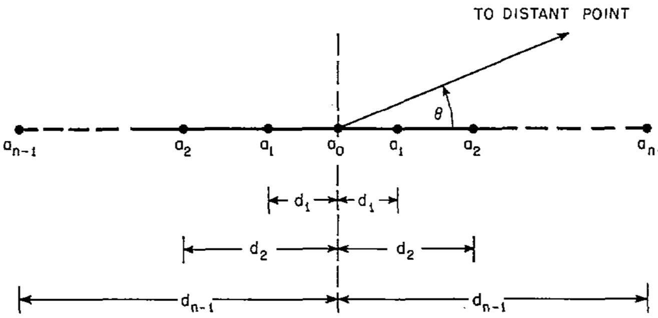

Fig. 1. A nonuniformly spaced symmetrical array.

is then found at u_i^0, $i=0, 1, \cdots, n$ with $a \leq u_0^0 < u_1^0 < \cdots < u_n^0 \leq b$ by setting $d/due^0(u)=0$. The values of $e^0(u)$ at these extrema are, of course, not equal since this is only an initial arbitrary interpolation. The next step is to assume that

$$E^1(u) = a_0^1 + 2\sum_{1=1}^{n-1} a_i^1 \cos b_i u \tag{9}$$

with a_i^1 as the new coefficients to be determined by equalizing

$$e^1(u) = f(u) - E^1(u) \tag{10}$$

at u_i^0, $i=0, 1, \cdots, n$ such that the following will be true:

$$e^1(u_i^0) = (-1)^i \epsilon_1, \quad i = 0, 1, \cdots, n. \tag{11}$$

Equation (11) consists of a system of $(n+1)$ equations which are just enough to solve for a_i^1 and ϵ_1. The set of coefficients a_i^1 so obtained, though satisfying (11), does not necessarily assure that $e^1(u)$ vary with equal ripples since $e^1(u)$ now has a new set of extrema points

$$u_i^1(a \leq u_0^1 < u_1^1 < \cdots < u_n^1 \leq b).$$

From these points, we shall again try to equalize the deviation function until a final set of extrema points, u_i^k, $i=0, 1, \cdots, n$ are obtained such that the values of $e^k(u) = f(u) - E^k(u)$ at these points are equal in magnitude to a certain accuracy but with signs alternately plus and minus. The superscript k signifies the kth iteration after an initial starting. The iterative process is proved to be convergent so long as the conditions for a Chebyshev system have been satisfied [15], [16].

If a broadside Gaussian pattern

$$f(u) = \exp(-Au^2)$$

is chosen as the desired pattern and since it practically vanishes after a point B (see Fig. 2), all the sidelobes of the synthesized pattern $E^k(u)$ will be approximately at the same level represented by 20 log $|\epsilon_k|$. It is noted that the technique discussed here can also be generalized for arrays of nonisotropic elements with the element pattern represented by $W(u)$ provided $W(u) \geq 0$. The result will then have minimized:

$$\max W(u) \mid f(u) - E(u) \mid$$
$$= \max \mid W(u)f(u) - W(u)E(u) \mid, \quad 0 \leq u \leq \pi$$

which may be considered as a maximum weighted deviation. Now, $W(u)E(u)$ represents the synthesized pattern with nonisotropic elements, and $W(u)f(u)$ can be considered as a new desired pattern.

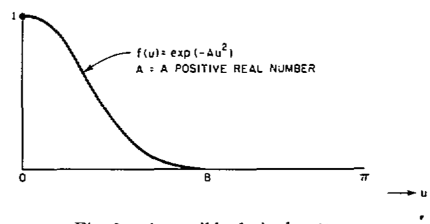

Fig. 2. A possible desired pattern.

The serious limitations of this proposed method of synthesizing nonuniformly spaced arrays lies on the question how the choice of n and b_k should be made such that a Chebyshev system is formed. A systematic way of trying to answer it by numerical examples is being investigated. It is hoped that useful results will be published in detail soon.

M. T. Ma
Central Radio Propagation Lab.
National Bureau of Standards
Boulder, Colo.

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Radiation Resistance and Irreversible Power of Antennas in Gyroelectric Media

