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PREFERENCES OVER SOLUTIONS TO THE BARGAINING PROBLEM

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Abstract

There are several solutions to the Nash bargaining problem in the literature. Since various authors have expressed preferences for one solution over another, we find it useful to study preferences over solutions in their own right. We identify two sets of appealing axioms on such preferences that lead to unanimity in the choice of solution. Thus bargainers may be able to reach agreement on which solution to employ. Under the first set of axioms, the Nash solution is preferred to any other solution, while under the second set, a new solution, which we call the weighted linear solution, is best.

Preferences over Solutions to the Bargaining Problem*

Kim C. Border† Uzi Segal‡

1 Introduction

In the Nash [7, 8] approach to bargaining, a bargaining game is described by a pair (S, d) where $S \subset \mathbb{R}^2$ is compact and convex and $d \in S$. Elements of S are interpreted as vectors of the von Neumann–Morgenstern utilities of the two players. The point d is the disagreement point, that is, a vector of utilities that either player can unilaterally enforce. A solution is a function f that assigns to each game (S, d) a point in S. Of course many solutions are possible, but Nash proposed internal consistency conditions on the values of a solution across different games. The only solution to satisfy his axioms is called the Nash bargaining solution, and is defined by

$$\mathcal{N}(S, d) = \operatorname{argmax}_{x \in S, x \geqq d}(x_1 - d_1)(x_2 - d_2).$$

Other notions of consistency can be imposed on a solution. Kalai and Smorodinsky [6], for example, offered another set of axioms, with another notion of internal consistency, and got a different solution. It is therefore natural to ask, given that different notions of consistency exist, which is a better solution. We assume that there exist preferences over solutions satisfying certain sets of axioms and explore whether they admit maximal elements, and if so, what they are.

The assumptions we make on the preference order over solutions can be interpreted in more than one way. One interpretation is that the two bargainers hire an arbitrator to

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make choices for them. The arbitrator will of course have a preference order over solutions that embodies his own notions of fairness, and the bargainers will have not necessarily be able to know in advance what those preferences are. Furthermore, both parties must agree to bind themselves to arbitration. We can interpret our axioms on preferences as properties that both bargainers can accept in an arbitrator, subject to the constraint that both must agree to hire one. For instance, while my coauthor's preference over solutions is simply to maximize his own payoff, and my preference is to maximize mine, we both know that neither will agree to impose such a requirement on an arbitrator.

Another approach is to interpret the preference ordering as somehow representing the bargainers' actual preferences and values. In general, aggregation of preferences is a problem because people usually have different preferences. However, when all relevant agents agree that a certain policy is the best, this problem becomes moot. So an alternative interpretation of our assumptions that each of the two players actually has such preferences. These two preference relations do not have to be the same, but it may nevertheless happen that both players will agree that a certain solution is best. Under this interpretation, we must find grounds to justify an individual having such preferences.

Since a player must choose a solution without any knowledge of what game(s) he will actually play or who his opponent will be, the choice of a solution is a decision problem under risk (or perhaps ignorance). On the other hand we also expect that a player's preferences will embody his notions of fairness. Thus we impose three kinds of conditions on preferences: conditions taken from the literature on decision making under risk (such as the independence axiom or mixture symmetry); conditions about fairness (such as our disagreement indifference and unevenness indifference axioms); and a monotonicity axiom, which just asserts that utility is a good thing. We also impose the technical condition of continuity in order to guarantee a utility representation of the preferences.

We consider two sets of axioms on preferences, although many more are conceivable. The first set of axioms implies that the Nash solution is the best, and we give a full description of all preference relations satisfying this set. The second axiomatization implies another best solution, which we call the weighted linear solution, and here too we prove that it is the best element of all preference relations satisfying our axioms. We present these two models in Sections 2 and 3. We conclude with a further discussion of the interpretation of our results and compare them to other models in the literature in Section 4. The theorems are proved in the appendices.

2 The Nash Bargaining Solution

For our purposes, a two-person *bargaining game* is represented by a compact and convex subset S of $\mathbb{R}^2_+$ such that S is disposable, that is, $[x \in S$ and $x \geqq y] \Rightarrow y \in S$. (We use the following orders on vectors. $x \geqq y$ means $x_i \geqslant y_i$ for all i and $x \gg y$ means $x_i > y_i$ for all i). Each point $x = (x_1, x_2) \in S$ corresponds to a utility allocation for the two players in which player 1 receives x_1 and player 2 receives x_2. We assume throughout that the disagreement point is $(0,0)$.

Let $\mathcal{G}$ denote the set of all games that are included in $[0,K]\times[0,K]\subset\mathbb{R}^2_+$ and contain the point (κ,κ). All we require is that $\kappa>0$, but it may be as close to zero as one wishes. A *solution* is a function $f\colon\mathcal{G}\to\mathbb{R}^2_+$ satisfying $f(S)\in S$ for every $S\in\mathcal{G}$. In this section we assume that solutions are continuous in the Hausdorff metric, given by

$$\begin{aligned}\rho(S,T) &= \max\{\sup_{x\in S} d(x,T), \sup_{x\in T} d(x,S)\}\\ &= \inf\{\varepsilon>0 : S\subset N_\varepsilon(T) \text{ and } T\subset N_\varepsilon(S)\},\end{aligned}$$

where $d(x,T)=\inf_{y\in T}\|x-y\|$ and $N_\varepsilon(T)=\{x\in\mathbb{R}^2_+ : d(x,T)<\varepsilon\}$. The space $\mathcal{G}$ of games is compact in the Hausdorff metric.[1]

Since games are convex and disposable, the outer boundary determines the distance between two games. If each point on the boundary of S is within ε neighborhood of a boundary point of T and vice-versa, then the Hausdorff distance between S and T is no more than ε, and conversely.

A solution is continuous if it is a continuous function from $\mathcal{G}$ to $\mathbb{R}^2_+$. Both Nash's [7] and Kalai-Smorodinsky's [6] solutions are continuous. Let $\mathcal{F}$ denote the set of continuous solutions. Then $\mathcal{F}$ is a metric space under the metric

$$d(f,g)=\sup_{S\in\mathcal{G}}\|f(S)-g(S)\|.$$

This metric defines the topology of uniform convergence of solutions on $\mathcal{G}$, and $\mathcal{F}$ is separable [1, Lemma 3.72].

Two solutions can be mixed as follows. For $f,g\in\mathcal{F}$ and $\alpha\in[0,1]$, the solution $\alpha f+(1-\alpha)g$ assigns game $S\in\mathcal{G}$ the outcome $\alpha f(S)+(1-\alpha)g(S)$. All games S are assumed to be convex so this mixture is well defined. Since points in S are interpreted as vectors of von Neumann–Morgenstern utility levels, one can interpret the solution $\alpha f+(1-\alpha)g$ as a lottery that yields the solution f with probability α and the solution g with probability $1-\alpha$. This is because ex ante, each player's utility is the same under the mixed solution and under the lottery.

On $\mathcal{F}$ we assume the existence of a quasiorder $\succcurlyeq$, that is, $\succcurlyeq$ is reflexive, total, and transitive. Consider the following assumptions.

C (Continuity) $\succcurlyeq$ is a closed subset of $\mathcal{F}\times\mathcal{F}$.

[1]This is because the space $\mathcal{K}$ of compact convex subsets of $[0,K]\times[0,K]$ is a compact metric space under the Hausdorff metric [1, Theorem 5.43], and it is straightforward to show that $\mathcal{G}$ is a closed subset of $\mathcal{K}$. Incidentally, this is why we need the assumption that all games in $\mathcal{G}$ contain the point (κ,κ). The set of games for which there only exists some strictly positive point is σ-compact, but not compact, which creates problems later.

M (Monotonicity) If $f(S) \geqq g(S)$ for all $S \in \mathcal{G}$ and $f(T) \gg g(T)$ for some T, then $f \succ g$.

DI (Disagreement Indifference) If for all $S \in \mathcal{G}$, either $f_1(S) = 0$ or $f_2(S) = 0$ (or both), then $f \sim \mathbf{0}$ ($\mathbf{0}$ is the solution that gives both players always zero).

