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THE MAXIMAL NUMBER OF REGULAR TOTALLY MIXED NASH EQUILIBRIA

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Abstract

Let $S = \prod_{i=1}^{n} S_i$ be the strategy space for a finite n-person game. Let $(s_{10}, \ldots, s_{n0}) \in S$ be any strategy n-tuple, and let $T_i = S_i - \{s_{i0}\}$, $i = 1, \ldots, n$. We show that the maximum number of regular totally mixed Nash equilibria to a game with strategy sets S_i is the number of partitions $\mathcal{P} = \{P_1, \ldots, P_n\}$ of $\cup_i T_i$ such that, for each i, $\#P_i = \#T_i$ and $P_i \cap T_i = \emptyset$. The bound is tight, as we give a method for constructing a game with the maximum number of equilibria.

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The Maximal Number of Regular Totally Mixed Nash Equilibria

by

Richard D. McKelvey and Andrew McLennan

1 Introduction

Harsanyi's (1973) theorem states that, for generic (that is, outside an exceptional set whose closure has Lesbesgue measure zero) payoffs of a normal form, there are finitely many Nash equilibria, all of them 'regular.' Roughly, an equilibrium is regular if the strategies assigned no probability have strictly suboptimal expected payoffs, and the derivative of those equilibrium conditions that are satisfied with equality is nonsingular. Harsanyi's methods do not suffice to show that there is a maximal number of regular equilibria, but this is a consequence of Bezout's theorem. For any regular equilibrium, the implicit function theorem implies that nearby payoffs have nearby regular equilibria, so the maximal number of regular equilibria is attained on an open subset of the set of payoffs.

A Nash equilibrium is *totally mixed* if every pure strategy is assigned positive probability. Any Nash equilibrium gives rise to a totally mixed equilibrium of the smaller game obtained by eliminating all unused pure strategies, so the maximal number of regular totally mixed Nash equilibria is a lower bound for the maximal number of regular Nash equilibria.

Let $S = \prod_{i=1}^{n} S_i$ be the strategy space for a finite n-person game. Let $(s_{10}, \ldots, s_{n0}) \in S$ be any strategy in n-tuple, and let $T_i = S_i - \{s_{i0}\}$, $i = 1, \ldots, n$. We show that the maximum number of regular totally mixed Nash equilibria to a game with strategy sets S_i is the number of partitions $\mathcal{P} = \{P_1, \ldots, P_n\}$ of $\cup_i T_i$ such that $\#P_i = \#T_i$ and $P_i \cap T_i = \emptyset$ for all i.

This result has implications for the complexity of the various computational problems arising out of the concepts of noncooperative game theory, most obviously for the problem of enumerating all equilibria. In §5 we provide closed form upper and lower bounds on the number of partitions described in the preceding paragraph. Roughly, the lower bound implies that the maximal number of regular totally mixed Nash equilibria grows rapidly with the number of agents and the numbers of pure strategies they can choose from. For example, fixing the number of pure strategies for each agent, the maximal number is exponential in the number of agents, while if we fix the number of agents, the maximal number is exponential in the minimum number of pure strategies for any one player. Here we illustrate these results in the following table giving the maximal number for normal forms in which each of n agents has k pure strategies to choose from.

		k							
		2	3	4	5	6	7	8	9
	2	1	1	1	1	1	1	1	1
	3	2	10	56	346	2252	15184	104960	739162
	4	9	297	13833	748521	4.4×10^7	2.8×10^9	1.8×10^{10}	1.2×10^{13}
	5	44	13756	6.7×10^6	4.0×10^9	2.7×10^{12}	1.9×10^{15}	1.5×10^{18}	1.2×10^{21}
n	6	265	925705	5.7×10^9	4.5×10^{13}	4.1×10^{17}	4.2×10^{21}		
	7	1854	8.5×10^7	7.8×10^{12}	9.6×10^{17}				
	8	14833	1.0×10^{10}	1.6×10^{16}					
	9	133496	1.6×10^{12}						
	10	1.3×10^6							

Table 1.1

Maximum generic number of totally mixed Nash equilibria
for an n-person game, where each player has k pure strategies

Another immediate consequence of our characterization is that if S_i is the largest strategy set, then a necessary and sufficient condition for there to exist payoffs for which there is a totally mixed regular Nash equilibrium is that $(\#S_i - 1) \leq \sum_{j\neq i}(\#S_j - 1)$.

The remainder of the paper is organized as follows. In §2 we show how to transform the usual expression of the conditions of totally mixed Nash equilibrium into a form that is well suited for our analysis. The transformed system developed in §2 eliminates one variable for each player and eliminates the requirements that probabilities add to one and are positive. We show that the transformed system has the same maximal number of regular solutions as the original system. The transformed algebraic system consists only of equations, and makes sense as a system of equations in several complex variables with complex coefficients. This allows us in §3 to apply a remarkable theorem of Bernstein (1975) (who built on the work of Kushnirenko (1975)) to obtain a combinatoric characterization of the number of *complex* regular solutions of the transformed system for payoffs that are generic in the space of *complex* payoffs. Since each regular equilibrium can be extended locally (in the space of complex payoffs) to a branch of the equilibrium correspondence, this number is automatically an upper bound on the maximal number of regular totally mixed equilibria for real payoffs.

