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STRATEGYPROOF ALLOCATION OF A SINGLE OBJECT

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Abstract

The problem of allocating a single indivisible object to one of several selfish agents is considered, where monetary payments are not allowed, and the object is not necessarily desirable to each agent. It is shown that ordinality and positive responsiveness together are necessary and sufficient conditions for strategyproofness, which implies that efficient social choice functions are not strategyproof. However, any Pareto-optimal, ordinal social choice function is strategyproof. A Gibbard-Satterthwaite-type impossibility result is established for nonbossy mechanisms. Thus, the best the planner can do without monetary transfers is to give the object to an agent who desires it, but whose valuation of the object may not be the highest among the agents, using a mechanism that is either dictatorial or bossy. It is also shown that all strategyproof, nonbossy, and Pareto-optimal social choice functions are serial dictatorships.

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1 Introduction

We examine the problem of allocating a single indivisible object to one of several selfish agents who may or may not desire the object, using a strategyproof mechanism. The objective is to give away the object without receiving any monetary payments, according to criteria such as efficiency, using a "nice" (e.g. nondictatorial) mechanism.

In the context of trading, Myerson and Satterthwaite (1983) have studied the problem of selling an indivisible object when there is a single buyer, and Makowski and Mezzetti (1993) examined the same problem with many buyers, both in the Bayesian framework.

The problem of allocating a single object without any monetary transfers was considered by Glazer and Ma (1989) in the complete information framework. They constructed multistage mechanisms with unique subgame perfect equilibrium outcomes. These outcomes are efficient in the sense that the agent with the highest valuation gets the object, without any monetary transfers being made at equilibrium. In this study, we assume that the planner cannot employ monetary transfers in order to aid the allocation process, even if payments were only to be made out of equilibrium. Kim and Ledyard (1994) examined the allocation of a single object in the Bayesian framework, and found that it is impossible to design an ex post efficient Bayesian incentive compatible mechanism for allocating the object, where the agents only know the distribution of other agents' valuations of the object. They also considered the case of balanced transfers. If only Pareto-optimality is required where the welfare function depends on the allocation of the object and on the payments made between the potential recipients and the supplier of the object, balanced transfers would be acceptable. However, in contexts that are not marketlike (e.g. within a company) or where it is not politically viable to require any compensations (e.g. where traditionally the object is allocated without any compensations and the potential recipients cannot be coerced to pay), no transfers of any form are accepted. This is the case we examine in this study, so that transfers are not allowed even if they were to balance. The above mentioned two papers are also different in that

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they require that the agents have some information (complete information in the case of Glazer and Ma (1989)) about other agents' preferences. Since in this study mechanisms are required to be strategyproof, that is, the mechanism has to ensure that honest behavior is a dominant strategy for every agent and preference profile, it is not necessary for the agents to have any information about the others' preferences. Of course, it is assumed that the agents know their own valuation of the object, and both the agents and the planner know the set of admissible preferences.

In our model the outcomes are n-dimensional if there are n agents, each dimension representing the alloted bundle of private goods for some agent. The agents are assumed to be selfish, which means that each agent i only cares about the ith component of the outcome, so that preferences other than indifference are ruled out between any two outcomes for any agent when the agent's component of the two outcomes are the same. When there is only a single indivisible private good to allocate, further domain restrictions are not necessary, since the agents may evaluate the object in any way. To keep the proofs simple, we also assume that the agents cannot be indifferent between obtaining and not obtaining the object. That is, since an agent's utility does not change if she does not obtain the object, we assume that the the agents have a non-zero value for the object. This, however, is not necessary. The results easily generalize to the domain where indifferences are allowed, if we modify some of the definitions appropriately.

In this paper we attempt to analyze the tradeoffs between strategyproofness and efficiency, and examine the implications of strategyproofness when it is combined with (some form of) efficiency. We also investigate the possibility of avoiding dictatorship without losing strategyproofness, which has been a much explored subject in various contexts since the famous impossibility result of Gibbard (1973) and Satterthwaite (1975). Strategyproofness has been extensively studied in environments where outcomes are public in nature (for surveys of this literature see Muller and Satterthwaite (1986) and Sprumont (1995)). However, given the selfishness assumption, these rules do not apply to our model. In particular, it should be noted that the outcome space in our model is not a product domain as defined in Le Breton and Sen (1995), since the components of the outcomes depend on each other given the feasibility constraints. The studies that examine private goods allocation problems tend to focus on divisible goods so that a structure natural to preferences over private goods (e.g., continuity, nonsatiation, convexity, etc.) is imposed on the preferences (see, for example, Dasgupta et al. (1978), Zhou (1991), and Barbera and Jackson (1995)). The most closely related results to ours can be found in Dasgupta et al. (1978), Ritz (1983), and Satterthwaite and Sonnenschein (1981). These will be discussed in the course of the exposition. Finally, let us note that the concept of non-bossiness, which turns out to be central to our analysis, was introduced by Satterthwaite and Sonnenschein (1981), and was used consequently by Ritz (1983), Olson (1991), and Barbera and Jackson (1995) among others.

