Disentanglement Cost of Quantum States

Mario Berta[1] and Christian Majenz[2]

[1]*Institute for Quantum Information and Matter, California Institute of Technology, Pasadena*
[2]*Department of Mathematical Sciences, University of Copenhagen, Copenhagen*

We show that the minimal rate of noise needed to catalytically erase the entanglement in a bipartite quantum state is given by the regularized relative entropy of entanglement. This offers a solution to the central open question raised in [Groisman *et al.*, PRA 72, 032317 (2005)] and complements their main result that the minimal rate of noise needed to erase all correlations is given by the quantum mutual information.

arXiv:1708.00360v1 [quant-ph] 1 Aug 2017

Introduction. Quantifying and classifying quantum correlations is of fundamental importance in quantum information theory [14]. Motivated by Landauer's erasure principle [15], one way to quantify the correlations present in a bipartite quantum state ρ_{AB} is to measure the amount of noise that is required to erase them. In that respect, Groisman *et al.* [12] showed that the optimal asymptotic rate of local noise to bring ρ_{AB} close to a product $\sigma_A \otimes \sigma_B$ is given by the *quantum mutual information*

$$I(A:B)_\rho := D(\rho_{AB}\|\rho_A \otimes \rho_B) = \inf_{\sigma \in \mathrm{PR}} D(\rho_{AB}\|\sigma_A \otimes \sigma_B) \tag{1}$$

where $D(\rho\|\sigma) := \mathrm{Tr}\left[\rho(\log\rho - \log\sigma)\right]$ is the quantum relative entropy, and $PR(A:B)$ denotes the set of product states in $A:B$. Hence, the quantum mutual information quantifies the total amount of correlations in bipartite states – including both the quantum and classical ones. This finding was generalized in various directions, including a catalytic analysis of the one-shot case [16], the study of tripartite correlations [3, 26, 27], as well as the study of coherence [20] and more general symmetries [25]. However, it remained open how to quantify the optimal asymptotic rate of local noise to bring ρ_{AB} close to a separable state $\sum_j p_j \sigma_A^j \otimes \sigma_B^j$. In particular, it was unclear if a quantity defined in such a way can be the basis of a proper entanglement measure.

In this note, we solve the problem and give a precise mathematical model for erasing entanglement in bipartite states where the optimal asymptotic rate of local noise needed to get close to a separable state is given by the *regularized relative entropy of entanglement*.

Entanglement Measures. The *relative entropy of entanglement* is given by [22]

$$E(A:B)_\rho := \inf_{\sigma \in \mathrm{SEP}} D(\rho\|\sigma)\,, \tag{2}$$

where $\mathrm{SEP}(A:B)$ denotes the set of separable states in $A:B$. Since the relative entropy of entanglement is in general not additive on tensor product states the *regularized relative entropy of entanglement* [24] is defined as

$$E^\infty(A:B)_\rho := \lim_{n\to\infty} \frac{1}{n} E(A:B)_{\rho^{\otimes n}} \tag{3}$$

This quantity has an operational interpretation in composite asymmetric quantum hypothesis testing as the asymptotic exponential rate of mistakenly identifying ρ_{AB} instead of a state separable in $A:B$ [6]. As a corresponding one-shot analogue based on the smooth max-relative entropy [10]

$$D^\varepsilon_{\max}(\rho\|\sigma) := \inf_{\bar\rho \approx_\varepsilon \rho} \inf\left\{2^\lambda :\ \lambda \cdot \sigma - \bar\rho \geq 0\right\} \tag{4}$$

with $\bar\rho \approx_\varepsilon \rho$ in purified distance [21],

we have the *smooth max-relative entropy of entanglement* [9]

$$E^\varepsilon_{\max}(A:B)_\rho := \inf_{\sigma \in \mathrm{SEP}} D^\varepsilon_{\max}(\bar\rho_{AB}\|\sigma_{AB})\,. \tag{5}$$

This is a smoothed version of the logarithm global robustness of entanglement [23]. All the quantities $E(A:B)_\rho$, $E^\infty(A:B)_\rho$, and $E^\varepsilon_{\max}(A:B)_\rho$ define proper entanglement measures with useful mathematical properties as requested by axiomatic entanglement theory (see, e.g., [4, 7] for an overview).

