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COMPETITIVE SOLUTIONS AND UNIFORM COMPETITIVE SOLUTIONS FOR COOPERATIVE GAMES

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Abstract

The competitive solution and the uniform competitive solution are different solution concepts for cooperative games. The first was introduced by McKelvey, Ordeshook and Winer (1978), the second is proposed in the present paper, however, it is deriving from the same original ideas.

A (uniform) competitive solution is a finite configuration of effective payoff vectors, each of them being associate to a coalition. This configuration satisfies two fundamental requirements: the internal stability and the external stability. Despite the existing differences, the competitive solutions and the uniform competitive solutions have some common properties and are also related to some other classical solution-concepts.

The uniform competitive solution is a modified version of the competitive solution. Some disadvantages of the earlier concept are removed and the existence theorems for very large classes of both transferable utility and non-transferable utility games are provided.

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1 Introduction

The competitive solution was proposed by McKelvey, Ordeshook and Winer [3] as an alternative to classical solution concepts for cooperative games, to explain better some situations in political science. For a more complete motivation behind this concept we refer to [3], [4] and [5].

Although some properties of the competitive solution were shown, general existence results are missing, except for those situations when the competitive solution are related to other solution concepts (core and stable sets). However, the examples analyzed in the above cited papers, would suggest that the existence of competitive solutions may be extended to larger classes of games, especially to the simple games which are important in the modeled political phenomena.

We begin our study in section 2 by defining three solution concepts deriving from the same common idea: the competitive solution, the strong competitive solution (both introduced in [3]) and the uniform competitive solution. We discuss their common properties and we clarify the relationship among them, pointing out the existing differences. Section 3 presents some special properties of these concepts for the transferable utility (TU) cooperative games. Section 4 provides existence theorems for a complete uniform competitive solution for different standard models of TU games. The method of proof of the main theorem allow us to rediscover Shapley's theorem for the core of convex

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games as a particular case of the existence of a uniform competitive solution. The last section presents the existence theorem of the uniform competitive solution for the non-transferable utility (NTU) case. It is also proved that simple games admit such solutions.

2 Competitive Solutions

First, we wlll explain the notation used in the paper. Everywhere in this paper, N is the (finite) set of players: $N = \{l, 2, \ldots, n\}$ and 2^N is the set of coalitions, i.e., the set of all subsets of N. If $C \subseteq N, C \neq \emptyset$ then $\mid C \mid$ stands for the cardinality of C and the notation R^C will be used instead of $R^{|C|}$. If $x \in R^N$ we write $x(C)$ for the sum $\sum_{k \in C} x^k$ of the components indexed in C, and we will use both the symbols x^C and $pr_C x$ for the vector $(x^k)_{k \in C}$, formed by the components of x indexed in C. Also, for any subset A of R^N and for any coalition C we define $pr_C A = \{x^C \mid x \in A\}$.

If x, y are in R^C, we write $x \geqq y$ if $x^k \geq y^k$ for all $k \in C$: we write $x \geq y$ if $x \geqq y$ but $x \neq y$ and we write $x > y$ if $x^k > y^k$ for all k. Some ambiguities may appear when $\mid C \mid$ can be 1. In this case, $x \geq y$ must be interpreted as $x > y$, but we will always keep the usual meaning of the relation $\geq$ when we refer to scalars.

The usual representation of a general cooperative game in characteristic function form is a pair (N, v), where v, the characteristic function, is defined on 2^N and assigns to each nonempty coalition C a subset of R^C. Note that for simple games $v(C)$ may be empty even when $C \neq \emptyset$. As usual we interpret an element of $v(C)$ as a payoff vector which the coalition C can guarantee for its members even if the players not in C disagree with him. An x belonging to $v(N)$ represents an n-payoff vector which may be achieved by all players at the end of a possible play. Consequently, $x \in v(N)$, or more general, $x \in pr_C v(N)$, when $x \in R^C$,is called the *effectiveness condition.*

Now we are ready to define the solution concepts. Let (N, v) be a cooperative game in the characteristic function form.

Definition 1. The pair (x, C) is a proposal if $C \subseteq 2^N, C \neq \emptyset, x \in v(C)$ and $x \in pr_C v(N)$.

Now, let $\mathcal{K} = \{(y_i, C_i): i = 1, 2, \ldots, m\}$ be a finite set of proposals.

Definition 2. $\mathcal{K}$ is a competitive solution (c.s.) if:

(1) there are no i, j such that $y_i^{C_i \cap C_j} > y_j^{C_i \cap C_j}$

(2) if (x, C) is a proposal and $x^{C \cap C_i} > y_i^{C \cap C_i}$ for some i, then there exists $j \in \{1, 2, \ldots, m\}$ such that $y_j^{C \cap C_j} > x^{C \cap C_j}$

(3) $C_i \neq C_j$ if $i \neq j$

Definition 3. $\mathcal{K}$ is a strong competitive solution (s.c.s.) if it satisfies (2), (3) and (4) where,

(4) there are no i, j such that $y_j^{C \cap C_j} \geq y_j^{C_i \cap C_j}$. Of course, a s.c.s. is a c.s. too.

Definition 4. $\mathcal{K}$ is an uniform competitive solution (u.c.s) if

(5) $y_i^{C \cap C_j} = y_j^{C_i \cap C_j}$ for every $i, j \in \{1, 2, \ldots, m\}$

(6) if (x, C) is a proposal and $x^{C \cap C_i} \geq y_i^{C \cap C_i}$ for some i, then there exist j and $k \in C \cap C_j$ such that $y_j^k > x^k$.

It follows from these definitions that a c.s. (s.c.s. or u.c.s.) is a configuration of proposals which is stable against the objections coming from outside and is also characterized by an internal stability condition. An important difference between the bargaining set of the Aumann-Maschler theory and the competitive solutions arises from the fact that the coalitions in the configurations are not necessarily disjoint. Consequently, a c.s. may only predict payoffs for a set of coalitions which seems to be profitable for the potential partners. The stability is expressed in terms of a counter objection against any objection. Related to this point, the concept of u.c.s. is intended to remove some ambiguities existing in the previous two solution concepts, respectively, c.s. and s.c.s. The situation illustrated in figure 1 is also emphasized in Example 2. Suppose that the proposal (x, C) is an objection against $\mathcal{K}$. This means that some players support this objection since x will improve their output. In all three definitions the supporting players are the pivotal players in $C \cap C_i$. A counter objection must be claimed by the players which would lose if the objection is accepted. But according to the definitions of c.s. and s.c.s. these players might be the same as the supporting players, i.e. $C \cap C_i = C \cap C_j$. In that situation one or more players may have two contradictory positions against the objection (x, C).

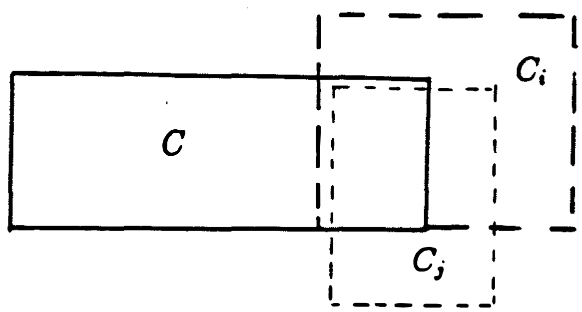
Condition (5) of the definition of u.c.s. makes impossible this situation. Consequently, any player must have one and only one position toward a given objection: he may support it, he may reject it or, he is indifferent.

Despite the existing differences, c.s., s.c.s. and u.c.s. have some remarkable common properties.

