrix}\epsilon\\ \mu\end{pmatrix}\frac{\partial}{\partial t}\left[\left(\frac{-g_{11}}{g_{00}}\right)^{1/2}\begin{pmatrix}\mu(\partial/\partial t)\lambda V\\ \epsilon(\partial/\partial t)\lambda U\end{pmatrix}\right] = 0. \tag{6, 7}$$

In addition, the BC's for V and U at any radial discontinuity $b = b_d$ of the constitutive parameters μ and ϵ, as implied by (2) and (3), are decoupled:

$$\left[\frac{1}{\epsilon}\frac{\partial}{\partial b}\lambda V\right] = 0, \quad [V] = 0, \tag{8}$$

$$[U] = 0, \quad \left[\frac{1}{\mu}\frac{\partial}{\partial b}\lambda U\right] = 0, \tag{9}$$

where $[y] \equiv y(b_d^+) - y(b_d^-)$. Here the assumptions of finite and continuous gravity $g_{00}, g_{11}, \lambda^2$ at b_d have been used. If the medium at $b \leq b_d$ happens to be perfectly conducting, the BC's reduce simply to

$\epsilon^{-1}(\partial/\partial b)(\lambda V)=0$ and $U=0$ at $b=b_d^+$.

C. Remarks on Debye Potentials, 4-Vector Potential, and Scalar Wave

Why does one need any potential as an intermediate artifice while the only quantities of physical interest are the EM fields which can be dealt with directly and exclusively? The answer is just to simplify the EM vector problem so as to provide insight into its physics and to make the mathematics easier to handle.

Now in a spherical gravity and medium, the Debye potentials V and U clearly and uniquely provide such a simplification by decomposing EM fields into completely independent scalar modes. They have the following advantages over A^μ: (a) They consist of two unknown functions, but A^μ consists of four. (b) Their equations are decoupled. However, the equations for A^μ are mixed badly, which even in a vacuum with $\epsilon\equiv 1\equiv\mu$ and curvature tensor $R_{\mu\nu}\equiv 0$ are coupled with each other by the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ in the $A^{\mu;\lambda}{}_{;\lambda}=0$. Notice that if and only if the space-time is flat, there exists[11] a coordinate system with $\Gamma^\mu_{\alpha\beta}=0$ in which $A^{\mu;\lambda}{}_{;\lambda}=0$ is decoupled.[12] (c) Their BC's are decoupled, but the BC's for A^μ at radial boundaries are mixed and one component of A^μ can excite others.

We must also point out that even in a vacuum with its $\mu\equiv\epsilon\equiv 1$, except for the case $\lambda'(b)[g_{00}/(-g_{11})]^{1/2}=\text{const}$, (6) and (7) for the Debye potentials are different from the scalar wave equation $\Phi^{;\mu}{}_{;\mu}=0$[13] which for geometry (1) becomes

$$\frac{\partial}{\partial b}\left[\lambda^2\left(\frac{g_{00}}{-g_{11}}\right)^{1/2}\frac{\partial}{\partial b}\Phi\right]+\frac{(-g_{00}g_{11})^{1/2}}{\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right)\Phi-\frac{\partial}{\partial t}\left[\lambda^2\left(\frac{-g_{11}}{g_{00}}\right)^{1/2}\frac{\partial}{\partial t}\Phi\right]=0\,. \tag{10}$$

Thus results on propagation and scattering of the scalar field Φ cannot be used for EM waves.[14] In fact, in the spherical problem, the equations for $\vec{E}$, $\vec{B}$, A^μ, V, and U are generally all different. None of them is the same as (10). Only in a flat vacuum do they (of course for Cartesian components for 3-vectors) all reduce to $\nabla^2-\partial^2/\partial t^2=0$ and become indistinguishable.