MS (Mixture Symmetry) For each $\alpha \in [0, 1]$,

$$f \sim g \text{ implies } \alpha f + (1-\alpha)g \sim \alpha g + (1-\alpha)f.^2$$

The continuity assumption is on the preference relation $\succcurlyeq$, and is different from the assumption that each of the solutions is a continuous function. Regarding monotonicity, note that if $f(T) \gg g(T)$, then continuity of solutions guarantees that $f \gg g$ on an open set of games. Also note that condition DI does not require that it is always the same player who receives zero. This condition may be rationalized on the following grounds. A player does not know whether his opponent is going to abide by the solution in any given game S. Clearly, the closer to zero his opponent's utility is (when better outcomes are possible) under a solution, the more likely his opponent is to deviate from the solution. This probability becomes a virtual certainty when his opponent's outcome is zero. This is because in that case his opponent has no incentive at all to play and may as well "punish" the first player. Thus any attempt to use a solution satisfying the hypotheses of condition DI is tantamount to using the zero solution.

One might argue that under this interpretation, the set $\mathcal{F}$ should not include solutions $f \neq \mathbf{0}$ such that for all S, $f_1(S)f_2(S) = 0$. In that case, we could adopt the following alternative version of DI: Let $f \in \mathcal{F}$ be such that for all S, $f(S) \gg 0$ and let $g: \mathcal{G} \to \mathbb{R}^2_+$ be such that for all S, $g_1(S)g_2(S) = 0$. Suppose further that $g^i \to g$ such that for each S, $g^i(S) \gg 0$. Then for $i \geqslant i^*$, $f \succcurlyeq g^i$. The original formulation of DI is preferable, as it is arguably easier to grasp. The DI condition may also be thought as a minimal fairness requirement, in that it requires some (even if highly unequal) positive division of the gains from cooperation, at least in some games.

If we interpret the axioms as characteristics of the arbitrator that both bargainers can accept, then player i's true preferences over solutions need not satisfy DI. Rather, the axiom represents his consent to the fact that he cannot impose his preferences on his opponent. Delegating authority to an arbitrator requires both players to agree on some ground rules, which by their nature ought to treat both bargainers symmetrically. DI satisfies this requirement.

The rationale for the MS condition is the same as the one used in decision theory (recall that $\alpha f + (1-\alpha)g$ is equivalent to the lottery that yields the solution f with probability α and the solution g with probability $1-\alpha$). This axiom claims that if $f \sim g$, then randomizing over the two may matter. However, given two states of the

[2] In [2], Chew, Epstein, and Segal refer to this property as strong mixture symmetry. However, they show that it is equivalent to what they called mixture symmetry, so we adopt this definition.

world with probabilities α and $1-\alpha$, it makes no difference whether one obtains the solution f in one state of the world or another. The mixture symmetry axiom is a weaker version of the usual independence axiom in decision theory (see Section 3), so any arguments in favour of independence also apply here. Mixture symmetry has the advantage that it allows for preferences that some choices be made according to chance (e.g., draft lotteries). However, it is also consistent with the betweenness assumption, namely that $f \sim g \Rightarrow f \sim \alpha f + (1-\alpha)g$ for all $\alpha \in [0,1]$.

Theorem 1 *If $\succcurlyeq$ satisfies conditions C, M, DI, and MS, then there is a measure μ on $\mathcal{G}$ with full support (that is, every open set in $\mathcal{G}$ has a positive measure), such that the utility function*

$$V(f) = \int_{\mathcal{G}} f_1(S) f_2(S)\, d\mu(S)$$

represents $\succcurlyeq$ on $\mathcal{F}$.

That is, solutions in $\mathcal{F}$ are ranked on a weighted average of the "Nash social welfare function," $w(x) = x_1 x_2$. This result immediately implies the next theorem.

Theorem 2 *If $\succcurlyeq$ satisfies conditions C, M, DI, and MS, then the Nash bargaining solution is the unique $\succcurlyeq$-best solution in $\mathcal{F}$.*

The formal proof of Theorem 1 is in Appendix A, but we shall explain the roles of the hypotheses here. In essence, we use a finite set $T_1, \ldots, T_n$ of games to construct a canonical preference relation on the subset $T_1 \times \ldots \times T_n$ of $\mathbb{R}^{2n}_+$. This preference relation is continuous and satisfies mixture symmetry. In [2] it is shown that for convex subsets of a finite dimensional space, any continuous preference order $\succcurlyeq$ satisfying mixture symmetry has a utility representation V of the following sort. The set may partitioned into three convex regions A, B and C, so that V is quadratic and quasiconcave on A, quadratic and quasiconvex on C, and satisfies betweenness on B. Furthermore $A \succcurlyeq B \succcurlyeq C$. Disagreement indifference rules out regions B and C. The problem is to extend this finite dimensional result to the infinite dimensional space $\mathcal{F}$. The canonical preference relation we construct has a quadratic utility, which we use to construct a probability measure on $\mathcal{G} \times \mathcal{G}$ which is supported by the pairs (T_i, T_j). We show that a unique limiting measure exists as $n \to \infty$. This allows us to show that there is a utility function of the form $V(f) = \int_{\mathcal{G}\times\mathcal{G}} f_1(S) f_2(T)\, d\mu(S,T)$. The assumption of disagreement indifference guarantees that the support of the limit measure is on the diagonal of $\mathcal{G} \times \mathcal{G}$, which gives rise then to the representation in Theorem 1. The full support is guaranteed by the monotonicity condition, which also figures in other subtle ways. The compactness of the space $\mathcal{G}$ is crucial to showing that a limiting measure exists, which is one reason we assume that (κ, κ) belongs to all games.[3] Of course, once we have the utility representation, since the Nash bargaining solution maximizes the integrand at each point, it is the best solution.

3 The Weighted Linear Solution

In this section we present a different set of assumptions about the preference relation $\succcurlyeq$, and get another (unique) solution that is better than any other solution. Here too, players may have different preferences $\succcurlyeq_i$, but will nevertheless agree what solution is best.

We assume in this section that all games S are strictly convex. That is, $x, y \in S \cap \mathbb{R}^2_{++}$ and $x \neq y$ imply that for all $\alpha \in (0,1)$, $\alpha x + (1-\alpha)y \in \mathrm{Int}(S)$. We no longer have to assume that $(\kappa, \kappa) \in S$, for all $S \in \mathcal{G}$, but we will assume that for all S, $S \cap \mathbb{R}^2_{++} \neq \varnothing$. Unlike the previous section, we assume here that a solution is any measurable (not necessarily continuous) function $f\colon \mathcal{G} \to \mathbb{R}^2$, and the set $\mathcal{F}$ is now the set of measurable solutions. Also, we replace mixture symmetry by the stronger independence assumption.

I (Independence) Fore every $f, g, h \in \mathcal{F}$ and every $\alpha \in (0,1]$,

$$f \succcurlyeq g \text{ if and only if } \alpha f + (1-\alpha)h \succcurlyeq \alpha g + (1-\alpha)h.$$

We use the following notation. For $S \in \mathcal{G}$, let

$$a^1(S) = \max\{x_1 : \exists x_2 \text{ such that } (x_1, x_2) \in S\},$$

and let

$$a^2(S) = \max\{x_2 : \exists x_1 \text{ such that } (x_1, x_2) \in S\}.$$

The three conditions of monotonicity, disagreement indifference, and independence are inconsistent. To see why, observe that by condition I, $f \sim g$ implies $f \sim \alpha f + (1-\alpha)g$ for all $\alpha \in [0,1]$. For every S, let $f(S) = (a^1(S), 0)$ and let $g(S) = (0, a^2(S))$. By DI, $f \sim g \sim \mathbf{0}$, hence $\frac{1}{2}f + \frac{1}{2}g \sim \mathbf{0}$. However, by monotonicity, $\frac{1}{2}f + \frac{1}{2}g \succ \mathbf{0}$. We therefore replace condition DI by a weaker condition, which we call unevenness indifference.