Proposition 1. If $\mathcal{K}$ is a c.s. (s.c.s. or u.c.s.) and $(y, C) \in \mathcal{K}$, then y is weak Pareto optimum in $v(C) \cap pr_c v(N)$.

Proof: To the contrary, let be $x \in v(C) \cap pr_C v(N)$ such that $x > y$. Then, there exists $(u, D) \in \mathcal{K}$ such that $u^{C \cap D} > x^{C \cap D}$ ($u^k > x^k$ for some $k \in C \cap D$, if $\mathcal{K}$ is u.c.s.). Hence, $u^{C \cap D} > y^{C \cap D} (u^k > y^k)$ contradicting (1) (respectively, (5)).

Figure 1



The following two propositions invoke the definition of the core of a cooperative game.

The *core* of the game (N, v) is the set $C(N, v) = \{u \in v(N) |$ there is no $C \subseteq N, C \neq \emptyset$ and $x \in v(C) \cap pr_C v(N)$ such that $x > u^C\}$.

The existence of a s.c.s. when $C(N, v) \neq \emptyset$ was been established in [3]. The same conclusion holds for u.c.s. A converse implication may be easily shown if the game admits a special type of s.c.s. (or u.c.s.)

Proposition 2. If $C(N, v) \neq \emptyset$, then, for every $u \in C(N, v)$, the set $\mathcal{K} = \{(u, N)\}$ is a s.c.s. and u.c.s.

Proposition 3. If $\mathcal{K} = \{(u, N)\}$ is a c.s. (respectively, u.c.s.) then $u \in C(N, v)$. Particularly, $C(N.v) \neq \emptyset$.

Proof: Indeed, if $u \notin C(N, v)$ then there exists a proposal (x, C) such that $x > u^C$. But, the last inequality is equivalent with $x^{C \cap N} > u^{C \cap N}$, which is impossible since $\{(u, N)\}$ is a c.s. (respectively, u.c.s.). As a consequence of the last two proposition we have:

$$C(N, v) = \{y | \{(y, N)\} \text{ is a c.s. (u.c.s.)}\}$$

The relationship between c.s. and stable sets was discussed in [3]. Now, two additional properties of u.c.s. can be stated.

Proposition 4. Let be $(y, C) \in \mathcal{K}$ and $\mathcal{K}$ a u.c.s. Then, y is Pareto optimal in $v(C) \cap pr_C v(N)$.

Proof: Suppose $x \in v(C) \cap pr_C v(N), x \geq y$. By (6), there exists $(u, D) \in \mathcal{K}$ and $k \in C \cap D$ such that $u^k > x^k$. Since $x^k \geq y^k$ this implies $u^k > y^k$, contradicting (5) of Definition 4.

Proposition 5. Let $k \in N$. Suppose $z^k = \sup(v(\{k\}) \cap pr_k v(N)) < \infty$. If $\mathcal{K}$ is a u.c.s. and $k \in C$ for some $(y, C) \in \mathcal{K}$, then $y^k \geq z^k$.

Proof: To the contrary, let $x \in v(\{k\}) \cap pr_k v(N), x^k > y^k$. Then, there exist $(u, D) \in \mathcal{K}$ with $k \in D$ such that $u^k > x^k$, i.e. $u^k > y^k$. Contradiction.

The next examples will clarify the relationship between c.s. and u.c.s. Also it will be shown that the Propositions 4 and 5 fail if $\mathcal{K}$ is c.s. (s.c.s.). At the same time, examples 1 and 4 illustrate some aspects of the competitive solutions which lead us to formulate additional requirements.

Example 1. $n = 3.v(\{i\}) = [0,1], i = 1,2,3; v(\{i,j\}) = \{(x^i, x^j) \in R^2_+ | x^i + x^j \leq 2\}$ for every pair $(i,j), i < j$ and $v(\{1,2,3\}) = \{(x^1, x^2, x^3) \in R^3_+ | x^1 + x^2 + x^3 \leq 3\}$. It is easy to recognize an additive TU game. Consequently, the core is nonempty consisting in one point. $C(N, v) = \{u\}$, where $u = (1,1,1)$.

From Proposition 2 it follows that $\mathcal{K}^1 = \{(u, N)\}$ is s.c.s. and u.c.s. Let us list some other sets of proposals:

$\mathcal{K}^2 = \{(1,\{1\}), (1,\{2\}), (1,\{3\})\}$

$\mathcal{K}^3 = \{((1,1),\{1,2\}), ((1,1),\{1,3\}), ((1,1),\{2,3\})\}$

$\mathcal{K}^4 = \mathcal{K}^1 \cup \mathcal{K}^2,$

$\mathcal{K}^5 = \mathcal{K}^2 \cup \mathcal{K}^3,$

$\mathcal{K}^6 = \mathcal{K}^3 \cup \mathcal{K}^4.$

$\mathcal{K}^2, \mathcal{K}^4, \mathcal{K}^5$ and $\mathcal{K}^6$ as well as $\mathcal{K}^1$ are s.c.s. and u.c.s. too. But $\mathcal{K}^3$ is a u.c.s. which is not c.s. To see this, consider the proposal (x, N), where $x = (1.5, 1.5, 0)$. It is easy to see that (x, N) is an objection against the proposal $(u, \{1,2\}) \in \mathcal{K}$ but there doesn't exist a counter objection in the sense of Definition 2.

On the other hand, in this example the set of competitive solutions may be (partially) ordered by the set-inclusion relation. Consequently some c.s. could be minimal or maximal in respect with this order. In principle, it is easy to obtain a minimal c.s. (or u.c.s.) if at least one is available. But is not always clear how can be recognized the maximality. However, we shall show, at the end of the present section, that this is possible for the u.c.s.

Example 2. $n = 3.v(\{1\}) = [0, 1.1], v(\{2\}) = [0,1], v(\{3\}) = \{0\}, v(\{1,2\}) = \{(x^1, x^2) \in R^2_+ | x^1 + x^2 \leq 3\}$, $v(\{i,j\}) = \{(x^i, x^j) \in R^2_+ | x^i + x^j \leq 1\}$, if $(i,j) \neq (1,2)$, $v(\{1,2,3\}) = \{(x^1, x^2, x^3) \in R^3_+ | x^1 + x^2 + x^3 \leq 3\}$

$\mathcal{K} = \{((1,2),\{1,2\}),((2,1,0),N)\}$ is a s.c.s. but not u.c.s. ((5) of Definition 4 is violated). The situation depicted in Figure 1 may be recognized here. Player 1 supports the objection $(1.1,\{1\})$ against the proposal $((1,2),\{1,2\}) \in \mathcal{K}$. At the same time he rejects this objection by the counter objection $((2,1,0),N)$.

Note also that $y^1 < z^1 = 1.1$, if y is the payoff component of the proposal $((1,2),\{1,2\}) \in \mathcal{K}$, so that, the conclusion of Proposition 5 can't be extended to the case of c.s.

Remark: However, a weaker form of Proposition 5 can be stated for c.s. Let $\mathcal{K} = \{(y_1,C_1),\dots,(y_m,C_m)\}$ a c.s. Define $w \in R^N$ by $w^k = \max\{y_i^k | k \in C_i\}$. Then, it is easy to verify that $w^k \geq z^k$ for every $k \in N$.

Example 3. $n = 2.v(\{1\}) = v(\{2\}) = [0,1], v(\{1,2\}) = [0,2] \times [0,2]$. Obviously, $\mathcal{K} = \{((1.5,2),\{1,2\})\}$ is a s.c.s. but not u.c.s. But $y = (1.5,2)$ is not Pareto-optimal in $v(\{1,2\})$, so that Proposition 4 fails if u.c.s. is replaced by c.s.