III. EFFECTIVE INHOMOGENEITY AND SPECIAL CASES

Since the $\epsilon\rightleftarrows\mu$ substitution exchanges the equation for V and U, we discuss only V from here on. By separation of variables as usual, with $V\equiv\Psi_l(t,b)P_l^m(\cos\theta)e^{im\phi}$, where $m=-|l|$ to $|l|$ and integer $l\geqslant 1$,[15] (6) gives

$$\epsilon\frac{\partial}{\partial b}\left[\left(\frac{g_{00}}{-g_{11}}\right)^{1/2}\frac{1}{\epsilon}\frac{\partial}{\partial b}(\lambda\Psi_l)\right]-\epsilon\frac{\partial}{\partial t}\left[\left(\frac{-g_{11}}{g_{00}}\right)^{1/2}\mu\frac{\partial}{\partial t}(\lambda\Psi_l)\right]-\frac{(-g_{00}g_{11})^{1/2}}{\lambda^2}l(l+1)(\lambda\Psi_l)=0. \tag{11}$$

This is the basic equation for any EM wave propagating in a spherical gravity and spherical simple medium. Note that the slope $db/dt=\pm[g_{00}/(-g_{11})]^{1/2}/(\mu\epsilon)^{1/2}$ of the characteristics gives the velocity of radial propagation, slowed down to a non-null value as it should be by the medium's factor $(\mu\epsilon)^{-1/2}$. The sources are coupled into the fields by BC's as multipole coefficients. Also for $g_{00}=-g_{11}=1$ and $\lambda=b$, (4) to (7) and (11) reduce to those for angularly homogeneous simple media in flat space.

A. Radial Separable Inhomogeneity

Now if to $\{O\}$ the medium and certain ratios of the gravity are "static," i.e.,

$$\mu,\epsilon,\left(\frac{g_{00}}{-g_{11}}\right)^{1/2},\left(\frac{-g_{00}g_{11}}{\lambda^2}\right)^{1/2}\equiv\text{time-independent}, \tag{12}$$

then (11) can be separated into time harmonics. Thus by

$$V\equiv P_l^m(\cos\theta)e^{-im\phi}e^{-i\omega t}\left(\frac{h(b)}{\lambda(t,b)}\right),$$

the propagating of the Debye wave V reduces to the one-dimensional problem

$$\epsilon\frac{d}{db}\left[\frac{1}{\epsilon}\left(\frac{g_{00}}{-g_{11}}\right)^{1/2}\frac{d}{db}h\right]+\left[\omega^2\mu\epsilon\left(\frac{-g_{11}}{g_{00}}\right)^{1/2}-\frac{(-g_{00}g_{11})^{1/2}}{\lambda^2}l(l+1)\right]h=0 \tag{13}$$

for $0<b<\infty$. We now want to examine (13) by transforming it in two ways into a standard Helmholtz equation.

The first, transforming the dependent variable alone by $h(b) \equiv (\epsilon/\alpha)^{1/2} f(b)$, gives

$$f'' + k_l^{\,2}(b) f \equiv f'' + \left\{ \frac{1}{\alpha}\left(\frac{\omega^2 \mu\epsilon}{\alpha} - \frac{(-g_{00} g_{11})^{1/2}}{\lambda^2}\, l(l+1) \right) - \left(\frac{\epsilon}{\alpha}\right)^{1/2} \left[\left(\frac{\alpha}{\epsilon}\right)^{1/2} \right]'' \right\} f = 0 , \qquad (14)$$

where $\alpha \equiv [g_{00}/(-g_{11})]^{1/2}$. Equation (14) is the same as a radial Schrödinger equation with radial distance b, and with potential energy $v_s(b)$ and total energy E given by

$$\frac{2mE}{\hbar^2} \equiv \omega^2 , \qquad (15a)$$

$$\frac{2m v_s(b)}{\hbar^2} \equiv \left(\frac{\epsilon}{\alpha}\right)^{1/2} \left[\left(\frac{\alpha}{\epsilon}\right)^{1/2} \right]'' + \omega^2 \left(1 - \frac{\mu\epsilon}{\alpha^2} \right) - \frac{l(l+1)}{b^2}(1 + g_{11}) . \qquad (15b)$$

Notice that $v_s(b)$ contains a mixture of the medium's constitutive parameters, the gravity, the angular momentum, and the ω^2 that makes the propagation dispersive.