In §4 we construct a real payoff for the game in which all equilibria are regular, and there are as many (real) equilibria as are allowed by Bernstein's theorem. Thus it follows that this is the maximum possible number of regular totally mixed equlibria, and there is an open set of real payoffs on which this number is attained.

In §5 we provide a recursive combinatoric characterization of the number derived from the application of Bernstein's theorem to our problem, and this characterization is used to develop closed form upper and lower bounds. In §6 we illustrate our results by analysing in detail the simplest non-trivial example, the three person game where each player has two strategies.

2 A Sequence of Related Problems

We consider a finite player, finite strategy normal form game determined by a set of players $I = \{1, \ldots, n\}$, and finite nonempty disjoint sets of pure strategies $S_1, \ldots, S_n$. For each i let $d_i = \#S_i - 1$, and let $S_i = \{s_{i0}, \ldots, s_{id_i}\}$. Define $S = \prod_{i \in I} S_i$.

For each $i \in I$, let Σ_i be the set of all functions $\sigma_i : S_i \to \mathbb{R}$. Let $\Sigma = \prod_{i \in I} \Sigma_i$, and let $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$. We write $\sigma_{ik} = \sigma_i(s_{ik})$. The projection of $\sigma \in \Sigma$ onto Σ_{-i} will typically be denoted by σ_{-i}. We will often write s_i in place of the corresponding unit vector in Σ_i, s in place of the corresponding element in Σ, and so forth.

Let U be the set of multilinear (that is, linear in each agent's strategy) functions $u : \Sigma \to \mathbb{R}$, and $\mathbf{U} = U^I$. The specification of an element $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbf{U}$ is equivalent to the customary description of a payoff function for an n person normal form game, since u_i is determined by the vector $(u_i(s))_{s \in S}$ via the formula

$$u_i(\sigma) = \sum_{s \in S} \Big(\prod_{j \in I} \sigma_j(s_j)\Big)\, u_i(s).$$

We will only be concerned with totally mixed equilibria. The following is an algebraic expression of the condition that $\sigma \in \Sigma$ is a *totally mixed Nash equilibrium*:

$$
\begin{aligned}
&(\mathrm{a}_1) \quad 0 = u_i(s_{ik}, \sigma_{-i}) - u_i(s_{i0}, \sigma_{-i});\\
&(\mathrm{b}_1) \quad 0 < \sigma_{ik};\\
&(\mathrm{c}_1) \quad 1 = \sum_{k=0}^{d_i} \sigma_{ik}.
\end{aligned}
$$

(In this system, and in similar systems below, all equations are understood to hold for all values of the variables, i.e., for all $i \in I$ and $0 \leq k \leq d_i$ in (a_1) and (b_1), and all $i \in I$ in (c_1).)

We now recast (a_1) in a way that reduces the number of parameters by taking advantage of the fact that the equilibrium conditions depend only on the *differences* between the expected payoffs of the various pure strategies. Let $Y = \prod_{i \in I} \mathbb{R}^{d_i}$. Conditions (b_1) and (c_1) describes a $\sum_{i \in I} d_i$ dimensional manifold, which we call $\mathbf{\Delta}$. Let $V_i \subseteq U$ be the set of all multilinear maps on Σ_{-i}, and let $\mathbf{V} = \prod_i V_i^{d_i}$. An element of $\mathbf{V}$ may be interpreted as a map with domain $\mathbf{\Delta}$ and range Y. Let $\mathrm{proj} : \mathbf{U} \to \mathbf{V}$ be the linear surjection mapping a payoff $\mathbf{u}$ into a point $\mathbf{v} \in \mathbf{V}$ by the rule:

$$
v_i^k(\sigma) = u_i(s_{ik}, \sigma_{-i}) - u_i(s_{i0}, \sigma_{-i}).
$$

Thus (a_1) is equivalent to $\mathbf{v}(\sigma) = 0$, where $\mathbf{v} = \mathrm{proj}(\mathbf{u})$.

We say that σ, a totally mixed equilibrium for $\mathbf{u}$, is *regular* if σ is a regular point of $\mathbf{v} = \mathrm{proj}(\mathbf{u})$. (That is, $d\mathbf{v}_\sigma : T_\sigma \mathbf{\Delta} \to T_{\mathbf{v}(\sigma)} Y$ is surjective at σ, where $T_\sigma \mathbf{\Delta}$ and $T_{\mathbf{v}(\sigma)} Y$ are the tangent spaces of $\mathbf{\Delta}$ and Y at σ and $\mathbf{v}(\sigma)$ respectively.) Define $\mathbf{U}[k]$ to be the set of all $\mathbf{u} \in \mathbf{U}$ for which there are at least k regular totally mixed equilibria. It follows from the implicit function theorem that if σ is a regular equilibrium for $\mathbf{u}$, then nearby payoffs have nearby regular equilibria. So $\mathbf{U}[k]$ is an open set, for all k. Let k^* be the maximum k for which $\mathbf{U}[k]$ is not empty. (That such a k^* exists is a consequence of Bezout's theorem, and also of its generalization by Kushnirenko and Bernstein, which we present in §3.) We want to characterize k^*.