Example 4. $n = 3.v(\{i\}) = [0,1]$ for $i = 1,2,3.v(\{i,j\}) = \{(x^i,x^j) \in R^2_+ | x^i + x^j \leq k\}$ if $\{i,j,k\} = \{1,2,3\}.v(N) = \{(x^1+x^2+x^3) \in R^3_+ | x^1+x^2+x^3 \leq 3\}$.

Note that $C(N,v) = \emptyset$, but the game admits competitive solutions. For instance $\mathcal{K} = \{((2,1),\{1,2\}),(1,\{2\}),(1,\{3\})\}$ is simultaneously, s.c.s. and u.c.s. It is easy to see that $\mathcal{K}$ is not minimal and $\mathcal{K}' = \{((2,1),\{1,2\})\}$ is also s.c.s. (u.c.s.). It is important to note that $\mathcal{K}'$ doesn't predict the payoff for all the players, since the coalition components of the proposals in $\mathcal{K}'$ don't cover N. Such situation can't appear if we deal with other classical solution concepts (core, von Neumann-Morgenstern solutions, bargaining set, etc.). This example suggest us to introduce a stronger version of competitive solution. The formal definition follows:

Definition 5. The competitive solution (s.c.s. or u.c.s.) $\mathcal{K} = \{(y_i,C_i), i = 1,2,\dots,m\}$ is complete if $\cup C_i = N$.

$\mathcal{K}$ in the previous example is complete, but $\mathcal{K}'$ is not.

In the remainder of this section we will restrict our attention to complete uniform competitive solutions (c.u.c.s.). According to (5) we can define an "ideal" payoff vector $y \in R^N$, whose components are: $y^k = y_i^k$, for every i such that $k \in C_i$. Of course, from definition (4) it follows that $y^{C_i} \in v(C_i) \cap pr_{C_i}v(N)$ for every $i = 1,2,\dots,m$. Finally, from (6) it follows that if $x \in v(C) \cap pr_C v(N)$, for some $C \subseteq N$, then the inequality $x \geq y^C$ is impossible. In summary, we are led to an alternative definition of a c.u.c.s.

Definition 4 '. A complete uniform competitive solution is a pair $(y,\mathcal{C}), y \in R^N, \mathcal{C} \subseteq 2^N$ such that

(7) $\cup_{C \in \mathcal{C}} C = N$

(8) $y^C \in v(C) \cap pr_C v(N)$, for every $C \in \mathcal{C}$

(9) if $x \in v(C) \cap pr_C v(N)$, for some $C \in 2^N$, then $x \not\geq y^C$

Now, let y be any vector of R^N and denote by $\mathcal{C}(y) = \{C \in 2^N \mid y^C \in v(C) \cap pr_C v(N)\}$

Then,

Proposition 6. If $\mathcal{K} = (y, \mathcal{C})$ is a c.u.c.s. then $\mathcal{K}' = (y, \mathcal{C}(y))$ is a maximal c.u.c.s.

Usually, the game theoretical model is constrained to satisfy some additional properties. Most important results concerning both TU and NTU games are obtained for games having comprehensive characteristic functions.

A game (N, v) is *comprehensive* (or, its characteristic function is comprehensive) if, for each $C \in 2^N$ if $v(C) \neq \emptyset$, then $v(C) - R^C{}_+ \subseteq v(C)$.

Proposition 7. Suppose (N, v) comprehensive. Then, $(y, \mathcal{C}(y))$ is (maximal) c.u.c.s. if and only if $\bigcup\{C \mid C \in \mathcal{C}(y)\} = N$ and, for each $C \in \mathcal{C}(y), y^C$ is Pareto-optimal in $v(C) \cap pr_C v(N)$.

Proof: it simply follows from the remark that if $x \geq y^C$ for some coalition C and $x \in v(C) \cap pr_C v(N)$, then $y^C \in v(C) \cap pr_C v(N)$.

Remarks. The definition of u.c s is derived from the concept of c.s. The internal stability (condition (5)) is strengthened to eliminate some contradictory aspects in the original definition. According to this new property, the external stability requirement (6) was also redefined. The class of objections considered is wider (compare (6) with (2)), but the response is somewhat weakened.

It its second formalization (Definition 4'), the concept of c.u.c.s. coincides with the "aspiration" of Bennett and Zame ([2]), except the feasibility condition. Moreover, some of existing theorems in this paper coincide with results due to Bennett ([1]) and Bennett and Zame ([3]), if the assumptions related to the feasibility condition are removed. However, our results were obtained independently using different methods of proof.

3 Competitive Solutions of TU Games

The most general representation of a TU game in characteristic function form is (N, v):
(10) $v(C) = \{x \in R^C \mid x(C) \leq \mathcal{V}(C)\}$ where $\mathcal{V}$ is a real valued function defined on 2^N (called also characteristic function).

However, some other definitions of v are often employed by introducing some additional restrictions. We will refer in this section to models in which the set-valued mapping v satisfies boundedness conditions. Namely,

(11) $v(C) = \{x \in R^C_+ \mid x(C) \leq \mathcal{V}(C)\}$ or,

(12) $v(C) = \{x \in R^C \mid x^k \geq \mathcal{V}(\{k\}), k \in C, x(C) \leq \mathcal{V}(C)\}$

Note that the third case may be reduced to the second by taking $\mathcal{V}(\{k\}) = 0$ for all $k \in N$, but it is not always possible to transfer the significant properties from one form to the other.

Some other usual properties of $\mathcal{V}$ will be cited in the following.

(13) $C \subset D \Rightarrow \mathcal{V}(C) \leq \mathcal{V}(D)$ (Monotonicity)

(14) $C \cap D = \emptyset \Rightarrow \mathcal{V}(C) + \mathcal{V}(D) \leq \mathcal{V}(C \cup D)$ (Superadditivity)

(14′) $C \cap D = \emptyset \Rightarrow \mathcal{V}(C) + \mathcal{V}(D) = \mathcal{V}(C \cup D)$ (Additivity)

(15) $\mathcal{V}(C) + \mathcal{V}(D) \leq \mathcal{V}(C \cup D) + \mathcal{V}(C \cap D)$, for every $C, D \in 2^N$ (Convexity)

It is easy to see that if $\mathcal{V}(\emptyset) = 0$, then (15) $\Rightarrow$ (1) and if $\mathcal{V} \geq 0$, then (14) $\Rightarrow$ (13).

Remarks.

1. If the game (N, v) is defined as in (10), then,

(16) $C \subseteq D \Rightarrow v(C) \subseteq pr_C v(D)$.

Particularly, $v(C) \subseteq pr_C v(N)$ for all C. As an important consequence of this remark, we can remove the feasibility condition from the definition of competitive solutions, or of the core.

2. (16) remains valid in the case (11) iff $\mathcal{V}$ is monotonic.

3. The same conclusion still holds in the case (12) if $\mathcal{V}$ is superadditive (sufficient condition). Indeed, suppose $C \subset D$ and $x \in v(C)$. That is, $x \in R^C, x^k \geq \mathcal{V}(\{k\}), k \in C$ and $x(C) \leq \mathcal{V}(C)$. Extend x to a vector $\hat{x}$ in R^D taking $\hat{x}^k = \mathcal{V}(\{k\})$, for $k \in D \backslash C$. Then, $\hat{x}(D) = \sum_{k \in D \backslash C} \mathcal{V}(\{k\}) + \mathcal{V}(C) \leq \mathcal{V}(D)$. Hence, $\hat{x} \in v(D)$ and $\hat{x}^C = x$, i.e., $x \in pr_C v(D)$.

Now we will point out some properties of competitive solutions for TU games. If no other assumption is made, we will consider (N, v) in the case of (10).

