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Comment

Comment on "Strength and genericity of singularities in Tolman–Bondi–de Sitter collapse" and a note on central singularities ☆

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Abstract

It has been claimed in Phys. Lett. A 287 (2001) 53 that the Lemaitre–Tolman–Bondi–de Sitter solution always admits future-pointing radial time-like geodesics emerging from the shell-focussing singularity, regardless of the nature of the (regular) initial data. This is despite the fact that some data rule out the emergence of future pointing radial null geodesics. We correct this claim and show that, in general in spherical symmetry, the absence of radial null geodesics emerging from a central singularity is sufficient to prove that the singularity is censored.

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The central or shell-focussing singularity which occurs in the gravitational collapse of spherical dust in the presence of a positive cosmological constant has been studied in [1,2]. In the latter paper, it was shown that the existence or otherwise of radial null geodesics emerging from the singularity depends on the initial data in much the same way as this dependence occurs in the asymptotically flat case. However, in [1], it is claimed that there are future pointing radial time-like geodesics emerging from the singularity for *all* regular initial data. This is in contrast to the asymptotically flat case and is somewhat surprising, given that radial null geodesics are the 'fastest' causal geodesics available [3], and so have the best chance of emerging from a singularity (this statement is made more rigorous below). We show here that the analysis of [1] is incomplete, and thus one of the results asserted—that the central singularity is always visible along radial time-like geodesics, regardless of the initial data—has not been proven. We then give a counter-example and a general result, which show that the assertion is incorrect.

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The space–time in question is the marginally bound Lemaitre–Tolman–Bondi with line element given by [6]

$$ds^2 = -dt^2 + R'^2\,dr^2 + R^2\,d\Omega^2, \tag{1}$$

where $' \equiv \partial_r$, and $d\Omega^2$ is the canonical metric of the unit two-sphere. The field equations for dust in the presence of a positive cosmological constant Λ yield

$$R^3(t,r) = \frac{6m}{\Lambda}\sinh^2 T(t,r), \tag{2}$$

$$T(t,r) = \frac{\sqrt{3\Lambda}}{2}\big[t_c(r) - t\big], \tag{3}$$

where $m = m(r)$ and $t_c(r)$ are functions determined by the initial data, which are imposed at time $t = 0$. We consider the collapsing situation, thus $\dot{R} < 0$. In the present case, there is no rebounce [4] and the dust sphere necessarily collapses to a singularity with zero radius ($R = 0$) at time $t = t_c(r)$; the singularity at $r = 0$ is called the shell-focussing singularity; we will simply refer to it as "the singularity". We can exploit coordinate freedom to set $R(0, r) = r$ and so obtain

$$t_c(r) = \frac{2}{\sqrt{3\Lambda}}\sinh^{-1}\left(\sqrt{\frac{\Lambda r^3}{6m}}\right).$$

The following derivatives are of relevance for the analysis that follows:

$$R' = R\left(\frac{m'}{3m} + \sqrt{\frac{\Lambda}{3}}\,t_c'\coth T\right), \tag{4}$$

$$R'' = \frac{R'^2}{R} + R\left(\frac{m''}{3m} - \frac{1}{3}\left(\frac{m'}{m}\right)^2\right) + R\sqrt{\frac{\Lambda}{3}}\left(t_c''\coth T - \frac{\sqrt{3\Lambda}}{2}\frac{t_c'^2}{\sinh^2 T}\right), \tag{5}$$

$$\dot{R}' = -\sqrt{\frac{\Lambda}{3}}\coth T\,R' + \frac{\Lambda}{2}R\frac{t_c'}{\sinh^2 T}. \tag{6}$$

The overdot denotes differentiation with respect to t. The equations governing a radial time-like geodesic with tangent $K^a = dx^a/d\tau$ (where τ is proper time) may be written as

$$K^t = \pm\sqrt{1 + R'^2\big(K^r\big)^2}, \tag{7}$$

$$\dot{K}^r R' + 2K^r\dot{R}' + \frac{K^r}{K^t}\big(K^r\big)'R' + \frac{(K^r)^2}{K^t}R'' = 0. \tag{8}$$