The second, changing the independent variable by $db^* \equiv \epsilon(-g_{11}/g_{00})^{1/2} db$ or $b^* \equiv \int \epsilon(-g_{11}/g_{00})^{1/2} db + \text{const}$,[16] gives

$$\frac{d^2}{db^{*2}} h + k_l^{*2}(b^*) h \equiv \frac{d^2}{db^{*2}} h + \left(\frac{\omega^2 \mu}{\epsilon} - \frac{g_{00}}{\lambda^2 \epsilon^2}\, l(l+1) \right) h = 0 . \qquad (16)$$

The range $-\infty < b^* < \infty$ in (16) may cover only part of the whole range $0 < b < \infty$, and thus may exclude and require separate treatment for the connective behavior of V at the limits of the range b^*. To investigate the whole range $0 < b < \infty$, (14) is preferable.

Before giving some examples, it is interesting and worthwhile to note that there is a medium which can "nullify" the gravity. From (13), if

$$\mu = \epsilon \equiv A \frac{(g_{00})^{1/2}}{\lambda} \exp\left(\int db\, \frac{(-g_{11})^{1/2}}{\lambda} \right), \qquad (17a)$$

$$r^* \equiv A \exp\left(\int db \frac{(-g_{11})^{1/2}}{\lambda} \right), \quad A \equiv \text{const} > 0 \qquad (17b)$$

then h satisfies

$$\frac{d^2}{dr^{*2}} h + \left(\omega^2 - \frac{l(l+1)}{r^{*2}} \right) h = 0 . \qquad (18)$$

Thus for the simple inhomogeneous medium (17a) surrounding $\{O\}$, the EM wave is formally reduced to the ordinary propagation in a flat vacuum with (r^*, θ, ϕ) as spherical coordinate by using (17b) and (18).

Now, we give some simple examples of (14).

(a) Schwarzschild point-mass geometry:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{1 - 2m/r} - r^2 d\Omega^2 .$$

For observer $\{O\}$ at fixed θ, ϕ and fixed r at $r > 2m$, fixed t at $r < 2m$,

$$k_l^{\,2}(r) = \left(\frac{\omega^2 \mu\epsilon}{(1 - 2m/r)^2} - \frac{l(l+1)}{r(r-2m)} + \frac{m(2r - 3m)}{r^2 (r - 2m)^2} \right) - \frac{1}{2}\left(\frac{3(\epsilon')^2}{2\epsilon^2} - \frac{\epsilon''}{\epsilon} - \frac{\epsilon' 2m}{\epsilon r (r - 2m)} \right) . \qquad (19a)$$

(b) Nordström point mass-charge geometry:

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 - \frac{dr^2}{1 - 2m/r + q^2/r^2} - r^2 d\Omega^2 .$$

For observer $\{O\}$ at fixed θ, ϕ and fixed r at $r^2 - 2mr + q^2 > 0$, fixed t at $r^2 - 2mr + q^2 < 0$, if such regions exist,

$$k_l^{\,2}(r) = \left(\frac{\omega^2 \mu\epsilon}{(1 - 2m/r + q^2/r^2)^2} - \frac{l(l+1)}{r^2 - 2mr + q^2} + \frac{2mr^3 - 3r^2(m^2 + q^2) + 6mq^2 r - 2q^4}{r^2 (r^2 - 2mr + q^2)^2} \right) - \frac{1}{2}\left(\frac{3\epsilon'^2}{2\epsilon^2} - \frac{\epsilon''}{\epsilon} - \frac{\epsilon'}{\epsilon}\, \frac{2(mr - q^2)}{r(r^2 - 2mr + q^2)} \right) . \qquad (19b)$$