Disentanglement Cost. We are interested in the amount of local noise needed to catalytically erase the entanglement in a bipartite quantum state. For this purpose, a randomizing map is generated by an ensemble of local unitaries $(U_A^i \otimes U_B^i)$ as

$$\Lambda^M_{A:B}(\cdot) := \frac{1}{M}\sum_{i=1}^M \left(U_A^i \otimes U_B^i\right)(\cdot)\left(U_A^i \otimes U_B^i\right)^\dagger\,. \tag{6}$$

It is called *ε-disentangling* if there exist a state $\omega_{A'B'} \in \mathrm{SEP}(A':B')$ such that

$$\inf_{\sigma \in \mathrm{SEP}} P\left(\Lambda^M_{AA':BB'}(\rho_{AB} \otimes \omega_{A'B'}), \sigma_{ABA'B'}\right) \leq \varepsilon \tag{7}$$

with $\sigma_{AA'BB'} \in \mathrm{SEP}(AA':BB')$. Here, we think of $\omega_{A'B'}$ as a catalytic resource state that is already separable to start with but has to be kept separable by the randomizing map (cf. catalytic decoupling [16]). The *one-shot ε-disentanglement cost* $C^\varepsilon_{\mathrm{SEP}}(A:B)_\rho$ is then defined as the minimal number $\log M$ such that Eq. (7) holds. We are particularly interested in the asymptotic behavior in the limit of many copies $\rho_{AB}^{\otimes n}$ and vanishing error $\varepsilon \to 0$, which we call the *disentanglement cost* of

quantum states:

$$C_{\mathrm{SEP}}(A:B)_\rho := \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C^{\varepsilon}_{\mathrm{SEP}}(A:B)_{\rho^{\otimes n}}\,. \tag{8}$$

Main Result. We find that the ε-disentanglement cost is given by the smooth max-relative entropy of entanglement and hence that the disentanglement cost is given by the regularized relative entropy of entanglement.

Theorem 1. *Let ρ_{AB} and $1\geq\varepsilon\geq\delta>0$. Then, we have*

$$E^{\varepsilon}_{\max}(A:B)_\rho \leq C^{\varepsilon}_{\mathrm{SEP}}(A:B)_\rho \tag{9}$$

$$\leq E^{\varepsilon-\delta}_{\max}(A:B)_\rho + \log\frac{1}{\delta}+1 \tag{10}$$

as well as $C_{\mathrm{SEP}}(A:B)_\rho = E^{\infty}(A:B)_\rho$.

This offers a solution to the central open question raised in Groisman *et al.* [12] and automatically establishes the disentanglement cost of quantum states as a proper *entanglement measure* – since it inherits all mathematical properties from the regularized relative entropy of entanglement. Note, however, that we do not show the disentanglement cost being equal to the entropy injected into the system as conjectured by Groisman *et al.* but to the relative entropy of entanglement as suggested in [13].

For pure states $|\psi\rangle_{AB}$ we get $E^{\infty}(A:B)_\psi = H(A)_\psi$ – the entropy of the Schmidt spectrum – whereas the quantum mutual information measuring the total correlations is equal to $2H(A)_\psi$. For the one-shot setting we find that

$$H^{\varepsilon}_{\max}(A)_\psi \leq C^{\varepsilon}_{\mathrm{SEP}}(A:B)_\psi \leq H^{\varepsilon-\delta}_{\max}(A)_\rho + \log\frac{1}{\delta}+1 \tag{11}$$

$$\text{with } H^{\varepsilon}_{\max}(A)_\rho := \inf_{\bar\rho\approx_\varepsilon\rho} 2\log\mathrm{Tr}\left[\sqrt{\bar\rho}\right]$$

the smooth max-entropy. Furthermore, we find with [16] that the amount of noise needed to erase all correlations in a pure state $|\psi\rangle_{AB}$ is given by two times the cost function from Eq. (11) – which is in exact analogy to the asymptotic case.

