Form the scalar product $\langle \mathbf{n}, \delta\boldsymbol{\xi}\rangle$ of Theorem 1.

$$\langle \mathbf{n}, \delta\boldsymbol{\xi}\rangle = -\int_0^T dt \left\langle \mathbf{n}, e^{-At}B\frac{\partial c}{\partial \mathbf{n}}\,\delta\mathbf{n}\right\rangle - \langle Ae^{-A^T}\theta, \mathbf{n}\rangle\delta T - \langle e^{-A^T}B\mathbf{c}(\mathbf{n}, T), \mathbf{n}\rangle\delta T. \quad (34)$$

Substituting (27) into (34) and noting that the matrices A and e^{-At} commute, one obtains

$$\langle \delta\boldsymbol{\xi}, \mathbf{n}\rangle = -\delta T[-L[c(\mathbf{n}, T)] + \langle A\boldsymbol{\theta}, e^{-A'^T}\mathbf{n}\rangle + \langle B\mathbf{c}(\mathbf{n}, T), e^{-A'T}\mathbf{n}\rangle]. \quad (35)$$

But $\boldsymbol{\theta} = \boldsymbol{x}(T)$, $e^{-A^T}\mathbf{n} = \boldsymbol{p}(T)$ and so

$$\langle \delta\boldsymbol{\xi}, \mathbf{n}\rangle = -\delta T[-L[c(\mathbf{n}, T)] + \langle A\boldsymbol{x}(T), \boldsymbol{p}(T)\rangle + \langle B\mathbf{c}(\mathbf{n}, T), \boldsymbol{p}(T)\rangle] = -\delta T H^*(T) = -\delta T H^*. \quad (36)$$

Thus we have found that

$$\langle \delta\boldsymbol{\xi}, \mathbf{n}\rangle = -H^*\delta T. \quad (37)$$

If the response time T is fixed, $\delta T = 0$. If the response time T is free $H^* = 0$ according to (10); in either case

$$\langle \delta\boldsymbol{\xi}, \mathbf{n}\rangle = 0, \qquad \text{Q.E.}D. \quad (38)$$

Comments

Since the proof of Theorem 1 involved the use of the necessary conditions provided by the maximum principle, it should be clear that Theorem 1 represents a necessary condition.

The result derived has been shown to be true for time-optimal systems[3,4] ($L[\boldsymbol{u}(t)] = 1$); for time-optimal systems the set $S(\boldsymbol{\xi})$ is the boundary of a closed, bounded and convex set $\hat{S}(\xi)$ so that the tangent plane is a support plane. This fact has been used to derive iterative techniques for the computation of the time-optimal control. The results presented in this communication represent the first step toward the development of iterative techniques for other optimal systems. If the set $S(\xi)$ is the boundary of a convex set then the tangent plane at ξ is a support plane and if the point ξ is regular then the initial costate $\mathbf{n}$ is unique and so the optimal control is unique. However, in general, there exist nonunique extremal controls and the iterative techniques will probably be quite complex. If $S(\xi)$ is smooth, Theorem 1 can be developed in a way quite different from the proof given here by extending a result given by Rozenoer.[5]

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On Cancellations, Controllability and Observability

This communication is concerned with *cancellations, controllability* and *observability* of *linear time-invariant differential systems* of *finite order.*

In a recent letter, Kwakernaak and Polak[1] discussed the reducibility of the order of linear time-invariant differential systems of finite order. Zadeh and Desoer[2] in their book, have also touched upon this question. They assert that a necessary and sufficient condition for controllability is that no cancellations occur in the differential equation. Gilbert,[3] investigating the distinct eigenvalue case, showed that the transfer function has some cancellations if the system is either not completely controllable or not completely observable.

In this communication we attack the general case when the "A-matrix" has *not necessarily distinct eigenvalues.* We find *necessary and sufficient* conditions for cancellations in the transfer function. Our approach uses direct matrix theory proofs and does not require the diagonalization of the system.

Consider the following linear time-invariant system S:

$$S: \quad \begin{aligned} \dot{x} &= Ax + bu \qquad x(0) = x_0 \quad &(1)\\ y &= c'x + \delta u \quad &(2) \end{aligned}$$

where

x—an n vector,[4] is the state of S with x_0 the initial state at time $t=0$;
y—a scalar, is the output;
A—is a constant $n \times n$ matrix;
b, c—are constant n vectors;
δ—is a scalar.