Disagreement indifference assumes that giving zero to one player is bad, and moreover, it does not matter who that player is. One possible rationale for this assumption is that if preferences over solutions reflect preferences for fairness, then solutions that always leave at least one player with nothing are worse than any other solution. Preferences over

[3] Otherwise, it might turn out that the utility of a solution f is determined by limits of values of $f(S)$ as S decreases to $\{(0,0)\}$. There are of course other assumptions that could be used to rule out this implausible sort of preference order, but we feel our choice is as good as any other.

such solutions are insensitive to the allocation the other player receives. The following assumption, which is weaker than disagreement indifference, assumes that the preference relation over solutions is sensitive to each player's payoff, even when the other player receives zero.

U (Unevenness Indifference) For every S, let $f^*(S) = (a^1(S), 0)$ and $g^*(S) = (0, a^2(S))$. Then $f^* \sim g^*$. Moreover, if $f \in \mathcal{F}$ is such that for all $S \in \mathcal{G}$, either $f(S) = (a^1(S), 0)$, or $f(S) = (0, a^2(S))$, then $f \succcurlyeq f^*$.

The first part of this condition states indifference between person 1 and person 2 being a dictator. The second part suggests that "mixed dictatorship," in the sense that each person is a dictator only for some games, cannot be worse than pure dictatorship.

Definition 1 *The weighted linear solution $W: \mathcal{G} \to \mathbb{R}^2$ is given by*

$$W(S) = \text{argmax}_{(x_1,x_2)\in S} a^2(S)x_1 + a^1(S)x_2.$$

The weighted linear solution enjoys the following property.

Theorem 3 *Suppose that the preference relation $\succcurlyeq$ satisfies conditions C, M, I, and U. Then the weighted linear solution is the $\succcurlyeq$-best solution in $\mathcal{F}$.*

This theorem says that the best solution is obtained by maximizing a linear social welfare function. Such social welfare functions were first axiomatized by Harsanyi [5].[4] Our result differs from the standard linear function in two significant ways. First, the coefficients are not fixed, and vary from one game to another. Second, we offer a justification of the different weights individuals' utilities receive. The weight of each individual is the maximal utility his opponent may receive. It suggests that society should pay higher consideration to the well being of its worse-off member.

It is also interesting to compare the weighted linear solutions with the one offered by Kalai and Smorodinsky [6]. Using our notation, their solution is the function $f \in \mathcal{F}$, given by $f(S) = \text{argmax}_{x\in S} \min\{a^2(S)x_1, a^1(S)x_2\}$. It is well known that the $\min\{\alpha_1 u_1, \alpha_2 u_2\}$ social welfare function is the extreme case of giving the weakest parts of society special consideration, whereas the additive function $\alpha_1 u_1 + \alpha_2 u_2$ is the extreme opposite case, where no one gets special consideration because of his position. This is true, however, only if the coefficients α_i are fixed. In our case, since they depend on the game S, we get the opposite results. The weighted linear solution gives more weight to the weaker of the two players (as measured by the maximal utility level they can achieve). The solution offered by Kalai and Smorodinsky does just the opposite, as it inflates the utility of the weaker player by multiplying it by the index of the other player.

[4]Observe that the Nash solution can be viewed as maximizing a quadratic social welfare function (Epstein and Segal [4]).

4 Some Remarks on the Literature

Recently, Rubinstein, Safra, and Thomson [9] analyzed the bargaining problem in light of the recent non-expected utility literature and presented an alternative model and axioms. They start with a certain set A of (physical) alternatives and a point D. Each of the two players has preferences over lotteries over elements of A. Given the utility functions u_1 and u_2 of the two players, points in A, and the point D, are transformed into points in $\mathbb{R}^2$. Formally, $d(u_1, u_2; D) = (u_1(D), u_2(D))$ and $S(u_1, u_2; A) = \{(u_1(a), u_2(a)) : a \in A\}$. The authors point out that there are two possible interpretations of the bargaining model.

According to the first, the set A and the point D are fixed, say $A = A^*$ and $D = D^*$. The bargaining problem is to decide what point in $S(u_1, u_2; A^*)$ to choose for each pair of utilities (u_1, u_2), given that the disagreement point is $d(u_1, u_2; D^*)$. If f is a solution to that problem, then a physical outcome of the solution is a point a in A^* such that $(u_1(a), u_2(a)) = f(S(u_1, u_2; A^*))$. Rubinstein, Safra, and Thomson use this interpretation. Note that if u_1 and u_2 are von Neumann–Morgenstern utility functions, then the invariance with respect to utility transformations axiom is almost trivial—the outcome (in A^*) should depend on preferences, and not on the particular choice of a utility function.

In this paper we adopt an alternative approach which holds the players (with their preferences and utility functions) fixed, and lets the set A vary (we assumed a fixed point of disagreement). Formally, let u_1^* and u_2^* be two given von Neumann–Morgenstern utility functions. What rule should be used to determine how much utility will each of the two players get for different sets A? This representation of the problem fits better into a social choice context. Note that Nash's axiom of invariance with respect to utility transformations does not fit into this interpretation, because it analyses rescaling of the utility functions, which in this approach are assumed to be fixed. (Nash's [7] axiom requires that if S' is obtained from S by taking linear transformations of the two utility functions, then the value of the solution at S' should be obtained by taking the same linear transformation of the value of the solution at S). Rescaling utility functions is possible in our approach, but leads to much weaker results. Suppose that instead of the utility function u_i, player i uses the utility function $\alpha_i u_i$. Then for each set A we get a new set of corresponding utility vectors. Formally, it defines an isomorphism $\mathcal{G} \mapsto \mathcal{G}_\alpha$, where for $S \in \mathcal{G}$, $S \mapsto S_\alpha = \{(\alpha_1 x_1, \alpha_2 x_2) : (x_1, x_2) \in S\}$. It is natural to require that possible solutions will be transformed in the same way. Formally, $\mathcal{F} \mapsto \mathcal{F}_\alpha$, where $f_\alpha(S_\alpha) = (\alpha_1 f_1(S), \alpha_2 f_2(S))$. Unlike Nash's axiom, this is a transformation of solutions (which are functions), and not values of a given solution for different games. All it says is that the choice of points out of the (different) sets A should depend on the players' preferences, and not on the chosen utility functions. A similar argument also implies that one should not assume symmetry. A symmetric set S depends on a specific choice of utility functions by the players and does not necessarily represent a true symmetric situation (see also [9]).

In this paper we assume that players have preferences over all possible solution concepts. A similar idea is employed by van Damme [10]. In his model players can choose a solution concept, and then bargain with their opponent, provided they adhere to their (perhaps different) solutions. Making some restrictions on the set of solutions players may adopt, he proves that Nash solution constitutes the unique equilibrium of the game induced by this procedure. In his approach, the Nash solution is the outcome of strategic behavior, and not necessarily the players' preferred solution. We are interested in the players' preferred solution, and we axiomatize preferences leading to agreement between players about such an optimal solution.

One motivation for this paper is the fact that both Nash's and Kalai-Smorodinsky's have a lot of appeal. Unfortunately, the two systems of axioms leading to these solutions are inconsistent. It is therefore natural to have preferences over such systems, and therefore over solutions. One might argue, however, that we replaced systems of axioms about solutions by systems of axioms on preferences over solutions. And since two such systems are presented above, we may once again face the same problem, namely, which one do we prefer. This objection is faulty. Nash and Kalai and Smorodinsky axiomatize the notion of consistency of a solution. Clearly, both offer reasonable definitions of consistency, and players may like both. We, on the other hand, axiomatize individual preferences over solutions. Standard models of consumer theory argue that each agent has one preference relation which is part of his characteristics, and is not an element of choice. Players may therefore prefer the axioms of Nash to those of Kalai–Smorodinsky, but it is meaningless for a player to prefer the axioms of Section 2 to those of Section 3.