Proof of Thm. 1. We first derive the converse direction – i.e. the lower bound in Thm. 1 – using standard entropy inequalities. To show the one-shot converse in Eq. (9) we begin by observing that tensoring a separable state does not change the smooth max-relative entropy of entanglement[1] and thus it suffices to show the converse for disentangling maps without catalysts. Let therefore $\Lambda^{M}_{A:B}$ be a disentangling randomizing map for ρ_{AB}, that is, there exists $\sigma_{AB}\in\mathrm{SEP}(A:B)$ such that $P\left(\Lambda^{M}_{A:B}(\rho_{AB}),\sigma_{AB}\right)\leq\varepsilon$. Next, define a classically maximally correlated state

$$\gamma_{X_aX_b} := \frac{1}{M}\sum_{i=1}^{M}|i\rangle\langle i|_{X_a}\otimes|i\rangle\langle i|_{X_b} \tag{12}$$

and the controlled unitaries V_{AX_a} and W_{BX_b} such that

$\mathrm{tr}_{X_aX_b}\left[\rho'_{ABX_aX_b}\right] = \Lambda^{M}_{A:B}(\rho_{AB})$ for the state

$$\rho'_{ABX_aX_b} := V_{AX_a}\otimes W_{BX_b}\left(\rho_{AB}\otimes\gamma_{X_aX_b}\right)V^{\dagger}_{AX_a}\otimes W^{\dagger}_{BX_b}. \tag{13}$$

By Uhlmann's theorem, there exists an extension $\sigma_{ABX_aX_b}$ of σ_{AB} such that $P\left(\rho'_{ABX_aX_b},\sigma_{ABX_aX_b}\right)\leq\varepsilon$ with the X_a- and X_b-registers classical in the same basis as in Eq. (12). Additionally, the extension can be chosen such that $\Pi_{X_aX_b}\sigma_{ABX_aX_b}\Pi_{X_aX_b} = \sigma_{ABX_aX_b}$, where $\Pi_{X_aX_b}$ is the projector onto the maximally correlated subspace, i.e. onto the support of $\gamma_{X_aX_b}$. Now, we bound

$$\begin{aligned}
E^{\varepsilon}_{\max}(A:B)_\rho =& E^{\varepsilon}_{\max}(AX_a:BX_b)_{\rho\otimes\gamma}\\
=& E^{\varepsilon}_{\max}(AX_a:BX_b)_{\rho'}\\
\leq& E^{\varepsilon=0}_{\max}(AX_a:BX_b)_{\sigma}\\
\leq& D^{\varepsilon=0}_{\max}(\sigma_{ABX_aX_b}\|\sigma_{AB}\otimes\gamma_{X_aX_b})\\
\leq& \log M\,.
\end{aligned} \tag{14}$$

The first two inequalities follow from picking two particular points in the minima defining $E^{\varepsilon}_{\max}$ and the last inequality follows from the matrix inequality

$$\begin{aligned}
\sigma_{ABX_aX_b} \leq& \sigma_{AB}\otimes\Pi_{X_aX_b}\\
=& M\cdot\sigma_{AB}\otimes\gamma_{X_aX_b},
\end{aligned} \tag{15}$$

which follows from Lemma 3.1.9 in [18]. This proves Eq. (9).

For the asymptotic expansion, we then use the composite quantum Stein's lemma from [6, Prop. II.1] and [9, Thm. 1]

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}E^{\varepsilon}_{\max}(A:B)_\rho = E^{\infty}(A:B)_\rho\,. \tag{16}$$

We note that asymptotic converses for similar scenarios were also shown in [13].

For the achievability part – i.e. the upper bound in Thm. 1 – we invoke a tool that was introduced as the *convex splitting lemma* by Anshu *et al.* [1]. We need a special case of the their main lemma which is as follows.