(c) Conformally flat geometry with any $\alpha(t, r)$:

$$ds^2 = e^{\alpha(t,r)} (dt^2 - dr^2 - r^2 d\Omega^2).$$

For observer $\{O\}$ at fixed θ, ϕ, and r,

$$k_l^{\,2}(r) = \omega^2 \mu\epsilon - \frac{l(l+1)}{r^2} - \frac{1}{2}\left(\frac{3\epsilon'^2}{2\epsilon^2} - \frac{\epsilon''}{\epsilon} \right) . \qquad (19c)$$

B. Non-Time-Separable Media and Gravity

Time and radial variables cannot be separated if (12) does not hold. This may be caused by the gravity of a spherically moving medium, the time change of μ and ϵ, and the chosen motion of $\{O\}$.

For example, an expanding Friedmann universe filled with uniform "dust" has the conformal flat factor $e^{\alpha(t,r)} = (1 - A/\tau)^4$ and $\tau = (t^2 - r^2)^{1/2}$.[17] But if we include the dust's permittivity $\epsilon(t, r) = \epsilon_0(1 - A/\tau)^{-6}$ as corresponding to the expanding proper volume, (19c) cannot apply. As another example, for convenience of examining wave behaviors near $r = 2m$ in the gravity of a point mass m, we can use the radially accelerated $\{O\}$ in the Kruskal coordinate $\{v, u, \theta, \phi\}$.[18] Then for these $\{O\}$, (11) becomes

$$\frac{\partial^2}{\partial u^2}(r\Psi_l) - \left(\frac{32m^3}{r^3} e^{-r/2m} l(l+1) + \frac{\partial^2}{\partial v^2}\right)(r\Psi_l) = 0, \tag{20}$$

where $r(v, u)$ is defined by

$v^2 - u^2 = e^{r/2m}(1 - r/2m)$, and $r\Psi_l$ cannot be separated for v and u.

IV. REFLECTION COEFFICIENT AND RELATED FORMULAS

To examine the reflected or scattered field of an incident l wave propagating in the inhomogeneous $k_l^2(b)$ of (14), we can use the well-known Ricatti equation.[19] This enables us to avoid solving for the field itself and to obtain the reflection easily by numerical method. To do so for (14), we assume $k_l^2 \equiv K^2 + \xi_l(b)$ with constant $K^2 > 0$ and $\xi_l(b) \to 0$ as $b \to \infty$, and introduce a cutoff distance at b_c such that

$$|\xi_l(b)/K^2| \ll 1 \quad \text{for} \quad b \geq b_c\,. \tag{21}$$

Now consider a one-dimensional wave f in such a medium that $k^2 = K^2$ at $b > b_c$ and $k^2 \equiv k_l^2(b)$ at $b < b_c$. An incident wave e^{-iKb} produces a reflection $R_l e^{iKb}$ at $b > b_c$. Then with the aid of (14) the function defined by[20]

$$\mathfrak{R}_l(b) \equiv \frac{ik_l f + f'}{ik_l f - f'} \tag{22}$$

obeys the Ricatti equation

$$\mathfrak{R}_l' = \frac{-k_l'}{2k_l}(1 - \mathfrak{R}_l^2) - 2ik_l\mathfrak{R}_l\,, \quad 0 < b < b_c \tag{23}$$

and the connection condition at any k^2 discontinuity

$$\mathfrak{R}_l(+) = \frac{[k_l(+) - k_l(-)] + [k_l(+) + k_l(-)]\mathfrak{R}_l(-)}{[k_l(+) + k_l(-)] + [k_l(+) - k_l(-)]\mathfrak{R}_l(-)}\,. \tag{24}$$

This $\mathfrak{R}_l(b)$ gives the reflection coefficient R_l by

$$R_l = \mathfrak{R}_l(b_c^+)e^{-2iKb_c} \tag{25}$$

if an appropriate boundary value of $\mathfrak{R}_l$ at some $b = b_i < b_c$ is used to start the integration of (23) from b_i outward to b_c. Similar procedure applies to $k_l^{*2}(b^*)$ and (16).

