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1 Introduction

In this paper, I construct a Liapunov function for Nash equilibria for finite $n - person$ games in normal form. This function is useful for computation of Nash equilibria, since it converts the problem into a standard minimization problem. It provides an alternative to existing computational methods, which are based either on $n - person$ extensions of the algorithm of Lemke and Howson [1961] (eg., Wilson [1971] and Rosenmüller [1971]), or on methods for finding the fixed point of the best response correspondence, such as simplicial division algorithms (eg., Todd [1976], and Van der Laan et al. [1987]). This work is also related to that of Brown and von Neumann [1950], and Rosen [1964], who construct differential equation systems for solving certain classes of games.[1]

2 Definitions and Results

Consider a finite $n - person$ game in normal form: There is a set $N = \{1, \ldots, n\}$ of players , and for each player $i \in N$ a *strategy set* $S_i = \{s_{i1}, \ldots, s_{iJ_i}\}$, consisting of J_i pure strategies. For each $i \in N$, we are given a *payoff function*, $M_i : S \to \mathbb{R}$, where $S = \prod_{i \in N} S_i$.

Let Δ_i be the set of probability measures on S_i. Elements of Δ_i are of the form $p_i : S_i \to \mathbb{R}$ where $\sum_{s_{ij} \in S_i} p_i(s_{ij}) = 1$, and $p_i(s_{ij}) \geq 0$ for all $s_{ij} \in S_i$. We use the notation $p_{ij} = p_i(s_{ij})$. So Δ_i is isomorphic to the J_i dimensional simplex $\Delta_i = \{p_i = (p_{i1}, \ldots, p_{iJ_i}) : \sum_j p_{ij} = 1, p_{ij} \geq 0\}$. We write $\Delta = \prod_{i \epsilon N} \Delta_i$, and let $J = \sum_{i \epsilon N} J_i$. So $\Delta \subseteq \mathbb{R}^J$. We denote points in Δ by $p = (p_1, \ldots, p_n)$, where $p_i = (p_{i1}, \ldots, p_{iJ_i}) \in \Delta_i$. We

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[1]Brown and von Newmann construct a differential equation system that is globally convergent for finite two person, zero sum games. The work of Rosen is for general $n - person$ games. His conditions for global convergence require a condition of diagonal strict concavity on the matrix of cross derivatives of the payoff functions, which would not in general be satisfied for finite $n - person$ games.

use the abusive notation s_{ij} to denote the strategy $p_i \in \Delta_i$ with $p_{ij} = 1$. We use the shorthand notation $p = (p_i, p_{-i})$. Hence, the notation (s_{ij}, p_{-i}) represents the strategy where i adopts the pure strategy s_{ij}, and all other players adopt their components of p.

The payoff function is extended to have domain Δ by the rule $M_i(p) = \sum_{s\in S} p(s)M_i(s)$, where $p(s) = \prod_{i\in N} p_i(s_i)$. A vector $p = (p_1, \ldots, p_n) \in \Delta$ is a *Nash Equilibrium* if for all $i \in N$, and all $p_i \in \Delta_i$, $M_i(p_i, p_{-i}) \leq M_i(p)$.

Now define the function $v : \Delta \to \mathbb{R}$ by

$$v(p) = \textstyle\sum_{i\in N} \sum_{s_{ij}\in S_i} \{max[M_i(s_{ij}, p_{-i}) - M_i(p), 0]\}^2. \tag{1}$$