Lemma 2 (Convex split)**.** *Let ρ,σ be quantum states and $N=\left\lceil D^{\zeta}_{\max}(\bar\rho\|\sigma)/\xi\right\rceil$ with $\xi\geq 0,\zeta>0$. Then, we have*

$$P\left(\frac{1}{N}\sum_{i=1}^{N}\rho_i\otimes\sigma_{i^c}^{\otimes(N-1)},\sigma^{\otimes N}\right)\leq\zeta+\xi\,, \tag{17}$$

where ρ_i sits in the i-th register and $i^c := [1,\ldots,N]\backslash i$.

[1] The argument is the same as for the relative entropy of entanglement [24] and based on the monotonicity under quantum operations.

Hence, for any state ρ_{AB} and $\sigma_{AB} \in \mathrm{SEP}(A:B)$ we can choose $\log N = D_{\max}^{\varepsilon-\delta}(\rho_{AB}\|\sigma_{AB}) + \log\frac{1}{\delta} + 1$ such that

$$P\left(\frac{1}{M}\sum_{i=1}^{M}\rho_{A_iB_i}\otimes\sigma_{\tilde{A}\tilde{B}\setminus(A_iB_i)}^{\otimes(M-1)}, \sigma_{\tilde{A}\tilde{B}}^{\otimes M}\right) \leq \varepsilon \tag{18}$$

for $\tilde{A}\tilde{B} := A_1\cdots A_M B_1\cdots B_M$ with $A_1B_1 := AB$ and $A_iB_i \cong AB$ for $i = 2,\ldots,A_M$. Now, the idea is to use the catalytic resource state $\sigma_{\tilde{A}\tilde{B}\setminus(A_1B_1)}^{\otimes(M-1)} \in \mathrm{SEP}(\tilde{A}\setminus A_1 : \tilde{B}\setminus B_1)$ together with the ensemble of local unitaries for $i = 1,\ldots,N$ given by

$$U_{\tilde{A}}^i \otimes U_{\tilde{B}}^i := (1i)_{\tilde{A}} \otimes (1i)_{\tilde{B}}\,, \tag{19}$$

where $(1i)$ denotes the unitary that swaps registers $1 \leftrightarrow i$ on $\tilde{A}$ and $\tilde{B}$, respectively. Optimizing over all $\sigma_{AB} \in \mathrm{SEP}(A:B)$ then gives the one-shot achievability in Eq. (10). Finally, the asymptotic expansion of the upper bound follows as in Eq. (16) which concludes the proof of Thm 1. □

Extensions. Groisman *et al.* [12] show that for their setting of going to product states one can also achieve the quantum mutual information by alternatively replacing the model of coordinated random local unitary channels as in Eq. (6) to only local unitary channels $\Lambda_A^M(\cdot) := \frac{1}{M}\sum_{i=1}^{M} U_A^i(\cdot)\left(U_A^i\right)^\dagger$ and not making use of any (product state) catalytic assistance. Whereas maps as in Eq. (6) and catalytic assistance – separable states in our case – seem necessary to obtain the tight result presented in the previous sections, it is nevertheless insightful to compare our result with other models. In particular, the model of local unitary channels $\Lambda_A^M(\cdot)$ can be related to *catalytic decoupling*, where the noisy operation to ensure closeness to product states is given by a *partial trace map* over a system of asymptotic rate size $\frac{1}{2}I(A:B)_\rho$ [16]. This can be done in our case as well, albeit not in the exact same optimal way as for local unitary channels. Namely, to implement the coordinated local random unitary channel from Eq. (6), a classically correlated state $\gamma_{X_aX_b}$ has to be used as an ancillary system, half of which has to be discarded afterwards on both sites A and B. More precisely for

$$\mu_{\bar{A}\bar{B}} := \rho_{AB}\otimes\omega_{A'B'}\otimes\gamma_{X_aX_b} \text{ with } \omega_{A'B'} \in \mathrm{SEP}(A':B') \tag{20}$$