Proposition 8. Let $\mathcal{K}$ be a c.s. (u.c.s.). Then, for each proposal $(y, C) \in \mathcal{K}$ we have $y(C) = \mathcal{V}(C)$. Particularly, y is Pareto-optimal in $v(C)$. The same conclusions remain valid in the case (11) if $\mathcal{V}$ is monotonic and in the case (12) if $\mathcal{V}$ is superadditive.

Proof: to prove first statement suppose that $y(C) \neq \mathcal{V}(C)$. Of course, in this case, $\varepsilon = \mathcal{V}(C) - y(C) > O$. Define $u \in R^C$ by $u^k = y^k + \varepsilon/|C|, k \in C$. Then, $u(C) = \mathcal{V}(C)$ i.e., $u \in v(C)$ and $u > y$. Then,

(a) If $\mathcal{K}$ is c.s., there is $(z, D) \in \mathcal{K}$ such that $z^{C\cap D} > u^{C\cap D}$, and, therefore $z^{C\cap D} > y^{C\cap D}$, contradicting (1).

(b) If $\mathcal{K}$ is u.c.s., there are $(z, D) \in \mathcal{K}$ and $k \in C\cap D$ such that $z^k > u^k > y^k$ contradicting (5).

For (11) and (12) the conclusions of the proposition follow from the remarks 2 and 3 above.

Proposition 9. Let be a u.c.s. If $(y, C) \in \mathcal{K}$ and $k \in C$, then $y^k \geq \mathcal{V}(\{k\})$. Consequently, $\mathcal{V}(C) \geq \sum_{k\in C} \mathcal{V}(\{k\})$. The same conclusion holds in case (11) if $\mathcal{V}$ is monotonic and in case (12) if $\mathcal{V}$ is superadditive.

Proof: it simply follows from Propositions 8 and 5 and from Remarks 2 and 3.

The following lemma will be used in the proofs of Section 4

Lemma 1. Let $\mathcal{K}$ be a c.u.c.s and a pair (x, D) where $x(D) < \mathcal{V}(D)$. Then, there exist $(y, C) \in \mathcal{K}$ such that $C \cap D \neq \emptyset$ and $k \in C \cap D$ for which $y^k > x^k$.

Proof: Let $(u, E) \in \mathcal{K}$, such that $D \cap E \neq \emptyset$. Suppose $x^{D\cap E} \geqq u^{D\cap E}$. Define $z \in R^D$ by $z^t = x^t + \varepsilon/|D|, t \in D$, where $\varepsilon = \mathcal{V}(D) - x(D) > 0$.. Obviously, $z(D) = x(D) + \varepsilon = \mathcal{V}(D)$, therefore, $z \in v(D)$. But $z^{D\cap E} > x^{D\cap E} \geqq u^{D\cap E}$ and then, by Definition 4, there exist a proposal $(y, C) \in \mathcal{K}$ and $k \in C \cap D$ such that $y^k > z^k > x^k$.

As a consequence of Proposition 8 and Lemma 1 we can obtain an algebraically characterization of c.u.c.s. For convenience we will use Definition 4′.

Denote by $\mathcal{M} = \{C \in 2^N | \mathcal{V}(C) \geq \sum_{i\in C} \mathcal{V}(\{i\})\}$. Of course, if $\mathcal{V}$ is superadditive then $\mathcal{M} = 2^N \setminus \{\emptyset\}$.

Proposition 10. The pair $(y, \mathcal{C})(y \in R^N)$ is a c.u.c.s. if and only if $\cup\{C | C \in \mathcal{C}\} = N, y(C) \geq \mathcal{V}(C)$ for all $C \subset N, C \neq \emptyset$ and $y(C) = \mathcal{V}(C)$ for $C \in \mathcal{C}$. Moreover, $\mathcal{C} \subseteq \mathcal{M}$. The same conclusions holds in the case (11) and (12) if monotonic, respectively, superadditive.

Corollary. Let $y \in R$. Suppose that $\cup\{C|C \in \mathcal{C}(y)\} = N$. Then $(y, \mathcal{C}(y))$ is a maximal c.u.c.s. if and only if $y(C) = \mathcal{V}(C)$ for all $C \in \mathcal{C}(y)$ and $y(C) > \mathcal{V}(C)$ for all $C \in \mathcal{M}\backslash\mathcal{C}(y)$.

Remark. If $\mathcal{K}$ is a complete c.s. and w is defined as in Section 3, then $w(C) \geq \mathcal{V}(C)$ for all $C \in 2^N$.

Now, let us recall some properties of the classical von Neumann-Morgenstern theory of TU games. Suppose that $(N, v), (N, v')$ are two TU games having $\mathcal{V}$, respectively, $\mathcal{V}'$ as the corresponding real-valued characteristic functions.

Definition. The games $(N, v), (N, v')$ are strategically equivalent if there exist the scalar $\alpha > 0$ and the vector $a \in R^N$ such that $\mathcal{V}'(C) = \alpha\mathcal{V}(C) + a(C)$, for every $C \subset N$.

Proposition 10′. Let $(N, v), (N, v')$ be strategically equivalent. Then, $\mathcal{K} = \{(y_i, C_i)|i = 1, \ldots, m\}$ is a c.s. (s.c.s. or u.c.s.) if and only if $\mathcal{K}' = \{(y_i', C_i)|i = 1, \ldots, m\}, y_i' = y_i + a_i^C$, is a c.s. (s.c.s., respectively u.c.s.) of the game (N, v'). The same conclusion holds for the game in the case (12). It still remains valid in case (11) if $a = 0$ (i.e. if $\mathcal{V}'(\mathcal{C}) = \alpha\mathcal{V}(\mathcal{C})$ for every $C \subseteq N, \alpha > 0$).

Proof: First, we note that $y \in v(C) \Leftrightarrow y' \in v'(C), y' = \alpha y + a^c$ in the cases (10) and (12), but the same relation holds in case (11) if and only if $a = 0$. Now, the proof of Proposition is trivial since the mapping $h^c : v(C) \to v'(C), h^C(y) = \alpha y + a^C$ preserves the order relation between the payoffs of the players in C.

4 Existence Theorems for TU Games

As a direct consequence of Proposition 2, if a TU game has a nonempty core it admits s.c.s. and u.c.s. Then, any sufficient conditions for the existence of the core is sufficient condition for the existence of s.c.s. and u.c.s. Particularly, balanced games and convex games admit a s.c.s. and u.c.s. Note also a simple situation when the existence of competitive solutions is guaranteed.

Proposition 11. Every TU two person game admits a s.c.s. and u.c.s.

Now, we will restrict our attention to u.c.s. We will prove the existence of a c.u.c.s. for any TU game satisfying (10) and we will deduce sufficient conditions in the other two cases. The method of proof is inspired by the Peleg paper [6] and is, essentially, the reduced-game method.

Theorem 1. For every TU game with v defined by (10) there exists a c.u.c.s.

Proof: By induction on $n = |N|$. For $n = 1$ the truth of the theorem is obvious;

$(\mathcal{V}(1), \{1\})$ is a c.u.c.s. Suppose the theorem valid for every game having at most $n-1$ players and let be $(N, v), |N| = n$.

Denote by $M = N\backslash\{n\}$ and define the reduced game (M, v_M) by its real-valued characteristic function $\mathcal{V}_M$;

$$\mathcal{V}_M(C) = \max\{\mathcal{V}(C), \mathcal{V}(C \cup \{n\}) - z^n\}, C \subseteq M$$

where $z^n = \mathcal{V}(\{n\})$.