Although the results in this paper are of interest on their own, the elementary and intuitive proofs also offer some insight into more general aspects of the problem of allocating indivisible private goods by using strategyproof mechanisms. Throughout the paper, special care is taken to emphasize which results are specific to the single object allocation problem due to its simple structure, which is helpful in identifying others that are potential candidates for generalization.

The paper is organized as follows. The notation and basic definitions are introduced in Section 2. In Section 3, necessary and sufficient conditions for a social choice function to be strategyproof are derived, which are specific to the single object allocation problem. Efficiency and Pareto-optimality are analyzed in Section 4, and the relationship between strategyproofness and Pareto-optimality of a social choice function is established. In Section 5, we investigate the implications of strategyproofness on the desirability of the mechanism. In particular, it is shown that a Gibbard-Satterthwaite-type impossibility is escaped in the model we examine. However, if nonbossiness is also ruled out, which typically arises in the context of private goods allocation problems, then the analog of the Gibbard-Satterthwaite theorem holds. The special case where there are only two agents is also considered in this section. In Section 6, we characterize the set of strategyproof, nonbossy, and Pareto-optimal social choice functions.

2 Definitions and Notation

There are $n \geq 2$ agents and one object to be allocated among the agents. Let N denote the set of n agents. An *outcome* $x = (x^1, \ldots, x^n)$ is such that $x^i \in \{0, 1\}$, where

$$x^i = \begin{cases} 1 & \text{if the object is given to agent } i \\ 0 & \text{otherwise,} \end{cases}$$

$\forall i \in N$. Clearly, an outcome x is *feasible* if at most one agent gets the object, i.e., if $\sum_{i \in N} x^i \leq 1$. Note that the object is not necessarily awarded. The set of feasible outcomes is denoted by X.

Let θ^i denote the value that agent i places on the object. We assume that $\theta^i \in \Re \setminus \{0\}, \forall i \in N$, that is, the agents cannot be indifferent between obtaining and not obtaining the object. Let Θ^i be the set of admissible values for each agent i, i.e., $\Theta^i = \Re \setminus \{0\}$. The set of values for all agents is $\Theta = \times_{i \in N} \Theta^i$. Let $\theta \in \Theta$ be a profile of the agents, and $\theta^{-i} \in \Theta^{-i}$ be a profile of all the agents except for agent i.

Each agent i is assumed to be *selfish*, that is, agent i only cares about the ith component of x. Thus, the value of outcome x to agent i with value θ^i is $x^i\theta^i$. For notational convenience, we define a utility function for each agent i by $U(x^i, \theta^i) = x^i\theta^i, \forall x \in X, \forall i \in N$.

The criteria regarding the desired rules of the outcome are embodied in social choice functions, functions that assign exactly one outcome to any preference profile of the agents. When strategyproofness is required, attention is restricted to direct mechanisms, mechanisms that ask the agents to report their own preferences, due to the well-known revelation principle. Therefore, a direct mechanism that implements a social choice function will mirror the social choice function, in the sense that the outcome of the mechanism will coincide with the outcome prescribed by the social choice function for each preference profile. Thus, the criteria applied to the mechanisms apply to the social choice functions as well.

Definition 1 A *social choice function (SCF)* is a function $f : \Theta \mapsto X$.

Let $f^i(\theta)$ denote the assignment prescribed to agent i by f at θ.

Definition 2 An SCF f is *strategyproof* if $\forall \theta \in \Theta$, $\forall i \in N$, $\forall \tilde{\theta}^i \in \Theta^i$, $U(f^i(\theta), \theta^i) \geq U(f^i(\tilde{\theta}^i, \theta^{-i}), \theta^i)$. *If* $\exists i \in N$ such that $U(f^i(\theta), \theta^i) < U(f^i(\tilde{\theta}^i, \theta^{-i}), \theta^i)$ for some $\theta \in \Theta, \tilde{\theta}^i \in \Theta^i$, then we say that f is *manipulable* and agent i can *manipulate* it.

3 Necessary and Sufficient Conditions for Strategy-proofness

First we define two characterisitics of an SCF f, *ordinality* and *positive responsiveness*, which together are necessary and sufficient conditions for f to be strategyproof in our context.

Definition 3 An SCF f is *ordinal* if $\forall \theta \in \Theta, \forall i \in N, \forall \tilde{\theta}^i \in \Theta^i$ such that $\theta^i, \tilde{\theta}^i > 0$ or $\theta^i, \tilde{\theta}^i < 0$, $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i})$.