Appendix A: Proof of Theorem 1

The proof of Theorem 1 is divided into numerous lemmas. Given $T \in \mathcal{G}$, for $S \in \mathcal{G}$ define $\alpha_T(S) = \max\{\lambda : \lambda T \subset S\}$. That is, $\alpha_T(S)T$ is the largest multiple of T that fits in S. In particular, $\alpha_T(T) = 1$. Also note that if $S, T \in \mathcal{G}$, then $\frac{\kappa}{K} \leqslant \alpha_T(S) \leqslant \frac{K}{\kappa}$. The mapping $S \mapsto \alpha_T(S)$ is continuous for any T.

We now temporarily fix T and for each $x \in T$ we construct a solution $\Phi_{T,x}$ by

$$\Phi_{T,x}(S) = \alpha_T(S)x.$$

By definition, $\alpha_T(S)x \in S$, so $\Phi_{T,x}$ is truly a solution. Furthermore, observe that $\Phi_{T,x}(T) = x$.

Lemma 1 *For every $S, T \in \mathcal{G}$ and for every $x \in T$, the solution $\Phi_{T,x}$ satisfies,*

$$\|x - \Phi_{T,x}(S)\| \leqslant 4\frac{K}{\kappa}\rho(S,T). \tag{1}$$

The proof of this lemma is a tedious exercise in elementary geometry, so we defer it to Appendix C.

The above construction induces an isomorphism between T and a subset of solutions.

Lemma 2 *The isomorphism $x \mapsto \Phi_{T,x}$ from T into $\mathcal{F}$ is continuous and one-to-one. It preserves mixtures in the sense that $\Phi_{T,\lambda x+(1-\lambda)y} = \lambda\Phi_{T,x} + (1-\lambda)\Phi_{T,y}$, and is monotonic (that is, if $x \gg y$, then $\Phi_{T,x}(S) \gg \Phi_{T,y}(S)$ for all S). Finally, if $x_i = 0$, then $\Phi_{T,x}$ gives player i the outcome 0 for all S.*

Proof. Continuity follows from continuity of $\alpha_T(S)$. Mixture preservation follows from

$$\begin{aligned}
(\lambda\Phi_{T,x} + (1-\lambda)\Phi_{T,y})(S) &= \lambda\Phi_{T,x}(S) + (1-\lambda)\Phi_{T,y}(S) \\
&= \lambda\alpha_T(S)x + (1-\lambda)\alpha_T(S)y \\
&= \alpha_T(S)(\lambda x + (1-\lambda)y) \\
&= \Phi_{T,\lambda x+(1-\lambda)y}(S).
\end{aligned}$$

The other claims are obvious. ■

Our aim is to approximate arbitrary solutions by convex combinations of solutions of this form. In order to do this, we need a partition of unity with some special properties. So given n, let $Q_n = \{z \in \mathbb{R}^n_+ : \text{at most one } z_i = 0\}$, and define $\lambda^n = (\lambda^n_1, \ldots, \lambda^n_n) \colon Q_n \to \mathbb{R}^n_+$ by

$$\lambda^n_i(z_1, \ldots, z_n) = \begin{cases} \dfrac{z_i^{-n}}{\sum_{j=1}^n z_j^{-n}} & \text{if each } z_j > 0 \\ 1 & \text{if } z_i = 0 \\ 0 & \text{if } z_j = 0 \text{ for some } j \neq i. \end{cases}$$

Observe that λ^n is continuous on Q_n and is a partition of unity, that is, $\sum_{i=1}^n \lambda^n_i(z) = 1$ for each $z \in Q_n$.

We now define the convex combinations. Given a vector $\mathbf{T}$ of distinct games $\mathbf{T} = (T_1, \ldots, T_n)$ and a vector $\mathbf{x}$ of points $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_i \in T_i$ for each i, define the solution $\Phi_{\mathbf{T},\mathbf{x}}$ by

$$\Phi_{\mathbf{T},\mathbf{x}}(S) = \sum_{i=1}^n \lambda^n_i(\rho(S, T_1), \ldots, \rho(S, T_n)) \Phi_{T_i, x_i}(S).$$

Observe that $\Phi_{\mathbf{T},\mathbf{x}}$ is continuous, $\Phi_{\mathbf{T},\mathbf{x}}(S) \in S$ for each S, and $\Phi_{\mathbf{T},\mathbf{x}}(S) = x_i$ whenever $S = T_i$.

Our next claim is that solutions of this form are dense in $\mathcal{F}$. For this, fix a countable dense subset $\{T_1, T_2, \ldots\}$ of $\mathcal{G}$. Let f be a fixed solution and define the solution

$$\Phi^f_n = \Phi_{T_1, \ldots, T_n; f(T_1), \ldots, f(T_n)}.$$

Observe that by construction, $\Phi^f_n(T_i) = f(T_i)$, $i = 1, \ldots, n$.

Lemma 3 *For each $f \in \mathcal{F}$, $\Phi^f_n \xrightarrow[n \to \infty]{} f$ uniformly on $\mathcal{G}$.*

Proof: Since f is continuous and $\mathcal{G}$ is compact, f is uniformly continuous on $\mathcal{G}$. Therefore for each $\varepsilon > 0$, there is some $\delta(\varepsilon) > 0$ such that $\rho(S, T) < \delta(\varepsilon)$ implies $\|f(S) - f(T)\| < \varepsilon$.

To simplify notation, suppress the f and just write Φ_n. Now let $\varepsilon > 0$ be given. Set $\eta = \min\{\frac{\kappa}{12K}\varepsilon, \delta(\frac{\varepsilon}{3})\}$. Choose N_0 large enough so that for $n \geqslant N_0$, for every $T \in \mathcal{G}$, there is some $T_i \in \{T_1, \ldots, T_n\}$ with $\rho(T, T_i) < \frac{\eta}{2}$. (This can be done since $\{T_1, T_2, \ldots\}$ is dense.) Choose N_1 large enough so that for $n \geqslant N_1$, $\frac{n}{2^n}\sqrt{2}K < \frac{\varepsilon}{3}$, and set $N = \max\{N_0, N_1\}$. Choose an arbitrary $S \in \mathcal{G}$. Then recalling that $\sum_{i=1}^n \lambda^n_i = 1$, we get

$$\begin{aligned} &\|f(S) - \Phi_n(S)\| \\ &= \|f(S) - \sum_{i=1}^n \lambda^n_i(\rho(S, T_1), \ldots, \rho(S, T_n)) \Phi_{T_i, f(T_i)}(S)\| \\ &= \|\sum_{i=1}^n \lambda^n_i(\rho(S, T_1), \ldots, \rho(S, T_n))(f(S) - \Phi_{T_i, f(T_i)}(S))\| \\ &\leqslant \sum_{i=1}^n \lambda^n_i(\rho(S, T_1), \ldots, \rho(S, T_n)) \|f(S) - \Phi_{T_i, f(T_i)}(S)\| \qquad (2) \end{aligned}$$

Now

$$\|f(S) - \Phi_{T_i,f(T_i)}(S)\| \leqslant \|f(S) - f(T_i)\| + \|f(T_i) - \Phi_{T_i,f(T_i)}(S)\|. \tag{3}$$

So break up the sum in (2) into two parts. Let $A = \{i : \rho(S, T_i) < \eta\}$ and $B = A^c$. Then for $i \in A$, equation (3) implies

$$\|f(S) - \Phi_{T_i,f(T_i)}(S)\| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