To get the solution for y in terms of the input u we take the Laplace transforms of (1) and (2)

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}bU(s) \quad (3)$$

$$Y(s) = c'X(s) + \delta U(s). \quad (4)$$

Elimination of $X(s)$ from (3) and (4) yields

$$Y(s) = c'(sI - A)^{-1}x_0 + c'(sI - A)^{-1}bU(s) + \delta U(s). \quad (5)$$

Note that $(sI - A)^{-1}b$ is an n vector, the components of which are rational functions of s. More precisely,

$$(sI - A)^{-1}b = \operatorname{col}\left(\frac{p_1(s)}{\Delta(s)}, \cdots, \frac{p_n(s)}{\Delta(s)}\right) \quad (6)$$

where $p_i(s)$ $i = 1, \cdots, n$ is some polynomial in s, and

$$\Delta(s) = \det(sI - A). \quad (7)$$

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[1] H. Kwakernaak and E. Polak, "On the reduction of the system $\dot{x} = Ax + Bu$, $y = c'x$ to its minimal equivalent," IEEE Trans. on Circuit Theory, vol. CT-10, pp. 530–531; December, 1963.

[2] L. A. Zadeh and C. A. Desoer, "Linear System Theory," McGraw-Hill Book Company, New York, N. Y., p. 500; 1963.

[3] E. G. Gilbert, "Controllability and observability in multivariable control systems," *J. SIAM Control*, ser. A, vol. 2, no. 1, 1963.

[4] All vectors are column vectors. Row vectors are written as the transpose of column vectors.

Similarly,

$$c'(sI - A)^{-1} = \left(\frac{q_1(s)}{\Delta(s)}, \cdots, \frac{q_n(s)}{\Delta(s)}\right) \quad (8)$$

where $q_i(s)$ $i = 1, \cdots, n$ is some polynomial in s.

Let us define what we mean by *cancellations.*

Definition: The vector $(sI - A)^{-1}b$ has no cancellation if, and only if, the polynomials $p_1(s)$, $p_2(s)$, $\cdots$, $p_n(s)$, $\Delta(s)$ have no common factor.

A similar definition applies also to the row vector $c'(sI - A)^{-1}$.

Observing (5), one might be led *intuitively* to the following conclusions:

1) if $(sI - A)^{-1}b$ has a cancellation then the system cannot be controlled in the "direction" of the cancelled mode
2) if $c'(sI - A)^{-1}$ has a cancellation then this cancelled mode cannot be observed in the output y.

These intuitive observations 1) and 2) will be expressed precisely in Theorems I and II, respectively.

Theorem I

A necessary and sufficient condition that the system S be completely controllable is that $(sI - A)^{-1}b$ have no cancellations.

Theorem II

A necessary and sufficient condition that the system S be completely observable is that $c'(sI - A)^{-1}$ have no cancellations.

Before proceeding to prove Theorems I and II let us remind the reader of the definitions of *complete controllability* and *complete observability* and also of a characterization for each one. All these definitions and characterizations are due to Kalman.[5]

Definition of complete controllability (c.c.): A system S is said to be c.c. if, and only if, for all x_0 there exists a time $T < \infty$ and an input $u_{[0,T]}$ such that $x[T; x_0, 0; u_{[0,T]}] = 0$.[6]

Definition of complete observability (c.o.): A system S is said to be c.o. if, and only if, for all initial state x_0 there exists a time $T < \infty$ such that the knowledge of $y_{[0,T]}$ determines x_0 uniquely.

A Characterization of c.c.: S is c.c. if, and only if,

$$\det(b, Ab, \cdots, A^{n-1}b) \neq 0 \quad (9)$$

A Characterization of c.o.: S is c.o. if, and only if,

$$\det(c, A'c, \cdots, (A')^{n-1}c) \neq 0. \quad (10)$$

The proofs of Theorems I and II evolve directly from the following lemma.

Let

G—be an $n \times n$ matrix
f—an n vector
λ—a scalar variable such that $(xI - G)^{-1}$ exists.

[5] R. E. Kalman, "On the general theory of control systems," *Proc. of the First International Congress of the Inter. Fed. of Auto. Control*, Moscow, U.S.S.R.; 1960.

[6] $x[T; x_0, 0; u_{[0,T]}]$ denotes the state of S at time T if the initial state at time $t=0$ was x_0 and an input $u_{[0,T]}$ was applied to S.