and $\bar{A}\bar{B} := \bar{A}_1\bar{A}_2\bar{B}_1\bar{B}_2 := AA'X_aBB'X_b$ there exist $\sigma_{\bar{A}_1\bar{B}_1} \in \mathrm{SEP}(\bar{A}_1 : \bar{B}_1)$ and a local unitary $U_{\bar{A}}\otimes U_{\bar{B}}$ such that

$$P\left(\mathrm{Tr}_{\bar{A}_2\bar{B}_2}\left[\left(U_{\bar{A}}\otimes U_{\bar{B}}\right)\mu_{\bar{A}\bar{B}}\left(U_{\bar{A}}\otimes U_{\bar{B}}\right)^\dagger\right], \sigma_{\bar{A}_1\bar{B}_1}\right) \leq \varepsilon \tag{21}$$

for $\log|\bar{A}_2| + \log|\bar{B}_2| = E_{\max}^{\varepsilon-\delta}(A:B)_\rho + \log\frac{1}{\delta} + 1$. We conclude that the straightforward translation of the disentangling protocol introduced here to two-sided catalytic decoupling leads to a cost twice the one obtained from the converse bound in the case of disentangling. It would be interesting to explore the decoupling to separable states notion as in Eq. (21) further.

Moreover, we might extend our results to analyze tripartite quantum correlations as well. Here, for tripartite states ρ_{ABC} we can define *locally recovered states* by

$$(\mathcal{I}_B\otimes\mathcal{R}_{C\to AC})(\rho_{BC}) \text{ with } \mathcal{R}_{C\to AC} \text{ local quantum channels.} \tag{22}$$

States ρ_{ABC} such that there exists $\mathcal{R}_{C\to BC}$ with $(\mathcal{I}_B\otimes\mathcal{R}_{C\to AC})(\rho_{BC}) = \rho_{ABC}$ are called *quantum Markov* [17] but in general ρ_{ABC} is far from its recovered states. A measure for the local recoverability is the relative entropy of recovery

$$D(A;B|C)_\rho := \inf_{\mathcal{R}_{C\to BC}} D\big(\rho_{ABC}\|(\mathcal{I}_B\otimes\mathcal{R}_{C\to AC})(\rho_{BC})\big) \tag{23}$$

and its regularized version $D^\infty(A;B|C)_\rho$ [5, 19]. The latter quantity has an operational interpretation in composite asymmetric quantum hypothesis testing as the asymptotic exponential rate of mistakenly identifying ρ_{ABC} instead of a corresponding locally recovered state $(\mathcal{I}_B\otimes\mathcal{R}_{C\to BC})(\rho_{AC})$ [8]. Moreover, it was recently shown that $D^\infty(A;B|C)_\rho \neq D(A;B|C)_\rho$ [11].

We can now ask for the amount of noise needed to catalytically transform the state into a corresponding locally recovered version thereof. For this purpose we again define a randomizing map Λ_{ABC}^M as in Eq. (6) but now with tripartite local unitaries $\left(U_A^i\otimes U_B^i\otimes U_C^i\right)$. Such maps are called *$\varepsilon$-recovery-degrading* if there exists a locally recovered state $\omega_{A'B'C'} = (\mathcal{I}_{B'}\otimes\mathcal{R}_{C'\to A'C'})(\rho_{B'C'})$ such that

$$\inf_{\mathcal{R}_{CC'\to AA'CC'}} P\big(\Lambda_{AA'BB'CC'}^M(\rho_{ABC}\otimes\omega_{A'B'C'}), (\mathcal{I}_{BB'}\otimes\mathcal{R}_{CC'\to AA'CC'})(\rho_{BC}\otimes\rho_{B'C'})\big) \leq \varepsilon\,. \tag{24}$$