This is because the first term on the right hand side of (3) is less than or equal to $\frac{\varepsilon}{3}$ by the definition of $\delta(\varepsilon)$, and the second term comes from Lemma 1. Also observe that there is some $i^* \in A$ with $\rho(S, T_{i^*}) < \frac{\eta}{2}$. Regarding the terms in B, since $S \in \mathcal{G}$ is bounded above by (K, K), the distance $\|f(S) - \Phi_{T_i,f(T_i)}(S)\|$ is bounded above by $\sqrt{2}K$, which is independent of S. Then using the definition of λ^n in (2), we have

$$\begin{aligned}
&\|f(S) - \Phi_n(S)\| \\
\leqslant\ & \sum_{i=1}^{n} \lambda_i^n(\rho(S, T_1), \ldots, \rho(S, T_n))\|f(S) - \Phi_{T_i,f(T_i)}(S)\| \\
\leqslant\ & \underbrace{\frac{\sum_{i\in A} \rho(S, T_i)^{-n}}{\sum_{i=1}^{n} \rho(S, T_i)^{-n}}}_{\leqslant 1}(\frac{\varepsilon}{3} + \frac{\varepsilon}{3}) + \frac{\sum_{i\in B} \rho(S, T_i)^{-n}}{\sum_{i=1}^{n} \rho(S, T_i)^{-n}}\sqrt{2}K \\
\leqslant\ & (\frac{\varepsilon}{3} + \frac{\varepsilon}{3}) + \frac{n\eta^{-n}}{\sum_{i=1}^{n} \rho(S, T_i)^{-n}}\sqrt{2}K \\
\leqslant\ & (\frac{\varepsilon}{3} + \frac{\varepsilon}{3}) + \frac{n\eta^{-n}}{(\frac{\eta}{2})^{-n} + \sum_{i\neq i^*} \rho(S, T_i)^{-n}}\sqrt{2}K \\
\leqslant\ & (\frac{\varepsilon}{3} + \frac{\varepsilon}{3}) + \frac{n\eta^{-n}}{(\frac{\eta}{2})^{-n}}\sqrt{2}K \\
\leqslant\ & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{n}{2^n}\sqrt{2}K \\
\leqslant\ & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}
\end{aligned}$$

for $n \geqslant N$. Thus Φ_n converges uniformly to f. ■

We next define an ordering $\succeq_n$ on $T_1 \times T_2 \cdots \times T_n$, where $T_1, \ldots, T_n$ are the first n elements of the fixed dense set $\{T_1, T_2, \ldots\}$, by

$$\mathbf{x} \succeq_n \mathbf{y} \iff \Phi_{T_1,\ldots,T_n;\mathbf{x}} \succcurlyeq \Phi_{T_1,\ldots,T_n;\mathbf{y}}$$

It follows from Lemma 2 that the relation $\succeq_n$ on $T_1 \times T_2 \cdots \times T_n$ is a monotonic convex continuous quasiorder, and satisfies mixture symmetry.

In the sequel we say that the function W of n variables is quadratic if it is of the form

$$W(x_1, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} b_ix_i + c$$

Lemma 4 *The preference $\succeq_n$ is represented by a utility function of the form*

$$V_n(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_{i1}x_{j2})p_{ij},$$

where $p \geqq 0$ and $\sum_{i,j=1}^{n} p_{ij} = 1$.

Proof: It follows from arguments in [2, 3], that $\succeq_n$ has a continuous utility V_n on $T_1 \times \cdots \times T_n$, which can be partitioned into three convex regions A, B and C, so that V is quadratic and quasiconcave on A, quadratic and quasiconvex on C, and satisfies betweenness on B. Furthermore $A \succeq_n B \succeq_n C$. We will rule out areas B and C by showing that the coordinate axes are indifference sets, which is compatible only with a quasiconcave quadratic representation.

Let $\mathbf{x}^i \in T_1 \times T_2 \cdots \times T_n$ satisfy $x_{m2} = 0$ for all $m = 1, \ldots, n$, $x_{m1} = 0$ for all $m \neq i$, and $x_{i1} = \max\{x_1 : x \in T_i\} \geqslant \kappa > 0$. It follows that $\Phi_{\mathbf{T},\mathbf{x}^i}(S)_2 = 0$ for all S. So by condition DI, $\Phi_{\mathbf{T},\mathbf{x}^i} \sim \Phi_{\mathbf{T},0}$, which implies that the $i1$-axis is a $\succeq_n$-indifference set. This shows that every axis is an indifference set, which rules out the regions B and C.

Thus $\succeq_n$ has a representation of the form

$$V_n(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{2}\sum_{\ell=1}^{2}(x_{ik}x_{j\ell})p_{ikj\ell} + \sum_{i=1}^{n}\sum_{k=1}^{2} x_{ik}b_{ik}, \tag{4}$$

where p satisfies $p \geqq 0$ (because of monotonicity) and $\sum_i \sum_j \sum_k \sum_\ell p_{ikj\ell} = 1$. (The sum cannot be zero, since V_n is not linear). We now show that the b_{ik} terms in the representation in equation (4) are zero. Let $\mathbf{x}^i$ be as above. Then by condition DI, $\Phi_{\mathbf{T},\mathbf{x}^i} \sim \Phi_{\mathbf{T},0}$, so $V_n(\mathbf{x}^i) = 0$, for all values of x_{i1}. This implies $p_{i1i1} = b_{i1} = 0$. Similar arguments show that each $p_{jkjk} = b_{jk} = 0$.

Next we show that if $i \neq j$ and $k = \ell$, then $p_{ikj\ell} = 0$. Let $\mathbf{x}^{ij} \in T_1 \times T_2 \cdots \times T_n$ satisfy $x^{ij}_{m2} = 0$ for $m = 1, \ldots, n$, $x^{ij}_{m1} = 0$ for $m \notin \{i, j\}$, $x^{ij}_{m1} = \max\{x_1 : x \in T_m\}$ for $m \in \{i, j\}$. Again $\Phi_{\mathbf{T},\mathbf{x}^{ij}}(S)_2 = 0$ for all S, so as above $V_n(\mathbf{x}^{ij}) = 0$, which implies $p_{i1j1} = 0$. Similarly each $p_{i2j2} = 0$. ■

This construction defines a discrete probability measure π_n on $\mathcal{G} \times \mathcal{G}$ by $\pi_n(\{(T_i, T_j)\}) = p_{ij}$ for each (i, j). This measure has the property that

$$V_n(x) = \int_{\mathcal{G}\times\mathcal{G}} f_1(S)f_2(T)\, d\pi_n(S,T) \text{ whenever } f_k(T_i) = x_{ik}$$

for $i = 1, \ldots, n$ and $k = 1, 2$. So abuse notation slightly and define $V_n \colon \mathcal{F} \to \mathbb{R}$ by

$$V_n(f) = \int_{\mathcal{G}\times\mathcal{G}} f_1(S)f_2(T)\, d\pi_n(S,T)$$

and note that $V_n(f) = V_n(\Phi_n^f)$. This means that

$$V_n(f) \geqslant V_n(g) \iff (f(T_1), \ldots, f(T_n)) \succeq_n (g(T_1), \ldots, g(T_n)).$$

Since $\mathcal{G}$ is compact, the set of probability measures on $\mathcal{G} \times \mathcal{G}$ is compact (in the topology of weak convergence of measures [1, Theorem 12.10]), so there is a subsequence of π_n converging to a limit π, which is also a probability measure on $\mathcal{G} \times \mathcal{G}$. Define

$$V(f) = \int_{\mathcal{G} \times \mathcal{G}} f_1(S) f_2(T) \, d\pi(S, T).$$

Note that if $f_m \to f$ uniformly, then $\int f_{m1} f_{m2} \, d\pi_{n_m} \to \int f_1 f_2 \, d\pi$, see [1, Corollary 12.6]. That is, $V_{n_m}(f_m) \to V(f)$. In particular, $V_{n_m}(\Phi_{n_m}^f) \to V(f)$.

