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JUDGING THE RATIONALITY OF DECISIONS IN THE PRESENCE OF VAGUE ALTERNATIVES

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Abstract

The standard framework of the decision theory is subjected to partial revision in regard to the usage of the notion of *alternative*. An approach to judging the rationality of decision-maker's behavior is suggested for various cases of incomplete observability and/or controllability of alternatives. The approach stems from the conventional axiomatic treatment of rationality in the general choice theory and proceeds via modifying the description of alternative modes of behavior into a generalized model that requires no explicit consideration of alternatives. The criteria of rationality in the generalized decision model are proposed. For the conventional model in the choice theory, these criteria can be reduced to the well known criteria of the regularity (binariness) of choice functions. Game and economic examples are considered.

Key words: decision theory; rational choice; alternatives; rationality criteria.

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0. Introduction

This paper reconsiders the criteria for the rationality of decisions that were elaborated in the works on the general choice theory by K.Arrow(1959,1963), A.Sen(1970,1971) and their followers, and expands these criteria to some unconventional yet more realistic situations with *vague* alternatives.

Those are the situations where the alternatives faced by the decision-maker (DM) are only partly controllable by DM and/or only partly observable by the investigator-analyst who tries to judge the rationality of DM's behavior. (We will consider different versions of DM below, such as game players or consumers). The extreme case of vagueness of alternatives for the analyst is a situation when neither the alternative modes of behavior themselves nor the implemented mode are accessible to an outside observation. In such a case the analyst may judge the rationality of the behavior only by juxtaposing, on the one hand, the potential abilities and opportunities of DM in various situations of the decision making, and on the other hand, the results achieved. The development and analysis of a generalized model for such a case is presented in the main part of this paper. The first, preliminary part is devoted to the discussion and illustrative analysis of a number of examples and particular problems.

The conventional approach to judging the rationality of decisions that are more general than simple pairwise comparisons of given alternatives, represents the following. It is usually assumed that, after seeing all alternative modes of behavior, one judges the rationality of the implemented mode of behavior based on the observation of the mode selected among all possible modes. That is, that the analyst who observes the behavior of or/and sets up real or *Gedanken* experiments with DM has to

(i) know the decision made eventually by DM;

(ii) see (mentally) the whole "space" of alternative modes of behavior;

(iii) see the "boundaries" of the set of admissible alternatives in this space for each specific choice act;

(iv) possess some basic principles for judging the rationality of the observed behavior based on the above data (plus perhaps something else).

Usually the items (i),(ii),(iii) are not being discussed (Schwartz (1986) is one exception) – they are included into the statement of the problem, and the analyst's attention is focused then on the item (iv). Concerning the latter, there are two opposite ways of reasoning: (A) *direct*, or *external*, and (B) *indirect*, or *internal*. In the type "A" reasoning, an *a priori* criterion for the appreciation and/or comparison of alternatives is assumed; this may be, e.g., an "objective" value (worth, utility etc.) of each alternative. Then, the basis for judging the rationality of the behavior is whether the alternative chosen by DM is the best one according to this criterion. In the type "B" reasoning, the investigator has no *a priori* criterion for judging the values, mutual advantages etc. of the alternatives. In such a case, the only way to judge the rationality of the behavior is to watch the results of the decisions made in different situations, to juxtapose and compare them, and to look whether they are in accordance with our own notion of a logically consistent behavior. This approach has led to the general (abstract) choice theory, in which the postulates of rational behavior are introduced in an axiomatic form. Such an abstract analysis of the choice rationality naturally produces constructive ways of revealing "subjective" values or advantages of alternatives; it is the subject of the theory of revealing preferences that will be discussed below in detail.

To show the essence of an approach based on indirect, external judgements about the rationality, let us consider the simplest kind of decision making – the choice based on a pairwise comparison of alternatives. It is usually accepted that by choosing the alternative a from the pair $\{a, b\}$, the alternative b from the pair $\{b, c\}$ and the alternative c from the pair $\{a, c\}$, the decision-maker simply manifests irrationality and the lack of logic of his/her behavior. Indeed, such a violation of *the transitivity of the revealed preferences* contradicts the hypothesis that DM does indeed maximize some value (utility) of alternatives, a scalar function $v(x)$ $(x = a, b, c)$. Thus, we conclude that DM is irrational in the *narrow sense*. At the same time the observed behavior is consistent with the hypothesis that DM follows the maximization according to a binary preference relation *"x is better than y"* (which in this case constitutes the cycle "a is better than b, b is better than c, c is better than a"). Note that the latter cycle implies the impossibility of a (nonempty) rational choice between the three alternatives a, b, c i.e., the necessity of *refusal from the choice*. By admitting an empty choice as an equally eligible choice, we shall treat the very possibility of representation of choice via the optimization according to a binary relation as the rationality in the *broad sense*. The exact criteria of rationality for this case based on the external observation of the choice will be given below.

Furthermore, if the subject chooses the alternative a from the pair $\{a, b\}$ but b from the triple $\{a, b, c\}$, this manifests an irrationality that is conventionally referred to as a violation of the *axiom of independence of irrelevant alternatives* (IIA). Such a violation implies impossibility to explain the choice via the optimization over some preference. Indeed, according to the first experiment, a is better than b but the opposite should be true in the second experiment; so we obtain the external irrationality in the broad sense.

The above reasoning concerns item (iv) of the list above, the basis for judging the rationality of a behavior. Here we have adopted the conventional concept of the choice rationality (Arrow, 1959, 1963; Sen, 1970,1971) both in the narrow sense, as optimization of a utility function, and in the broad sense, as optimization over a binary preference relation. This concept has been critically analyzed in a number of works (see, e.g., Fishburn, 1970; Schwartz, 1972; Plott, 1973; Aizerman and Malishevski, 1981; and the survey by Aizerman, 1985). It was shown that for various widely spread modes of choice some of the axioms introduced by Arrow, Sen and their successors (different combinations of those produce criteria with a different degrees of rationality) cannot be satisfied together: some may actually fail. The idea of satisfying one axiom at a time has led to the nonconventional *quasirational* mechanisms of choice. Such mechanisms focus on the choice of alternatives undominated according to some extended dominance relations; the latter are more complex than usual binary relations on the fixed alternative space. We shall touch upon this nonconventional theory below. In this paper we make the next step and concentrate on the criticism of the items (i), (ii), (iii) of the analyst's list; for the sake of simplicity, we take as a basis the classical criteria and the conventional concept of the choice rationality.

These issues are discussed in Section 1 below. The remaining, constructive part of the paper deals with formal models and their analysis, and is arranged as follows. In Section 2, the conventional model of the general choice theory is reviewed and a summary of rationality conditions is given in a convenient form. The main of those is the axiom of revealed preferences or, in another aspect, the axiom of independence of irrelevant alternatives. Section 3 presents three examples in which the rationality criteria are applied to situations with vague alternatives. In Section 4 a new model of *decisions with hidden alternatives* is proposed, and corresponding rationality axioms are introduced. Analysis of this model reveals the inner structure of rationality. The conclusive Section 5 presents a sketch of further possible generalizations.

1. Informal discussion of the problem

We shall depart, step by step, from conventional statements and from the standard concept. Recall that we have already abandoned the item (iv) of the analyst's list (optimization approach based on value functions or preference relations), but we have still kept the preceding items (i), (ii), (iii). Now we shall gradually abandon the item (iii) (observability of the admissible alternative set), then (ii) (observability of the total alternative space), and finally (i) (observability of the alternative(s) chosen).

We start with item (iii), the notion of admissible (or feasible) alternative sets. Apparently, the principal source of this notion and of the choice theory itself has been the theory of consumption. The prototype of the abstract admissible set in that theory was the notion of the budget set as a subset in the space of commodity bundles considered as the totality of all conceivable alternatives for a consumer. Now we shall use this particular case to demonstrate an ambiguity of the seemingly clear notion of admissible sets. The budget set is the set of all commodity bundles whose cost does not exceed the value of the consumer's budget (income); we shall go to the formal notations a little later. The very idea that all those and only those commodity bundles that belong to the budget set are available to the consumer seems to be justified when one speaks about a long-run term, say ten years. But for a short-run term, say a month, the boundaries of the admissible set become hazy. Indeed, the consumer may exceed his budget, e.g., using a seller's credit or a bank loan; in such a case the virtual admissible set turns out to be wider than the primary budget set. As another possibility, the consumer may judge as inadmissible even those commodity bundles which cost less but not too much less than his income. In such a case the virtual subjective admissible set becomes narrower than the initial budget set.

In both cases the resulting behavior, deemed rational by the consumer, may look irrational to the external observer. To demonstrate this, let us start with a "polite guest" story which seem to have been described first by Corbin and Marley, 1974. The following quote is from the review lecture by Sen, 1992:

> "Suppose the person is choosing between slices of cakes offered to him" and "he is trying to choose as large a slice as possible, subject to not picking the very largest, because he does not want to be taken as greedy, or because he would like to follow a social convention or a principle learned at his mother's knees: "never pick the largest slice".(...) If the three slices in decreased order were z, y, x, then he is behaving exactly correctly according to that principle" [when choosing x from $\{x, y\}$ and y from $\{x, y, z\}$ which violates the IIA axiom].

In this example the best choice is not the choice of the best. That is, the virtual admissible alternative set for the person, i.e., the set of alternatives allowed both "physically" and "psychologically" becomes a different, narrower set than an externally observable set of feasible alternatives.

Returning to the example of the choice under bounded budget, it is clear that qualitatively the behavior of the consumer who saves money, and thus makes the subjective admissible set narrower than the budget set, will be similar to that in the above Sen's example. More formally, present the standard model of consumer's behavior as the solution of the utility maximization problem

$$\max u(x_1, \ldots, x_n) \tag{1}$$

$$\sum p_i x_i \le I,\ x_i \ge 0,\ i = 1, \ldots, n,$$

where x_i and p_i are the quantity and the price of i-th good, respectively, and I is the consumer's income. (Another, nonconventional model will be considered in an axiomatic form in Section 3). Here the standard admissible set for the consumer is his budget set $B \subset R^n$:

$$B(p_1, \ldots, p_n; I) = \{\mathbf{x} \geqq 0 \mid \sum p_i x_i \le I\}. \tag{2}$$

The modified behavior of the consumer-saver can be described by the model

$$\max v(x_1, \ldots, x_n; s) \tag{3}$$

$$\sum p_i x_i + s \le I,\ x_i \ge 0,\ i = 1, \ldots, n; s \ge 0,$$

where s is the level of consumer's savings.