In recent years, many investigators have been working on the problem of calculating the radiation resistance of a dipole antenna immersed in an anisotropic medium [1]. The central difficulty of their method of calculation is that it yields an infinite value for the radiation resistance. They have attributed this infinity to the infinitesimal size of the source, and have suggested that if the source were of finite spatial extent the difficulty would not arise. It is our contention that the difficulty is of a more basic nature and is not due to the size of the source but to the method of calculation. The purpose of this letter is to show that if the radiation resistance is calculated with proper conformity to the thermodynamical laws of reversibility and irreversibility, the value of the radiation resistance will turn out to be finite. Clearly, radiation resistance is on the same footing as ordinary circuit resistance in the sense that they both are measures of irreversible power, and hence in calculating radiation resistance it is necessary that only the irreversible part of the power be used. Accordingly, we shall construct an expression for the irreversible part of the power emitted by a source, and show that the expression so constructed is finite and hence leads to a finite value for the radiation resistance. To construct the required expression, we recall that in the case of an accelerating point electron in vacuum, the combination of half the retarded minus half the advanced field is free from singularity [2] and corresponds to the irreversible power radiated by the electron [3]. We shall extend this idea of taking a combination field to the case of a monochromatic source $\mathrm{Re}(\boldsymbol{J}e^{-i\omega t})$ radiating into a lossless anisotropic medium.

Our starting point is the conventional expression

$$P = -\frac{1}{2}\mathrm{Re}\int \boldsymbol{J}^*(\boldsymbol{r})\cdot\boldsymbol{E}^{\mathrm{out}}(\boldsymbol{r}, \boldsymbol{B}_0)dV \tag{1}$$

for the time-average power P. Here $\boldsymbol{E}^{\mathrm{out}}$ is the electric field of the outgoing wave generated by the source current $\boldsymbol{J}$, and $\boldsymbol{B}_0$ is the externally applied magnetic field which produces the anisotropy of the surrounding medium. Clearly $\boldsymbol{E}^{\mathrm{out}}$ can be written as the following identity

$$\begin{aligned}\boldsymbol{E}^{\mathrm{out}}(\boldsymbol{B}_0) = {} & \tfrac{1}{4}[\boldsymbol{E}^{\mathrm{out}}(\boldsymbol{B}_0) + \boldsymbol{E}^{\mathrm{out}}(-\boldsymbol{B}_0) \\ & - \boldsymbol{E}^{\mathrm{in}}(\boldsymbol{B}_0) - \boldsymbol{E}^{\mathrm{in}}(-\boldsymbol{B}_0)] \\ & + \tfrac{1}{4}[\boldsymbol{E}^{\mathrm{out}}(\boldsymbol{B}_0) - \boldsymbol{E}^{\mathrm{out}}(-\boldsymbol{B}_0) \\ & - \boldsymbol{E}^{\mathrm{in}}(\boldsymbol{B}_0) + \boldsymbol{E}^{\mathrm{in}}(-\boldsymbol{B}_0)] \\ & + \tfrac{1}{4}[\boldsymbol{E}^{\mathrm{out}}(\boldsymbol{B}_0) + \boldsymbol{E}^{\mathrm{out}}(-\boldsymbol{B}_0) \\ & + \boldsymbol{E}^{\mathrm{in}}(\boldsymbol{B}_0) + \boldsymbol{E}^{\mathrm{in}}(-\boldsymbol{B}_0)] \\ & + \tfrac{1}{4}[\boldsymbol{E}^{\mathrm{out}}(\boldsymbol{B}_0) - \boldsymbol{E}^{\mathrm{out}}(-\boldsymbol{B}_0) \\ & + \boldsymbol{E}^{\mathrm{in}}(\boldsymbol{B}_0) - \boldsymbol{E}^{\mathrm{in}}(-\boldsymbol{B}_0)],\end{aligned} \tag{2}$$

where $\boldsymbol{E}^{\mathrm{in}}$ is the electric field of the incoming wave associated with $\boldsymbol{J}$. Substituting (2) into (1), we see that the first two terms of ex-

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pression (2) yield the irreversible part P^{irr} of P, and the last two terms yield the reversible part P^{rev} of P, in agreement with the requirement that on reversing the positive sense of time and simultaneously the positive sense of $\boldsymbol{B}_0$ the irreversible part P^{irr} remain unchanged and the reversible part P^{rev} change its sign. Since it is only P^{irr} that enters the calculation for the radiation resistance, we discard the last two terms of (2). We note that the first two terms reduce to the combination $\frac{1}{2}[\boldsymbol{E}^{\text{out}}(\boldsymbol{B}_0)-\boldsymbol{E}^{\text{in}}(\boldsymbol{B}_0)]$. This combination satisfies everywhere the homogeneous equation

$$\nabla \times \nabla \times (\boldsymbol{E}^{\text{out}} - \boldsymbol{E}^{\text{in}}) - \omega^2\mu_0\boldsymbol{\varepsilon}(\boldsymbol{B}_0) \cdot (\boldsymbol{E}^{\text{out}} - \boldsymbol{E}^{\text{in}}) = 0, \quad (3)$$

where $\boldsymbol{\varepsilon}(\boldsymbol{B}_0)$ is the hermitian dyadic that characterizes the lossless anisotropic medium, and hence is free from singularity. Thus we conclude that P^{irr} is finite provided, of course, that the volume integral of $|\boldsymbol{J}|$ itself is bounded.