Like before we can think of $\omega_{A'B'C'}$ as a catalytic resource state that is already locally recovered to start with but has to be kept locally recovered by the randomizing map (cf. conditional decoupling [3]). The *non-recoverability cost* denoted by $C_{\mathrm{REC}}(A;B|C)_\rho$ is then defined as the minimal rate $\frac{1}{n}\log M$ needed for ε-recovery-degrading in the limit of asymptotically many copies $\rho_{ABC}^{\otimes n}$ and vanishing error $\varepsilon \to 0$. Using again the convex split lemma (Lem. 2) and the framework in [6] for the asymptotic expansion it is straightforward to see that non-recoverability cost is upper bounded by the regularized relative entropy of recovery

$$C_{\mathrm{REC}}(A;B|C)_\rho \leq D^\infty(A;B|C)_\rho\,. \tag{25}$$

It would be interesting to understand if this is also tight. We remark that the upper bound in Eq. (25) is in contrast to other recent work about conditional

decoupling of quantum information by Berta *et al.* [3] and Wakakuwa *et al.* [26, 27]. The fundamental difference is that our final states are locally recovered, i.e. of the form $(\mathcal{I}_B \otimes \mathcal{R}_{C\to AC})(\rho_{BC})$, but are not themselves (approximately) locally recoverable – whereas this is demanded in all of these alternative models.

Conclusion. We have presented a model for catalytic erasure of entanglement in quantum states and showed the optimal asymptotic rate of noise needed is given by the regularized relative entropy of entanglement. This establishes the disentanglement cost of quantum states as a proper entanglement measure. We left open a few questions about extensions to catalytic decoupling models as well as to tripartite quantum correlations in terms of the non-recoverability cost. Finally, our proofs make crucial use of the convex splitting lemma (Lem. 2) by Anshu *et al.* [1] and it would be interesting to better understand the consequences of this technique in quantum information theory.

Note. Our main result Thm. 1 was also derived in the independent work [2].

Acknowledgments. We thank the Institute for Mathematical Sciences at the National University of Singapore for hosting the stimulating workshop Beyond I.I.D. in Information Theory during which part of this work was done. MB acknowledges funding by the SNSF through a fellowship, funding by the Institute for Quantum Information and Matter (IQIM), an NSF Physics Frontiers Center (NFS Grant PHY-1125565) with support of the Gordon and Betty Moore Foundation (GBMF-12500028), and funding support form the ARO grant for Research on Quantum Algorithms at the IQIM (W911NF-12-1-0521). CM acknowledges financial support from the European Research Council (ERC Grant Agreement no. 337603), the Danish Council for Independent Research (Sapere Aude) and VILLUM FONDEN via the QMATH Centre of Excellence (Grant No. 10059).

[1] A. Anshu, V. K. Devabathini, and R. Jain. Quantum message compression with applications. arXiv: 1410.3031.

[2] A. Anshu, M.-H. Hsieh, and R. Jain. Quantifying resource in catalytic resource theory. *to appear on arXiv.org on the same day.*

[3] M. Berta, F. G. S. L. Brandao, C. Majenz, and M. M. Wilde. Deconstruction and conditional erasure of quantum correlations. arXiv: 1609.06994.

[4] F. G. S. L. Brandão, M. Christandl, and J. Yard. Faithful Squashed Entanglement. *Comm. Math. Phys.*, 306(3):805–830, 2011. DOI: 10.1007/s00220-011-1302-1.

[5] F. G. S. L. Brandão, A. W. Harrow, J. Oppenheim, and S. Strelchuk. Quantum conditional mutual information, reconstructed states, and state redistribution. *Phys. Rev. Lett.*, 115:050501, 2015. DOI: 10.1103/PhysRevLett.115.050501.

[6] F. G. S. L. Brandão and M. B. Plenio. A Generalization of Quantum Stein's Lemma. *Commun. Math. Phys.*, 295(3):791–828, 2010. DOI: 10.1007/s00220-010-1005-z.

[7] M. Christandl. *The Structure of Bipartite Quantum States - Insights from Group Theory and Cryptography.* PhD thesis, 2006. Available online: http://arxiv.org/abs/quant-ph/0604183.

[8] T. Cooney, C. Hirche, C. Morgan, J. P. Olson, K. P. Seshadreesan, J. Watrous, and M. M. Wilde. Operational meaning of quantum measures of recovery. *Phys. Rev. A*, 94:022310, 2016. DOI: 10.1103/PhysRevA.94.022310.