The b_i and the appropriate $\mathfrak{R}(b_i)$ are determined by the $k_l^2(b)$ near that b_i and by consideration of the physics of the problem. In general, we have to find the two limiting independent solutions of f there, and impose a physical condition to select their ratio. If a singularity of k_l^2 at b_i is strong enough to reflect everything so that $f(b_i) = 0$, or if a perfectly conducting spherical surface is there, then $R(b_i) = -1$.

The R_l is related to the usual phase shifts δ_l by

$$\delta_l = \lim_{b_c \to \infty} \frac{1}{2i} \ln[(-1)^{l+1}R_l] \tag{26}$$

and $\lim|R_l|^2$ at $b_c \to \infty$ is the reflected energy of the l wave. For an incident plane EM wave $\vec{E} = \vec{e}_{(x)}e^{i(Kz-\omega t)}$ at $z = -\infty$, the total scattering or extinction cross section,[21] expressed by R_l and Q_l, is

$$\sigma_{tot} = \frac{\pi}{K^2}\,\mathrm{Re}\left(\lim_{b_c \to \infty} \sum_{l=1}^{\infty} (2l+1)[2 - (-1)^{l+1}(R_l + Q_l)]\right), \tag{27}$$

where Q_l stands for the reflection of magnetic wave U. The elastic scattering cross section σ_{sc} is

$$\sigma_{sc} = \frac{\pi}{2K^2}\sum_1^{\infty}(2l+1) \times \lim_{b_c \to \infty}[|1 + (-1)^l R_l|^2 + |1 + (-1)^l Q_l|^2], \tag{28}$$

which equals σ_{tot} if k_l^2 is real.

If $\xi_l(b)$ of (21) vanishes slower than or at the same rate as b^{-1} at $b \to \infty$, then δ_l, σ_{sc}, and the forward scattering amplitude all diverge.[22] For this case $|R_l|^2$ still converges and gives the reflected energy if $\mathrm{Im}(k_l^2) > 0$. But, although δ_l diverges, the differential cross section $d\sigma_{sc}/d\Omega$ of a scalar scattering in nonforward directions is finite and can be expressed by the δ_l's. In trying to get such an expression for the EM scattering considered, we need the relation

$$\sum_1^{\infty} \frac{2l+1}{l(l+1)}[\tau_l(\cos\theta) + \pi_l(\cos\theta)] = 0, \quad \theta \neq 0 \tag{29}$$

where $\tau_l \equiv P_l^1(\cos\theta)/\sin\theta$ and $\pi_l \equiv (d/d\theta)[P_l^1(\cos\theta)]$. To prove (29), we first express π_l in terms of $P_l(\cos\theta)$ and $P_l^1(\cos\theta)$, and then use the recurrence formula

$$\frac{2l+1}{l(l+1)}(1 - x^2)^{1/2}P_l^1(x) = P_{l+1}(x) - P_{l-1}(x) \tag{30}$$

to rewrite (29). Then (29) is equivalent to

$$\sum_1^{\infty}(2l+1)P_l(\cos\theta) = 0, \quad \theta \neq 0 \tag{31}$$

which is readily seen to be true by expanding $\delta(1 - \cos\theta)$ on $\{P_l^m(\cos\theta)\}$. Then with (29) and the well-known Mie scattering expression,[23] we obtain