Definition 4 An SCF f satisfies *positive responsiveness (PR)* if $\forall \theta \in \Theta, \forall i \in N$, $\forall \tilde{\theta}^i \in \Theta^i$ (a) $\theta^i > 0, \tilde{\theta}^i < 0$ and $f^i(\theta) = 0$ imply that $f^i(\tilde{\theta}^i, \theta^{-i}) = 0$, and (b) $\theta^i < 0, \tilde{\theta}^i > 0$ and $f^i(\theta) = 1$ imply that $f^i(\tilde{\theta}^i, \theta^{-i}) = 1$.

Proposition 1 *An SCF is strategyproof if, and only if, it is ordinal and PR.*

Proof:

Strategyproofness $\Rightarrow$ ordinality
Let f be strategyproof and not ordinal. Then $\exists \theta \in \Theta, i \in N$, and $\tilde{\theta}^i \in \Theta^i$ such that we have either $\theta^i, \tilde{\theta}^i > 0$ or $\theta^i, \tilde{\theta}^i < 0$ and $f^i(\theta) \neq f^i(\tilde{\theta}^i, \theta^{-i})$. Since $f^i(\bar{\theta}) \in \{0, 1\}, \forall i \in N, \forall \bar{\theta} \in \Theta$, we can assume, without loss of generality, that $f^i(\theta) = 1$ and $f^i(\tilde{\theta}^i, \theta^{-i}) = 0$. Given that f is strategyproof, $\theta^i > 0$, otherwise agent i would report $\tilde{\theta}^i$ and get 0 which she would prefer to 1 if $\theta^i < 0$. Similarly, f's strategyproofness implies that $\tilde{\theta}^i < 0$. We have reached a contradiction.

Strategyproofness $\Rightarrow$ PR
Let f be strategyproof and not PR. Then $\exists \theta \in \Theta, i \in N$ and $\tilde{\theta}^i \in \Theta^i$ such that $\theta^i > 0, \tilde{\theta}^i < 0, f^i(\theta) = 0$, and $f^i(\tilde{\theta}^i, \theta^{-i}) = 1$. Since f is strategyproof, we must have $U(0, \theta^i) \geq U(1, \theta^i)$, which implies that $\theta^i < 0$, a contradiction.

Ordinality and PR $\Rightarrow$ strategyproofness
Let f be ordinal, PR, and manipulable. Then $\exists\theta \in \Theta, i \in N$, and $\tilde{\theta}^i \in \Theta^i$ such that $U(f^i(\theta), \theta^i) < U(f^i(\tilde{\theta}^i, \theta^{-i}), \theta^i)$. Then either (a) $f^i(\theta) = 0, f^i(\tilde{\theta}^i, \theta^{-i}) = 1$, and $\theta^i > 0$, or (b) $f^i(\theta) = 1, f^i(\tilde{\theta}^i, \theta^{-i}) = 0$, and $\theta^i < 0$. However, PR implies that $\tilde{\theta}^i > 0$ for (a) and $\tilde{\theta}^i < 0$ for (b). Therefore, f is not ordinal in either case, which is a contradiction.□

We would like to remark here that ordinality and PR together are equivalent to the well-known IPM condition for strategyproofness (see, for example, Laffont and Maskin (1982)). The proof of this is straightforward and is left to the reader. We will work with the ordinality and PR properties, since they better facilitate the following analysis, which will be clear throughout this paper. For now, let us say that, for the allocation of a single object, the condition for strategyproofness has been split into an independence property (ordinality) and a monotonicity property (PR). This is useful, because the independence property, ordinality, is very intuitive and is easily checked, and therefore it helps in ruling out manipulable SCF's. Although in the voting context, where the outcomes are of a political nature, cardinal valuations may not make sense, for resource allocation problems they are of importance. It is usually implicitly assumed in the implementation literature that only ordinal preferences can be elicited when monetary payments are not used.[1] For private goods allocation problems, this informational constraint has considerable consequences. While a somewhat trivial condition, ordinality has important implications for the efficiency of strategyproof mechanisms. This will be discussed in the next section.

4 Efficiency

Given the necessity of ordinality for strategyproofness, it follows immediately that it is not possible to design an *efficient* strategyproof mechanism, a mechanism which assigns the object to the agent who values it most. The same is shown in the Bayesian framework by Kim and Ledyard (1994, Theorem 1). However, ordinality implies more than that. It rules out any interpersonal utility level comparisons, and therefore, less stringent efficiency criteria, such as assigning the object to an agent whose value for it is within the $k (k \leq n)$ highest positive values, or even assigning it to an agent whose value is not the lowest among the positive valuations, cannot be implemented. Therefore, we resort to Pareto-optimality as a criterion of efficiency, given that Pareto-optimal SCF's may satisfy ordinality. We use the stronger notion of Pareto-optimality, which is more appropriate in this context than the other, weaker version. In fact, the weak version of Pareto-optimality, which only requires that no other feasible outcome should be strictly preferred to the prescribed outcome by all agents at any profile, is automatically satisfied, as long as there are at least three agents. However, the strong Pareto-optimality condition is still very weak, in the sense that it typically allows for several different outcomes.