Since $|M| = n-1$, the game (M, v_M) has a c.u.c.s., say $\mathcal{K}_M = \{(\bar{y}_i, C_i) | i = 1, \dots m\}$.

Denote by

(16)

$$C_i' = \begin{cases} C_i \text{ if } \mathcal{V}(C_i \cup \{n\}) - z^n < \mathcal{V}(C_i) \\ C_i \cup \{n\} \text{ if } \mathcal{V}(C_i \cup \{n\}) - z^n \geq \mathcal{V}(C_i) \end{cases}$$

(17)

$$y_i = \begin{cases} \bar{y}_i \text{ if } C_i' = C_i \\ (\bar{y}_i, z^n) \text{ if } C_i' = C_i \cup \{n\} \end{cases}$$

$$\mathcal{K} = \{(y_i, C_i') | i = 1, \dots, m\}$$

(18)

$$\mathcal{K}_N = \begin{cases} \mathcal{K}, \text{ if there exists } i \text{ such that } C_i' \ni n \\ \mathcal{K} \cup \{(z^n, \{n\})\}, \text{ otherwise.} \end{cases}$$

(for convenience, we will write (y_{m+1}, C'_{m+1}) for $(z^n, \{n\})$).

We are ready to prove that $\mathcal{K}_N$ is a c.u.c.s. of (N, v). Of course, $\cup C_i' = N, y_i \in v(C_i')$ for all i and $y_i \in pr_{C_i'} v(N)$ (see Remark 1). To verify (5) suppose $(y_i, C_i'), (y_j, C_j') \in \mathcal{K}_N$ and $y_i^k > y_j^k$ for some $k \in C_i' \cap C_j'$. We claim that $k \neq n$. Indeed, if $C_i' = C_i$ then $n \notin C_i' \cap C_j'$ and if $n \in C_i' \cap C_j'$, then $y_i^n = y_j^n = z^n$.

Then the initial assumption implies that $\bar{y}_i^k > \bar{y}_j^k$ for some $k \in C_i \cap C_j$, contradicting the properties of $\mathcal{K}_M$.

Let us verify (6). Let $C \subset N, u \in v(C)$ and suppose that $u^{C \cap C_i'} \geq y^{C \cap C_i'}$ for some i. We analyze three possibilities:

(i) if $n \notin C$, then $u \in v_M(C)$ (and $u \in pr_C v_M(M)$) and $u^{C \cap C_i} \geq \bar{y}^{C \cap C^i}$. Since $\mathcal{K}_M$ is a u.c.s. of (M, v_M) then there exist $j \leq m$ and $k \in C \cap C_j$ such that $\bar{y}_j^k > u^k$. I.e., $y_j^k > u^k$ for some $k \in C \cap C_j'$.

(ii) If $C = \{n\}$ then $u^n \leq z^n$ and the initial assumption fails.

(iii) Suppose that $C = Q \cup \{n\}$ where $\emptyset \neq Q \subseteq M$. Again there are three cases:

a) $u^n < z^n$. Then $y_j^n > u^n$ for every j for which $n \in C_j'$.

b) $u^n > z^n$. Then, $u(Q) = u(C) - u^n < \mathcal{V}(C) - z^n \leq \mathcal{V}_M(Q)$. Hence, $u^Q \in v_M(Q)$.

From Lemma 1, there exist j and $k \in Q \cap C_j$ such that $\bar{y}_j^k > u^k$. That is, $y_j^k > u^k$ for some $k \in C \cap C_j$.

c) $u^n = z^n$. Of course, if $n \in C_i'$ then $y_i^n = z^n$. Therefore, the initial assumption implies that $u^{Q \cap C_i} \geq \bar{y}^{Q \cap C_i}$. Since $u(Q) \leq \mathcal{V}(Q \cup \{n\}) - z^n \leq \mathcal{V}_M(Q)$ it follows that $u^Q \in v_M(Q)$. But $\mathcal{K}_M$ is a c.u.c.s. of (M, v_M) so that there exist j and $k \in Q \cap C_j$ such that $\bar{y}_j^k > u^k$, or, equivalently, $y_j^k > u^k$ for $k \in C \cap C_j'$.

Corollary 1. If $\mathcal{V}$ is convex and $\mathcal{V}(\emptyset) = 0$, then the game (N, v) has a c.u.c.s. of the form $\mathcal{K} = \{(y, N)\}$. $\mathcal{K}$ is a s.c.s. too.

Proof: Firstly, note that the proof of Theorem 1 is constructive. A c.u.c.s. may be obtained in n steps, each step extending an existing c.u.c.s. to a solution for a game having one more player. This extension may be made on two ways: by adding a new proposal to the existing solution or keeping the same number of proposals. In the last case, according to (16) and (17), a new player is added to some coalition-components of the proposals in the previous solution. The process starts at the step 1 with the solution of a one person game. Of course, this solution has one proposal. At the second step this solution would be extended to a one-proposal solution too, if (16) pick on the second alternative. It is easy to see that this happens if the corresponding characteristic function is superadditive. In summary, if at each step, the real-valued characteristic function of the current game is superadditive, then the final solution will consist of a single proposal. The completeness implies that the corresponding coalition is N.

Now, it is sufficient to prove that if $\mathcal{V}$ satisfies the assumptions of the corollary, then $\mathcal{V}_M$ is also convex. Obviously, since $\mathcal{V}(\emptyset) = 0$, the convexity implies the superadditivity (we can always $\mathcal{V}_M(\emptyset) = 0$, by definition).

Let $C, D \subset M$. We are going to prove that $\mathcal{V}_M(C)+\mathcal{V}_M(D) \leq \mathcal{V}_M(C\cup D)+\mathcal{V}_M(C\cap D)$.

Since it is obviously verified if one of the two coalitions is $\emptyset$ or M, we can consider both C and D different of $\emptyset$ and M. Since $\mathcal{V}$ is superadditive, if $n \in C$ if follows that $\mathcal{V}(C)+\mathcal{V}(\{n\}) \leq \mathcal{V}(C\cup\{n\})$. Hence, $\mathcal{V}_M(C) = \mathcal{V}(C\cup\{n\})-z^n$. Then, $\mathcal{V}_M(C)+\mathcal{V}_M(D) = \mathcal{V}(C\cup\{n\})+\mathcal{V}(D\cup\{n\})-2z^n$. Since $\mathcal{V}$ is convex, we obtain finally, $\mathcal{V}_M(C)+\mathcal{V}_M(D) \leq \mathcal{V}(C\cup D\cup\{n\})+\mathcal{V}((C\cap D)\cup\{n\})-2v^n = \mathcal{V}_M(C\cup D)$.

Corollary 2. (Shapley, [7]) Suppose $\mathcal{V}$ is convex and $\mathcal{V}(\emptyset) = 0$. Then $C(N, v) \neq \emptyset$.

Proof: It simply follows from the previous Corollary and Proposition 3.

Corollary 3. Suppose $\mathcal{V}$ is subadditive (i.e. $\mathcal{V}(C\cup D) \leq \mathcal{V}(C)+\mathcal{V}(D)$ if $C\cap D = \emptyset$). Then $\mathcal{K}_0 = \{(u_0^k, \{k\}) | k \in N\}$ where $u_0 = (\mathcal{V}(\{1\}), \mathcal{V}(\{2\}), \ldots, \mathcal{V}(\{n\}))$ is a c.u.c.s. Moreover, $\mathcal{K}_0$ is a complete s.c.s. too.