$$\left(\frac{d\sigma_{sc}}{d\Omega}\right)_{\theta\neq 0} = \frac{\cos^2\phi}{4}\left|\sum_{1}^{\infty}\frac{2l+1}{l(l+1)}\left[e^{2i(\delta_l-\delta_1)}\tau_l(\cos\theta)+e^{2i(\eta_l-\delta_1)}\pi_l(\cos\theta)\right]\right|^2 + \frac{\sin^2\phi}{4}\left|\sum_{1}^{\infty}\frac{2l+1}{l(l+1)}\left[e^{2i(\delta_l-\delta_1)}\pi_l(\cos\theta)+e^{2i(\eta_l-\delta_1)}\tau_l(\cos\theta)\right]\right|^2, \quad (32)$$

where η_l is the phase shift for l partial waves of magnetic mode U. In (32) we used the assumption that if μ and ϵ taper off at $b\rightarrow\infty$ slower than $1/b$, they do so at the same rate. This formula (32) is analogous to the well-known expression for the spherical scattering of a scalar field.[24]

---

*This work is supported by the U. S. Air Force Office of Scientific Research under Grant No. AFOSR-70-1935.

†A second paper concerning a specific application of this method is being prepared.

[1]See, e.g., P. Moon and D. E. Spencer, *Field Theory Handbook* (Springer, Berlin, 1961).

[2]For a discussion in flat space, see, e.g., A. Nisbet, Proc. Roy. Soc. (London) A240, 375 (1957); M. Carmeli, J. Math. Phys. 10, 1699 (1969).

[3]M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1970), 4th ed., p. 634.

[4]See C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (University of Maryland Press, College Park, Md., 1971), Chap. 22; also J. L. Synge, *General Relativity* (Interscience, New York, 1960), p. 256. The signature (+ − − −) is used here.

[5]Of course, Eq. (1) can be simplified to contain only two metric functions in a new frame of spherical coordinates by various choices such as rescaling time and/or radial labeling or setting up a radial motion. We purposefully retain (1) to formally cover all these choices.

[6]T. C. Mo, J. Math. Phys. 11, 2589 (1970), Sec. 4-A.

[7]Geometrized MKS units with $c=1$, $G=1$, $K=1$ are used here. For retrieval to MKS see attached table in T. C. Mo, Radio Sci. 6, 673 (1971).

[8]Here for simplicity we use linear media with pointwise constitution relations. Linear time and radial dispersiveness can be included and treated similarly.

[9]Born and Wolf, Ref. 3.

[10]Mo, Ref. 6, Sec. 2.

[11]V. Fock, *Theory of Space, Time and Gravitation* (Pergamon, New York, 1964), 2nd ed., pp. 146–150.

[12]Their Cartesian components are decoupled in any simple medium. See Ref. 6.

[13]For radiation and propagation of Φ, see, e.g., R. H. Price, Ph.D. thesis, Caltech, 1971 (unpublished).

[14]R. A. Matzner, J. Math. Phys. 9, 163 (1968).

[15]Notice that the $l=0$ field, be it a wave or a static field, is excluded by the Debye potential representations. This means no EM monopole radiation and the static spherical Coulomb field should be added (if it exists) separately.

[16]This is the medium modified version of a standard transform used in T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).

[17]Fock, Ref. 11, Sec. 94.

[18]Misner, Thorne, and Wheeler, Ref. 4, Chap. 21.

[19]V. A. Ambartsumian, J. Phys. USSR 8, 65 (1944); R. Bellman and R. Kalaba, in *Electromagnetic Wave Propagation*, edited by M. Desirant and J. L. Michiels (Academic, New York, 1960), p. 243.

[20]R. W. Latham, Can. J. Phys. 46, 1463 (1968).

[21]H. C. van de Hulst, *Light Scattering By Small Particles* (Wiley, New York, 1957), p. 127.

[22]L. D. Landau and E. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1958), Sec. 106.

[23]van de Hulst, Ref. 21.

[24]Landau and Lifshitz, Ref. 22, p. 401.