From the linearity of Maxwell's equations we can write

$$\begin{aligned} \boldsymbol{E}^{\text{out}}(\pm\boldsymbol{B}_0) &= i\omega\mu_0 \int \boldsymbol{\Gamma}^{\text{out}}(\pm\boldsymbol{B}_0)\cdot\boldsymbol{J}dV \\ \boldsymbol{E}^{\text{in}}(\pm\boldsymbol{B}_0) &= i\omega\mu_0 \int \boldsymbol{\Gamma}^{\text{in}}(\pm\boldsymbol{B}_0)\cdot\boldsymbol{J}dV \end{aligned} \quad (4)$$

in terms of the dyadic Green's functions which satisfy

$$\nabla \times \nabla \times \boldsymbol{\Gamma}^{\text{out}}(\pm\boldsymbol{B}_0) - \omega^2\mu_0\boldsymbol{\varepsilon}(\pm\boldsymbol{B}_0)\cdot\boldsymbol{\Gamma}^{\text{out}}(\pm\boldsymbol{B}_0) = \boldsymbol{u}\delta(\boldsymbol{r} - \boldsymbol{r}')$$

$$\nabla \times \nabla \times \boldsymbol{\Gamma}^{\text{in}}(\pm\boldsymbol{B}_0) - \omega^2\mu_0\boldsymbol{\varepsilon}(\pm\boldsymbol{B}_0)\cdot\boldsymbol{\Gamma}^{\text{in}}(\pm\boldsymbol{B}_0) = \boldsymbol{u}\delta(\boldsymbol{r} - \boldsymbol{r}') \quad (5)$$

where $\boldsymbol{u}$ = unit dyadic. Since

$$\boldsymbol{\varepsilon}(-\boldsymbol{B}_0) = \boldsymbol{\varepsilon}^*(\boldsymbol{B}_0),$$

(5) implies that

$$\boldsymbol{\Gamma}^{\text{in}}(\pm\boldsymbol{B}_0) = [\boldsymbol{\Gamma}^{\text{out}}(\mp\boldsymbol{B}_0)]^*. \quad (6)$$

With the aid of (4) and (6) we thus obtain from (1), and the first two terms of (2), the required expression

$$\begin{aligned} P^{\text{irr}} = \frac{\omega\mu_0}{4} \text{Re} \int \{&\boldsymbol{J}^*(\boldsymbol{r}) \\ &\cdot \text{Im}\,[\boldsymbol{\Gamma}^{\text{out}}(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{B}_0) + \boldsymbol{\Gamma}^{\text{out}}(\boldsymbol{r}, \boldsymbol{r}', -\boldsymbol{B}_0)] \\ &\cdot \boldsymbol{J}(\boldsymbol{r}')\}\, dVdV' \\ + \frac{\omega\mu_0}{4} \text{Im} \int \{&\boldsymbol{J}^*(\boldsymbol{r}) \\ &\cdot \text{Re}\,[\boldsymbol{\Gamma}^{\text{out}}(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{B}_0) - \boldsymbol{\Gamma}^{\text{out}}(\boldsymbol{r}, \boldsymbol{r}', -\boldsymbol{B}_0)] \\ &\cdot \boldsymbol{J}(\boldsymbol{r}')\}\, dVdV'. \end{aligned} \quad (7)$$

In the special case of an oscillating dipole of moment $\boldsymbol{p}$, we have $\boldsymbol{J} = -i\omega\boldsymbol{p}\delta(\boldsymbol{r})$, and hence (7) becomes

$$P^{\text{irr}} = \frac{\omega^3\mu_0}{4}\boldsymbol{p}\cdot\text{Im}\,[\boldsymbol{\Gamma}^{\text{out}}(0, 0, \boldsymbol{B}_0) + \boldsymbol{\Gamma}^{\text{out}}(0, 0, -\boldsymbol{B}_0)]\cdot\boldsymbol{p}, \quad (8)$$

which is clearly an even function of $\boldsymbol{B}_0$ as it should be. When $\boldsymbol{B}_0 = 0$ (isotropic media), or when $\boldsymbol{B}_0 = \infty$ (uniaxial crystals), (8) reduces to the conventional (1).

If the anisotropy is due to the uniform motion $\boldsymbol{v}$ of the medium instead of an externally applied magnetic field, $\boldsymbol{B}_0$, (7), with $\boldsymbol{B}_0$ replaced by $\boldsymbol{v}$, gives the irreversible power when the dyadic Green's functions are appropriately calculated. In the case of an oscillating dipole in a moving medium (8) yields a value for P^{irr} that agrees with a previous result obtained in a different manner [4].

K. S. H. Lee
C. H. Papas
Calif. Inst. Tech.
Pasadena, Calif.

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