Consider a special case of the modified utility function v:

$$v(\mathbf{x}, s) = \begin{cases} u(\mathbf{x}) & \text{if } s \ge \alpha, \\ -\infty & \text{if } s < \alpha, \end{cases} \tag{4}$$

where $\alpha = \text{const} > 0$ - the least subjectively admissible level of savings (assurance level). This describes the case where the consumer maximizes his genuine utility function under the budget constraint, with the additional requirement that the level of savings be not less than the admissible threshold. In such a case, if we replace I by $I' = I - \alpha$, the problem (3) becomes equivalent to the problem (1), and we find ourselves in the position of the Sen's cake-chooser, with seemingly irrational external behavior, from the external observer standpoint. Namely, the optimal solution $(\mathbf{x}^*; s) = (x_1^*, \ldots, x_n^*, s)$ of the problem (3) has as its x-th component the optimal solution $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ of the problem (1) with I replaced by $I' = I - \alpha$, and (under the natural condition of monotonic increasing of u in x_i's) $\sum p_i x_i^* = I'$. Then, accepting the new budget constraint I' instead of I in the modified problem (3) will definitely lead to the consumer's changing his decision $\mathbf{x}^*$, in spite of its seeming admissibility even in the new budget set $B(p, I')$.

Moreover, a similar result will appear in a "smooth" case when v depends on s continuously but increases in s sharply enough when s is small. Then the s-component s^* of the optimal solution of (3) will become positive, and the qualitative picture will obviously remain the same. In the framework of the standard model (1), from the external observer's point of view the consumer choice of the commodity bundle $\mathbf{x}^*$ will look irrational. Note that in the smooth case it is more difficult to interpret the situation just as narrowing of the admissible alternative set. Indeed, formally we can still reduce the problem (3) to (1), by replacing I in (1) by $I' = I - s^*$ and taking $u(\mathbf{x}) = v(\mathbf{x}; s^*)$. But the parameter s^* here will be known only *a posteriori*. Since the consumer chooses the quantities $\mathbf{x} = (x_1, \ldots, x_n)$ and s simultaneously, it is hardly right to say that he selects the best among alternatives $\mathbf{x} \in B(p, I - s^*)$. Rather, one could say that the consumer considers as alternatives other objects, namely the composite bundles $(x_1, \ldots, x_n; s)$.

Then the constraint in (3) yields the admissible set of the bundles; with such treatment of the alternative space and the feasible set in it, the behavior of the consumer starts looking rational.

Note that by now we have encroached upon the very nature of the space of alternatives , i.e., not only upon item (iii) but also item (ii) of the analyst's list. Indeed, the lack of clarity, on the decision-maker's part, of what the real alternatives are is rather typical. This problem is especially acute when the possible outcomes of one's behavior, viz., the resulting states (or trajectories), depend not only on one's decisions, but on other external influences as well.

In game-theoretical terms we can speak about dependencies of the results upon other *players*, or in particular upon the *nature* (which reflects factors beyond the person's control). One can see the important manifestation of this problem in the difference between *actions* of a player and *outcomes* of the play. The main interest of the player is in the play outcome value; but, since he has to make choices between actions (modes of behavior) rather than directly between feasible outcomes, he is forced to also assign some subjective values to the actions themselves. Such "imputed values" of actions, to use the economic-theoretical term, can become tied to the outcome values in a rather sophisticated manner, due to the incomplete observability/controllability of factors and/or outcomes; this can also lead to a seeming irrationality of the player's actions. "In general, it is hard to specify precisely the strategy sets available to the players" (Luce and Raiffa, 1957, § 1.4). A thorough discussion of some paradoxes in the decision logic, arising due to an insufficiently clear distinction between actions and outcomes, is held in Maher, 1987. Here we shall confine ourselves to two game-theoretic examples.

Game example 1. Consider an antagonistic game with the payoff matrix $\|v_{ij}\|$ of the form

$$\begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 0 & 4 \end{bmatrix} \tag{5}$$

where v_{ij} is the reward of the player 1 after the move (action) $i \in I$ by player 1 and move $j \in J$ by player 2; I and J are the sets of possible actions of the players 1 and 2 respectively; here $I = \{1, 2, 3\}$, $J = \{1, 2\}$. Let player 2 make his choice first, and player 1 - second, after learning the choice of player 1. Then the pair (i^*, j^*) of optimal moves of both players forms the *Stackelberg equilibrium*, with player 2 being the *leader* and player 1 the *follower*. In the equilibrium, the components i^* and j^* yield the inner max in i and the outer min in j in the expression

$$\min_{j \in J} \max_{i \in I} v_{ij}. \tag{6}$$

It is easy to see that for the payoff matrix $\|v_{ij}\|$ of the form (5) $j^* = 1$ and $i^* = 1$. If one changes the matrix (5) by deleting its third line, one obtains new optimal actions $j^{*\prime} = 2$

and $i^{*'} = 2$. Thus, if one observes only the behavior of player 1, one will see that in the case of possible alternative actions $\{1,2,3\}$ player 1 will choose the action 1, whereas in the case of the set $\{1,2\}$ he will choose action 2. This runs counter to the IIA axiom.

Game example 2. Consider the following payoff matrix for the game of DM with nature:

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} \tag{7}$$

Let DM (in the role of player 1) use the criterion of the minimax regret (risk) after Savage (see, e.g., Luce and Raiffa, 1957). Following this criterion, we first take the initial payoff matrix $\|v_{ij}\|$ and convert it into the regret matrix $\|w_{ij}\|$, $i \in I$, $j \in J$, using the formula

$$w_{ij} = \max_{l \in I} v_{lj} - v_{ij} \tag{8}$$

The player chooses such an action i^* which yields

$$\min_{i \in I} \max_{j \in J} w_{ij} \tag{9}$$

For the matrix $\|v_{ij}\|$ of the form (7) the corresponding regret matrix $\|w_{ij}\|$ is

$$\begin{bmatrix} 2 & 2 \\ 0 & 3 \\ 3 & 0 \end{bmatrix} \tag{10}$$

The optimal action of DM (player 1) will be $i^* = 1$. Now, delete DM's action 3 from the admissible set. Then the new payoff matrix is obtained by deleting the last line from (7); the corresponding regret matrix $\|w'_{ij}\|$ becomes

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{11}$$

(rather than the corresponding submatrix of (10)). So the new optimal action by DM is $i^{*'} = 2$, again contrary to the IIA axiom. The above is a modification of the argument by Chernoff, 1954 (see also Luce and Raiffa, 1957, § 13.2); it will be used again in the sequel.

Thus, there are two points of view on the notion of alternatives: (a) alternatives are externally observable actions of the player, and (b) alternatives are consequences of the actions, as perceived by the player. In (b), the values of alternatives are the player's subjective estimates $e(i)$ of predicted consequences of his actions $i \in I$. As the above examples of two games demonstrate, DM's behavior can be rational from his subjective standpoint but irrational from the external standpoint. In Example 1, $e(i) =$

v_{ij^*} where $j^* = \mathrm{argmin}_{j\in J}\, u_j$ with $u_j = \max_{l\in I} v_{lj}$; in Example 2, $e(i) = -\max_{j\in J} w_{ij} = \min_{j\in J}(v_{ij} - u_j)$. It is easy to see that the "technical" reason of the observed irrationality is the dependence of the estimate $e(i)$ not only on the action i but also on the set of all admissible actions I; so a more correct notation for e should be $e(i; I)$. We call the scalar value function of such form a *pseudoscale* of estimates, as distinct from a genuine *scale*, in accordance with which the estimate e of the alternative i must depend only on i.

The seemingly irrational behavior in the situation above can be explained as follows. The observer presupposes that the decision-maker compares the actual values of the observed alternatives independent of the *context of comparison*, while such a dependence can actually exist. In the game examples above it was the admissible alternative set I itself that played the role of the comparison context for alternative actions $i, i', \ldots$. In more general cases the role of the comparison context for alternatives $x, y, \ldots$ may be given not (or not only) to the admissible alternative set X but to some other *experiment conditions* E, and so the value of the alternative x may have the generalized form of a pseudoscale $v = v(x; E)$. As an example of such a representation, replace the simplest consumer problem (1) by the equivalent problem without constraints, using the Lagrange–Kuhn–Tucker multiplier λ:

$$\max_{x_1,\ldots,x_n \geq 0} \left(u(x_1, \ldots, x_n) - \lambda \sum p_i x_i\right). \tag{12}$$

Then, the nonnegative orthant R^n_+ becomes the admissible alternative set, independent of the variables $p_1, \ldots, p_n, I$. But at the same time the modified value function becomes parametrically dependent on p, I, and so it is a pseudoscale.

Furthermore, it is easy to "explain" an arbitrary behavior of DM in the framework of the broad rationality, i.e. in terms of preference relations, if one accepts that a preference relation xRy ("x is preferred to y") depends on experiment conditions E (in particular, on the admissible alternative set X). Then R will be a *pseudorelation* rather than a proper binary relation on the alternative space (more precisely a pseudorelation of the form $xR(E)y$ is a ternary relation between x, y and E). The approach to rationality in terms of pseudoscales or pseudorelations, i.e., in terms of alternative absolute or relative values depending on experimental conditions, can lead to nontrivial results. This will be so if one does not allow the absolute arbitrariness in this dependency and confines himself to some natural types of such dependencies. This results in "quasirational" modes of behavior, to be discussed in Section 2.

The examples above illustrate ambiguity of the notion "alternative". This leads us to extend or even completely change the alternative space. In conclusion, we shall demonstrate the possibility and necessity of such changes in the framework of the highly formalized standard model in the abstract choice theory. In the model, a set U of objects x, y, ... called *alternatives* is given; in every act of choice, some set $X \subseteq U$ is presented and interpreted as an admissible alternative set. The act of choice amounts to the selection of one of the alternatives x^* among all $x \in X$. A fundamental notion of the choice theory is the choice function c that puts into correspondence to a given X the alternative chosen from $X : x^* = c(X)$. Also considered in the general choice theory is the extended concept of choice function, a set-to-set mapping C, which puts into correspondence to the set X some of its subsets, $X^* \subseteq X$; so we have $X^* = C(X)$. The subset X, called the *choice set*, is usually treated as the set of the chosen or, more precisely, eligible to be chosen, alternatives. In the case of a (conventionally) rational choice, the set $C(X)$ consists of all those and only those alternatives which are better (or not worse) than each alternative in X. The case of *single* choice $c(X)$ we began with, is formally reduced to a special case when $C(X)$ is a singleton $\{c(X)\}$. Generally the set $C(X)$ can contain more than one element, $|C(X)| > 1$: call it *hyperchoice*; or it may contain no elements at all, $C(X) = \emptyset$: call it *hypochoice*. The latter is often considered as the sign of irrationality of the choice – a view which we will not share.