Proof: The Proof of Theorem 1 remains still valid if (16) is replaced by

(16′)

$$C_i' = \begin{cases} C_i \text{ if } \mathcal{V}(C_i\cup\{n\}) - z^n \leq \mathcal{V}(C_i) \\ C_i\cup\{n\} \text{ if } \mathcal{V}(C_i\cup\{n\}) - z^n > \mathcal{V}(C_i) \end{cases}$$

Then, if $\mathcal{V}$ is subadditive, $C_i' = C_i$ for $i = 1, \ldots, m$ and, consequently, $\mathcal{K}_N$ is obtained from $\mathcal{K}_M$ by adding a new proposal. Note that the coalition of this proposal is a singleton. If the above mentioned situation appears to each step, then the constructive process discussed in the proof of Corollary 1 end by $\mathcal{K}_0$. All what we must do is to prove that $\mathcal{V}_M$ is subadditive if $\mathcal{V}$ is. Obviously. this is true because if $\mathcal{V}$ is subadditive, then $\mathcal{V}_M$ is the restriction of $\mathcal{V}$ to the subsets of M.

Remark 1. If $\mathcal{V}$ is additive. then the conclusions of Corollary 1 and Corollary 3 hold. But it is easy to see that, in this case. for any set $\mathcal{C}$ of coalitions which cover N, the pair $(u_0, \mathcal{C})$ is a c.u.c.s. (see Definition 4′ and Example 1).

Remark 2. As it was shown. if $C(N, v) \neq \emptyset$. then there exist (strong) competitive solutions of the form $\{(u, N)\}$ with $u \in C(N, v)$. Also. if the game is subadditive, it has c.c.s. But we can prove that a TU game has not c.c.s., except these above mentioned situations. More precisely. we will prove the following statement:

if $C(N, v) = \emptyset$ and $\mathcal{V}$ is not subadditive. then the game (N, v) has no c.c.s.

Proof: Let the game (N, v) and suppose that $\mathcal{K} = \{(y_C, C) | C \in \mathcal{C}\}$ is a c.c.s. Denote by $\mathcal{L} = \{k | \{k\} \in \mathcal{C}\}$ and let w be the corresponding "ideal payoff vector" (see remark of Example 2). Three cases will be discussed;

(a) $\emptyset \neq \mathcal{L} \neq N$.

Pick an $k \in \mathcal{L}$ and $m \notin \mathcal{L}$. Define $x \in R^N$ by:

$$x^t = \begin{cases} w^t + \varepsilon, \text{ if } t \neq m \\ \mathcal{V}(N) - \sum_{t \neq m} x^t, \text{ if } t = m \end{cases}$$

for an arbitrary positive ε.

Of course, $x \in v(N)$, i.e., (x, N) is a proposal. Obviously, $x^k > y^k_{\{k\}}$ but there are no $C \in \mathcal{C}$ such that $y^C_C > x^C$.

(b) $\mathcal{C} = \emptyset$. Pick an $k \in N$ such that there exists $C \in \mathcal{C}, C \not\ni k$. (This is possible if $\mathcal{C} \neq \{N\}$. To the contrary, $C(N, v) \neq \emptyset$). Define $x \in R^N$ by:

$$x^t = \begin{cases} w^t + \varepsilon, \text{ if } t \neq k \\ \mathcal{V}(N) - \sum_{t \neq k} x^t, \text{ if } t = k \end{cases}$$

for an arbitrary positive $\in$.

Again, (x, N) is a proposal and $x^C > y^C_C$. Moreover, if $C' \in \mathcal{C}$, then $y^{C'}_C \not> x^{C'}$.

(c) $\mathcal{L} = N$. This means that $(\mathcal{V}\{k\}), \{k\}) \in \mathcal{K}$ for every k. If $\mathcal{V}$ is not subadditive, then there exists $C \in 2^N$ such that $\mathcal{V}(C) > \sum_{k \in C} \mathcal{V}(\{k\})$, and it is easy to find $x \in \mathcal{V}(C)$ such that $x^k > \mathcal{V}(\{k\})$ for all $k \in C$. At the same time, for each $C \in \mathcal{C}$ it is necessary that $u^k_C = \mathcal{V}(\{k\})$ for every $k \in C$. So, again the properties of the c.s. are contradicted.

Now, we will show how we can extend Theorem 1 to cover the TU game satisfying (11) or (12). As we have seen in Section 3, we were constrained to impose some additional assumptions to $\mathcal{V}$ in order to satisfy the feasibility condition required by the definitions of competitive solutions.

Theorem 2. Suppose $\mathcal{V} : 2^N \to R$, monotonic and $\mathcal{V}$ defined by (11). Then the game (N, v) has a c.u.c.s.

Proof: Firstly, note that if $\mathcal{V}$ is monotonic, then $\mathcal{V}_M$ is too. Consequently, if $C \subset D$ then $v(C) \subset pr_C v(D)$ and this statement holds also for the reduced game. In other words, the feasibility condition is satisfied at every step in the proof of Theorem 1. It is easy to see now that the proof of Theorem 2 follows the same way as for Theorem 1.

Remark. Since $\mathcal{V}$ is nonnegative, $v(C) \neq \emptyset$ for every $C \in 2^N$. If we remove this assumption, then the proof of Theorem 2 could fail, because it is possible that $v(C) = \emptyset$ for some C.

However, Theorem 2 remains true even if $\mathcal{V}$ is not necessarily positive. The conclusion will follow from the main result of the next section.

Theorem 2'. Suppose $\mathcal{V} : 2^N \to R$, monotonic and $\mathcal{V}(N) \geq 0$. If v is defined by (11), then the game (N, V) has a c.u.c.s.

Theorem 3. Suppose $\mathcal{V}$ superadditive and v defined by (12). Then the game (N, v) has a c.u.c.s.

Proof: (N, v) is strategically equivalent to the game (N, v'), whose corresponding real-valued characteristic function is

$$\mathcal{V}'(C) = \mathcal{V}(C) - u_0(C), C \subset N, (u_0 = (\mathcal{V}(\{k\})_{k \in N}))$$

Obviously, $v'(C) = \{x \in R_+^C | x(C) \leq \mathcal{V}'(C)\}$.

Since $\mathcal{V}$ is superadditive, $\mathcal{V}'$ is monotonic and nonnegative. Indeed, if $C \subset D$, then,

$$\mathcal{V}'(C) - \mathcal{V}'(D) \doteq \mathcal{V}(C) - \mathcal{V}(D) + u_0(D \backslash C) \leq \mathcal{V}(C) + \mathcal{V}(D \backslash C) - \mathcal{V}(D) \leq 0$$

Consequently, by Theorem 2, (N, v') has a c.u.c.s. The conclusion of Theorem 3 follows now from Proposition 10.

Remark. The result established in Theorem 2 is contained in Theorem 3.1. of Bennett ([1]). The monotonicity assumed here is needed only for the effectiveness requirement.

5 Existence Theorems for NTU Games

Everywhere in this section, a game is defined by a triple (N, W, v), where $W \subseteq 2^N \backslash \{\emptyset\} W \neq \emptyset$, is the set of winning coalitions and v assigns to each $C \in 2^N$ a subset of R_+^C which is nonempty iff $C \in W$.

Theorem 4. Let the game (N, W, v) satisfy the following assumptions

(19) $C \in W, C \subset C' \Rightarrow C' \in W$

(20) $v(C)$ is compact (nonvoid) in R_+^C for every $C \in W$

(21) $x \in v(C), 0 \leqq y \leqq x \Rightarrow y \in v(C)$ (Comprehensivity)

(22) $C \in W, C \subset D \Rightarrow v(C) \subset pr_C v(D)$ (Weak monotonicity)

(23) $C \in 2^N, k \notin C, x \in v(C \cup \{k\})$ and $(x^C, a) \in v(C \cup \{k\})$ for some $0 \leq a < x^k \Rightarrow$ there exists $y \in v(C \cup \{k\})$ such that $y^C > x^C, y^k = a$.