Lemma 5 *If $f \succcurlyeq g$, then $V(f) \geqslant V(g)$. In particular, $f \sim g$ implies $V(f) = V(g)$.*

Proof. First suppose $f \succ g$. Since $\Phi_n^f \to f$ and $\Phi_n^g \to g$, for large enough n we have $\Phi_n^f \succ \Phi_n^g$, so $V_n(\Phi_n^f) > V_n(\Phi_n^g)$. Therefore $V(f) \geqslant V(g)$. By continuity of V and $\succcurlyeq$, we have $f \succcurlyeq g$ implies $V(f) \geqslant V(g)$. ■

Lemma 6 *The limit measure π is supported by the diagonal $\Delta = \{(S, S) : S \in \mathcal{G}\}$ of $\mathcal{G} \times \mathcal{G}$.*

Proof. Let $G \times H \subset \mathcal{G} \times \mathcal{G}$ be an open region with $\Delta \cap (G \times H) = \varnothing$ and $\pi(G \times H) > 0$. Note that this implies that $G \cap H = \varnothing$. Pick $(\bar{S}, \bar{T}) \in G \times H$. Let $\lambda: \mathcal{G} \to \mathbb{R}$ be a continuous function satisfying $\lambda(\bar{S}) = 1$, $1 > \lambda(U) > 0$ for all $U \in G$ with $U \neq \bar{S}$ and $\lambda(U) = 0$ for all $U \in G^c$. Similarly let $\alpha: \mathcal{G} \to \mathbb{R}$ be a continuous function satisfying $\alpha(\bar{T}) = 1$, $1 > \alpha(U) > 0$ for all $U \in H$ with $U \neq \bar{T}$ and $\alpha(U) = 0$ for all $U \in H^c$. Let h be the solution that gives everything to 1, that is $h_1(U) = \max\{x_1 : x \in U\}$ and $h_2(S) = 0$ for all S. Similarly let g give everything to 2. Set $f(U) = \lambda(U)h(U) + \alpha(U)g(U)$. Then f is a continuous solution with $\{(S, T) : f_1(S)f_2(T) > 0\} = G \times H$. But by condition DI, $f \sim 0$, so by Lemma 5, $V(f) = V(0) = 0$. Thus $\pi(G \times H) = 0$. This implies $\pi(\Delta) = 1$. ■

Set $\mu(G) = \pi(G \times G)$ for every Borel subset G of $\mathcal{G}$. Then clearly

$$V(f) = \int_{\mathcal{G}} f_1(S) f_2(S) \, d\mu(S).$$

Lemma 7 *The measure μ has full support.*

Proof: Let G be an open subset of $\mathcal{G}$. Let N denote the Nash bargaining solution (any positive solution will do), and let $\lambda: G \to \mathbb{R}$ be a continuous function satisfying $1 \geqslant \lambda(S) > 0$ for $S \in G$ and $\lambda(S) = 0$ for $S \in G^c$. Let g be the solution defined by $g(S) = \lambda(S)N(S)$. Then by monotonicity, $g \succ 0$. Thus for some $\varepsilon > 0$ small enough the solution f with $f_k(S) = \varepsilon \leqslant \kappa$, $k = 1, 2$ is a solution on $\mathcal{G}$ satisfying $g \succ f$. Therefore by Lemma 5, $V(g) \geqslant V(f)$. Now $V_n(f) \to V(f)$, and $V_n(f) = \varepsilon^2$ for all n, so $V(g) \geqslant \varepsilon^2 > 0$, which implies $\mu(G) > 0$. Thus μ has full support. ■

We can now finally prove Theorem 1.

Proof: [Proof of Theorem 1] We already know that $f \succcurlyeq g$ implies $V(f) \geqslant V(g)$. It only remains to show that $f \succ g$ implies $V(f) > V(g)$. Now if $f \succ g$, it follows by condition DI that $f_1(S)f_2(S) > 0$ for some S. (Otherwise $f \sim \mathbf{0}$.) Also, for some $1 > \alpha \geqslant 0$, we have $\alpha f \sim g$. But since μ has full support, we get $V(f) > V(\alpha f) \geqslant V(g)$, so $V(f) > V(g)$. ■

Appendix B: Proof of Theorem 3

Lemma 8 *Let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{G}$ be such that $\mathcal{G}_2 = \mathcal{G} \backslash \mathcal{G}_1$. Let $f^1, g^1, f^2, g^2 \in \mathcal{F}$ such that on $\mathcal{G}_1$, $f^i = g^i$, $i = 1, 2$, and on $\mathcal{G}_2$, $f^1 = f^2$ and $g^1 = g^2$. Then $f^1 \succcurlyeq f^2 \Longleftrightarrow g^1 \succcurlyeq g^2$.*

Proof: For every S, $\frac{1}{2}f^1 + \frac{1}{2}g^2 = \frac{1}{2}f^2 + \frac{1}{2}g^1$. By condition I, $f^1 \succcurlyeq f^2$ iff $\frac{1}{2}f^1 + \frac{1}{2}g^1 \succcurlyeq \frac{1}{2}f^2 + \frac{1}{2}g^1 = \frac{1}{2}f^1 + \frac{1}{2}g^2$ iff $g^1 \succcurlyeq g^2$. ■

Lemma 9 *Let f^* and g^* be as in condition U, let $\mathcal{G}_1, \mathcal{G}_2$ be as in Lemma 8, and let $f \in \mathcal{F}$ such that on $\mathcal{G}_1$, $f = f^*$ and on $\mathcal{G}_2$, $f = g^*$. Then $f^1 \sim f^*$.*

Proof: Define $f^1 = f^*$, $g^2 = g^*$, and $g^1 = f$. Also, let $f^2 = g^*$ on $\mathcal{G}_1$ and $f^2 = f^*$ on $\mathcal{G}_2$. By Lemma 8, $f^* \succcurlyeq f^2 \Longleftrightarrow f \succcurlyeq g^*$. By condition U, $f^2, f \succcurlyeq f^* \sim g^*$. Hence $f \sim f^*$. ■

For $S \in \mathcal{G}$ and $x \in S$, let L be the line through $(0,0)$ and x and let M be the chord $[(a^1(S), 0), (0, a^2(S))]$. Denote the intersection point of these two lines d. Denote the length of the chord $[a, b]$ by $\theta[a, b]$ and define $\alpha = \alpha(x, S) = \theta[(0,0), x]/\theta[(0,0), d]$, $\beta = \beta(x, S) = \theta[(0, a^2(S)), d]/\theta[M]$. Clearly, $\alpha \in [0, 2]$ and $\beta \in [0, 1]$. In the sequel, for $a \in \mathbb{R}$, $[a]$ denotes the largest integer not bigger than a.

For $x \in S$, define $\langle x \rangle^n$ and $\langle d \rangle^n$ by

1. $\langle d \rangle^n$ is in M, and satisfies $\theta[(0, a^2(S)), \langle d \rangle^n]/\theta[M] = [n\beta(x, S)]/n$.

2. $\langle x\rangle^n$ is on the line through $(0,0)$ and $\langle d\rangle^n$, and satisfies $\theta[(0,0),\langle x\rangle^n]/\theta[(0,0),\langle d\rangle^n] = [n\alpha(x,S)]/n$.

Of course, $\langle x\rangle^n$ is not necessarily in S. Define $\mathcal{F}_n$ to be the set of solutions in $\mathcal{F}$ such that for every $S \in \mathcal{G}$, $\langle f(S)\rangle^n \in S$. Observe that $\mathcal{F}_n \subset \mathcal{F}_{k\cdot n}$ for $k = 1, \ldots$.

Define a function $\varphi^n: \mathcal{F}_n \to \mathcal{F}_n$ by $\varphi^n(f)(S) = \langle f(S)\rangle^n$. Denote $\langle f\rangle^n = \varphi^n(f)$.