In the general case of set-valued choice, including both hyperchoice and hypochoice, the very term "alternative" as applied to the primary objects $x, y, \ldots \in U$ stops to be justifiable, since its original meaning implies both mutually exclusive and mutually complementary opportunities. Indeed, with the hypochoice the new unforeseen possibility is realized, viz., the empty choice. In different interpretations this may mean either refusal from a real choice or the return to the *status quo* situation which was not present before in the set of considered alternatives. We thus might formally include the additional alternative into the admissible set; the newly added alternative denotes the aforementioned state "refusal from choice" or "status quo". To comment upon the situation, let us cite S.J.Lec: "To venture on the indecision, one has to be a man of decision". As for the primary alternatives, in the case of the hypochoice they do not exhaust all the opportunities for choice and so are not mutually complementary.

On the other hand, in the case of the hyperchoice, alternatives are not mutually exclusive: formally it looks as if all objects forming the choice set $C(X)$ are chosen simultaneously, and sometimes all the options from $C(X)$ can be actually realized simultaneously. One example may be the choice of dishes from the dinner menu X, when $C(X)$ is the chosen set of dishes. In cases like this it is reasonable to treat as alternatives faced by the decision-maker the sets of objects taken as a whole rather than primary objects themselves. From the formal standpoint this implies that it is not the initial object set U, but the set of its subsets 2^U which plays the role of the alternative space.

Thus, even on the abstract level of the formal model of the choice an ambiguity may arise in what are the alternatives. This leads to a necessity of modifying not only the notion of admissible sets but the alternative space as well. An informal discussion of such modifications is given in Schwartz, 1986 (§1.2). A formal analysis of both such cases above, hypochoice and hyperchoice, will be done in Section 3.

2. The conventional model of abstract decision making in the choice theory and the rationality criteria

Following the conventions of the choice theory, take the set U of all conceivable *objects of choice* (for the reasons mentioned above, we abandon the term "alternative" from the very beginning). The set U may be arbitrary (for simplicity one may consider a finite set). In the act of choice some admissible set of objects $X \subseteq U$ is presented to DM for the choice. Denote by $\mathcal{U}$ the family of all nonempty sets X that may be presented for the choice and call it the *family of presentations*. In general, $\mathcal{U}$ may be an arbitrary subset of the set $2^U \backslash \{\emptyset\}$; we shall consider below some special types of families $\mathcal{U}$. To each presentation $X \in \mathcal{U}$ an object $x^* \in X$ or a subset $X^* \subseteq X$ are put in correspondence; either of these is called *choice from* X and is denoted $c(X)$ or $C(X)$, respectively. In the first case we use the term *single choice*, in the second case *set-wise choice*. The functions $c : \mathcal{U} \longrightarrow U$ and $C : \mathcal{U} \longrightarrow 2^U$ are called *choice functions*, single and set-wise respectively.

Definition 1 We shall say that the choice function c (or C, respectively) is *rational in the narrow sense*, if there exist a linearly ordered set L and a mapping $v : U \longrightarrow L$ such that for every $X \in \mathcal{U}$

$$c(X) = \arg\max_{x \in X} v(X) \tag{13}$$

or, respectively,

$$C(X) = \operatorname{Arg}\max_{x \in X} v(X). \tag{14}$$

Note that the latter, more general definition (14) may be rewritten as

$$C(X) = \{x \in X \mid \forall y \in X : v(x) \geq v(y)\}. \tag{15}$$

Definition 2 We shall say that the choice function c (or C, respectively) is *rational in the broad sense*, or *rational* for brevity, if there exists a binary relation R on U such that $c(x)$ is the unique greatest element x^* in X by R, i.e.,

$$c(X) = x^* \quad \text{such that} \quad \forall y \in X : x^* R y \tag{16}$$

or, respectively, C(X) is the set of all the greatest elements in X by R, i.e.

$$C(X) = \{x \in X \mid \forall y \in X : xRy\}. \tag{17}$$

Remark 1 In the definitions above, function $v(x)$ has the meaning of a *value* of the object x from DM's point of view, and xRy is a (nonstrict) *preference relation*: x is "as good as" or "not worse than" y. The representability of choice functions by means of their *rationalization* in the form (16) or (17) via some preference relation is sometimes called *regularity*, or *binariness* of choice functions.

Remark 2 For single choice functions c in Defs 1 and 2 to be well defined it is needed that the maximal element in (13) and the greatest element in (16) are unique; this uniqueness is the presupposition for these definitions.

Remark 3 By taking the logical negation in Def 2, one can rewrite the definitions (16) and (17) in the equivalent forms

$$c(X) = x^* \quad \text{such that} \quad \bar{\exists} y \in X : yPx \tag{18}$$

and, respectively,

$$C(X) = \{x \in X \mid \bar{\exists} y \in X : yPx\}, \tag{19}$$

where $P = \bar{R}^{-1}$ is conversely complementary to R (i.e., xPy iff not yRx), which is usually interpreted as a *strict preference* relation "better than". Sometimes, when R is interpreted more as "as good as" than "not worse than", another definition of P via R is used: $P = R \& \bar{R}^{-1}$ (i.e., P is the asymmetrical part of R). These two expressions for P via R coincide iff $\bar{R}^{-1} \Longrightarrow R$, i.e. iff R is a complete relation (for all $x, y \in U$ either xRy or yRx), which is equivalent to $P = \bar{R}^{-1}$ being an asymmetric relation (i.e., for all $x, y \in U$ both xPy and yPx must not hold together). Incompleteness of R in (16) or (17) corresponds to the lack of asymmetry of P in (18) or (19), respectively. The case when both xPy and yPx hold at the same time contradicts to usual interpretation of P as a strict preference and demands a more broad treatment as a rather abstract *dominance relation*. Formally we may admit even the lack of irreflexivity for a dominance relation P, i.e., an *autodominance* xPx for some x. The reasons for this will be discussed in Section 3.

Remark 4 It is evident that a choice function rational in the narrow sense must be rational in the broad sense. Indeed, the narrow rationality (15) is the particular case of the broad rationality when the corresponding R is a (nonstrict) weak order, i.e., a complete transitive relation. Namely, in such a case, given R in Def 2, one can take as L in Def 1 the set of equivalence classes of elements of U in the weak order R, this set being linearly ordered by the corresponding factor-relation for R. Conversely, given v in Def 1, take R in Def 2 as $xRy \Longleftrightarrow v(x) \geq v(y)$. The issue of the type of a scale v, and in particular of numerical representability of preferences is a very important problem in the decision theory, but it is beyond the limits of this paper.

Remark 5 The notions of the narrow and broad rationality in the case of single choice functions are more close to each other than one might expect based on their definitions. Namely, if a presentation family U is complete, i.e., it contains all nonempty subsets of U, then a single choice function c on $\mathcal{U}$ is rational (in the broad sense) if and only if it is rational in the narrow sense. Indeed, the only type of binary relations R which can generate a single-value choice by the optimizational rule (16) on all nonempty $X \subseteq U$ is a nonstrict *strong*, or *linear*, order, i.e., an antisymmetric ($xRy \& yRx \Longrightarrow x = y$) complete transitive relation. This is easy to verify by considering the choice on pairs $\{x, y\}$ and triples $\{x, y, z\}$. Moreover, the same is still true for the case of a "3-complete" family $\mathcal{U}$ containing all pairs and triples of objects. This is the reason for many simplifications in the case of single choice functions and "sufficiently complete" presentation families. It is also the reason why the general choice theory studies mainly set-wise choice functions: they enrich the theory by allowing, in particular, divergence between narrow and broad rationality, with many intermediate "degrees of rationality", at the expense of both technical difficulties and less clarity in using the notion "alternative". In this paper, a different way to enrich reasoning is used – that of considering essentially incomplete presentation families. As will become clear later, this approach is very important for applications.

Definition 3 We shall call a representation family $\mathcal{U} \subseteq 2^U$: *complete*, if $\mathcal{U} = 2^U \backslash \{\emptyset\}$; $\cup$-*complete*, if $\mathcal{U}$ is closed under unions, i.e., when $\{X^\nu\}_{\nu \in N} \subseteq \mathcal{U}$ then $\bigcup_{\nu \in N} X^\nu \in \mathcal{U}$; *pairwise*-$\cap$-*complete*, if $\mathcal{U}$ is closed under nonempty pairwise intersections, i.e., when $X, X' \in \mathcal{U}$ and $X \bigcap X' \neq 0$ then $X \bigcap X' \in \mathcal{U}$; *finitely complete* or *k-complete*, if every finite set, or respectively every set with the cardinality no more than k, belongs to U. Finally, we shall call $\mathcal{U}$ *irreducible*, if each $X \in \mathcal{U}$ is irreducible under $\cup$-operation in the semilattice sense, i.e., if in every covering $\{X^\nu\}_{\nu \in N}$ of X there exists $X^{\nu *}$ such that $X \subseteq X^{\nu *}$.

In the choice-theoretical literature, mainly complete or at least finitely complete representation families are dealt with; besides, we shall make use of other types including arbitrary families $\mathcal{U}$.

Now we shall formulate some properties that are typically expected from a rational choice. These have been developed in a long line of works from the pioneering papers of Samuelson (in the context of consumer choice), Chernoff (statistical decisions), Nash (a bargaining model), via formalization in terms of abstract choice functions by Uzawa, Arrow, and Sen et al., up to various recent extensions and generalizations. We shall list only those properties that are necessary for the present paper. They will be mainly properties of single choice functions c (following their real historical origin) which are more transparent than their generalized forms for set-wise choice functions C.

We shall start with two properties that are both historically and conceptually the primary ones: the *Weak Axiom of Revealed Preferences*, WARP, introduced by Samuelson, 1938, 1950, and the axiom of *Independence of Irrelevant Alternatives*, IIA, which, in the version used here, originates in Nash, 1950, Chernoff, 1954, and Radner and Marschak, 1954 (the latter refer to the K.J.Arrow's idea) (see also Luce and Raiffa, 1957; Arrow, 1959; Sen, 1969). This axiom is often called Chernoff's axiom or α-axiom by Sen's classification. In view of an ambiguity in the treatment of IIA (see, e.g., comments in Luce, 1959, and also Ray, 1973; Hansson, 1973; Karni and Schmeidler, 1976) we shall give this axiom a neutral name ***heredity***, H (which reflects one important feature of it). Thus, start with conditions on single choice functions.

Definition 4 We shall say that a single choice function c on a presentation family $\mathcal{U}$ satisfies the ***heredity*** property, H (or IIA axiom, in traditional terms, or Postulate 4 in Chernoff, 1954, or axiom α in Sen, 1969), if for every $X, X' \in \mathcal{U}$ such that

$$X' \subseteq X, \tag{20}$$

holds

$$x = c(X),\ x \in X' \Longrightarrow x = c(X'). \tag{21}$$

The meaning of the definition is that an object that is the best in a large set is even more so the best in a smaller set; hence the property "to be the best" is hereditary. We will exploit this idea in a generalized form in the sequel.