For for $i \in N$ and $1 \leq j \leq J_i$, define

$$g_{ij}(p) = \{max\ [M_i(s_{ij}, p_{-i}) - M_i(p), 0]\}^2. \tag{2}$$

Then we can write

$$v(p) = \sum_{i\in N} \sum_{1\leq j\leq J_i} g_{ij}(p). \tag{3}$$

Let T be the tangent space of Δ. So $T \subseteq \prod_{i\epsilon N} \mathbb{R}^{J_i} = \mathbb{R}^J$, and any $y \in T$ is of the form $y = (y_1, \ldots, y_n)$, where $y_i \in \mathbb{R}^{J_i}$ satisfies $\sum_{j=1}^{J_i} y_{ij} = 0$ for all $i \in N$. Define X to be a $J \times n$ matrix whose j^{th} column is a vector $x_j = (x_{j1}, \ldots, x_{jn}) \in \mathbb{R}^J$, with $x_{ji} \in \mathbb{R}^{J_i}$ satisfying $x_{ji} = (0, \ldots, 0)$ if $i \neq j$, and $x_{ji} = (1, \ldots, 1)$ if $i = j$. Then T is the $J - n$ dimensional subspace of $\mathbb{R}^J$ defined by $T = \{y \in \mathbb{R}^J : X'y = 0\}$. Let $P = (I - X(X'X)^{-1}X')$, where I is the $J \times J$ identity matrix. Thus P is a positive definite, idempotent projection operator which maps $\mathbb{R}^J$ into T. For any $y \in \mathbb{R}^J$, $Py \in T$ is the projection of y into T. For any function $f : \mathbb{R}^J \to \mathbb{R}$, write $\bigtriangledown f(p) = \frac{\partial f(p)}{\partial p}$.

Now, based on v, we can define a dynamical system:

$$\dot{p} = -P \bigtriangledown v(p), \tag{4}$$

Thus, $\dot{p}$ is the projection of the negative gradient of v at p into the tangent space T.

Lemma 1 *The function $v : \Delta \to \mathbb{R}$ defined in* (1) *is a Liapunov function for* (4), *whose zeros coincide with the Nash equilibria of the finite game $M : \Delta \to \mathbb{R}^n$.*

Proof: First note that $v(p)$ is a non-negative function that is zero if and only if p is a Nash equilibria for the game with payoff function M.

Next, we show that v is everywhere differentiable. To see this, we will show that each g_{ij} is everywhere differentiable. Fix i and j. For convenience, we drop the subscripts on g, writing $g = g_{ij}$. Now pick arbitrary $p \in \Delta$. There are two cases.

Case 1: $M_i(s_{ij}, p_{-i}) \neq M_i(p)$.

Find a neighborhood, $N(p)$ of p such that $M_i(s_{ij}, p'_{-i}) \neq M_i(p')$ for all $p' \in N(p)$. Then if $M_i(s_{ij}, p_{-i}) < M_i(p)$, it follows that this inequality holds for all $p' \in N(p)$. So $g(p') = 0$ for all $p' \in N(p)$. Hence g is differentiable at p since it is a constant function on $N(p)$. On the other hand, if $M_i(s_{ij}, p_{-i}) > M_i(p)$, then $g(p') = (M_i(s_{ij}, p'_{-i}) - M_i(p'))^2$ for all $p' \in N(p)$. Again, g is differentiable at p since it is the square of the difference of two functions, each of which is differentiable at p.

Case 2: $M_i(s_{ij}, p_{-i}) = M_i(p)$.

Pick any unit length vector $y \in \mathbb{R}^J$, and let $\bigtriangledown_y$ denote the directional derivative in the direction y. Then

$$\bigtriangledown_y g(p) = \lim_{t \to 0} \frac{g(p+ty) - g(p)}{t} = \lim_{t \to 0} \frac{g(p+ty)}{t}.$$

Now for all t,

$$0 \leq \frac{g(p+ty)}{t} \leq \frac{1}{t} \{M_i(s_{ij}, (p+ty)_{-i}) - M_i(p+ty)\}^2.$$

Taking limits, we get

$$0 \leq \bigtriangledown_y g(p) \leq \bigtriangledown_y \{M_i(s_{ij}, p_{-i}) - M_i(p)\}^2.$$

But for all $k \in N, and\ 1 \leq l \leq J_k$, using $M_i(s_{ij}, p_{-i}) = M_i(p)$, we get

$$\frac{\partial}{\partial p_{kl}} [M_i(s_{ij}, p_{-i}) - M_i(p)]^2$$

$$\begin{aligned} &= 2[M_i(s_{ij}, p_{-i}) - M_i(p)] \frac{\partial}{\partial p_{kl}} [M_i(s_{ij}, p_{-i}) - M_i(p)] \\ &= 0. \end{aligned}$$

Hence $\bigtriangledown_y \{M_i(s_{ij}, p_{-i}) - M_i(p)\}^2 = 0$, implying that $\bigtriangledown_y g(p) = 0$. Hence, g is differentiable at p.