Definition 2 *The two solutions $f, g \in \mathcal{F}$ are said to be equivalent to each other if for every $S \in \mathcal{G}$, $a^2(S)f_1(S)+a^1(S)f_2(S) = a^2(S)g_1(S)+a^1(S)g_2(S)$. This relation is denoted fIg.*

Fact 1 *Let $f, g \in \mathcal{F}_n$. If fIg, then $\langle f\rangle^n I \langle g\rangle^n$.*

Fact 2 *Let $f \in \mathcal{F}_n$. Then $\langle f\rangle^{k\cdot n} \xrightarrow[k\to\infty]{} f$.*

Lemma 10 *Let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{G}$ such that $\mathcal{G}_2 = \mathcal{G}\backslash\mathcal{G}_1$ and let $f, g \in \mathcal{F}$ such that on $\mathcal{G}_1$, $f = g$, for $S, S' \in \mathcal{G}_2$, $\alpha^* := \alpha(f(S),S) = \alpha(f(S'),S') = \alpha(g(S),S) = \alpha(g(S'),S')$, $\beta_1^* := \beta(f(S),S) = \beta(f(S'),S')$ and $\beta_2^* := \beta(g(S),S) = \beta(g(S'),S')$. Then $f \sim g$.*

Proof: Suppose first that $\alpha^* \leqslant 1$. By Lemma 9, $f^* \sim \tilde{g}$, where $\tilde{g} = f^*$ on $\mathcal{G}_1$, and $\tilde{g} = g^*$ on $\mathcal{G}_2$. By condition I, $f^1 := \beta_1^* f^* + (1-\beta_1^*)\tilde{g} \sim g^1 := \beta_2^* f^* + (1-\beta_2^*)\tilde{g}$. As before, let $\mathbf{0}$ be the zero solution. Again by condition I, $f^2 := \alpha^* f^1 + (1-\alpha^*)\mathbf{0} \sim g^2 := \alpha^* g^1 + (1-\alpha^*)\mathbf{0}$. On $\mathcal{G}_2$, $f^2 = f$ and $g^2 = g$. (Clearly, if $\alpha(f(S),S) = \alpha(g(S),S)$, and $\beta(f(S),S) = \beta(g(S),S)$, then $f(S) = g(S)$). Also, on $\mathcal{G}_1$, $f^2 = g^2$. Therefore, by Lemma 8, $f \sim g$.

Suppose now that $\alpha^* > 1$. Note that on $\mathcal{G}_1$, $f^1 = g^1$. Let $f^3, g^3 = \alpha^* f^* + (1-\alpha^*)\mathbf{0}$ on $\mathcal{G}_1$, and on $\mathcal{G}_2$, let $f^3 = f^1$ and $g^3 = g^1$. By Lemma 8, $f^3 \sim g^3$. Also, let $f^4 = g^4 = f^*$ on $\mathcal{G}_1$, and on $\mathcal{G}_2$, let $f^4 = f$ and $g^4 = g$. Now $f^3 = (1/\alpha^*)f^4 + [1-(1/\alpha^*)]\mathbf{0}$, while $g^3 = (1/\alpha^*)g^4 + [1-(1/\alpha^*)]0$, hence $f^4 \sim g^4$. Again by Lemma 8, $f \sim g$. ■

Lemma 11 *If $\langle f\rangle^n I \langle g\rangle^n$, then $\langle f\rangle^n \sim \langle g\rangle^n$.*

Proof: let

- $\mathcal{G}_i^1 = \{S : \alpha(\langle f\rangle^n(S),S) = \alpha(\langle g\rangle^n(S),S) = i/n\}$, $i = 0,\ldots,2n$
- $\mathcal{G}_j^2 = \{S : \beta(\langle f\rangle^n(S),S) = j/n\}$, $j = 0,\ldots,n$
- $\mathcal{G}_k^3 = \{S : \beta(\langle g\rangle^n(S),S) = k/n\}$, $k = 0,\ldots,n$

Let $m^* = (2n+1)(n+1)^2$. For $m = 1, \ldots, m^*$, let $\mathcal{G}_m = \mathcal{G}_i^1 \cap \mathcal{G}_j^2 \cap \mathcal{G}_k^3$, where $m = i(n+1)^2 + j(n+1) + k + 1$. Of course, for some m, $\mathcal{G}_m$ may be empty. For $m = 0, \ldots, m^*$, define $f^m = g^m = \mathbf{0}$ on $\cup_{i>m}\mathcal{G}_i$, and on $\cup_{i\leqslant m}\mathcal{G}_i$, $f^m = \langle f \rangle^n$ and $g^m = \langle g \rangle^n$. Note that $f_{m^*} = \langle f \rangle^n$ and $g_{m^*} = \langle g \rangle^n$. We prove by induction that for all $m = 0, \ldots, m^*$, $f^m \sim g^m$. The claim is trivially true for $m = 0$. Suppose it holds for m, and prove for $m+1$.

Define a solution h such that on $\cup_{i \neq m+1}\mathcal{G}_i$, $h = f^{m+1}$, and on $\mathcal{G}_{m+1}$, $h = g^{m+1}$. By Lemma 10, $h \sim f^{m+1}$. Also, it follows by Lemma 8 and the induction hypothesis that $h \sim g^{m+1}$. Therefore, $f^{m+1} \sim g^{m+1}$. ■

Suppose fIg. By Fact 1, $\langle f \rangle^{k \cdot n} I \langle g \rangle^{k \cdot n}$, therefore $\langle f \rangle^{k \cdot n} \sim \langle g \rangle^{k \cdot n}$. Note also that $\mathcal{F}_n \to \mathcal{F}$. By Fact 2 and the continuity of $\succeq$, it follows that $f \sim g$. Theorem 3 now follows from condition M.

Appendix C: Proof of Lemma 1

We start with some simple facts.

Fact 3 *If $\alpha_T(S) < 1$, then $\alpha_T(S \cap T) = \alpha_T(S)$.*

Fact 4 $\rho(S \cap T, T) \leqslant \rho(S, T)$.

Proof. We have to show that for $x \in T$, $d(x, S \cap T) \leqslant d(x, S)$. Let $x \in T \backslash S$ and $y \in S$ such that $d(x, S) = d(x, y)$. If $y \in S \cap T$, then $d(x, S \cap T) = d(x, S)$. Suppose $y \in S \backslash T$. Then since S is disposable, and by the triangle inequality, $d(x, S) < d(x, y)$, a contradiction. ■

We now turn to the proof of the Lemma 1. We break the analysis into cases. For each of these cases we show that equation (1) is satisfied.

Case 1: We consider first games S and T with $\alpha_T(S) < 1$. Let $x^* \in T$ be such that $\|x - \Phi_{T,x}(S)\| = (1 - \alpha_T(S))\|x\|$ is maximized over T at x^*. Note that $x^* \in \text{argmax}_{x \in T}\|x\|$ and that x^* is on the boundary of T. We shall replace S and T by sets S^* and T^* satisfying the following three requirements.

1. $x^* \in T^*$.
2. $\alpha_{T^*}(S^*) = \alpha_T(S)$.
3. $\rho(S^*, T^*) \leqslant \rho(S, T)$.

To simplify notation, replace $\alpha_T(S)$ by simply α. Also, for $x, y \in \mathbb{R}^2$, let $\ell(x,y)$ denote the line through these two points. The slope of a line H is denoted $\sigma(H)$.

By Facts 3 and 4, the three requirements are satisfied if $T^* = T$ and $S^* = S \cap T$. We can thus assume that $S \subset T$. By the definition of $\alpha_T(S)$, there is a point $y \in T$ such that αy is on the outer boundary of S. Of course, y is on the boundary of T. Assume without loss of generality that $y_1 \leqslant x_1^*$ and $y_2 \geqslant x_2^*$.

Let H be a supporting line to S at αy. Denote the area bounded by H and the two axes by S_1, and let $S_2 = T \cap S_1$. Note that $\alpha_T(S_2) = \alpha$, but $\rho(S_2, T) \leqslant \rho(S, T)$.

Let T^* be the minimal convex disposable set containing y, x^*, and (κ, κ), and let $S^* = S_2 \cap T^*$. Clearly, $\alpha_{T^*}(S^*) = \alpha$, but $\rho(S^*, T^*) \leqslant \rho(S, T)$. Observe that for all x on the boundary of T^*, $d(x, S^*) \leqslant d(y, S^*)$. This is because $\alpha x^* \in S^*$, hence $\sigma(H) \geqslant \sigma(\ell(y, x^*))$. On the other hand, the boundary of T^* to the left of y is either the horizontal segment $A = [(0, y_2), y]$ $(y_2 \geqslant \kappa)$, or it is the two segments $B = [(0,\kappa), (\kappa,\kappa)]$ and $C = [(\kappa,\kappa), y]$. In the first case, no point in A can be further away from H than y. In the second case, since $(\kappa, \kappa) \in S^*$, it follows that $B \subset S^*$, and since $\sigma(H) \leqslant \sigma(\ell((\kappa,\kappa), \alpha y))$, it follows that $y = \operatorname{argmax}_{x \in C} d(x, H)$. We thus obtain

$$\rho(S^*, T^*) = d(y, S^*) \tag{5}$$

There are now four subcases.