A different but closely related property (as we shall see later) is formulated as follows (Samuelson, 1938, 1950; Houthakker, 1950).

Definition 5 We shall say that a single choice function c satisfies the *weak axiom of revealed preference*, WARP, if for every $X, X' \in \mathcal{U}$ holds

$$x = c(X),\ x' = c(X'),\ x \in X',\ x' \in X \Longrightarrow x = x'. \tag{22}$$

To interpret WARP and explain the term "revealed preference," transform (22) into the following equivalent form. Let us treat the case $x = c(X)$, $y \in X$, as the binary relation "x is (revealingly) preferable to y"; denote it xR_cy. Then (22) means that $xR_cx' \& x'R_cx \Longrightarrow x = x'$. At the same time, if one considers the case $x = c(X)$, $y \in X$, $y \neq x$, as the binary relation "x is (revealingly) strictly preferable to y" and denotes it xP_cy, then (22) is equivalent to the impossibility of a cycle of the form $xP_cx' \& x'P_cx$. And one more equivalent formulation for WARP has the form $P_c \Rightarrow \overline{R}_c^{-1}$.

Remark 6 Note that, in general, revealed nonstrict and strict preference relations R_c and P_c may satisfy neither the equality $P_c = \bar{R}_c^{-1}$ nor the equality $P_c = R_c \& \bar{R}_c^{-1}$ in Remark 3; one can guarantee only the implications $R_c \& \bar{R}_c^{-1} \Longrightarrow P_c \Longrightarrow R_c$. But since WARP means $P_c \Longrightarrow \bar{R}_c^{-1}$, the previous chain of implications yields $P_c = R_c \& \bar{R}_c^{-1}$. Moreover, the latter equality turns out to be still another equivalent formulation of WARP.

Definition 6 (Houthakker, 1950; see also Richter, 1966, 1971; Hansson, 1968; Suzumura, 1977, et al.) We shall say that a single choice function c satisfies the *strong axiom of revealed preferences*, SARP, if for every $X^1, \ldots, X^n \in \mathcal{U}, n > 1$,

$$x^i = c(X^i), x^i \in X^{i+1}, i = 1, \ldots, n \Longrightarrow x^1 = \ldots = x^n \tag{23}$$

(where $X^{n+1} = X^1$).

WARP is the particular case of SARP with $n = 2$. The equivalent reformulation of SARP in terms of the strict preference P_c is: there exist no $x^1, \ldots, x^n \in U$, $n > 1$, such that $x^1 P_c x^2, \ldots, x^n P_c x^1$ (a cycle of revealed strict preferences).

The axioms of revealed preferences are oriented toward the rationality in the narrow sense (see below). Now we shall introduce additional axioms oriented toward the broad rationality. The first is γ-axiom due to Sen's classification (Sen, 1971; see also Chernoff, 1954, Postulate 10); keeping in mind its further extension to more general models, we give it a neutral term "concordance".

Definition 7 We shall say that a single choice function c satisfies the *concordance* condition C, if for every set $X \in \mathcal{U}$ and for every its *decomposition in* $\mathcal{U}$, viz., a set family $\{X^\nu\}_{\nu \in N} \subseteq \mathcal{U}$ (where the index set N is arbitrary, perhaps infinite) such that

$$X = \bigcup_{\nu \in N} X^\nu, \tag{24}$$

holds

$$(\forall \nu \in N : x = c(X^\nu)) \Longrightarrow x = c(X). \tag{25}$$

A generalized formulation which combines concordance (γ-axiom) with the heredity (α-axiom) was given in Mirkin, 1974:

Definition 8 We shall say that a single choice function c satisfies the *concordant heredity* condition, CH, if for every set $X \in \mathcal{U}$ and for every its *covering in* $\mathcal{U}$, viz., a set family $\{X^\nu\}_{\nu \in N} \subseteq \mathcal{U}$ such that

$$X \subseteq \bigcup_{\nu \in N} X^\nu, \tag{26}$$

holds

$$(\forall \nu \in N : x = c(X^\nu))\&(x \in X) \Longrightarrow x = c(X). \tag{27}$$

The last axiom of this series is taken from Richter, 1966, 1971.

Definition 9 We shall say that a single choice function c satisfies the *Richter's Axiom*, RA, if for every $X \in \mathcal{U}$

$$(\forall y \in X : xR_c y) \Longrightarrow x = c(X). \tag{28}$$

Remark 7 It is easy to see that owing to the definition of R_c the converse implication in (28) is always true, and so the Richter's axiom may be presented in the following equivalent form:

$$x = c(X) \Longleftrightarrow (\forall y \in X : xR_c y). \tag{29}$$

Now we shall write a number of statements concerning interrelations between the above axioms under various assumptions on the presentation family $\mathcal{U}$, using auxiliary Def 3. Some of these statements are apparently known (although at times I am unable to give the primary references) or can be easily obtained from the very definitions. Nevertheless, for the sake of completeness, I will give a summary of statements and my own system of shortened proofs. We start with the correspondence between two fundamental conditions, IIA (i.e., H) and WARP.

Proposition 1 For an arbitrary $\mathcal{U}$, WARP implies IIA. Conversely, when $\mathcal{U}$ is pairwise-$\cap$-complete, then IIA implies WARP.

Indeed, let WARP be fulfilled. Then, with the condition (20) in IIA, let us apply the formulation of WARP; the result will yield (21), as required. Conversely, let IIA be true and $\mathcal{U}$ be pairwise-$\cap$-complete. Take $X, X' \in \mathcal{U}$ that satisfy the premise in (22) and consider $X'' = X \bigcap X'$ which belongs to $\mathcal{U}$ by assumption. Then, owing to IIA as applied to the pair X, X'', we have $c(X'') = x$, and owing to IIA as applied to X', X'', we have $c(X'') = x'$; hence $x = x'$, which satisfies the conclusion in (22), and so WARP is true. Q.E.D.

Thus, IIA and WARP are intimately close: WARP is an apparent strengthening of IIA, and with a pairwise-$\cap$-complete family $\mathcal{U}$ they are simply equivalent. SARP is a further strengthening of WARP; the purpose of it will be clarified in what follows.

Now we state interrelations among the rest of axioms together with WARP and H (i.e., IIA).

Proposition 2 For an arbitrary $\mathcal{U}$, WARP implies CH.

Indeed, let WARP be true, and under the premise of CH, let the left part in (27) be fulfilled, and assume that $x^* = c(X)$. Take a set $X^{\nu *} \in \{X^\nu\}$ such that $x^* \in X^{\nu *}$. Then, x, x^* and $X, X^{\nu *}$ satisfy the premise of WARP, hence $x^* = x$. Q.E.D.

The converse implication in Proposition 2 is generally not true: if $\mathcal{U} = \{X, X'\}$, where $X = \{x, y, z\}, X' = \{y, z, w\}$, and $c(X) = y, c(X') = z$, then CH is fulfilled but WARP fails.

Proposition 3 For an arbitrary $\mathcal{U}$, CH implies both H and C. Conversely, when $\mathcal{U}$ is $\cup$-complete, then H & C implies CH.

Indeed, let CH be fulfilled. With the condition (20) in the premise of H, consider the one-member family $\{X\}$ as a covering family for X'. Then, owing to CH, $c(X') = c(X)$. As for C, it is simply a special case of CH when the covering is the decomposition. Conversely, let both H and C be true and $\mathcal{U}$ be $\cup$-complete. Take an arbitrary $X \in \mathcal{U}$ and its arbitrary covering $\{X^\nu\}_{\nu \in N} \subseteq \mathcal{U}$, and consider $X^\cup = \bigcup_{\nu \in N} X^\nu$ which belongs to $\mathcal{U}$ by assumption. Then under the premise in (25) we obtain $c(X^\cup) = x$ due to C, and finally, if $x \in X$, then $c(X) = x$ due to H, hence (25) in CH is true. Q.E.D.

Thus, CH is generally a strengthening of the conjunction H & C, and with $\cup$-complete family $\mathcal{U}$ they are equivalent.

Proposition 4 If $\mathcal{U}$ is irreducible, then H is equivalent to CH.

Indeed, let H be true, and let (26) and the premise of (27) in Def 8 of CH be fulfilled. Then due to the irreducibility of $\mathcal{U}$ there exists $X^{\nu*} \in X^\nu$ such that $X \subseteq X^{\nu*}$, and owing to H the conclusion of (27) is true. The converse implication of H from CH has been established in Proposition 3.

Proposition 5 For an arbitrary $\mathcal{U}$, RA is equivalent to CH.

To prove this statement, it is sufficient to show the following.

Lemma 1 For any $x \in X \in \mathcal{U}$, the left-hand side in (28) is equivalent to the existence of a covering family $\{X^\nu\}_{\nu \in N}$ for the given X, such that the left-hand side in (27) is fulfilled.

Proof. Given $x \in X \in \mathcal{U}$, let the left-hand side in (27) be true for some $\{X^\nu\}$ covering X. Then each $y \in X$ belongs to some set X^y from the family $\{X^\nu\}$, with $x = c(X^y)$, hence xR_cy, and so the left-hand side in (28) is true. Conversely, let the left-hand side in (28) be fulfilled for given $x \in X \in \mathcal{U}$. Then, for every $y \in X$ due to the very definition of R_c, there exists $X^y \in \mathcal{U}$ such that $x = c(X^y), y \in X^y$. So we obtain a desired covering family $\{X^y\}_{y \in X}$ for the given $X \ni x$, which satisfies the left-hand side in (27). $\bigtriangledown$

Thus, RA is in fact an explicit expression for the implicit idea of decomposing rational choice which underlies the CH axiom and the primary H & C axioms. To elucidate this idea, let us transform the condition H into its equivalent form which is a straight conversion, or a "mirror reflection," of the condition C:

Definition 4′ We shall say that a single choice function c satisfies the *heredity* condition, H, if for every $X \in \mathcal{U}$ and for its every decomposition $\{X^\nu\}_{\nu \in N}$ in $\mathcal{U}$, i.e. for every family of sets from $\mathcal{U}$ satisfying (24), and such that a given $x \in X$ belongs to every $X^\nu, \nu \in N$, holds

$$x = c(X) \Longrightarrow (\forall \nu \in N : x = c(X^\nu)). \tag{30}$$

On noticing that every subset $X' \subseteq X$ together with X itself forms a decomposition of X, it is easy to see that Def 4′ is equivalent to Def 4. As an immediate consequence of (25) and (30), we obtain

Proposition 6 For an arbitrary $\mathcal{U}$, the conjunction H & C is equivalent to the following property HC: for every $X \in \mathcal{U}$, for every $x \in X$, and for every decomposition $\{X^\nu\}_{\nu \in N}$ of X such that $x \in \cap_{\nu \in N} X^\nu$, the following *Condorcet-Sen condition* holds:

$$x = c(X) \Longleftrightarrow (\forall \nu \in N : x = c(X^\nu)). \tag{31}$$

Now everything is prepared to present short and transparent proofs of principal criteria of choice function rationality.