Finally, it follows that $\dot{v}(p) = Dv(p)(\dot{p}) = - \bigtriangledown v(p) P \bigtriangledown v(p) \leq 0$, since P is positive definite. So v is a Liapunov function for the dynamical system (4).

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3 Computation

We want to minimize $v(p)$ subject to the constratins that $\sum_j p_{ij} = 1$ and $p_{ij} \geq 0$ for all i, j. For the purposes of computation, it will be convenient to impose the constraints as penalty functions, yielding a revised version of the objective function:

$$w(p) = v(p) + \sum_j \{min[p_{ij}, 0]\}^2 + \sum_{i \epsilon N} (1 - \sum_j p_{ij})^2.$$

Define

$$Q(p) = \{(i, j) : M_i(s_{ij}, p_{-i}) > M_i(p)\},$$

Then we can write

$$v(p) = \sum_{(i,j) \epsilon Q(p)} [M_i(s_{ij}, p_{-i}) - M_i(p)]^2.$$

So the elements of $\bigtriangledown w(p)$ are given by

$$\frac{\partial w(p)}{\partial p_{i_1 j_1}} = 2 \sum_{(i,j) \epsilon Q(p)} [M_i(s_{ij}, p_{-i}) - M_i(p)][M_i(s_{ij}, s_{i_1 j_1}, p_{-ii_1}) - M_i(s_{i_1 j_1}, p)]$$

$$+2min[p_{i_1 j_1}, 0] - 2(1 - \sum_j p_{i_1 j}),$$

where

$$M_i(s_{ij}, s_{i_1 j_1}, p_{-ii_1}) = 0 \text{ if } i = i_1.$$

The elements of the Jacobian are given by

$$\frac{\partial w(p)}{\partial p_{i_1 j_1} p_{i_2 j_2}}$$

$$= 2 \sum_{(i,j) \epsilon Q(p)} \{[M_i(s_{ij}, p_{-i}) - M_i(p)] \cdot$$

$$[M_i(s_{ij}, s_{i_1 j_1}, s_{i_2 j_2}, p_{-ii_1 i_2}) - M_i(s_{i_1 j_1}, s_{i_2 j_2}, p_{-i_1 i_2})]$$

$$+[M_i(s_{ij}, s_{i_2 j_2}, p_{-ii_2}) - M_i(s_{i_2 j_2}, p_{-i_2})] \cdot$$

$$[M_i(s_{ij}, s_{i_1 j_1}, p_{-ii_1}) - M_i(s_{i_1 j_1}, p_{-i_1})\}$$

$$+\begin{cases} 4 & \text{if } i_1 = i_2, p_{ij} < 0 \\ 2 & \text{if } i_1 = i_2 \text{ and } (j_1 \neq j_2 \text{ or } p_{ij} > 0) \end{cases}$$

The above first and second derivative information can be used to implement any of a number of standard algorithms for finding the minimum of a function of several variables.

The author has written a computer algorithm in C, based on the above objective function, to find mixed strategy equilibria for an arbitrary finite $n - person$ normal form game. A copy of the program can be obtained from the author. Two different gradient search methods are implemented, namely Newton's method and the Davidon-Fletcher-Powell (DFP) conjugate gradient algorithm. The algorithm uses "off the shelf" minimization routines from Press et. al. [1988]. Experience shows that the DFP method generally works better than Newtons method. Since the objective function is not twice differentiable, and Newton's method assumes this, the superior convergence speed of Newton does not seem to be obtained in practice. Further, there are the usual problems that Newton's method does not satisfy global convergence. So it needs a good starting point to even guarantee convergence.

Note that it was not established in the above characterization that all minima to the function v are global minima. Hence, there may be local minima to the objective function that are not global minima, and hence not Nash. So it is important to check the value of the objective function after convergence, to verify that the point found is indeed a Nash equilibrium. Computational experience shows that there can be local minima. However, there do not seem in general to be so many local minima as to make the algorithm useless in finding Nash equilibria. When local minima exist, they seem to have small sinks, and a new choice of starting point will typically avoid the local minimum. All local minima that I have found to date are boundary points, i. e., points without full support. I have not yet found any examples of interior local minima which are not also zeroes, and conjecture that there are none. The next section gives some preliminary results along these lines.