Definition 5 An SCF f is *Pareto-optimal* if $\forall\theta \in \Theta$, there does not exist $y \in X$ such that $U(y^i, \theta^i) \geq U(f^i(\theta), \theta^i), \forall i \in N$, and for some $j \in N, U(y^j, \theta^j) > U(f^j(\theta), \theta^j)$.

[1]An explicit discussion of this issue with regard to Nash-implementation can be found, for example, in Maskin (1986).

If an SCF f is Pareto-optimal then $\forall i \in N, \forall \theta \in \Theta, f^i(\theta) = 1$ implies that $\theta^i > 0$. Therefore, a Pareto-optimal SCF also satisfies *individual rationality*, where f is individually rational if $\forall i \in N, \forall \theta \in \Theta, \theta < 0$ implies that $f^i(\theta) = 0$. Individual rationality alone, however, is satisfied by an imposed mechanism, for example, in which the object is never awarded. Pareto-optimality, on the other hand, implies *citizen sovereignty.*

Definition 6 An SCF f satisfies *citizen sovereignty (CS)* if $\forall x \in X, \exists \theta \in \Theta$ such that $f(\theta) = x$.

Now we show that any Pareto-optimal and ordinal SCF is strategyproof.

Proposition 2 *If an SCF is ordinal and Pareto-optimal then it is strategyproof.*

Proof: Notice that if an SCF violates PR then $\exists i \in N$ and $\theta \in \Theta$ such that $\theta^i < 0$ and $f^i(\theta) = 1$. This, however, implies that f is not Pareto-optimal. Therefore, Pareto-optimality implies PR, which, together with Proposition 1, yields the required result.□

From the above proof it is also clear that if PR is violated, then individual rationality does not hold either. Thus, any individually rational ordinal SCF is strategyproof.

We would like to point out that the above result only holds in the context of the single object allocation problem, and it does not generalize to more complex problems, for example, where there is more than one object to allocate. This is illustrated in Example 1. Given the proof of Proposition 2, this should not come as a surprise. The proof is based on a relationship between Pareto-optimality and PR, namely, that any Pareto-optimal SCF also satisfies PR. Since Pareto-optimality is an intraprofile property (i.e., it can be determined whether the outcome is Pareto-optimal for a given profile), while PR is an interprofile property, this relationship is clearly due to the simple structure of our problem, and cannot hold in general. Before the example is provided, we need an appropriate generalization of the ordinality property for the case where there is more than one object to allocate. Note that in this case each $x \in X$ is a matrix, and each $\theta^i \in \Theta^i$ is a vector.

Definition 7 An SCF f is *ordinal* if $\forall \theta \in \Theta, \forall i \in N, \forall \tilde{\theta}^i \in \Theta^i$ such that $\forall x, y \in X, U(x^i, \theta^i) > U(y^i, \theta^i) \Leftrightarrow U(x^i, \tilde{\theta}^i) > U(y^i, \tilde{\theta}^i)$, $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i})$.

Example 1 *An ordinal, Pareto-optimal, and manipulable SCF for allocating more than one object.*

Let there be two agents and two objects to be allocated among them. Then the two agents have strict preferences over the elements of the set $\{a, b, ab, 0\}$, where a and b are the two objects, ab indicates the allocation to an agent when the agent gets both objects, and 0 denotes the allocation to an agent when she doesn't get anything. Consider the following SCF. If both agents prefer a to b or if both prefer b to a, give agent 1 her first choice, and then give agent 2 her first choice from the remaining object(s). Otherwise, if the two agents' preference orderings are not the same over a and b, then give agent 2 her first choice, and then agent 1 her first choice from the remaining object(s). This SCF is Pareto-optimal and ordinal. However, it is not strategyproof. Consider the reported preferences $(ab, b, a, 0)$ for agent 1, and $(ab, a, b, 0)$ for agent 2. Since 1 prefers b to a and 2 prefers a to b, agent 2 gets her first choice, ab, and agent 1 gets 0. However, agent 1 can manipulate the outcome by reporting $(ab, a, b, 0)$ and obtaining ab, her first choice, instead of 0, her last choice.□