Theorem 1 (Richter, 1971). With an arbitrary $\mathcal{U}$, for a single choice function c to be rational it is necessary and sufficient that c satisfies RA.

Proof. If c satisfies RA, then it is obviously rational since it is already represented by (29) in the form (16). Conversely, if c is rational, i.e., satisfies (16) with some R, then it is easy to verify directly that c satisfies RA. Indeed, in such a case $(\forall y \in X : xR_c y) \Longrightarrow (\forall y \in X\ \exists X^y \in \mathcal{U} : x = c(X^y), y \in X^y) \Longrightarrow (\forall y \in X : xRy) \Longrightarrow x = c(X)$, i.e., the converse implication in (29) (from right to left) is true; the direct implication, as it was already said, is evident. $\bigtriangledown$

The following are versions of the fundamental theorem of Sen, 1971, where the conjunction of H & C is proposed as the criterion of choice rationality under a complete $\mathcal{U}$.

Theorem 2 (Mirkin,1979). With an arbitrary $\mathcal{U}$, for a single choice function c to be rational it is necessary and sufficient that c satisfies CH.

This theorem is now obtained as a direct corollary from Theorem 1 and Proposition 5.

Theorem 3 With an arbitrary $\mathcal{U}$, for a single choice function c to be rational, WARP is sufficient.

This theorem follows from Theorem 2 and Proposition 2.

Theorem 4 With an arbitrary $\mathcal{U}$, for a single choice function c to be rational it is necessary, and if $\mathcal{U}$ is $\cup$-complete, it is also sufficient that c satisfies H & C.

This theorem is a direct corollary from Theorem 2 and Proposition 3.

Theorem 5 If $\mathcal{U}$ is 2-complete, then for a single choice function c to be rational it is necessary and sufficient that c satisfies H & C.

Proof. Let c satisfy H & C. Then, owing to Proposition 6, the equivalence (31) holds for any two-element sets $X^\nu \in \mathcal{U}$ as well. So we obtain the *Condorcet equation*

$$x = c(X) \iff (\forall y \in X : x = c(\{x, y\})) \qquad (32)$$

which in fact rationalizes c by the relation $xRy \iff x = c(\{x,y\})$. Conversely, let c be rationalized by some R in (16). Then fulfilling the conditions H and C follows immediately from their definitions. $\bigtriangledown$

As a corollary from Theorems 2 and 5, we obtain, in addition to Proposition 2, one more case of $\mathcal{U}$ that provides the equivalence of conditions CH and H & C:

Proposition 7 If $\mathcal{U}$ is 2-complete, then conditions CH and H & C are equivalent.

The following theorem is a corollary from Theorem 2 and Proposition 4.

Theorem 6 If $\mathcal{U}$ is irreducible, then for a single choice function c to be rational it is necessary and sufficient that c satisfies H.

Now we shall present some criteria of narrow rationality (Def. 1).

Theorem 7 (Richter, 1966, 1971; Suzumura, 1977) With an arbitrary $\mathcal{U}$, for a single choice function c to be rational in the narrow sense, it is necessary and sufficient that c satisfies SARP.

The proof of this theorem is relatively difficult, and since we do not need it in the sequel, we omit it (see, e.g., Richter, 1971, Corollary 1 from Theorem 8).

The following theorem is a version of Theorem 3 in Arrow, 1959.

Theorem 8 With a 3-complete $\mathcal{U}$, for a single choice function c to be rational in the narrow sense, it is necessary and sufficient that c satisfies WARP.

To prove this theorem for single-valued choice, consider the following statement which can be extracted from the reasoning in Remark 5.

Lemma 2 If $\mathcal{U}$ is 3-complete, then for single choice functions the (broad) rationality is equivalent to the narrow rationality.

We can now see that sufficiency in Theorem 8 follows from Theorem 3 and Lemma 2. As for necessity, it easily follows from the definitions. $\bigtriangledown$

The given proof of Theorem 8 is one of the few among the preceding statements and proofs, which essentially exploits the single-valuedness of choice. The rest of those are rather easily (often almost literally) extendable onto the general multiple (set-wise) choice. Now we shall consider some other statements where the single-valuedness of choice is vitally important.

Proposition 8 With an arbitrary $\mathcal{U}$, for single choice functions H implies C.

The proof of this proposition is completely parallel to that of Proposition 2 above.

Theorem 9 (Uzawa, 1956) If $\mathcal{U}$ is 3-complete, then for a single choice function c on $\mathcal{U}$ to be rational, and, equivalently, to be narrow rational, it is necessary and sufficient that c satisfies H (i.e., IIA).

Proof. By virtue of Propositions 7 and 8, under 3-completeness of $\mathcal{U}$ we have H $\Longleftrightarrow$ H & C $\Longleftrightarrow$ CH. By virtue of Theorem 2, this is equivalent to the rationality, and owing to Lemma 2 also to the narrow rationality of single choice functions on $\mathcal{U}$. $\bigtriangledown$

From collation of Theorems 1,2,5,8 and 9, we obtain, as a strengthening of Proposition 1, a deeper connection of IIA, i.e., H, with other conditions on single choice functions c on 3-complete families $\mathcal{U}$.

Proposition 9 If $\mathcal{U}$ is 3-complete, then for single choice functions on $\mathcal{U}$ the conditions H, WARP, SARP, CH, H & C and RA are equivalent.

Finally, here is one more particular type of families $\mathcal{U}$ in which the condition H is equivalent to rationality of single choice functions.

Theorem 10 If $\mathcal{U}$ is pairwise-$\cap$-complete, then for a single choice function c to be rational it is necessary and sufficient that c satisfies H.

Proof. Sufficiency follows from Theorem 3 and Proposition 1, and necessity from Theorem 2 and Proposition 3. ∇

Now we shall briefly touch upon the general case of multiple (set-wise) choice. The forms of expressions, as used above for single choice, can in fact be applied word by word to this general case as well. What is needed is the replacement of the equality $x = c(X)$ by the inclusion $x \in C(X)$. With such replacement, all the definitions of conditions H (formerly IIA), C, CH, RA (and HC in Proposition 6), as well as Theorems 1-6 and 8 are still valid together with their proofs given *mutatis mutandis*; the same relates to Propositions 1 (except the converse part of its statement) and 2-7. As for the definitions of WARP and SARP in the set-wise case, they need a more careful approach, and with their appropriate formulations the statements of Theorems 7 and 8 still remain valid (see, e.g., Suzumura, 1976); but we do not need them here. All we need in the sequel are the very ideas of abstract versions of "independence of irrelevant alternatives" (or heredity H), of concordance C, and of their amalgamation, concordant heredity CH. In fact, these conditions remain in the general case without changes, except for the replacement of $x = c(X)$ by $x \in C(X)$.

The following theorem will help elucidate the role of conditions H and C taken separately.[1] Here the general case of set-wise choice is considered, so in the definitions of H and C one should take $x \in C(X)$ instead of $x = c(X)$ (though one may continue to deal with the particular case, a single choice $x = c(X)$).

Theorem 11 For an arbitrary $\mathcal{U}$, a choice function C satisfies H or, respectively, C iff it is representable in the *semirational* form

$$C(X) = \{x \in X \mid \forall y : xR(X)y\},\ X \in \mathcal{U}, \tag{33}$$

with a pseudorelation $R(X)$ decreasing (respectively, increasing) in X, in the sense that if $X' \subseteq X$, then $xR(X)y \Rightarrow$ (resp., $\Leftarrow$)$xR(X')y$.

[1]This theorem, established by the author, has been published in a collection of papers which is hardly accessible to a Western reader; its equivalent version in English was quoted in the survey by Aizerman, 1986.

This theorem will follow from the results of Section 4. It is clear that the conventional rational case (17) is the case when a pseudorelation $R(X)$ is indeed a true relation, i.e., it is in fact independent on X. This means that formally $R(X)$ is both (nonstrictly) decreasing and increasing in X. (Caution: for an arbitrary $\mathcal{U}$, out of cases given in Theorems 4 and 5, even under H & C there may be no such a relation R. Then pseudorelations $R(X)$ in representations of the form (33) for H and for C must be different).

It is necessary to remark that extention of some rationality conditions from a single choice onto a multiple choice can be performed in different ways. Consider the case of the extention of independence of irrelevant alternatives. Above such an extention was carried out by simply replacing $x = c(X)$ by $x \in C(X)$; this leads to the following set-wise form of the condition H: if $X, X' \in \mathcal{U}$ and $X' \subseteq X$, then

$$C(X) \bigcap X' \subseteq C(X'). \tag{34}$$

However, this is not the only natural way of extending IIA from a single onto a multiple choice. Among other possibilities we shall choose and use in the sequel one from Chernoff, 1954 – Postulate 5*; Jamison and Lau, 1973 – Axiom 2; Aizerman and Malishevski, 1981 – Condition O; Schwartz, 1986 – Condition W7; Bordes, 1986 – "Strong Superset Condition":

Definition 10 We shall say that a set-wise choice function C on $\mathcal{U}$ satisfies the condition of *outcasting of rejected alternatives*, O, if for every $X, X' \in \mathcal{U}$ such that $X' \subseteq X$, holds

$$X^* = C(X),\ X^* \subseteq X' \Longrightarrow X^* = C(X'). \tag{35}$$

This definition in fact treats the choice set $C(X)$ as a whole, as a direct counterpart of the choosen element $c(X)$ in the single-choice formulation of IIA (see (20), (21) above). This treatment of the condition O will be examined in Section 4.

3. Examination of behavior rationality in some models

Here we shall consider three particular models of decision making: one describes consumer choice and two others represent the cases of hypochoice and hyperchoice in the abstract choice model given in Section 1.

I. Let us start with a nonconventional model of consumer choice under rationing of consumption. This model was proposed by E.M.Braverman, 1976, as an attempt to describe the former Soviet-type *economics of shortages*. In this model the variables which affect the consumer behavior are the given limitations on the feasible commodity quantities. The prices and income values may still exist, but they ought to be fixed and so they may be removed from the explicit description of the person's behavior as a function of variable parameters.