Define $\mathbf{V}[k]$ to be the set of all $\mathbf{v} \in \mathbf{V}$ such that there are at least k regular points of $\mathbf{v} : \mathbf{\Delta} \to Y$ which are solutions of the equation $\mathbf{v}(\sigma) = 0$. It follows from the above discussion that $\mathbf{V}[k] = \mathrm{proj}(\mathbf{U}[k])$ and $\mathbf{U}[k] = \mathrm{proj}^{-1}(\mathbf{V}[k])$. Since proj is an open map, each $\mathbf{V}[k]$ is open, and the maximum k for which $\mathbf{V}[k]$ is non empty is equal to k^*. Thus our problem reduces to finding $\mathbf{v} \in \mathbf{V}$ for which there is the maximum number of regular solutions, $\sigma \in \boldsymbol{\Delta}$, of the system of equations $\mathbf{v}(\sigma) = 0$.

Now note that the components of such a $\mathbf{v}$ are homogeneous polynomials. Therefore, the truth value of the condition $\mathbf{v}(\sigma) = 0$ is unaffected by replacing any σ_i with $\lambda_i \sigma_i$ for any nonzero scalar λ_i. From this point of view (c_1) is a normalization specifying a particular representative of each n-dimensional space of solutions of (a_1). But it will be more convenient in the analysis to use a different normalization, namely $\sigma_{i0} = 1$, since in this way we eliminate one variable for each agent.

For each i let $T_i = \{s_{i1}, \ldots, s_{id_i}\} = S_i - \{s_{i0}\}$. Define $\mathcal{T}_i$ to be the set of functions from T_i into $\mathbb{R}$. Typical elements of $\mathcal{T}_i$ will be denoted by τ_i. We write $\tau_{ik} = \tau(s_{ik})$. Define $\mathcal{T} = \prod_i \mathcal{T}_i$.

Define W_i to be the space of multi-affine maps $w : \mathcal{T} \to \mathbb{R}$ which are constant on $\mathcal{T}_i$. So W_i is the space of maps that are of degree 0 in each component of τ_i and of degree 1 in each component of each τ_j, for $j \neq i$. Then set $\mathbf{W} = \prod_i W_i^{d_i}$. Now let $\Phi : \mathbf{V} \to \mathbf{W}$ be the mapping which takes $\mathbf{v}$ into $\mathbf{w}$ via the rule

$$w_i^k(\tau_1, \ldots, \tau_n) = v_i^k((1, \tau_1), \ldots, (1, \tau_n))$$

Using the homogeneity of $\mathbf{v}$, Φ defines a bijection between $\mathbf{V}$ and $\mathbf{W}$: for any $\mathbf{w} \in \mathbf{W}$, $\mathbf{v} = \Phi^{-1}(\mathbf{w})$ is defined by $v_i^k(\sigma) = (\prod_i \sigma_{i0}) w_i^k(\phi(\sigma))$, where $\phi(\sigma) = (\phi_1(\sigma_1), \ldots, \phi_n(\sigma_n))$ is defined by $\phi_{ik}(\sigma_i) = \frac{\sigma_{ik}}{\sigma_{i0}}$ for $1 \leq i \leq d_i$.

Write $\mathcal{T}_+ = \{\tau \in \mathcal{T} : \tau_{ik} > 0 \text{ for all } i, k\}$. Then restricting ϕ to domain $\boldsymbol{\Delta}$, we have that $\phi : \boldsymbol{\Delta} \to \mathcal{T}_+$ is a bijection. We can write, $\mathbf{w}(\tau) = (\prod_i \sigma_{i0}(\tau))^{-1} \mathbf{v}(\phi_{-1}(\tau))$. But then $\mathbf{w}(\tau) = 0$ if and only if $\mathbf{v}(\phi^{-1}(\tau)) = 0$. But for any τ with $\mathbf{v}(\phi^{-1}(\tau)) = 0$, $d\mathbf{w}_\tau = (\prod_i \sigma_{i0}(\tau))^{-1} d\mathbf{v}_{\phi^{-1}(\tau)} \circ d\phi_\tau^{-1}$. Since $d\phi_\tau^{-1}$ is of full rank, it follows that τ is a regular point of $\mathbf{w}$ if and only if $\phi^{-1}(\tau)$ is a regular point of $\mathbf{v}$.

We say that $\tau \in \mathcal{T}$ is a *solution* of $\mathbf{w}$ if $\mathbf{w}(\tau) = 0$. A solution τ of $\mathbf{w}$ is *regular* if it is a regular value of $\mathbf{w} : \mathcal{T} \to Y$. Define $\mathbf{W}[k]$ to be the set of all $\mathbf{w} \in \mathbf{W}$ such that there are at least k regular solutions of $\mathbf{w}$.