[9] N. Datta. Max- relative entropy of entanglement, alias log robustness. *Int. J. Quantum Inf.*, 7(2):475–491, 2009.

[10] N. Datta. Min- and Max- Relative Entropies and a New Entanglement Monotone. *IEEE Trans. on Inf. Theory*, 55(6):2816–2826, 2009. DOI: 10.1109/TIT.2009.2018325.

[11] H. Fawzi and O. Fawzi. Relative entropy optimization in quantum information theory via semidefinite programming approximations. arXiv: 1705.06671.

[12] B. Groisman, S. Popescu, and A. Winter. Quantum, classical, and total amount of correlations in a quantum state. *Phys. Rev. A*, 72(3):032317, 2005. DOI: 10.1103/PhysRevA.72.032317.

[13] M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, and B. Synak-Radtke. Local versus nonlocal information in quantum-information theory: Formalism and phenomena. *Phys. Rev. A*, 71:062307, 2005. DOI: 10.1103/PhysRevA.71.062307.

[14] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. *Rev. Mod. Phys.*, 81:865–942, 2009. DOI: 10.1103/RevModPhys.81.865.

[15] R. Landauer. Irreversibility and heat generation in the computing process. *IBM J. Res. Dev.*, 5:183, 1961. DOI: 10.1147/rd.53.0183.

[16] C. Majenz, M. Berta, F. Dupuis, R. Renner, and M. Christandl. Catalytic decoupling of quantum information. *Phys. Rev. Lett.*, 118:080503, 2017. DOI: 10.1103/PhysRevLett.118.080503.

[17] D. Petz. Sufficient Subalgebras and the Relative Entropy of States of a von Neumann Algebra. *Commun. Math. Phys.*, 105(1):123–131, 1986. DOI: 10.1007/BF01212345.

[18] R. Renner. *Security of Quantum Key Distribution.* PhD thesis, ETH Zurich, 2005. arXiv: quant-ph/0512258.

[19] K. P. Seshadreesan and M. M. Wilde. Fidelity of recovery, squashed entanglement, and measurement recoverability. *Phys. Rev. A*, 92:042321, 2015. DOI: 10.1103/PhysRevA.92.042321.

[20] U. Singh, M. N. Bera, A. Misra, and A. K. Pati. Erasing quantum coherence: An operational approach. arXiv: 1506.08186.

[21] M. Tomamichel, R. Colbeck, and R. Renner. Duality Between Smooth Min- and Max-Entropies. *IEEE Trans. on Inf. Theory*, 56(9):4674–4681, 2010. DOI: 10.1109/TIT.2010.2054130.

[22] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight. Quantifying entanglement. *Phys. Rev. Lett.*, 78:2275–

2279, 1997. DOI: 10.1103/PhysRevLett.78.2275.

[23] G. Vidal and R. Tarrach. Robustness of entanglement. *Phys. Rev. A*, 59:141–155, 1999. DOI: 10.1103/PhysRevA.59.141.

[24] K. G. H. Vollbrecht and R. F. Werner. Entanglement measures under symmetry. *Phys. Rev. A*, 64:062307, 2001. DOI: 10.1103/PhysRevA.64.062307.

[25] E. Wakakuwa. Symmetrizing cost of quantum states. *Phys. Rev. A*, 95:032328, 2017. DOI: 10.1103/PhysRevA.95.032328.

[26] E. Wakakuwa, A. Soeda, and M. Murao. The cost of randomness for converting a tripartite quantum state to be approximately recoverable. *IEEE Trans. Inf. Theory*, 63(8):5360–5371, 2017. DOI: 10.1109/TIT.2017.2694481.

[27] E. Wakakuwa, A. Soeda, and M. Murao. Markovianizing cost of tripartite quantum states. *IEEE Trans. Inf. Theory*, 63(2):1280–1298, 2017. DOI: 10.1109/TIT.2016.2639523.