Lemma: $(\lambda I - G)^{-1} f$ *has a cancellation if, and only if,* $\det[f, Gf, \cdots, G^{n-1} f] = 0$.

Proof:

1) Necessity

Let

$$h \triangleq (\lambda I - G)^{-1} f. \tag{11}$$

Form the following identities:

$$\begin{aligned} h &= h \\ \lambda h &= Gh + f \\ \lambda^2 h &= G^2 h + Gf + \lambda f \\ &\vdots \\ \lambda^n h &= G^n h + G^{n-1} f + \lambda G^{n-2} f + \cdots \\ &\quad + \lambda^{n-1} f. \end{aligned} \tag{12}$$

We define the constants α_i $i = 0, \cdots, n$ by the following identity:

$$\det(\lambda I - G) = \sum_{i=0}^{n} \alpha_i \lambda^i. \tag{13}$$

Next, multiply the first equation of (12) by α_0, the second by α_1, and so on; summation of the $n+1$ equations gives

$$\sum_{i=0}^{n} \alpha_i \lambda^i h = \sum_{i=0}^{n} \alpha_i G^i h + \sum_{i=0}^{n-1} \lambda^i \sum_{k=0}^{n-i-1} \alpha_{k+i+1} G^k f. \tag{14}$$

Now, by the Caley-Hamilton theorem[7]

$$\sum_{i=1}^{n} \alpha_i G^i = 0.$$

Also, let us introduce the notation

$$v_i \triangleq \sum_{k=0}^{n-i-1} \alpha_{k+i+1} G^k f. \tag{15}$$

Thus (14) reduces to

$$h = \frac{1}{\sum_{i=0}^{n} \alpha_i \lambda^i} \sum_{i=0}^{n-1} \lambda^i v_i. \tag{16}$$

By hypothesis (16) has a cancellation so that the numerator of the right side of (16) *must* have the following form:

$$\sum_{i=0}^{n-1} \lambda^i v_i = (\lambda - \lambda_j) \sum_{i=0}^{n-2} \lambda^i w_i \tag{17}$$

where λ_j is a scalar (in fact, an eigenvalue of G), and w_i $i = 0, \cdots, n-2$ are some vectors.

By equating the coefficients of λ^i $i = 0, \cdots, n-1$ in the *identity* (17) we get the following set of equations:

$$\begin{aligned} v_0 &= -\lambda_j w_0 \\ v_1 &= w_0 - \lambda_j w_1 \\ v_2 &= w_0 - \lambda_j w_2 \\ &\vdots \\ v_{n-1} &= w_{n-2}. \end{aligned} \tag{18}$$

We now multiply the first equation of (18) by $\lambda_j^0 (=1)$, the second by λ_j, the third by λ_j^2, etc., and add up all n equations. We get

$$\sum_{i=0}^{n-1} \lambda_j^i v_i = 0. \tag{19}$$

Referring to (14) we see that (19) implies that

$$\sum_{i=0}^{n-1} G^i f \left(\sum_{k=0}^{n-i-1} \alpha_{i+k+1} \cdot \lambda^k \right) = 0. \tag{20}$$

Since the coefficient of $G^{n-1} f$ in (20) is $\alpha_n \cdot \lambda^0$ which is nonzero [in fact, refer to (13) to see that $\alpha_n = 1$], (20) implies that the vectors $f, Gf, \cdots, G^{n-1} f$ are linearly dependent and hence

$$\det[f, Gf, \cdots, G^{n-1} f] = 0.$$

2) Sufficiency

By hypothesis there exist constants $\gamma_0, \cdots, \gamma_{n-1}$, not all zero, so that

$$\sum_{i=0}^{n-1} \gamma_i G^i f = 0. \tag{21}$$

But by (11) $f = (\lambda I - G) h$ so that (21) transforms into

$$(\lambda I - G) \sum_{i=0}^{n-1} \gamma_i G^i h = 0. \tag{22}$$

Note that in deriving (22) from (21) we used the fact that the matrices G^i and $(\lambda I - G)$ commute. Now, for λ such that $\det(\lambda I - G) \neq 0$, (22) implies that

$$\sum_{i=0}^{n-1} \gamma_i G^i h = 0. \tag{23}$$

Next, multiply the first n equation of (12) as follows: the first one by γ_0, the second by γ_1, etc. and add up all equations. Using (23) we get

$$\sum_{i=0}^{n-1} \gamma_i \lambda^i \cdot h = \sum_{i=0}^{n-2} \lambda^i \sum_{k=0}^{n-i-2} \gamma_{i+k+1} G^k f \tag{24}$$

which can also be written as

$$h = \frac{1}{\sum_{i=0}^{n-1} \gamma_i \lambda^i} \sum_{i=0}^{n-1} \lambda^i \sum_{k=0}^{n-i-2} \gamma_{i+k+1} G^k f. \tag{25}$$

The denominator of (25) clearly indicates that a cancellation occurred. This completes the proof of the Lemma.