Therefore, our problem is equivalent to finding $\mathbf{w} \in \mathbf{W}$ for which there is the maximum number of regular solutions, $\tau \in \mathcal{T}_+$, of the system

$$(a_2) \quad 0 = \mathbf{w}(\tau);$$

$$(b_2) \quad 0 < \tau_{ik}.$$

Now for any fixed $\tau_0 \in \mathcal{T}$ we can define a bijection from $\mathbf{W}$ into itself by the rule $\mathbf{w}_{\tau_0}(\tau) = \mathbf{w}(\tau + \tau_0)$. Clearly τ is a regular solution of $\mathbf{w}$ if and only if $\tau - \tau_0$ is a regular solution of $\mathbf{w}_{\tau_0}$. Thus any regular solution τ of $\mathbf{w}$ for which $\tau > \tau_0$ corresponds to a regular solution $\tau - \tau_0$ of $\mathbf{w}_{\tau_0}$ for which $\tau - \tau_0 > 0$.

For any finite set of regular solutions of (a_2), say $\{\tau^1, \ldots, \tau^k\}$, it is possible to find τ_0 such that (b_2) is satisfied by each $\tau^j - \tau_0$. The consequence of this is that if the set of $\mathbf{w} \in \mathbf{W}$ having a certain number of regular solutions of (a_2) has a nonempty interior, then so does the set of $\mathbf{w}$ having that number of regular solutions of (a_2), and (b_2). Hence, condition (b_2) can be dropped. Summarizing,

Lemma 2.1: *For any k, $\mathbf{U}[k]$ is nonempty if and only if $\mathbf{W}[k]$ is nonempty.*

For any $\mathbf{w} \in \mathbf{W}$, as long as the set of regular solutions, $\{\tau^1, \ldots, \tau^k\}$, of $\mathbf{w}$ can be computed, the proof of lemma 2.1 provides a means of constructing a corresponding normal form game $\mathbf{u} \in \mathbf{U}$ with k regular Nash equilibria: Find $\tau_0 \in \mathcal{T}$ such that $\tau^j > \tau_0$ for all k. Then any $\mathbf{u} \in \mathrm{proj}^{-1} \circ \Phi^{-1}(\mathbf{w}_{\tau_0})$ will have k regular solutions at $\{\sigma^1, \ldots, \sigma^k\}$, where $\sigma^j = \phi^{-1}(\tau^j - \tau_0)$. Section 4 provides a method to find a $\mathbf{w} \in \mathbf{W}[k^*]$ for which the τ^j can be computed. Together with the above argument, this gives a constructive method to find a normal form game with the maximum number of regular Nash equilibria.

Bernstein's (1975) theorem, which we apply in §3, considers solutions in which no variable vanishes. Thus we take our goal to be the characterization of the maximal number of regular solutions of the system

$$(*) \qquad \begin{aligned} &(\mathrm{a}_3) \quad 0 = \mathbf{w}(\tau); \\ &(\mathrm{b}_3) \quad 0 \neq \tau_{ik}. \end{aligned}$$

3 The Number of Complex Nash Equlibria

Since it contains no inequalities, the system $(*)$ makes sense over the field of complex numbers, and in this section we study it in that context. For each $i \in I$, let $\tilde{\mathcal{T}}_i$ be the space of functions $\tilde{\tau} : T_i \to \mathbb{C}$. Let $\tilde{\mathcal{T}} = \prod_{i \in I} \tilde{\mathcal{T}}_i$, and $\tilde{\mathcal{T}}_{-i} = \prod_{j \neq i} \tilde{\mathcal{T}}_j$. Let $\tilde{W}_i$ be the space of multiaffine maps $\tilde{w} : \tilde{\mathcal{T}} \to \mathbb{C}$ which are constant on $\tilde{\mathcal{T}}_i$, and let $\tilde{\mathbf{W}} = \prod_{i \in I} \tilde{W}_i^{d_1}$ For each i let $\tilde{\mathcal{T}}_i^* = \{ \tilde{\tau}_i \in \tilde{\mathcal{T}}_i : \tilde{\tau}_i(s_i) \neq 0$ for all $s_i \in T_i \}$. A *complex equilibrium* for $\tilde{\mathbf{w}} \in \tilde{\mathbf{W}}$ is a $\tilde{\tau} \in \tilde{\mathcal{T}}$ satisfying $(*)$. As in the real case, a complex equilibrium $\tilde{\tau}$ for a payoff $\tilde{\mathbf{w}}$ is *regular* if $d\tilde{\mathbf{w}}(\tilde{\tau}) \in L(\tilde{\mathcal{T}}, \mathbb{C}^{d_1 + \cdots + d_n})$ is surjective.

For all $i \in I$ and $k = 1, \ldots, d_i$, the exponent vectors $\alpha \in \prod_{i=1}^n \mathbb{Z}^{T_i}$ such that $\tau^\alpha = \prod_{i=1}^n \prod_{j=1}^{c_i} \tau_{ij}{}^{\alpha_{ij}}$ has a (potentially) nonzero coefficient in w_i^k are precisely the elements of

$$\mathcal{A}_{ik} = \mathbf{0}_i \times \prod_{j \neq i} B_j \subseteq \prod_{i=1}^n \mathbb{Z}^{T_i},$$

where $\mathbf{0}_i \in \mathbb{Z}^{T_i}$ is the vector of 0's, and B_j is the set containing the origin and the standard unit basis vectors in $\mathbb{R}^{T_j}$.