Let $\mathbf{x} = (x_1, \ldots, x_n)$, as earlier, be a commodity bundle, and let $\mathbf{b} = (b_1, \ldots, b_n)$ be the vector of admissible upper bounds for consumption of corresponding goods, $\mathbf{b} \in R^n_+$. Following Braverman, 1976, with a slight modification, let us introduce the dependence of the commodity bundle consumed, $\mathbf{x}$, on the limitation vector $\mathbf{b} \in R^n_+$: $\mathbf{x} = \mathbf{f}(\mathbf{b})$, called *consumer choice function.* This function must obey the obvious condition

$$\mathbf{o} \leqq \mathbf{f}(\mathbf{b}) \leqq \mathbf{b}. \tag{36}$$

Definition 11 We shall say that the consumer choice function $\mathbf{f}(\mathbf{b})$ is *normal*, if for any $\mathbf{b}, \mathbf{b}'$ such that

$$f_i(\mathbf{b}) = b_i \Longrightarrow b_i' = b_i \tag{37}$$

$$f_i(\mathbf{b}) < b_i \Longrightarrow b_i' \geq f_i(\mathbf{b}) \tag{38}$$

holds

$$\mathbf{f}(\mathbf{b}) = \mathbf{f}(\mathbf{b}'). \tag{39}$$

This normality condition yields an axiomatic description of consumer choice.

Consider this model from the choice theory standpoint. For an outside observer $\mathbf{f}(\mathbf{b})$ plays the role of a choice function $c(X(\mathbf{b}))$ defined on a family $\mathcal{U} = \{X(\mathbf{b})\}_{b \in \mathcal{B}}$ of admissible sets in $U = R^n_+$:

$$X(\mathbf{b}) = \{\mathbf{x} \in R^n_+ \mid \mathbf{o} \leqq \mathbf{x} \leqq \mathbf{b}\}, \tag{40}$$

where $\mathcal{B}$ is a parameter set; in particular, but not necessarily, we can take $\mathcal{B} = R^n_+$. It is easy to see, that a) a family $\mathcal{U}$ is irreducible (Def.3) and b) a normal choice function $f(\mathbf{b})$ on $\mathcal{U}$ satisfies the condition H, i.e., independence of irrelevant alternatives. The latter is a direct corollary of normality (37)-(39) (and moreover, is equivalent to normality under continuity of $\mathbf{f}(\mathbf{b})$). To see the fulfillment of H it is enough to take $X(\mathbf{b}') \subseteq X(\mathbf{b})$, with $\mathbf{f}(\mathbf{b}) \in X(\mathbf{b}')$, which implies $\mathbf{f}(\mathbf{b}) \leqq \mathbf{b}' \leqq \mathbf{b}$. Therefore the premise (37) and (38) of normality is fulfilled, hence the conclusion (39) must be true, and so H is satisfied. Using Theorem 6, we obtain

Statement 1 The normal consumer choice in Braverman's model is rational for the outside observer, i.e., when considering $\mathbf{f}(\mathbf{b})$ as the single choice function $c(X(\mathbf{b}))$ given on a family $\mathcal{U} = \{X(\mathbf{b})\}_{b \in \mathcal{B}}$ of admissible sets of the form (40), with an arbitrary $\mathcal{B}$.

In other words, the consumer choice $f(\mathbf{b})$ here may be "explained" by means of optimization of the form (16) on $X(\mathbf{b})$ with some preference relation R on R^n_+ (or, equivalently, of the form (18) with some strict preference relation P on R^n_+).

We emphasize that above we meant rationality in the broad, not in the narrow sense. The latter is generally false for Braverman's model, which can be seen from the following example. Take $n = 3, b^1 = (\alpha, 1, 1), b^2 = (1, \alpha, 1), b^3 = (1, 1, \alpha)$ (with $p = (1, 1, 1)$ and $I = 2\alpha + 1$, if one wishes to take into account the budget constraint explicitly), and let $f(b^1) = (\alpha, \alpha, 1), f(b^2) = (1, \alpha, \alpha)$ and $f(b^3) = (\alpha, 1, \alpha)$. Then one can easily see that SARP is violated, hense the narrow rationality is impossible here.

Let us recall that in a monetary economy, even under the conditions of deficit, the consumer has to obey the budget constraint $\mathbf{p}f(\mathbf{b}) \leq I$, or, in terms of the budget set $B = B(\mathbf{p}, I)$ in (2) (remember that $\mathbf{p}$ and I are fixed), the condition

$$f(\mathbf{b}) \in B. \tag{41}$$

The presence of the budget constraint and perhaps other considerations may force the consumer to abandon the "best" corner point $\mathbf{b}$ (here we implicitly suppose that all commodities are desirable) in the set $X(\mathbf{b})$ admissible to him as seen by the outside observer, and to choose a point $f(\mathbf{b})$ which is typically $\leq \mathbf{b}$. From the consumer's standpoint, it is natural to consider as admissible the narrower sets of the form

$$X^I(\mathbf{b}) = X(\mathbf{b}) \bigcap B. \tag{42}$$

The budget constraint is essential only if $\mathbf{pb} > I$, i.e., when $X^I(\mathbf{b})$ is indeed smaller than $X(\mathbf{b})$.

Let us consider the following example. Let $n = 2, \mathbf{p} = (1, 1), I = 4$, and $\mathbf{b} = (3, 3), \mathbf{b}' = (4, 2), \mathbf{b}'' = (2, 4)$. Let $\mathbf{f}(\mathbf{b}) = (3, 1)$ and $\mathbf{f}(\mathbf{b}') = \mathbf{f}(\mathbf{b}'') = (2, 2)$. Then for the set $\mathcal{B} = \{\mathbf{b}, \mathbf{b}', \mathbf{b}''\}$ the normality condition is fulfilled, and hence, both for $\mathcal{U} = \{X(\mathbf{b})\}_{\mathbf{b}\in\mathcal{B}}$ and $\mathcal{U}^I = \{X^I(\mathbf{b})\}_{\mathbf{b}\in\mathcal{B}}$ the condition H is valid. However, because of

$$X^I(\mathbf{b}) \subseteq X^I(\mathbf{b}') \bigcup X^I(\mathbf{b}''), \tag{43}$$

the condition CH is violated (and even more so WARP also fails, which can be seen independently by comparing $X(\mathbf{b})$ and $X(\mathbf{b}')$). Therefore, due to Theorem 2, the described consumer choice is irrational when considering the sets $X^I(\mathbf{b})$ admissible.

The reason for the discrepancy between this conclusion and Statement 1 is that a possible "rational explanation" of consumer choice, as seen by the outside observer, requires a specific feature of the strict preference P in (18). Namely, the element $\mathbf{x} = (2, 2)$ belonging to each of $X(\mathbf{b}), X(\mathbf{b}'), X(\mathbf{b}'')$, and chosen in $X(\mathbf{b}')$ and $X(\mathbf{b}'')$ but rejected in $X(\mathbf{b})$, can be dominated in $X(\mathbf{b})$ only by some $y \in X(\mathbf{b}) \backslash (X(\mathbf{b}') \cup X(\mathbf{b}''))$, so that

$y \in X(\mathbf{b}) \setminus X^I(\mathbf{b})$. This means that y dominating x in fact is not the commodity bundle which might have been considered by the consumer as a feasible alternative. (I avoid the word "preferable" because WARP is violated here). So, from the consumer's point of view, the explanation of the choice by means of such a preference P is not valid.

This example raises the following question: is it possible to obtain any result about the rationality, based only on external observations of the admissible sets $X(\mathbf{b})$, without knowing the budget set B, and hence by knowing "true" feasible sets $X^I(\mathbf{b})$ only incompletely? The answer is "yes." To show this, let us consider a special but rather natural type of parameter sets $\mathcal{B}$.

Definition 12 We shall call a set $\mathcal{B}$ *rectangular* if it represents a Cartesian product $\mathcal{B} = \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ of some sets $\mathcal{B}_i$.

The examples of rectangular sets $\mathcal{B}$ would be $\mathcal{B} = R^n_+$, or more generally, $\mathcal{B} = \{\mathbf{b} \in R^n \mid \mathbf{o} \leqq \mathbf{b} \leqq \bar{\mathbf{b}}\}$. It is easy to see that rectangularity implies the following property: for every $\mathbf{b}', \mathbf{b}'' \in \mathcal{B}$ it is true that $\min\{\mathbf{b}', \mathbf{b}''\} \in \mathcal{B}$, where $\min\{\mathbf{b}', \mathbf{b}''\}$ denotes the n-vector with components $\min\{\mathbf{b}'_i, \mathbf{b}''_i\}$, $i = 1, \ldots, n$.

Lemma 3 If the parameter set $\mathcal{B}$ is rectangular, then the family $\mathcal{U} = \{X^I(\mathbf{b})\}_{\mathbf{b} \in \mathcal{B}}$ is pairwise-$\bigcap$-complete.

Proof. It is clear that $X(\mathbf{b}') \bigcap X(\mathbf{b}'') = X(\mathbf{b})$, where $\mathbf{b} = \min(\mathbf{b}', \mathbf{b}'')$, and furthermore $X^I(\mathbf{b}') \bigcap X^I(\mathbf{b}'') = X^I(\mathbf{b})$. Since due to rectangularity of $\mathcal{B}$, $\mathbf{b}', \mathbf{b}'' \in \mathcal{B}$ implies $\mathbf{b} \in \mathcal{B}$ then $\mathcal{U}$ is pairwise-$\bigcap$-complete. $\bigtriangledown$

Statement 2 Independent of the fixed but unknown budget set B, if the family $\mathcal{B}$ is rectangular then the normal consumer choice in Braverman's model is rational when considering $\mathbf{f}(\mathbf{b})$ as a single choice function $c(X^I(\mathbf{b}))$, given on a family $\mathcal{U}\{X^I(\mathbf{b})\}_{\mathbf{b} \in \mathcal{B}}$ of inner subjectively admissible sets.

Proof. By Theorem 9 and Lemma 3, and since $f(\mathbf{b})$ as a single choice function satisfies the condition H, $\mathbf{f}(\mathbf{b})$ must be rational. $\bigtriangledown$

We have thus considered a case, where we can assert the rationality of a person's behavior in spite of our lack of complete knowledge of the person's subjective view on admissible sets.