In [1], a proof of the existence of a solution of these equations emerging from the singularity is attempted by assuming the ansatze

$$t_{\mathrm{RTG}}(r) = t_0 + br^p, \tag{9}$$

$$R' = a_1 r^q, \tag{10}$$

$$\dot{R}' = a_2 r^{q-p}, \tag{11}$$

$$R'' = a_3 r^{q-1}, \tag{12}$$

$$K^r \propto (t - t_0)^\alpha r^\beta, \tag{13}$$

where $t_0 = t_c(0)$ is the time of the shell-focussing singularity and b, p, q are positive constants. It is then claimed that these are consistent with the geodesic equations and so indicate the existence of a solution representing a radial time-like geodesic (RTG) emerging from the singularity. Furthermore, it is claimed that this result follows independently of the initial data $m(r)$.

A vital part of this consistency check results in $p = 1 + q$. To see that this condition may fail, we consider the example

$$m(r) = m_0 r^3 + m_1 r^7. \tag{14}$$

The lower power here is required for regularity of the initial data, and the higher power ensures that that there are no radial null geodesics emerging from the singularity [2]. This choice is included in the class of mass functions $m(r)$ considered in [1]. Along an RTG emerging from the singularity, we must have $T \geqslant 0$ for r sufficiently small, with equality only at $r = 0$. Then examining the leading-order behaviour in

$$T(r) = \frac{\sqrt{3\Lambda}}{2}\big(t_c(r) - t_0 - br^p\big),$$

for the mass function given, we deduce that $p \geqslant 4$ and that

$$T(r) \sim T_0 r^4,$$

for some positive T_0 (the fact that $T \geqslant 0$ is vital here). The functional dependence here and below indicates evaluation along the geodesic. This asymptotic behaviour can be fed into the expressions above for R and its derivatives and yields

$$R(r) \sim R_0 r^{11/3}, \tag{15}$$

$$R'(r) \sim R_1 r^{8/3}, \tag{16}$$

$$\dot{R}'(r) \sim R_2 r^{-4/3}, \tag{17}$$

$$R''(r) \sim R_3 r^{5/3}. \tag{18}$$

Comparing with the ansatze above, we see that $q = 8/3$ and $p = 4$. However, this violates the consistency condition $p = q + 1$, indicating that such a solution cannot in fact exist. We note that two assumptions made here played a vital role: (i) the mass function $m(r)$ excludes radial null geodesics emerging from the singularity, and (ii) the RTG emerges into the regular region of space–time $T > 0$.

The crucial point that is missing in [1] is that the parameters b and p (cf. Eq. (28) in [1]) are *not* independent of the initial data, as implicitly assumed therein. In fact, we must have $p \geqslant n$ (where n signals the first non-vanishing derivative of the initial central density distribution, $\rho_n \equiv (\partial^n \rho/\partial r^n)_{r=0}$), or else the geodesic thus constructed will not belong in the space–time. When this inequality saturates, we obtain the additional constraint $0 < b < t_n$, where t_n is the first non-vanishing coefficient of a MacLaurin series for $t_c(r)$ (cf. Eq. (14) in [1]). From Eqs. (2), (9), to leading order in r we obtain, along the RTG's,

$$R \sim r^{2n/3+1} + \mathcal{O}\big(r^{p+2-n/3}\big). \tag{19}$$

This implies $q = 2n/3$, and thus the consistency relation $p = 1 + q$ reads $p = 1 + 2n/3$, which is formally the same as that obtained for outgoing radial null geodesics. The parameters p and q are then uniquely determined from the initial data, and must obey the constraint $p = 1 + q \geqslant n \Rightarrow n \leqslant 3$.

The statement in [1] (second paragraph) that the work of Deshingkar, Joshi and Dwivedi (DJD) [7] shows that "when one considers *time-like* radial geodesics, the singularity is found to be locally naked and Tipler strong for an infinite number of non-space-like geodesics, irrespective of the initial data" is partially incorrect: DJD show that the visibility of the singularity is initial data dependent and that if a naked singularity is formed then it is necessarily Tipler strong (wrt time-like radial geodesics).