Subcase 1.a: Suppose first that $\sigma(H) < -1$ and $y_1 \geqslant \kappa$. Now $\sigma(\ell(y, x^*)) \leqslant \sigma(H) < -1$, so given that $x^* \in \operatorname{argmax}_{x \in T^*} \|x\|$ it follows that y lies in the region bounded by the quarter circle of radius $\|x^*\|$ and the line of slope -1 through x^*. This line intersects this arc in exactly two places: x^* and its transpose (x_2^*, x_1^*) (which may coincide, in which case $x_1^* = x_2^*$ and $y = x^*$). In particular, $x_1^* \geqslant x_2^*$ and since y lies to the right of the transpose (x_2^*, x_1^*), we have $y_1 \geqslant x_2^*$. It also follows that $d(y, S^*) \geqslant (1-\alpha) y_1$, so by equation (5) we have $(1-\alpha) y_1 \leqslant \rho(S^*, T^*)$. Using $y_1 \geqslant \kappa$, and these other facts we get

$$\begin{aligned}
\|x^* - \Phi_{T,x^*}(S)\| &= (1-\alpha)\|x^*\| \\
&\leqslant (1-\alpha)\|(K, x_2^*)\| \\
&\leqslant (1-\alpha)\|(y_1, K)\| \\
&\leqslant \|(1, K/y_1)\| \rho(S^*, T^*) \\
&\leqslant \|(1, K/\kappa)\| \rho(S^*, T^*) \\
&\leqslant \sqrt{2}\frac{K}{\kappa}\rho(S, T).
\end{aligned}$$

Subcase 1.b: Suppose now that $\sigma(H) < -1$ and $y_1 < \kappa$. Since y is on the boundary of T^*, $y_2 \geqslant \kappa$. Observe that $\|x^*\| \geqslant \|y\|$. Let $s^2 = x_1^{*2} + x_2^{*2} = \|x\|^2$, and obtain

$$\sigma(H) \geqslant \sigma(\ell(y, x^*)) \geqslant \sigma(\ell(s/\sqrt{2}, s/\sqrt{2}), (s, 0)) = -\frac{1}{\sqrt{2} - 1}$$

Let H^* be the line through αy with slope $-1/(\sqrt{2}-1)$. Then, since $\sigma(H) < -1$,

$$d(y,S^*) \geqslant d(y,H^*) = \frac{(1-\alpha)(y_1+(\sqrt{2}-1)y_2)}{\sqrt{4-2\sqrt{2}}} \geqslant \frac{(1-\alpha)(\sqrt{2}-1)\kappa}{\sqrt{4-2\sqrt{2}}}$$

In this calculation, we use the fact that the distance between the line $Ax_1+Bx_2+C=0$ and the point (x_1^0, x_2^0) is

$$\frac{|(Ax_1^0+Bx_2^0+C)|}{\sqrt{A^2+B^2}} \tag{6}$$

Now

$$\begin{aligned} \|x^*-\Phi_{T,x^*}(S)\| &= (1-\alpha)\|x^*\| \\ &\leqslant (1-\alpha)\sqrt{2}K \\ &\leqslant \frac{\sqrt{2}K\sqrt{4-2\sqrt{2}}}{(\sqrt{2}-1)\kappa}\rho(S,T) \\ &< 4\frac{K}{\kappa}\rho(S,T) \end{aligned}$$

The last inequality follows from

$$\begin{aligned} \frac{\sqrt{2}\sqrt{4-2\sqrt{2}}}{(\sqrt{2}-1)} &= \sqrt{2}(\sqrt{2}+1)\sqrt{4-2\sqrt{2}} \\ &= \sqrt{8+4\sqrt{2}} < 4 \end{aligned}$$

Subcase 1.c: Suppose now that $\sigma(H) \geqslant -1$ and $y_2 \geqslant \kappa$. Since $\sigma(H) \geqslant -1$, we have $d(y,S^*) \geqslant (1-\alpha)y_2$. Since $y_2 \geqslant \kappa$, we have

$$\begin{aligned} \|x^*-\Phi_{T,x^*}(S)\| &= (1-\alpha)\|x^*\| \\ &\leqslant (1-\alpha)\|(K,y_2)\| \\ &\leqslant \rho(S^*,T^*)\|(K/y_2,1)\| \\ &\leqslant \rho(S^*,T^*)\|(K/\kappa,1)\| \\ &\leqslant \sqrt{2}\frac{K}{\kappa}\rho(S,T). \end{aligned}$$

Subcase 1.d: The remaining case is $\sigma(H) \geqslant -1$ and $y_2 < \kappa$. Let $H^* = \ell((\kappa,\kappa),\alpha y)$. Since $(\kappa,\kappa) \in S^*$, $\sigma(H^*) \geqslant \sigma(H)$. Therefore, $\rho(S,T) \geqslant d(y,S^*) \geqslant d(y,H) \geqslant d(y,H^*)$. Note that H^* is given by $-(\alpha y_2-\kappa)x_1+(\alpha y_1-\kappa)x_2-\alpha\kappa(y_1-y_2)=0$. Therefore, by using equation (6), we obtain

$$\rho(S,T) \geqslant d(y,H^*) = \frac{(1-\alpha)\kappa(y_1-y_2)}{\sqrt{(\alpha y_1-\kappa)^2+(\alpha y_2-\kappa)^2}}$$

Hence

$$\begin{aligned}
\|x^* - \Phi_{T,x^*}(S)\| &= (1-\alpha)\|x^*\| \\
&\leqslant (1-\alpha)\sqrt{2}K \\
&\leqslant \frac{\sqrt{(\alpha y_1-\kappa)^2+(\alpha y_2-\kappa)^2}}{\kappa(y_1-y_2)}\sqrt{2}K\rho(S,T)
\end{aligned} \tag{7}$$

Since αy is on the boundary of S, it follows that $\alpha \geqslant \kappa/y_1$. Also, by assumption, $\alpha \leqslant 1$. Since $h(\alpha) := (\alpha y_1 - \kappa)^2 + (\alpha y_2 - \kappa)^2$ is a convex function of α, it follows that for $\alpha \in [\kappa/y_1, 1]$, $h(\alpha) \leqslant \max\{h(\kappa/y_1), h(1)\}$. For $\alpha = \kappa/y_1$, we obtain

$$\frac{\sqrt{\left(\frac{\kappa}{y_1}y_1-\kappa\right)^2+\left(\frac{\kappa}{y_1}y_2-\kappa\right)^2}}{\kappa(y_1-y_2)} = \frac{1}{y_1} \leqslant \frac{1}{\kappa}$$

Now let $\alpha = 1$. Since $y_1 \geqslant \kappa$ and $y_2 \leqslant \alpha$, we obtain for $y \neq (\kappa, \kappa)$ that

$$\frac{\sqrt{(y_1-\kappa)^2+(y_2-\kappa)^2}}{\kappa(y_1-y_2)} \leqslant \frac{1}{\kappa}$$

Thus inequality (7) becomes

$$\begin{aligned}
\|x^* - \Phi_{T,x^*}(S)\| &\leqslant \frac{\sqrt{(\alpha y_1-\kappa)^2+(\alpha y_2-\kappa)^2}}{\kappa(y_1-y_2)}\sqrt{2}K\rho(S,T) \\
&\leqslant \sqrt{2}\frac{K}{\kappa}\rho(S,T)
\end{aligned}$$

Finally, if $y = (\kappa, \kappa)$, then since αy is on the boundary of S, $\alpha = 1$, and the lemma is trivially true.

Case 2: The analysis of the case $\alpha_T(S) > 1$ is similar, only we replace T with by S^*, and S by T^*. ■

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