Our next question is, which strategyproof mechanisms satisfy Pareto-optimality? In order to answer this question, we need the concept of *bossiness*, which was introduced by Satterthwaite and Sonnenschein (1981). An SCF is bossy if there exists at least one agent whose preferences can change in a way that the prescribed allocation is different to some other agent(s), but not to herself, while everyone else's preferences are unchanged. Intuitively, this is an undesirable property, given that the mechanism mirrors the SCF that it implements. This means that the agent who can change some other agent's allocation without changing her own may use her "power" by accepting a bribe or blackmailing. Note that bossiness is only a concern when indifference between outcomes is allowed. In particular, when private goods are being allocated and the agents are selfish, indifferences cannot be ruled out, so that a mechanism may allow agents to change the allocations for others without changing their own allocation. For a further discussion of bossiness see Ritz (1983).[2]

Definition 8 An SCF f is *bossy* if $\exists \theta \in \Theta, i \in N$, and $\tilde{\theta}^i \in \Theta^i$ such that $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i})$ and $f^j(\theta) \neq f^j(\tilde{\theta}^i, \theta^{-i})$ for some $j \in N$. An SCF f is *nonbossy* if it is not bossy. If $i \in N$ is such that $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i})$ and $\exists j \in N$ such that $f^j(\theta) \neq f^j(\tilde{\theta}^i, \theta^{-i})$ for some $\theta \in \Theta, \tilde{\theta}^i \in \Theta$, then we say that agent i is *bossy.*

In order to answer the earlier question, we prove Proposition 3, which says that a strategyproof, nonbossy, and CS SCF is Pareto-optimal.

Proposition 3 *If an SCF is strategyproof, nonbossy, and CS then it is Pareto-optimal.*

Proof: Let an SCF f be strategyproof, nonbossy, CS, and not Pareto-optimal. By Proposition 1, f is ordinal and PR. Since f is not Pareto-optimal, we have one of the following two cases:
(a) $\exists \theta \in \Theta$ such that $\theta^i > 0$ for some $i \in N$ and $f(\theta) = \mathbf{0}$.
(b) $\exists \tilde{\theta} \in \Theta$ such that $\tilde{\theta}^i < 0$ for some $i \in N$ and $f^i(\tilde{\theta}) = 1$.
Let's look at the two cases in turn.
(a) Since f satisfies CS, $\exists \tilde{\theta} \in \Theta$ such that $f^i(\tilde{\theta}) = 1$. Consider the sequence of profiles

[2]Ritz calls bossy social choice functions *corruptible*, and also defines corruptibility for social choice correspondences.

$$(\theta^1, \ldots, \theta^n)$$
$$\vdots$$
$$(\theta^1, \ldots, \theta^{j-1}, \tilde{\theta}^j, \ldots, \tilde{\theta}^n)$$
$$\vdots$$
$$(\tilde{\theta}^1, \ldots, \tilde{\theta}^n).$$

Let $i = 1$. Since f is nonbossy, the outcome either does not change when θ^j is replaced by $\tilde{\theta}^j$ in the above sequence of profiles, or $f^j(\theta^1, \ldots, \theta^{j-1}, \tilde{\theta}^j, \ldots, \tilde{\theta}^n) = 1$ for $j = 2, \ldots, n$. Thus, $f^1(\theta^1, \tilde{\theta}^{-1}) = f^i(\theta^i, \tilde{\theta}^{-i}) = 0$. Since $f^i(\tilde{\theta}) = 1, \tilde{\theta}^i < 0$, by ordinality. However, this violates PR.
(b) Since f satisfies CS, $\exists \theta \in \Theta$ such that $f(\theta) = \mathbf{0}$. Now we can repeat the first part of the argument in case (a) to get that $f^i(\theta^i, \tilde{\theta}^{-i}) = 0$. Since $f^i(\tilde{\theta}) = 1, \theta^i > 0$, by ordinality. However, this violates PR.□

We would like to note here that there is a related result in Dasgupta, Hammond and Maskin (1978, Theorem 3.3.1) for the domain where all strict orderings are admissible, although in a much more general framework. Their result does not require nonbossiness, since indifference between outcomes is ruled out.

5 A Gibbard-Satterthwaite-type Impossibility Result for Nonbossy Mechanisms

The focus of this section is whether a Gibbard-Satterthwaite-type result holds for the private goods domain when there is a single object to allocate. That is, we would like to know whether the strategyproof mechanisms used to implement the chosen SCF can be nondictatorial. First, we demonstrate by an example that a Gibbard-Satterthwaite-type impossibility is escaped in our context. The example is given for the case where there are at least three agents, since the two-agent case will be treated later.

Definition 9 An SCF f is *dictatorial* if $\exists i \in N$ such that $\forall \theta \in \Theta, \forall x \in X, f(\theta) = x$ only if $\forall y \in X, U(x^i, \theta^i) \geq U(y^i, \theta^i)$. Then i is called a *dictator* with respect to f.

Example 2 *A strategyproof, CS, and nondictatorial SCF for* $n \geq 3$.