Define $Q_{ik} = co(\mathcal{A}_{ik}) \subseteq \prod_{i=1}^n \mathbb{R}^{T_i}$. Then Q_{ik} is the *Newton polytope* of w_i^k.

Lemma 3.1 : *For for any* $\lambda = \{\lambda_{ik} \in \mathbb{R}_+ : i \in I, k = 1, \ldots, d_i\}$ *define* $Q_\lambda = \sum_{i \in I} \sum_{k=1}^{d_i} \lambda_{ik} Q_{ik}$. *For* $i, j = 1, \ldots, n$ *and* $k = 1, \ldots, d_i$ *let* λ_{ik}^j *be* 0 *if* $j = i$ *and* λ_{ik} *if* $j \neq i$. *For* $j \in I$ *let* $\lambda_{-j} = \sum_{i=1}^n \sum_{k=1}^{d_i} \lambda_{ik}^j$. *Then*

$$Q_\lambda = \prod_{j \in I} \lambda_{-j} \mathrm{co}(B_j).$$

Proof: Using the general fact that, for sets $E_1, E_2 \subset \mathbb{R}^k$ and $F_1, F_2 \subset \mathbb{R}^\ell$,

$$(E_1 + E_2) \times (F_1 + F_2) = (E_1 \times F_1) + (E_2 \times F_2) \subset \mathbb{R}^k \times \mathbb{R}^\ell,$$

we have

$$\begin{aligned} Q_\lambda = \sum_{i=1}^n \sum_{k=1}^{d_i} \lambda_{ik} Q_{ik} &= \sum_{i=1}^n \sum_{k=1}^{d_i} \lambda_{ik} \cdot \mathrm{co}(\mathbf{0}_i \times \prod_{j \neq i} B_j) = \mathrm{co}(\sum_{i=1}^n \sum_{k=1}^{d_i} \prod_{j=1}^n \lambda_{ik}^j B_j) \\ &= \mathrm{co}(\prod_{j=1}^n (\sum_{i=1}^n \sum_{k=1}^{d_i} \lambda_{ik}^j) B_j) = \mathrm{co}(\prod_{j=1}^n \lambda_{-i} B_j). \qquad \blacksquare \end{aligned}$$

Let $V_m(A)$ denote the m-dimensional volume of $A \subseteq \mathbb{R}^m$. Note that the d_j-dimensional volume of $\mathrm{co}(B_j)$ is $\frac{1}{d_j!}$. Let $d = d_1 + \ldots + d_n$. It now follows immediately that

Corollary 3.2 :

$$V_d(Q_\lambda) = \prod_{i \in I} V_{d_i}(\lambda_{-i} B_i) = \prod_{i \in I} \frac{1}{d_i!} (\sum_{j \neq i} \sum_{k=1}^{d_i} \lambda_{jk})^{d_i}. \qquad (**)$$

Bezout's theorem characterizes the generic (in the space of coefficients) number of roots, in complex d-dimensional projective space, of a system of d homogeneous polynomials. The papers of Kushnirenko (1975) and Bernstein (1975) give a combinatoric characterization of the generic number of complex roots of "sparse" systems of polynomials, where sparseness means that some of the coefficients are required to vanish, so that in effect one is considering a coordinate subspace of the space of coefficients considered by Bezout's theorem. Specifically, Bernstein's theorem states that the generic number of complex solutions of the system (*) is equal to the *mixed volume*, $\mathcal{M}(Q_1, \ldots, Q_n)$ of the sets Q_i, which is defined as the coefficient of $\lambda_{11} \cdot \ldots \cdot \lambda_{1d_1} \cdot \ldots \cdot \lambda_{n1} \cdot \ldots \cdot \lambda_{1d_n}$ in the expansion of (**) as a polynomial in the variables λ_{ik}. Let Ψ be the set of partitions $\mathcal{P} = \{P_1, \ldots, P_n\}$ of $T_1 \cup \ldots \cup T_n$ such that, for each i, $\#P_i = d_i$ and $P_i \cap T_i = \emptyset$. By expanding (**) it is easy to obtain

Theorem 3.3: *$\mathcal{M}(Q_1, \ldots, Q_n)$ is the number of elements of Ψ.*

4 A Real System with the Generic Number of Regular Solutions

In this section we consider the special case of system $(*)$ in which, for each i and $k = 1, \ldots, d_i$,

$$w_i^k(\tau_{-i}) = \prod_{j \neq i} a_{ij}^k(\tau_j), \tag{5.1}$$

where the $a_{ij}^k : \Sigma_j \to \mathbb{R}$ are nonzero affine functionals. For each $j = 1, \ldots, n$, let

$$L_j = \{\, a_{ij}^k : i \neq j, k = 1, \ldots, d_i \,\}.$$

Theorem 4.1: *If the functionals a_{ij}^k are in general position in the sense that, for any j and any subset $L \subset L_j$ of cardinality $q \leq d_j$, the affine subspace of $\mathcal{T}_j$ on which all elements of L vanish has dimension $d_j - q$, then the system $(*)$ has $\#(\Psi)$ solutions, each of which is regular.*