Then, the game (N, v) has a c.u.c.s.

Remark. Under assumption (21), the condition (23) may be restated: ((23′) $C \in 2^N, k \notin C, x \in v(C \cup \{k\}), 0 \leq a < x^k \Rightarrow$ there exists $y \in v(C \cup \{k\})$ such that $y^C > x^C, y^k = a$.

Proof of the Theorem: We will prove it into two steps. Firstly, consider a particular case.

Case 1.

(24) $W = 2^N \setminus \{\emptyset\}$

By induction on $n = |N|$. Of course, for $n = 1$ the theorem holds. ($v(\{1\})$ is nonempty, compact, so $\mathcal{K} = \{(z, \{1\})\}$ is a c.u.c.s., where $z = \max\{x \in v(\{1\})\}$). Suppose the theorem is also true for every game satisfying (19′)–(23) with at most $n-1$ players ($n \geq 2$). Now consider a game having n players. Denote by $z^n = \max\{x | x \in v(\{n\})\}$ and $M = N \setminus \{n\}$. Define:

$$v_M(C) = \begin{cases} \emptyset \text{ if } C = \emptyset \\ v(C) \cup \{x \in R^C_+ | (x, x^n) \in v(C \cup \{n\})\} \text{ for some } x^n \geq z^n, \text{ if } \emptyset \neq C \subset M \end{cases}$$

Of course, v_M is the characteristic function of an $n-1$ person game (M, W_M, v_M) for which $W_M = \{C \subset M | v_M(C) \neq \emptyset\} = 2^M \setminus \{\emptyset\}$. This game also satisfies the assumption of case 1. Since (19), (19′) and (20) are trivial, let us verify (21), (22) and (23).

Let $x \in v_M(C)$ and $y \in R^C_+, y \leqq x$. If $x \in v(C)$, then $y \in v(C)$ and therefore, $y \in v_M(C)$. If, for some $x^n \geq z^n, (x, x^n) \in v(C \cup \{n\})$, then, taking $y^n = z^n$ it follows that $(y, y^n) \in v(C \cup \{n\})$ and hence, $y \in v_M(C)$. Consequently, (21) is also satisfied by v_M.

Suppose $C \subset D \subset M$ and $x \in v_M(C)$. If $x \in v(C)$, then $x \in pr_C v(D)$ and, therefore, $x \in pr_C v_M(D)$. If, for some $x^n \geq z^n, (x, x^n) \in v(C \cup \{n\})$, then $(x, x^n) \in pr_{C \cup \{n\}} v(D \cup \{n\})$. Then, there exists $y \in v(D \cup \{n\})$ such that $y^C = x$ and $y^n = x^n \geq z^n$. Hence, $y^D \in v_M(D)$ and $y^C = x$. Consequently, $x \in pr_C v_M(D)$.

Finally, let us verify (23). Take $Q \subset M, k \in M \setminus Q, x \in v(C)$ $(C = Q \cup \{k\})$ and $0 \leq a < x^k$. There are two possibilities:

(i) if $x \in v(C)$, then there exists $y \in v(C)$ with $y^Q > x^Q, y^k = a$, i.e.,

$y \in v_M(C), y^{C\setminus\{k\}} > x^{C\setminus\{k\}}, y^k = a.$

(ii) if $(x, x^n) \in v(C \cup |n|)$ for some $x^n \geq z^n$, then we can find $y \in v(C \cup \{n\})$ such that $y^{Q\cup\{n\}} > x^{Q\cup\{n\}}, y^k = a$. Particularly, since $y^n > x^n \geq z^n$ it follows that $y^C \in v_M(C)$. Since $y^Q > x^Q$ the proof is complete.

Now, since (M, W_M, v_M) has $n-1$ players, it has a c.u.c.s., say $\mathcal{K}_M$:

$$\mathcal{K}_M = \{(\bar{y}_i, C_i) | i = 1, \ldots, m\}$$

Denote by

$$C_i' = \begin{cases} C_i \text{ if } \bar{y}_i \in v(C_i) \\ C_i \cup \{n\} \text{ if } \bar{y}_i \notin v(C_i) \text{ (but } (\bar{y}_i, y_i^n) \in v(C_i \cup \{n\}) \text{ for some } y_i^n \geq z^n) \end{cases}$$

$$y_i = \begin{cases} \bar{y}_i \text{ if } C_i' = C_i \\ (\bar{y}_i, z^n) \text{ if } C_i' = C_i \cup \{n\} \end{cases}$$

$$\mathcal{K} = \{(y_i, C_i') | i = 1, \ldots, m\}$$

$$\mathcal{K}_N = \begin{cases} \mathcal{K}, \text{ if } C_i' \ni n \text{ for some } i \\ \mathcal{K} \cup (y_{m+1}, C_{m+1}') \text{ otherwise} \end{cases}$$

(here $C_{m+1}' = \{n\}, y_{m+1} = z^n$).

We will prove that $\mathcal{K}_N$ is a c.u.c.s. of (N, W, v).

Of course, $\cup C_i' = N$ since $\cup C_i = M$. Also, $y_i \in v(C_i')$. Note also that the effectiveness of y_i (i.e., $y_i \in pr_{C_i} v(N)$) is a direct consequence of (22). It is easy to verify (5) too (see Theorem 1). Let us verify (6), or, equivalently, (9). Denote by y the n vector whose components are $y^k = y_i^k$, whenever $k \in C_i'$ and by $\bar{y}$ the $n-1$ vector whose components are defined by $\bar{y}^k = \bar{y}_i^k$ whenever $k \in C_i$.

Suppose that $x^C \geq y^C$ for an $x \in v(C) \cap pr_C v(N)$. Then:

(i) If $n \notin C$, it follows that $x \in v_M(C), x^C \geq \bar{y}^C$, which is impossible by the definition of $\mathcal{K}_M$.

(ii) If $C = \{n\}$, then $x \leq z^n = y^n$, so the initial assumption fails.

(iii) If $C = Q \cup \{n\}$, $\emptyset \neq Q \subset M$, we will examine two possibilities.

a) $x^n = z^n$, then, $x^Q \in v_M(Q)$, $x^Q \geq \bar{y}^Q$ contradicting (9) of $\mathcal{K}_M$.

b) $x^n > z^n$. Then, by (23), there exists $u \in v(C)$ such that $u^Q > x^Q$, $u^n = z^n$. Hence, $u^Q \in v_M(Q), u^Q > x^Q \geq \bar{y}^Q$ and again we contradict the properties of $\mathcal{K}_M$.

Case 2.

Now we consider the general case, when W is not necessarily $2^N \backslash \{\emptyset\}$.

Pick $T \in W$ such that $|T| = \min\{|C| | C \in W\}$. Take $z \in R_+^T$, Pareto-optimal in $v(T)$ and denote by $M = N \backslash T$. Define for each $C \subset M$,

$$v_M(C) = \begin{cases} \emptyset \text{ if } C = \emptyset \\ \cup_{S \subset T}\{x \in R_+^C | (x, z^S) \in v(C \cup S)\} \text{ if } \emptyset \neq C \subset M \end{cases}$$

Of course, $W_M = \{C \in M | v_M(C) \neq \emptyset\} = 2^M \backslash \{\emptyset\}$. Indeed, by (19), $C \cup T \in W$ and by (22) $z^T = z \in v(T) \subseteq pr_T v(C \cup T)$, so that, $\{x \in R_+^C | (x, z^T) \in v(C \cup T)\} \neq \emptyset$. Moreover, the triple (M, W_M, v_M) satisfies the assumptions of the theorem. Since the proof is the same as in case 1 we will not repeat it.