An additional comment concerns the parameters a_i and c_i, introduced in Eqs. (28)–(34) in [1]. Since R'' is obtained from R' by differentiation with respect to r along the geodesic, a_3 is linearly dependent on a_1: $a_3 = qa_1$. Similarly, $c_3 = (\alpha p + \beta)c_1$. We note that the constants c_i are not "free", since they must be fixed by consistency relations involving R and its derivatives. With the substitutions, the algebraic constraint $C(a_i, c_i) = 0$ reads $a_1c_2 + 2a_2c_1 = 0$. That is, for given initial data (whereby a_1 and a_2 are fixed), there is only one degree of freedom in the specification of the two c_i parameters (whose ratio is fixed).

As mentioned above, it can be shown that the absence of a radial null geodesic emerging from a central singularity is sufficient to guarantee censorship of the singularity, i.e., it rules out the existence of any causal geodesic emerging from the singularity. To see this, consider a general spherically symmetric space–time with line element

$$ds^2 = -e^{2\mu}\,dt^2 + e^{2\nu}\,dr^2 + R^2(r,t)\,d\Omega^2,$$

where $\mu = \mu(r,t)$, $\nu = \nu(r,t)$. Then the tangent to a causal geodesic satisfies

$$-e^{2\mu}\dot{t}^2 + e^{2\nu}\dot{r}^2 + \frac{L^2}{R^2} = \epsilon,$$

where the overdot represents differentiation with respect to an affine parameter, L is the conserved angular momentum, and $\epsilon = 0, -1$ for null and time-like geodesics, respectively. Thus, at any point on such a geodesic,

$$e^{2\mu}\dot{t}^2 \geqslant e^{2\nu}\dot{r}^2,$$

with equality holding *only* for radial null geodesics. On the t–r plane, this reads

$$\frac{dr}{dt} \leqslant e^{\mu-\nu}, \tag{20}$$

where we take the positive root for future pointing outgoing geodesics (we can use coordinate freedom to guarantee that t increases into the future globally, and $\partial_r R \geqslant 0$ in a neighbourhood of $R = 0$). We can read (20) as

$$\frac{dr_{\mathrm{CG}}}{dt} < \frac{dr_{\mathrm{RNG}}}{dt}, \tag{21}$$

where the subscripts represent causal (excluding radial null) geodesics and outgoing radial null geodesics, respectively. Now suppose that a CG $t = t_{\mathrm{CG}}(r)$ extends back to a central singularity located on the t–r plane at $r = 0$, $t = t_0$. Assume that the singularity is of the form $t = t_c(r)$ with $t_c(0) = t_0$ and that the regular region of space–time is $t < t_c(r)$. This is the case for the singularity studied above. Let p be any point on the CG, to the future of the singularity. Applying inequality (21) at p, we see that the RNG $t = t_{\mathrm{RNG}}(r)$ through p crosses CG from below and hence points $t_{\mathrm{RNG}}(r)$ on this RNG prior to p must lie to the *future* of points on

CG prior to p, in the sense that $t_{RNG}(r) > t_{CG}(r)$ for $r \in (0, r_*)$, where r_* corresponds to p. Thus, the RNG, which necessarily lies at $t < t_c(r)$, must extend back to $r = 0$ at time $t = t_0 = t_{CG}(0)$, and so must emerge from the singularity. The contrapositive of this result gives the censorship result mentioned above.

We conclude by emphasising that, whereas the analysis of [2]—wherein the general solution is derived and the singularity is analysed along radial null directions—is correct, that of [1] was incomplete, which led to the incorrect claim that the singularity is always locally naked along outgoing RTG's, regardless of the initial data. The assertion in [1] that the singularity is always Tipler strong along RTG's remains true, and is independent of the visibility. We have shown here that the emergence of outgoing RTG's from the singularity *is* dependent on the initial data, and thus the singularity is *not* always locally naked along RTG's. In particular, we have shown that initial data that precludes outgoing RNG's also forbids outgoing RTG's, in any spherically symmetric space–time.

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