Let $n = 3$. Let $(1, 2, 3)$ be a fixed ordering of the three agents. Consider the following SCF f. If there is an odd number of agents whose values are positive for the object, give the object to the first agent in the above fixed ordering whose value is positive. If there is an even number of agents whose values are positive for the object, give the object to the second agent in the above ordering whose value is positive. If all agents have negative values for the object then don't give the object to any one of them. Clearly, this SCF satisfies citizen sovereignty. On the basis of the following table containing all the different profiles based on preference orderings and the corresponding outcomes, it is easy to verify that f is also strategyproof and nondictatorial.

1	$\boxed{+}$	$+$	$+$	$-$	$\boxed{+}$	$-$	$-$	$-$
2	$+$	$\boxed{+}$	$-$	$+$	$-$	$\boxed{+}$	$-$	$-$
3	$+$	$-$	$\boxed{+}$	$\boxed{+}$	$-$	$-$	$\boxed{+}$	$-$

(In the table $+$ means a positive value and $-$ means a negative value for the object. The columns represent profiles and the rows represent agents. The outcome for each preference profile is indicated by the boxes.)

The example generalizes to $n > 3$. The SCF f, as defined above, clearly satisfies CS for any number of agents. Since no agent with a negative value will obtain the object, only agents with a positive value have any reason to manipulate. However, if an agent reports a negative value instead of a true positive one, which is the only way for her to change the allocation given f, then she will not obtain the object. Thus, f is strategyproof for $n > 3$ agents. To see that f is also nondictatorial for $n > 3$, consider the preference profile $(+, \ldots, +)$ for the n agents, i.e., where each agent's value is positive for the object. Then, if n is odd, agent 1 gets the object according to f, and if n is even, then agent 2 gets the object. Thus, agents $3, \ldots, n$ are not dictators. Notice, however, that if the preference profile $(+, \ldots, +, -)$ is reported then agent 2 gets the object if n is odd, and agent 1 gets the object if n is even. Therefore, agents 1 and 2 are not dictators with respect to f either.□

Notice, first, that the SCF does not only satisfy CS in the above example, but it is also Pareto-optimal. Secondly, note that f is bossy. For example, 2 is bossy when agents 1 and 3 both report $+$, and agent 3 is bossy when agents 1 and 2 both report $+$. The next proposition verifies that this observation is true in general, that is, any Pareto-optimal, strategyproof, and nonbossy mechanism is dictatorial.

Proposition 4 *If an SCF is strategyproof, nonbossy, and Pareto-optimal then it is dictatorial.*

Proof: First note that $j \in N$ is a dictator with respect to an SCF f if $f^j(\theta) = 1$ whenever $\theta^j > 0$, and if $f^j(\theta) = 0$ whenever $\theta^j < 0$ for $\theta \in \Theta$. Fix a strategyproof, nonbossy, and Pareto-optimal SCF f. Let $\theta \in \Theta$ be such that $\theta^i > 0, \forall i \in N$. Then $\exists j \in N$ such that $f^j(\theta) = 1$, by Pareto-optimality. Then $\forall i \neq j$ the ordinality of f (using Proposition 1) implies that $\forall \tilde{\theta}^i > 0, f^i(\tilde{\theta}^i, \theta^{-i}) = 0$, and PR implies (again, using Proposition 1) that $\forall \tilde{\theta}^i < 0, f^i(\tilde{\theta}^i, \theta^{-i}) = 0$. Therefore, $\forall i \neq j$, agent i cannot change the outcome for herself, as long as agent j's reported value is θ^j. Since f is nonbossy, $f^j(\theta^j, \tilde{\theta}^{-j}) = 1, \forall \tilde{\theta}^{-j} \in \Theta^{-j}$, including $\tilde{\theta}^i = \theta^i, \forall i \neq j$. Then $f^j(\tilde{\theta}) = 1, \forall \tilde{\theta}^j > 0, \forall \tilde{\theta}^{-j} \in \Theta^{-j}$, by ordinality. However, we also have $f^j(\tilde{\theta}) = 0, \forall \tilde{\theta}^j < 0, \forall \tilde{\theta}^{-j} \in \Theta^{-j}$, since f is Pareto-optimal. Therefore, j is a dictator with respect to f, and f is dictatorial.□

Given Proposition 3, Pareto-optimality can be replaced by citizen sovereignty in Proposition 5, which gives an analog to the Gibbard-Satterthwaite theorem for nonbossy mechanisms.

Corollary 1 *If an SCF is strategyproof, nonbossy, and CS then it is dictatorial.*

Corollary 1 shows that if no manipulations in the form of bossiness are allowed then the Gibbard-Satterthwaite theorem carries over to the situation where a single object is being allocated to selfish agents. Ritz (1983) proved a related result to Corollary 1. Theorem 3 of Ritz (1983) implies that any rational, nonbossy, strategyproof, and CS mechanism is dictatorial on the domain we examine. Our result does not require rationality. (See Muller and Satterthwaite (1986) on the importance of the rationality condition.)