II. Let us consider now the case of hypochoice in the abstract choice model sketched in Section 1. Let us admit that a given choice function $C : \mathcal{U} \to 2^U$ yields empty values on some subfamily $\mathcal{U}^\emptyset = \{X \in \mathcal{U} \mid C(X) = \emptyset\}$. Suppose for simplicity that on the rest of $\mathcal{U}$, i.e., on $\mathcal{U}^* = \mathcal{U} \backslash \mathcal{U}^\emptyset$, the function C is singleton-valued, so C is a single-or-empty choice function on $\mathcal{U}$, describing a single choice with the possibility of indecisiveness. As discussed in Section 1, the phenomenon of the empty choice, or abstaining from choice, can be considered a new *imagined* alternative; denote it $\oint$. Then we obtain the new, extended totality of alternatives $\bar{U} = U \bigcup \{\oint\}$, the new family of admissible representations $\bar{\mathcal{U}} = \{X \bigcup \{\oint\} \mid X \in \mathcal{U}\}$ and the new, purely single-valued choice function $\bar{c}$ on $\bar{\mathcal{U}}$ defined as

$$\bar{c}(X \cup \oint) = \begin{matrix} x & \text{if } C(X) = \{x\}, \\ \oint & \text{if } C(X) = \emptyset. \end{matrix} \tag{44}$$

Call it *surrogate* choice function. The family $\bar{\mathcal{U}}$ is evidently incomplete, even when $\mathcal{U}$ is complete, because $\bar{\mathcal{U}}$ contains no sets that do not include $\oint$.

The rationality of the initial function C does not guarantee the rationality of its *surrogate* $\bar{c}$. Indeed, consider the simplest case of two primary options: $U = \{x, y\}$, $\mathcal{U}$ is complete for U, and C on $\mathcal{U}$ is given by $C(\{x\}) = x$, $C(\{y\}) = \{y\}$, $C(\{x, y\}) = \emptyset$. It is the situation of Buridan's ass in front of two equivalent piles of hay. This choice function is rational: it is rationalized by the non-antisymmetric dominance relation P of the form xPy and yPx in (18). Nevertheless, the corresponding surrogate choice function $\bar{c}$ cannot be rational since $\bar{c}(\{x, y, \oint\}) = \oint$ but $\bar{c}(\{x, \oint\}) = x$, which contradicts IIA (H). Thus, Buridan's ass is eventually seen as irrational.

Now consider the general case and examine conditions of the rationality for surrogate choice functions $\bar{c}$. For this purpose the general criterion, the condition CH (Theorem 2), can be applied. The application of CH, i.e., (26), (27), is reduced to two separate cases:
a) $x \in U$ - then the criterion demands the satisfaction of CH, i.e., the rationality of the single-or-empty choice function C on $\mathcal{U}$;
b) $x = \oint$ - then the criterion is reduced to the following requirement: for all $X \in \mathcal{U}$ and $\{X^\nu\}_{\nu \in N} \subseteq \mathcal{U}^\emptyset$ such that $X \subseteq \cup_{\nu \in N} X^\nu$ one obtains $C(X) = \emptyset$, i.e., $X \in \mathcal{U}^\emptyset$.

Case (b) can be described in other terms as follows. Denote $U^\emptyset = \bigcup\{X \mid X \in \mathcal{U}^\emptyset\}$. Then the requirement of CH in (b) is equivalent to

$$C(X) = \emptyset \Longleftrightarrow X \subseteq U^\emptyset. \tag{45}$$

Thus we obtain

Statement 3 For the surrogate choice function $\bar{c}$ (44) to be rational it is necessary and sufficient that a) the original choice function C is rational, and b) the equation (45) holds.

To interpret equation (45), let us call $U^\emptyset$ the set of *conditionally unfit* alternatives. Then (45) means that the person refrains from (real) choice if and only if the admissible (real) alternative set consists entirely of conditionally unfit alternatives. Here the word "conditionally" implies that a conditionally unfit alternative might be chosen, when not all present alternatives are unfit. The example is a modification of the above Buridan's ass case, when an extra real alternative z is added and when $\mathcal{U} = \{\{x, y\}, \{x, z\}, \{y, z\}\}$ with $C(\{x,y\}) = \emptyset$, $C(\{x,z\}) = \{x\}$, $C(\{y,z\}) = \{y\}$. The primary function C on $\mathcal{U}$ is obviously rationalized by the following P: xPz, yPz, xPy, yPx, and the surrogate $\bar{c}$ on $\mathcal{U}$ is rationalized, in addition, by $zP\oint$. The alternatives x and y are conditionally unfit, and each of them is chosen in the corresponding context.

A stronger and more natural result is obtained when the family $\mathcal{U}$ is 1-complete, i.e., it includes all singletons. Note that usually "choice" from one-element sets is not considered, being implicitly treated as a degenerate case , especially when the strong rationality requirement does not allow empty choice. (The former Soviet "elections" on the principle "one seat – one candidate" were the real example of the "choice" of such kind). But if one accepts the possibility of empty choice, then the choice from singletons, *Hobson's choice* [2] becomes meaningful and can lead to useful results. Let us consider an alternative which is rejected when presented alone. In terms of the conventional rational choice the alternative x such that $C(\{x\}) = \emptyset$ must be *selfdominant* in the sense that xPx. Therefore such alternatives shall not be chosen in any other situation, when together with other arbitrary alternatives; so they may be called *absolutely unfit.* In spite of the ultimate simplicity, the inclusion of such alternatives into consideration may yield a useful extension of the traditional framework of choice rationality (a display of such a phenomenon can be found in the survey of Aizerman, 1985, and also, in an implicit form, in Sertel, 1988). Returning to the above construct, we can see that in case of 1-complete family $\mathcal{U}$ each conditionally unfit alternative turns out to be absolutely unfit: indeed, owing to (43),

$$x \in U^{\emptyset} \Longleftrightarrow C(\{x\}) = \emptyset, \tag{46}$$

and so $x \in U^{\emptyset} \Leftrightarrow (xPx \quad \text{or} \quad \oint Px)$ for every possible rationalization of $\bar{c}$. Thus, in the case of 1-completeness of $\mathcal{U}$ Statement 3 can be extended to

Statement 4 If $\mathcal{U}$ is 1-complete, then in addition to conditions of Statement 3, under rationality of $\bar{c}$ the following is true:

$$x \in U^{\emptyset} \Leftrightarrow \forall X \in \mathcal{U} : (x \not\in C(X)). \tag{47}$$

III. Now we shall consider the general case of set-wise choice function $C(X)$ when the condition $|C(X) = 1$ is not assumed. We shall call this general case *hyperchoice*, now

[2] *Hobson's choice*, the choice of taking either that which is offered or nothing; the absence of a real choice or alternative. [after Thomas Hobson (1544-1631), of Cambridge, England, who rented horses and gave his customer only one choice, that of the horse nearest the stable door]. (Webster's Dictionary)

using a wider meaning of this term, viz., not requiring that $|C(X)| > 1$ for each X, but admitting that $|C(X)| = 1$ (single choice) or even $|C(X)| = 0$ (empty choice) for some $X \in \mathcal{U}$. Following the approach described in Section 1, we will treat such a choice as that of ***composite objects***, viz., of initial object sets taken as a whole. From this standpoint, the presentation of a set $X \subseteq U$ of initial objects $x, y, \ldots \in X$ means the presentation of the set $\mathcal{X} = 2^X$ of mentioned composite objects $Y, Z, \ldots \subseteq X$.

For a given choice function $C : \mathcal{U} \to 2^U$ we introduce the ***surrogate*** single choice function $\bar{c} : \bar{\mathcal{U}} \to \bar{U}$, where $\bar{U} = 2^U$, $\bar{\mathcal{U}} = \{2^X \mid X \in \mathcal{U}\}$, and $\bar{c}(\mathcal{X}), \mathcal{X} \subseteq \mathcal{U}$, is defined as

$$\bar{c}(2^X) = C(X). \tag{48}$$

The set $C(X)$ in the right-hand side of (48) is treated as a point in the set $\bar{U} = 2^U$, this meaning that $\bar{c}$ is actually the single-valued choice function.

Let us examine the rationality of $\bar{c}$. Note that its domain, the family $\bar{\mathcal{U}}$, is incomplete (unless $\mathcal{U}$ contains only singletons): $\mathcal{X} \in \bar{\mathcal{U}}$ cannot contain some X without containing every $Y \subseteq X$ at the same time. However, $\bar{\mathcal{U}}$ possesses an important property:

Lemma 4 For an arbitrary $\mathcal{U}$, the corresponding surrogate family $\bar{\mathcal{U}}$ is always irreducible.

Proof. Let $\bar{X} \in \bar{\mathcal{U}}, \{\bar{X}^\nu\}_{\nu \in N} \subseteq \bar{\mathcal{U}}$ and $\bar{X} \subseteq \bigcup_{\nu \in N} \bar{X}^\nu$. Then $\bar{X} = 2^X$ and $\bar{X}^\nu = 2^{X^\nu}, \nu \in N$, for some $X, X^\nu \in \mathcal{U}$ ($\nu \in N$), with $2^X \subseteq \bigcup_{\nu \in N} 2^{X^\nu}$. The latter means that for some $X^* \in \{X^\nu\}$ holds $X \subseteq X^*$, hence $\bar{X} = 2^X \subseteq 2^{X^*} = \bar{X}^*$ for some $\bar{X}^* \in \{\bar{X}^\nu\}$. ∇

(It is easy to verify also that if $\mathcal{U}$ is pairwise-$\cap$-complete then so is $\bar{\mathcal{U}}$).

It follows from Lemma 4 and Theorem 6 that the surrogate single choice function $\bar{c}$ on $\bar{\mathcal{U}}$ is rational if and only if it satisfies the condition H. It is easy to see that the condition H for $\bar{c}$ on $\bar{\mathcal{U}}$ is actually equivalent to the condition O (Def. 10) for the original set-wise choice function C on $\mathcal{U}$. Thus we obtain

Statement 5 Let $C : \mathcal{U} \to 2^U$ be a set-wise choice function. For the corresponding surrogate single choice function $\bar{c}$ (48) on $\bar{\mathcal{U}}$ to be rational it is necessary and sufficient that C satisfies the condition O.

Let us interpret the fact of rationality of the surrogate choice function $\bar{c}$ in terms of the initial choice function C. This fact shall mean that there exists a binary relation $\mathcal{R}$ on 2^U, we shall call it *hyperrelation*, such that

$$C(X) = X^* \quad \text{such that} \quad \forall Y \subseteq X : X^* \mathcal{R} Y \tag{49}$$

(we assume that $C(X)$ in (49) is well defined, i.e., the corresponding $X^* \subseteq X$ is unique).

The expression (49) has been introduced in Aizerman and Malishevski, 1981, as the *hyperdominant choice mechanism.* Statement 5 above presents a reformulation (in a generalized form - for arbitrary $\mathcal{U}$) of Theorem 7 in Aizerman and Malishevski, 1981; it says that a choice function $C : \mathcal{U} \to 2^U$ is generable by a hyperdominant mechanism (49) if and only if it satisfies the condition O. Now we have interpreted such an ability to generate a choice function satisfying O on a different level, namely on the level of object bundles taken as composite alternatives. This interpretation was given in terms of a single choice of such composite alternatives.