Proof: At a solution τ, for each j at most d_j elements of L_j vanish. Since there are $\sum_i d_i$ functions w_i^k, in order for each to vanish, for each j exactly d_j elements of L_j must vanish at τ_j, and for each i and k there must be exactly one $j \neq i$ such that a_{ij}^k vanishes. Thus each solution τ induces a partition of $\{\, w_i^k : i = 1, \ldots, n, k = 1, \ldots, d_i \,\}$ into sets $U_1, \ldots, U_n$ where $w_i^k \in U_j$ if $a_{ij}^k(\tau_j) = 0$. Note that $\#(U_j) = d_j$ for each j and $v_i^k \in U_j$ implies that $j \neq i$. Conversely, given a partition with these properties, there is exactly one corresponding solution.

By varying τ_j in the intersection of the kernels of the other $a^{k'}_{i'j} \in L_j$ with $a^{k'}_{i'j}(\tau_j) = 0$, one can change the value of w^k_i (since no other $a^k_{ij'}$ vanishes at τ) without affecting any other $w^{k'}_{i'}$. Thus the image of the derivative of $\mathbf{w}$ includes each standard basis vector of $\mathbb{R}^{d_1+\ldots+d_n}$, whence τ is a regular solution. ■

5 The Function $\mathcal{L}_n$

In this section we study the asymptotic order of magnitude of $\#(\Psi)$, beginning with the following inductive characterization.

Proposition 5.1 : *For each $n = 1, 2, \ldots$, and each pair $\mathbf{c} = (c_1, \ldots, c_n)$, $\mathbf{d} = (d_1, \ldots, d_n)$ of vectors of nonnegative integers with $c_1 + \ldots + c_n = d_1 + \ldots + d_n$, let $\mathcal{L}_n(\mathbf{c}, \mathbf{d})$ be the number of partitions $\mathcal{P} = \{P_1, \ldots P_n\}$ of $T_1 \cup \ldots \cup T_n$ with $\#(P_i) = c_i$ and $T_i \cap P_i = \emptyset$, where $T_1, \ldots, T_n$ is a collection of pairwise disjoint sets in which $\#(T_i) = d_i$. Then*

$$\mathcal{L}_n : \{ (\mathbf{c}, \mathbf{d}) \in \mathbf{Z}^n \times \mathbf{Z}^n : \sum_i c_i = \sum_i d_i \} \to \mathbf{Z}_+$$

if the unique function satisfying the following conditions:

(a) $\mathcal{L}_n(\mathbf{0}, \mathbf{0}) = 1$;

(b) $\mathcal{L}_n(\mathbf{c}, \mathbf{d}) = 0$ *whenever* $(\mathbf{c}, \mathbf{d}) \notin \mathbf{N}^n \times \mathbf{N}^n$;

(c) *for each* $i = 1, \ldots, n$, *if* $d_i > 0$ *then*

$$\mathcal{L}_n(\mathbf{c}, \mathbf{d}) = \sum_{j \neq i} \mathcal{L}_n(\mathbf{c} - \mathbf{e}_j, \mathbf{d} - \mathbf{e}_i).$$

where $\mathbf{e}_i \in \mathbb{R}^n$ is the unit basis vector (that is, $e_{ij} = 1$ if $i = j$, and $e_{ij} = 0$ otherwise.)

Proof: In view of (c), induction on $c_1 + \ldots + c_n = d_1 + \ldots + d_n$ implies that there is at most one function satisfying (a) – (c). That $\mathcal{L}_n$ satisfies these conditions also follows from induction on $c_1 + \ldots + c_n = d_1 + \ldots + d_n$. Specifically, fixing i with $d_i > 0$ and $v \in T_i$, for each $j \neq i$ with $c_j > 0$ there is a one-to-one correspondence between partitions $\{P_1, \ldots P_n\}$ as above with $v \in P_j$ and partitions $\{W_1, \ldots, W_n\}$ of $T_1 \cup \ldots \cup T_i - \{v\} \cup \ldots \cup T_n$ with $\#(W_k) = c_k$ for $k \neq j$, $\#(W_j) = c_j - 1$, and $T_i \cap W_i = \emptyset$. ■

Remark 5.2 : *In fact the analysis in §3-4 extends easily to the generalization of system (*) in which the number d_i of equilibrium conditions for agent i may differ from the number c_i of strategic degrees of freedom possessed by agent i.*

The following properties of $\mathcal{L}$ are easily verified. They are, respectively, the expressions of the strategic principles that the presence or absence of a dummy (a player with one pure strategy and consequently no strategic freedom) and the numbering of the players are inconsequential.

Property 1: $\mathcal{L}_{n+1}((\mathbf{c},0),(\mathbf{d},0)) = \mathcal{L}_n(\mathbf{c},\mathbf{d})$.

Property 2: $\mathcal{L}_n$ is unaffected by the interchange of any (c_i, d_i) and (c_j, d_j).