Another immediate implication of Proposition 4, combined with Proposition 2, is the following.

Corollary 2 *If an SCF is ordinal, nonbossy, and Pareto-optimal then it is dictatorial.*

Now we turn to the case where there are only two agents, which is somewhat different from the general case. First, it is shown that for this special case any Pareto-optimal SCF is nonbossy.

Proposition 5 *If $n = 2$ and an SCF is Pareto-optimal then it is nonbossy.*

Proof: For any Pareto-optimal SCF f, we have $f^1(+,-) = 1, f^2(-,+) = 1$, and $f(-,-) = 0$ for possible profiles of the agents, based only on preference orderings. Furthermore, Pareto-optimality also requires that either $f^1(+,+) = 1$ or $f^2(+,+) = 1$. Now it is easy to check that f is nonbossy.□

If f is also strategyproof in the above proof then the ordinality of f implies that if $f^i(+,+) = 1$ for some profile $(+,+)$, where $i \in \{1,2\}$, then $f^i(+,+) = 1$ for any profile $(+,+)$. Thus, either 1 or 2 is a dictator with respect to f. Therefore, we can obtain the following two corollaries, both of which are also implied by earlier results.

Corollary 3 *If $n = 2$ and an SCF is strategyproof and Pareto-optimal then it is dictatorial.*

Corollary 3 follows from Propositions 4 and 5.

Corollary 4 *If $n = 2$ and an SCF is ordinal and Pareto-optimal then it is dictatorial.*

This corollary is implied by Corollary 3 and Proposition 2.

It follows from Corollary 1 that if an SCF is strategyproof, nondictatorial, and CS then it is bossy. Then Proposition 5 implies that the SCF is not Pareto-optimal when $n = 2$. An example of such an SCF f is given by $f(+,+) = (1,0), f(-,+) = (0,1)$, and $f(+,-) = f(-,-) = (0,0)$. Finally, it should be remarked that Corollary 4 does not generalize to the case where there is more than one object to allocate. This is illustrated by the SCF in Example 1, which is ordinal, Pareto-optimal, and nondictatorial. Example 1 is also a counterexample to the generalization of Corollary 2.

6 Serial Dictatorship

Unlike in the voting context, dictatorship alone does not characterize the set of strategyproof, nonbossy, and Pareto-optimal SCF's. When there are only two agents, a Pareto-optimal and dictatorial SCF is strategyproof and nonbossy. Therefore, we have a complete characterization for the two-agent case. This, however, is not true for more than two agents. Take, for example, a Pareto-optimal and dictatorial SCF such that agent 1 is the dictator with respect to f, and $f(0,8,7) = (0,1,0), f(0,7,7) = (0,0,1), f(-2,7,7) = (0,1,0)$. Clearly, f is not strategyproof, since agent 2 can manipulate it, and agent 1 is bossy, so it is not nonbossy either. We show next that the set of strategyproof, nonbossy, and Pareto-optimal SCF's is characterized by *serial dictatorships.* In our context,[3] a serial dictatorship is a mechanism in which the agents have priorities for the object in a predetermined order. That is, in a serial dictatorship, the object is awarded to the first agent in a fixed ordering of the agents who reports a positive value. Satterthwaite and Sonnenschein (1981) established a similar characterization result in a lot more general framework. However, they require the mechanisms to satisfy numerous differentiability conditions, and, although they don't require Pareto-optimality, some further conditions in addition to strategyproofness and nonbossiness are imposed on the mechanism in order to get serial dictatorships.

In the following, let σ denote a permutation of N.

Definition 10 An SCF f is a *serial dictatorship* if $\exists \sigma = (\sigma^1, \ldots, \sigma^n)$ such that $\forall \theta \in \Theta, \forall i \in N, f^{\sigma^i}(\theta) = 1$ if, and only if, $\theta^{\sigma^i} > 0$, and $\forall j \in N$, such that $j < i, \theta^{\sigma^j} < 0$. We then call σ the *hierarchy* associated with f.

Proposition 6 *An SCF is strategyproof, nonbossy, and Pareto-optimal if, and only if, it is a serial dictatorship.*

Proof: Suppose f is a serial dictatorship and the hierarchy associated with f is $\sigma = (1, \ldots, n)$. Then $\forall \theta \in \Theta, \forall i \in N, f^i(\theta) = 1$ implies that $\theta^i > 0$, and $\forall \theta \in \Theta, f(\theta) = \mathbf{0}$ implies that $\theta^i < 0, \forall i \in N$. Thus, f is Pareto-optimal. Now suppose that f is bossy.