Let $\mathcal{K}_M = \{(\bar{y}_i, C_i) | i = 1, \ldots, m\}$ a c.u.c.s. of (M, W_M, v_M). Of course, $\bar{y}_i \in v_M(C_i)$, that is, there exists $S_i \subset T_i$ such that $y_i = (\bar{y}_i, z^{S_i}) \in v(C_i \cup S_i)$. Denote by $C_i' = C_i \cup S_i$ and put $\mathcal{K} = \{((\bar{y}_i, z^{S_i}), C_i') | i = 1, \ldots, m\}$ and

$$\mathcal{K}_N = \begin{cases} \mathcal{K}, \text{ if } \cup S_i = T \\ \mathcal{K} \cup \{(z, T)\}, \text{ otherwise} \end{cases}$$

(for convenience we will denote (y_{m+1}, C_{m+1}') for (z, T))

$\mathcal{K}_N$ is a c.u.c.s. of (N, W, v).

Obviously, $y_i \in v(C_i')$ for all i and, by (22), $y_i \in pr_{C_i'} v(N)$. To verify (5) we proceed as in case 1. Let us verify (9). Suppose $C \in W, x \in v(C) \cap pr_C v(N)$ and $x \geq y^C$ ($y = (y^k)_{k \in N}$ where $y^k = \bar{y}_i^k$ if $k \in C_i$ for some $i \leq m$, and $y^k = z^k$ if $k \in T$).

Of course, $C = T$ or $C \cap M \neq \emptyset$.

If $C = T$, then $x \in v(T), x \geq z$, which is impossible since z is Pareto-optimal in $v(T)$.

Then assume $C = Q \cup S$, where $\emptyset \neq Q \subset M, S \subset T$. We have either $x^Q \geq \bar{y}^Q$ or $x^S \geq z^S$ ($\bar{y}$ is the restriction of y to M).

(i) Suppose $x^Q \geq \bar{y}^Q$. Clearly, by (21), $(x^Q, z^S) \in v(Q \cup S)$. Then, $x^Q \in v_M(Q)$ and the previous inequality contradicts the properties of $\bar{y}$.

(ii) Suppose $x^S \geq z^s$. Pick a $k \in S$ such that $x^k > z^k$. Since $(\bar{y}^Q, s^S) \in v(Q \cup S)$ it follows by (23), that there exists some $u \in v(Q \cup S)$ such that $u^k = z^k, u^Q > \bar{y}^Q, u^{S \setminus \{k\}} > x^{S \setminus \{k\}}$. Again by (21), $(u^Q, z^S) \in v(Q \cup S)$. Then $u^Q \in v_M(Q)$. Since $u^Q > \bar{y}^Q$, we contradict the definition of $\mathcal{K}_M$.

Corollary. Theorem 2′.

Proof: Of course, if $\mathcal{V}$ is monotonic and $\mathcal{V}(N) \geq 0$, then, $W = \{C | v(C) \neq \emptyset\}$ is not empty and is closed under the set-inclusion relation (i.e. satisfies (19)). All other assumptions of Theorem 4 are trivially verified by the TU game defined by (11).

Remark. The assumptions of theorem 4 are weaker than the usual conditions which imply the existence of the core. It is easy to see that the weak monotonicity used here is a weaker condition than convexity or superadditivity. However, (23) may be violated by NTU games satisfying other usual conditions. The following example shows that the existence of u.c.s. can't be guaranteed if (23) is removed.

Example 5. $n = 3$. $v(\{i\}) = [0, 1], i = 1, 2, 3; v(\{1, 2\}) = [0, 1.5] \times [0, 2]; v(\{1, 3\}) = [0, 2] \times [0, 1.5]; v(\{2, 3\}) = [0, 1.5] \times [0, 2]; v(\{1, 2, 3\}) = \{(x^1, x^2, x^3) \in R^3_+ | x^1 + x^2 + x^3 \leq 3.5\}$.

Note that this game does not have any competitive solution either.

Remark. The assumption (21) may be removed in Theorem 1 if we will impose (23) for the comprehensive closure of v. Indeed, if we define,

$$\bar{v}(C) = \begin{cases} \emptyset, \text{ if } v(C) \emptyset \\ \{y \in R^C_+ | \text{ there exists } x \in v(C) \text{ such that } y \leqq x\} \text{ otherwise,} \end{cases}$$

(the comprehensive closure of $v(C)$)

we are able to prove the following results.

Proposition 12. If $\mathcal{K} = \{(y_i, C_i) | i = 1, \ldots, m\}$ is a c.u.c.s. of (N, W), then it is a c.u.c.s. of (N, W, v) too.

Proof: It is sufficient to verify that $y_i \in v(C_i)$. In the contrary, there exists $x \in v(C_i) \subset \bar{v}(C_i), y \leq x$. But, then, there exists j and $k \in C_i \cap C_j$ such that $y^k_j > x^k \geq y^k_i$, which is impossible since $\mathcal{K}$ is u.c.s. of $(N, W, \bar{v})$.

Proposition 13. If v satisfies (20) and (22), then $\bar{v}$ also satisfies them. Moreover, $\bar{v}$ is comprehensive.

Proof: Since $\bar{v}(C)$ is obviously bounded, it is sufficient to prove that it is closed. Let $y_k \to y, y_k \in \bar{v}(C)$. Then, there exist $x_k \in v(C), y_k \leqq x_k$, for all k. Since $v(C)$ is compact we may assume that $x_k \to x \in v(C)$ (otherwise a subsequence will be considered). Clearly, $y \leqq x$, i.e., $y \in \bar{v}(C)$. To verify (22), let $y \in \bar{v}(C)$ and $C \subset D$. There is $x \in v(C)$ such that $y \leqq x$. Also, there is a $z \in v(D)$ with $x = z^C$. Then, $(y, z^{D \setminus C}) \leqq z$. Hence, $(y, z^{D \setminus C}) \in \bar{v}(D)$ and $y \in pr_C \bar{v}(D)$. Now, we can restate the theorem.

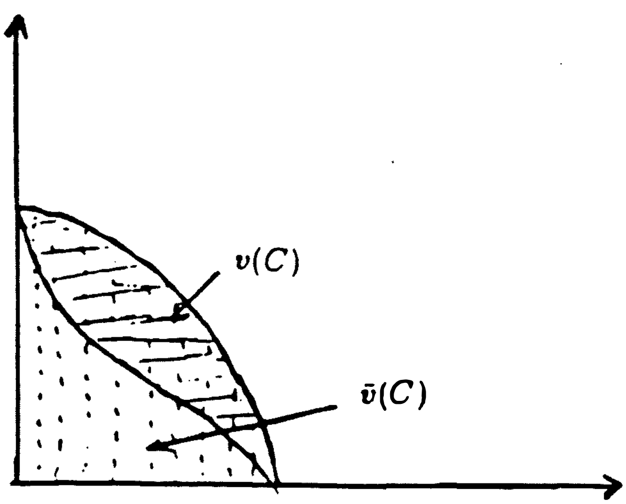
Theorem 1′. Suppose (N, W, v) satisfy (19), (20), (22), and

(23′) $C \in 2^N, k \notin C, x \in \bar{v}(C \cup \{k\}), 0 \leq a < x^k$ there exists $y \in \bar{v}(C \cup \{k\})$ such that $y^C > x^C, y^k = a$.

Then, the game (N, W, v) admits a c.u.c.s.

Remark. Theorem 1′ actually covers different situation. For instance, see Figure 2.

Figure 2



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