Proof of Theorem I

The proof of Theorem I follows immediately from the Lemma and the characterization (9).

Proof of Theorem II

Observe that $c'(A - sI)^{-1}$ has a cancellation if, and only if, $[c'(A - sI)^{-1}]'$ has a cancellation. Since

$$[c'(A - sI)^{-1}]' = (A' - sI)^{-1} c \tag{26}$$

we can use the Lemma and the characterization (10) to complete the proof of Theorem II.

To summarize, this communication has established an *s-plane characterization* of the notions of controllability and observability which were originally introduced in the *time domain*.

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On Mobility in Constrained Dynamical Systems

This communication describes a property of states of dynamical systems[1] somewhat analogous to Kalman's controllability.[2] The notion of *state mobility* is shown elsewhere[3] to be of significance in the design of closed loop (feedback) optimal control systems.

For the dynamical system

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)) \tag{1}$$

with $\mathbf{x}$ and f n vectors and $\mathbf{u}$ an m vector and the dot denoting differentiation with respect to time, we use the notation

$$\mathbf{x}(t) = \mathbf{\Phi}_u(t; \mathbf{x}(t_0)) \tag{2}$$

to mean the solution of (1), assuming its existence and uniqueness, which (1) defines at time t if the state of the system at t_0 is $\mathbf{x}(t_0)$ and the control $\mathbf{u}(\tau)$ is used for $\tau \in [t_0, t]$.

We define $\mathbf{x}(t)$ to be an admissible state of (1) if and only if $\mathbf{x}(t) \in X$, and $\mathbf{u}(t)$ to be an admissible control for (1) if and only if $\mathbf{u}(t) \in U$ for $t \in [t_0, t_1]$ (X and U are finite dimensional real spaces). Then we define $\mathbf{x}(t_0)$ to be a *mobile state* of (1) if and only if there exists a $t_1 > t_0$ and a $\mathbf{u}^*(t) \in U$ such that $\mathbf{x}(t_0) \in X$ and $\phi_{u^*}(t_1; \mathbf{x}(t_0)) \in X$ and further, there is no $t_2 \in [t_0, t_1]$ such that $\phi_{u^*}(t_2; \mathbf{x}(t_0)) \notin X$.

A similar definition holds for the discrete time case.

Intuitively this defines a mobile state to be an admissible state from which the system can be driven into an admissible state by use of an admissible control.

An example of a situation in which immobile states arise is given below. It should be observed that the presence or absence of such states is not predicted by existing theory and yet is clearly an interesting question in the design of optimal systems.[3] The example given is for a set of difference equations. Consider the system

$$S: \quad \begin{aligned} x_1(k+1) &= x_1(k) + x_2(k) + u_1(k) \\ x_2(k+1) &= x_1(k) + u_2(k) \end{aligned}$$

Define X and U by:

$$X: \{x_1(k): \ |x_1(k)| \leq R_1 \geq 0; \quad k = 0, 1, 2, \cdots\}$$

$$U: \{u_1(k): \ |u_1(k)| \leq R_2 \geq 0; \quad k = 0, 1, 2, \cdots\}$$

The mobile, immobile and inadmissible states are shown in Fig. 1. As a brief illustration of how the system could reach an immobile state, let $R_1 = 1.5$, $R_2 = 1.0$, the (admissible) initial state be $\mathbf{x}^T(0) = (1.2 \ \ 1.0)$, and the (admissible) initial control be $\mathbf{u}^T(0) = (-0.8 \ \ 0.)$. Then the (admissible but immobile) state into which the system is driven is $\mathbf{x}^T(1) = (1.4 \ \ 1.2)$. From this state there is no admissible control $\mathbf{u}$ (2) which will drive S into an admissible state.

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[1] R. E. Kalman, "Mathematical description of dynamical systems," *J. SIAM Control*, vol. 1, no. 2, 1963.

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