[3] For more on serial dictatorships, see Satterthwaite and Sonnenschein (1981).

Then $\exists i, j \in N, \theta \in \Theta$, and $\tilde{\theta}^i \in \Theta^i$ such that $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i})$ and $f^j(\theta) \neq f^j(\tilde{\theta}^i, \theta^{-i})$. This implies that $f^i(\theta) = f^i(\tilde{\theta}^i, \theta^{-i}) = 0$. Let $f^j(\theta) = 1$ and $f^j(\tilde{\theta}^i, \theta^{-i}) = 0$, without loss of generality. By Pareto-optimality, $\theta^j > 0$, so if $j < i$ then $f^j(\tilde{\theta}^i, \theta^{-i}) = 1$. However, if $j > i$ then $\tilde{\theta}^i < 0$, so that $f^j(\tilde{\theta}^i, \theta^{-i}) = 1$ in this case, too. This proves that f is nonbossy. It is straightforward to verify that f is ordinal. Therefore, given Proposition 2, it follows that f is strategyproof.

Now we prove the converse. Let f be strategyproof, nonbossy, and Pareto-optimal. Suppose $\exists \theta, \tilde{\theta} \in \Theta, i, j \in N, i \neq j$, such that $f^i(\theta) = 1, f^j(\tilde{\theta}) = 1, \tilde{\theta}^i > 0$, and $\theta^j > 0$. Let $\bar{\theta}^t < 0, \forall t \in N, t \neq i, j$. Since f is ordinal, $f^t(\bar{\theta}^t, \theta^{-t}) = 0, \forall t \in N, t \neq i, j$ such that $\theta^t < 0$. Since f is PR, $f^t(\bar{\theta}^t, \theta^{-t}) = 0, \forall t \in N, t \neq i, j$ such that $\theta^t > 0$. Then, by nonbossiness, $f^i(\theta^i, \theta^j, \bar{\theta}^{-i,j}) = 1$. A similar argument shows that $f^j(\tilde{\theta}^i, \tilde{\theta}^j, \bar{\theta}^{-i,j}) = 1$. Given that f is Pareto-optimal, $\theta^i > 0$ and thus $f^i(\tilde{\theta}^i, \theta^j, \bar{\theta}^{-i,j}) = 1$, by ordinality. Similarly, $\tilde{\theta}^j > 0$ by Pareto-optimality, and so $f^j(\tilde{\theta}^i, \theta^j, \bar{\theta}^{-i,j}) = 1$, by ordinality. This is a contradiction. Therefore, $\forall \theta, \tilde{\theta} \in \Theta, \forall i, j \in N$, if $f^j(\tilde{\theta}) = 1, \tilde{\theta}^i > 0$, and $\theta^j > 0$ then $f^i(\theta) = 0$. Thus, there exists an ordering of the agents, $\sigma = (\sigma^1, \ldots, \sigma^n)$, such that $\forall \theta \in \Theta, \forall i \in N$, if $f^{\sigma^i}(\theta) = 1$ then $\forall j \in N, j \neq i$, such that $\theta^{\sigma^j} > 0$, we have $j > i$. Then $\forall j \in N$ such that $j < i$, we have $\theta^{\sigma^j} < 0$. By Pareto-optimality, if $f^{\sigma^i}(\theta) = 1, \theta^{\sigma^i} > 0$. Now suppose that for some $i \in N$ and $\theta \in \Theta$ we have $\theta^{\sigma^i} > 0, \forall j \in N$ such that $j < i$ we have $\theta^{\sigma^j} < 0$, and $f^i(\theta) = 0$. Then Pareto-optimality implies that $\exists j \in N$ such that $\theta^{\sigma^j} > 0$ and $f^{\sigma^j}(\theta) = 1$, where $j > i$. However, in this case $\theta^{\sigma^i} < 0$, which is a contradiction. Therefore, it follows that f is a serial dictatorship.□

Given Propositions 3 and 2, we have the following corollaries.

Corollary 5 *An SCF is strategyproof, nonbossy, and CS if, and only if, it is a serial dictatorship.*

Corollary 6 *An SCF is ordinal, nonbossy, and Pareto-optimal if, and only if, it is a serial dictatorship.*

Clearly, Proposition 6 does not hold for more complex allocation problems.

A natural and convenient decentralization of a serial dictatorship is to ask the first agent in the hierarchy associated with the mechanism whether she wants the object. If she turns it down then we ask the second agent, etc., until one of the agents takes the object, or until we have asked each agent. This decentralized mechanism greatly reduces the informational requirements, while retaining the properties of a serial dictatorship. Notice that the ordering of the agents is exogeneously given in a serial dictatorship. Thus, the supplier of the object (or society) may determine the order. It can be set up as a priority ranking, incorporating some criteria of justice or other known characteristics of the individuals.

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