The speed of the algorithm is generally slower than other methods, and as is evident from the discussion in the preceeding paragraph, the algorithm may sometimes require judicious choice of a starting point. Nevertheless, there may be some situations in which the algorithm is better than existing methods. Namely, for $n - person$ games, when n is greater than 2, other methods are not capable of finding all of the Nash equilibria. The Lemke-Howson based algorithms are only capable of locating equilibria that are accessible from the the extraneous solution. The fixed point algorithms (based on this author's experience,) are apparently incapable of converging to equilibria where the best response dynamics lead away from the equilibrium (such as the mixed equilibria in a two person 2×2 coodination game). However, every isolated Nash equilibrium will have an open region around it where the value of the objective function v is strictly greater than 0. Hence, every Nash equilibrium will have a radius of convergence such that if one starts within this radius of the Nash equilibrium, the algorithm proposed in this paper will converge to that equilibrium.

4 On Interior Minima

This section gives a very preliminary result on the existence of interior local minima to v that are not global: When all of the functions g_{ij} are strictly quasiconcave, then it can be proven that there are no such local minima. Hence any interior local minimum must be a Nash equilibrium.

Lemma 2 *Let g_{ij} be strictly quasiconcave for each i, j, and satisfy $g_{ij}(p) > 0 \Rightarrow \bigtriangledown g_{ij}(p) \neq 0$ for all p in Δ. Let p^* be an interior point of Δ. I. e., $p^*_{ij} > 0$ for all i, j. Then if p^* is a local minimum of v, it must be a global minimum.*

Proof: Note that the tangent space T of Δ is of dimension $K = J - n$. Further, for any $p \in \Delta$, and $i \in N$, at most $(J_i - 1)$ of the functions $g_{ij}(p)$ can be strictly positive. Thus, at any point $p \in \Delta$, at most K of the functions $g_{ij}(p)$ can be strictly positive. Now let p be a local minimum of v, and assume that $v(p) \neq 0$. Let $Q = \{(i,j) : g_{ij}(p) > 0\}$. Since $v(p) \neq 0$ it follows that $Q \neq \emptyset$. Write $z = P \bigtriangledown v(p)$, and $z_{ij} = P \bigtriangledown g_{ij}(p)$ to be the projected gradients of v and g_{ij}, respectively at the point p. Clearly $z \in T$, and $z_{ij} \in T$ for all i, j. Also, if p is a local minimum, then from the Kuhn Tucker conditions, it follows that

$$0 = z = P \bigtriangledown v(p) = P \sum_{(i,j) \epsilon Q} \bigtriangledown g_{ij}(p) = \sum_{(i,j) \epsilon Q} z_{ij}$$

But, now the right hand side of the above equation consists of no more than K vectors which sum to 0. Hence, they span at most a $K - 1$ dimensional subspace of T. We can find a vector y in T with $y \cdot z_{ij} = 0$ for all $(i,j) \in Q$. But since each g_{ij} is strictly quasi concave and $z_{ij} \neq 0$, it follows for this y, and small enough ϵ, that $g_{ij}(p + \epsilon y) < g_{ij}(p)$. Since this is true for all $(i,j) \in Q$, it follows that $v(p + \epsilon y) = \sum_{(i,j) \epsilon Q} g_{ij}(p + \epsilon y) < \sum_{(i,j) \epsilon Q} g_{ij}(p) = v(p)$. So p is not a local minimum. It follows that we must have $v(p) = 0$. But then p is a global minimum, as we needed to prove.

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If $n = 2$, and $|S_i| = 2$ for each $i = 1, 2$, then it is easily shown that the g_{ij} satisfy the conditions of the above Lemma. Unfortunately, for $|S_i| > 2$, examples can be found where g_{ij} is not quasiconcave. Thus, at present, the above lemma is of limited usefulness.

5 References

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