For the analysis of the complexity of various algorithms associated with the Nash equilibrium concept one wishes to know the asymptotic order of magnitude of the number of equilibria. On the one hand we may apply (c) inductively to obtain

$$\mathcal{L}_n(\mathbf{c},\mathbf{d}) \leq (n-1)^{d_1+\ldots+d_{n-2}}.$$

(In fact this will coincide with the bound on the generic number of solutions resulting from Bezout's theorem.) The following lower bound is crude, in the sense that refinements seem possible, but it does provide the most basic information.

Theorem 5.3 : *If* $c_1 \leq c_2 \leq \ldots \leq c_n$, *then*

$$\mathcal{L}_n(\mathbf{c},\mathbf{d}) \geq (n-1)^{c_1} \cdot \ldots \cdot 2^{c_{n-2}}.$$

Proof: We say that $(\mathbf{c},\mathbf{d})$ is *tight* at i if $d_i = \sum_{j \neq i} c_j$. Note that, since $\sum_i c_i = \sum_i d_i$, if $(\mathbf{c},\mathbf{d})$ is tight at both i and j, then $c_k = 0$ for $k \neq i,j$, and in fact $d_k = 0$ for $k \neq i,j$ as well. Condition (c) of Proposition 5.1 then yields

$$\mathcal{L}_n(c_i\mathbf{e}_i + c_j\mathbf{e}_j, c_j\mathbf{e}_i + c_i\mathbf{e}_j)) = \mathcal{L}_n(c_i\mathbf{e}_i + (c_j-1)\mathbf{e}_j, (c_j-1)\mathbf{e}_i + c_i\mathbf{e}_j)$$

and by induction this quantity is precisely 1.

Let j be the first integer such that $c_j > 0$. If there is no index at which $(\mathbf{c},\mathbf{d})$ is tight, let k be any index such that $d_k > 0$. Otherwise let k be the index at which $(\mathbf{c},\mathbf{d})$ is tight. If, in the latter case, $d_k = 0$, then $j = k = n$, and $\mathcal{L}_n(\mathbf{c},\mathbf{d}) = 1$, which is in conformity with the claim. Thus we may assume that $d_k > 0$.

To begin with we consider the case $k \geq j$. The induction formula yields

$$\mathcal{L}_n(\mathbf{c},\mathbf{d}) = \sum_{\substack{i \geq j \\ i \neq k}} \mathcal{L}_n(\mathbf{c}-\mathbf{e}_i, \mathbf{d}-\mathbf{e}_k).$$

By induction we may assume that the claim holds for all n-tuples "smaller" (in the obvious partial order) than $(\mathbf{c}, \mathbf{d})$. Thus

$$\begin{aligned}\mathcal{L}_n(\mathbf{c}, \mathbf{d}) &\geq \sum_{\substack{i \geq j \\ i \neq k}} (n-j)^{c_j} \cdot \ldots \cdot (n-i)^{c_i - 1} \cdot \ldots \cdot 2^{c_n - 2} \\ &\geq (n-j) \cdot \left[(n-j)^{c_j - 1}(n-j+1)^{c_{j+1}} \cdot \ldots \cdot 2^{c_n - 2}\right].\end{aligned}$$

which is, of course, the desired inequality.

It remains only to consider the possibility that $k < j$, but in this case the condition $i \neq k$ is no longer imposed in the summation alone, with the consequence that the resulting lower bound is multiplied by $\frac{n-j+1}{n-j}$. ∎

Define $A(n, k+1) = \mathcal{L}_n(\mathbf{k}, \mathbf{k})$, where $\mathbf{k} \in \mathbb{R}^n$ is the vector of k's; this is the maximal generic number of equilibria for an n-person game in which each player has $k+1$ pure strategies. It is this function whose values are displayed in Figure 1.1. Also, define $A_1 = 0$, and for $n > 1$, $A_n = A(n, 2)$.

Proposition 5.4: *$A_2 = 1$, and for all $n > 2$,*

$$A_n = (n-1)(A_{n-1} + A_{n-2}).$$

Proof: That $A_2 = 1$ follows directly from Proposition 5.1. Define $L(j, k, l) = \mathcal{L}_n(\mathbf{c}, \mathbf{d})$ where, for each i, c_i, $d_i \in \{0, 1\}$, and

$$\begin{aligned} j &= \#\{i : (c_i, d_i) = (1,1)\} \\ k &= \#\{i : (c_i, d_i) = (1,0)\}\,. \\ l &= \#\{i : (c_i, d_i) = (0,1)\} \end{aligned}$$

Proposition 5.1 implies that

$$L(j, 1, 1) = L(j, 0, 0) + jL(j-1, 1, 1)$$

and

$$\begin{aligned} A_j = L(j,0,0) &= (j-1)L(j-2,1,1) \\ &= (j-1)L(j-2,0,0) + (j-1)(j-2)L(j-3,1,1) \\ &= (j-1)L(j-2,0,0) + (j-1)L(j-1,0,0) \\ &= (j-1)[A_{j-2} + A_{j-1}]. \quad ∎ \end{aligned}$$

Evidently A_n is the number of permutations of the integers from one to n without a fixed point; such permutations are called *derangements.* Induction shows that it has the closed form*

$$A_n = n! \cdot \sum_{j=0}^{n} \frac{-1^j}{j!}$$

and by comparing the right hand side with the power series expansion of the exponential function, evaluated at -1, one can show that

$$\lim_{n \to \infty} (\frac{A_n}{n!}) = e^{-1}.$$

6 A Three Person Example

In this section we explore the algebra of the simplest case in which there can be multiple totally mixed equilibria, namely three players, each of whom has two pure strategies. As we will see, already the calculations are rather complicated.

Let $S_i = \{0,1\}$, $i = 1,2,3$. Write $v^i_{h\ell} = v^1_i(h,\ell) = u_i(1,(h,\ell)) - u_i(0,(h,\ell))$, for $i = 1,2,3$, $h, \ell \in \{0,1\}$. Then equations $(a1)$ for an equilibrium become

$$\begin{aligned}
v^1_{11}\sigma_{21}\sigma_{31} + v^1_{10}\sigma_{21}\sigma_{30} + v^1_{01}\sigma_{20}\sigma_{31} + v^1_{00}\sigma_{20}\sigma_{30} &= 0\\
v^2_{11}\sigma_{11}\sigma_{31} + v^2_{10}\sigma_{11}\sigma_{30} + v^2_{01}\sigma_{10}\sigma_{31} + v^2_{00}\sigma_{10}\sigma_{30} &= 0\\
v^3_{11}\sigma_{11}\sigma_{21} + v^3_{10}\sigma_{11}\sigma_{20} + v^3_{01}\sigma_{10}\sigma_{21} + v^3_{00}\sigma_{10}\sigma_{20} &= 0.
\end{aligned}$$

For any interior solution, this system reduces to (a2):

$$\begin{aligned}
w^1_{11}\tau_2\tau_3 + w^1_{10}\tau_2 + w^1_{01}\tau_3 + w^1_{00} &= 0\\
w^2_{11}\tau_1\tau_3 + w^2_{10}\tau_1 + w^2_{01}\tau_3 + w^2_{00} &= 0\\
w^3_{11}\tau_1\tau_2 + w^3_{10}\tau_1 + w^3_{01}\tau_2 + w^3_{00} &= 0.
\end{aligned}$$

where we define $\tau_i = \tau_{i1} = \frac{\sigma_{i1}}{\sigma_{i0}}$ and $w^i_{h\ell} = w^1_i(h,\ell)$, $i = 1,2,3$, $h, \ell \in \{0,1\}$, where w^1_i is the function defined in §2. Then equations $(*)$ for an equilibrium become

This system is equivalent to

$$\begin{array}{cccl}
 & (\tau_2 - a_{12}) & (\tau_3 - a_{13}) & = \delta_1\\
(\tau_1 - a_{21}) & & (\tau_3 - a_{23}) & = \delta_2\\
(\tau_1 - a_{31}) & (\tau_2 - a_{32}) & & = \delta_3
\end{array}$$

* We are grateful to Michel leBreton for this and the following observations.

where

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} = - \begin{pmatrix} 0 & \frac{w^1_{01}}{w^1_{11}} & \frac{w^1_{10}}{w^1_{11}} \\ \frac{w^2_{01}}{w^2_{11}} & 0 & \frac{w^2_{10}}{w^2_{11}} \\ \frac{w^3_{01}}{w^3_{11}} & \frac{w^3_{10}}{w^3_{11}} & 0 \end{pmatrix},$$

$$\delta_i = \frac{-\begin{vmatrix} w^i_{00} & w^i_{01} \\ w^i_{10} & w^i_{11} \end{vmatrix}}{(w^i_{11})^2} = \frac{w^i_{01}w^i_{10} - w^i_{00}w^i_{11}}{(w^i_{11})^2} \qquad (i = 1, 2, 3).$$

It is tedious but straightforward to verify that there are two solutions, $\tau = (\tau_1, \tau_2, \tau_3)$ and $\tau' = (\tau'_1, \tau'_2, \tau'_3)$, to this system of equations, given by

$$\tau_i = \frac{d_i + \Delta^{1/2}}{2c_i}, \quad \tau'_i = \frac{d_i - \Delta^{1/2}}{2c_i},$$

where

$$\eta_i = a_{i+2,i} - a_{i+1,i},$$

$$c_i = \frac{\eta_1\eta_2\eta_3}{\eta_i} + \delta_i, \quad d_i = (a_{i+1,i} + a_{i+2,i})c_i - \eta_{i+1}\delta_{i+1} + \eta_{i+2}\delta_{i+2},$$

$$\Delta = (\eta_1\eta_2\eta_3 + \eta_1\delta_1 + \eta_2\delta_2 + \eta_3\delta_3)^2 + 4\delta_1\delta_2\delta_3.$$

(In the definitions of η_i and d_i the indices are to be interpreted as integers modulo 3.)

The solutions x^1 and x^2 are distinct if and only if $\Delta \neq 0$. They are real if and only if $\Delta \geq 0$; this will certainly be the case if any one of the three equations factor, in the sense that $\delta_i = 0$, in which case $\delta_1\delta_2\delta_3 = 0$. One can work out the conditions under which the components of the roots are positive, but they are difficult to state and interpret.

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