Note that the former problem of rationality for the hypochoice is now embedded into the latter more general problem for hyperchoice. The empty bundle $X^* = \emptyset$ here is treated as equally eligible, and the application of the rationality criterion (condition H for surrogate choice or, equivalently, O for original choice) immediately leads to the notions of conditionally unfit or, with 1-complete $\mathcal{U}$, absolutely unfit objects in the same manner as it has been done in the previous hypochoice analysis.

To complete, let us pose this additional question: under what conditions is a choice function $C : \mathcal{U} \to 2^U$ representable in the form of the *hyperscale optimization*

$$C(X) = X^* \quad \text{such that} \quad X^* = \arg\max V(Y), \tag{50}$$

where $V(Y)$ is a *hyperscale*, viz., a mapping $V : 2^U \to L$ (L is a linearly ordered set)?

The answer is easily obtained by means of Theorem 7, applying SARP to the surrogate choice function $\bar{c}$ (48), which in terms of the original C yields the following *hyper*-SARP formulation: for any $X^1, \ldots, X^n \in \mathcal{U}, n > 1$, holds

$$X^{*i} = C(X^i),\ X^{*i} \subseteq X^{i+1}, i = 1, \ldots, n \Rightarrow X^{*1} = \ldots = X^{*n}. \tag{51}$$

The final result can be formulated as follows:

Statement 6 With an arbitrary $\mathcal{U}$, for a set-wise choice function $C : \mathcal{U} \to 2^U$ to admit a hyperscale optimizational representation (50) it is necessary and sufficient that C satisfies hyper-SARP (51).

In this section we thus considered applications of the rationality criteria which have been elaborated earlier for the conventional choice-theoretical models. We applied those criteria to three problems while demonstrating the step-by-step departure from the common view of alternatives. In the first problem, Braverman's model, the admissible alternative sets were allowed to be partially unobservable. In the second problem, the hypochoice, the initial totality of options was augmented by an artificial "irreal" alternative. And in the third problem, the hyperchoice, we changed the very nature of the original totality of options by switching to the newly created totality of alternatives. The next step, the complete refusal from an explicit consideration of alternatives in the question of rationality of behavior, will be made in the following section.

4. A model of rational decisions with hidden alternatives

As mentioned in the Introduction, the common notion of decision making as of selecting an alternative among a set of admissible alternatives admits different modes of formalization. The most refined formalization, the abstract scheme in the choice theory, has been presented in Section 2. The key concept of this theory is the choice correspondence (function): the admissible alternative set $X \mapsto$ the chosen alternative(s) $c(X)$ $(C(X))$. It should be noted that already in earlier works on the choice theory some other types of correspondences had been considered, such as an explicit dependency $C = C(X; R)$ on the underlying preference relation R, or on the "profile" $R = (R_1, \ldots, R_n)$ of individual preferences R_i, when a collective choice is considered (Arrow, 1963; Sen, 1970), or a dependency on the variable composition of a group (Smith, 1973), and so on. Moreover, the decision problem itself can be formulated not as a selection among primary objects named alternatives but a selection among other, secondary objects such as, e.g., a group ordering of primary alternatives. In the latter case it is the orderings of primary alternatives that deserve the name of true alternatives in front of DM rather than primary ones. Nevertheless, the alternatives and dependences of the chosen alternatives on some "experiment conditions" are present in each typical statement of the decision problem.

Now we shall make an important step and abandon an explicit consideration of alternatives in the formulation of the problem. The prerequisite for such a step lies implicitly in the Samuelson's idea of revelation of preferences which serves as a source for an implicit judgement of the rationality of decisions. We take Samuelson's Weak Axiom of Revealed Preferences and give it an equivalent form that is more convenient for the explication of the idea needed: for any $X, X' \in \mathcal{U}$

$$X' \ni c(X) \Rightarrow \text{ either } c(X') = c(X) \text{ or } c(X') \notin X. \tag{52}$$

The premise (left-hand-side) of (52) means that the replacement of the former admissible set X by a new one X' is such that the former best choice $c(X)$ still remains admissible, hence it is natural to believe that the new position (opportunity scope) of DM is at least as good as the old one. The two possibilities considered in the right-hand side of (52) mean that the new position is, respectively, either not better than the old one - and so DM does not change his decision, or is even better - and then DM makes a new decision which could not have been made in the old position (so the new choice $c(X')$

is "revealingly better" than the old $c(X)$). Thus, the Samuelson's approach implies the idea of implicit comparison of worth of two opportunity scopes.

The formulation of WARP demands the explicit indication of the chosen alternatives. However, the corresponding formulation of its close variation, IIA (see Proposition 1), permits to avoid such indication in the premise of the statement. Indeed, the equivalent formulation of IIA parallel to (52) has the following form: for any $X, X' \in \mathcal{U}$

$$X' \supseteq X \Rightarrow \text{ either } c(X') = c(X) \text{ or } c(X') \notin X. \tag{53}$$

As compared with (52), the only change we introduce is the change of the premise: the new one, $X' \supseteq X$, does not require the condition $X' \ni c(X)$ since this is already warranted by the condition $X' \supseteq X$ itself. What remains is the next step: to abandon the explicit indication of the chosen alternatives in the right-hand side of (53) as well. For this purpose we shall formalize the notions of *opportunity scope* and *value of opportunity scope.*

As a particular case of an opportunity scope, one may consider the explicit indication of the set of the admissible alternatives, X, or of a parameter characterizing such a set. The example of the latter may be the classical budget set $B(\mathbf{p}, I)$ given by its parameters $\mathbf{p}, I$ or the nonclassical consumer admissible set in Braverman's model, $X(\mathbf{b})$, given by the parameter $\mathbf{b}$, etc. Moreover, an opportunity scope in general may be given implicitly by indicating the "experiment conditions" under which the decision maker shall demean. For example, the behavior of an undertaker is determined by the legislation under which he does his business. We can possess the exact and complete text of legislation but not know the alternative modes of behavior available to the businessman. Furthermore, we may estimate (or obtain by inquiry) the value of the current opportunity scope that is the best (highest) from DM's standpoint (e.g., an expected maximal profit). We denote the opportunity scope, independent of its nature, by X and save the notation $\mathcal{U}$ for the family of possible opportunity scopes; then we introduce the *scope value* as a function $v : \mathcal{U} \dashrightarrow L$, where L is a linearly ordered set. Thus, $v(X)$, $X \in \mathcal{U}$, is a scale of values of opportunity scopes, and we can now introduce the axiomatic conditions of behavior rationality in terms of the function v. Note that the well-known *indirect utility function*, which for a classical model of consumer of the form (1) is just the optimal value of the achievable utility: $v(\mathbf{p}, I) = \max_{\mathbf{x} \geqq 0 : \mathbf{p}\mathbf{x} \leqq I} u(\mathbf{x})$, may be considered an example of the scope value.

In what follows, we consider a simple model in which the opportunity scopes are finite sets of elements from some primary *opportunity space* U. Then the family $\mathcal{U}$ consists of some feasible sets of available point-wise opportunities, like in the case of alternative sets. We shall use the terminology from Def 3 when speaking about various types of families $\mathcal{U}$. As an example of a situation where an "opportunity space" is modelled as a space in the set-theoretical sense, we consider a simple case of legislation in which each "law" has a form of "bill of rights". Then the totality of all conceivable rights forms the set U, every possible "right" is considered as a point of U, every "law" is a finite set $X \subseteq U$, and usual set-theoretical notions and operations are meaningful, in the sense that they form new legislations from the old ones. Note that different $x \in U$ here are "rights", which create opportunities for admissible alternative modes of behavior, but in no case x's are alternatives themselves for DM's choice.

Let us introduce the first condition of rationality in terms of the scope value v, which is a direct analogue of IIA in the form (53).

Definition 13 We shall say that a scope value $v : \mathcal{U} \rightarrow L$ satisfies the *monotonicity* condition, M, if for any $X, X' \in \mathcal{U}$

$$X' \supseteq X \Rightarrow v(X') \geq v(X). \tag{54}$$

This condition looks quite convincing: the more possibilities, the better. However, sometimes one might dispute this thesis. Recall Buridan's ass: the appearance of the second pile of hay worsens his position. Another reason for a worsening situation after enlarging the opportunity scope can be, e.g., an additional expense on the choice procedure (information processing etc). A formal violation of the condition M is presented in Game example 2 in Section 1. There, the optimal expected gain $e(i^*)$ of DM, player 1 (viz., his regret with the opposite sign) for three possible actions $I = \{1,2,3\}$ is $e(1) = -\max_{j \in J} w_{1j} = -2$, but the optimal $e(i^{*\prime})$ for $I' = \{1,2\}$ is $e(2) = -\max_{j \in J} w'_{2j} = -1$. Thus, $e(i^{*\prime}) > e(i^*)$, i.e., the gain increases after narrowing the set of possible actions, which plays the role of the opportunity set for DM.

This example, perhaps, looks a little artificial since the elements of the virtual payoff matrix, viz., of the regret matrix, change when changing the set of DM's actions. To avoid this shortcoming, consider still another game example.

Game example 3. Consider the well-known game *the battle of the sexes*, (see, e.g., Luce and Raiffa, 1957), the bimatrix game with the payoff bimatrix

$$\begin{bmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{bmatrix} \tag{55}$$

where elements v_{ij} and u_{ij} of an ij-th entry (v_{ij}, u_{ij}) denote rewards of the players 1 and 2, respectively, when player 1 performs the action $i \in I$ and player 2 the action $j \in J$. Let us again use Stackelberg's approach, with player 2 being the leader and player 1 the follower. To describe the corresponding solution of the game, denote $i^o(j) = \arg\max_{j \in J} v_{ij}$. Then the optimal action of player 2 is $j^* = \arg\max_{i \in I} u_{i^o(j),j}$ and of player 1 is $i^* = \arg\max_{i \in I} v_{ij^*}$. It is easy to see that in the above game (where $I = \{1,2\}$ and

$J = \{1, 2\}$) we have $j^* = 2$, $i^* = 2$ and the optimal reward of player 1 is $v_{i^*j^*} = v_{22} = 1$. However, if one changes the game by canceling the second action of player 1, so that $I' = \{1\}$, then we obtain $j^* = 1, i^* = 1$, and DM's reward becomes $v_{11} = 2$. Thus, adding a second possible action for DM only decreases his reward.

Therefore, the condition M is not universally true, but is simply a plausible assumption for a class of behavioral situations. The following condition is meaningful for a rather wide class of situations, though it is more restrictive and less evident than monotonicity. This condition, as opposed to M, requires that if, while enlarging the opportunity scope, an increase of the scope value takes place, then this increase should be not too large. Namely, if the opportunity scope is decomposed into several parts (with possible overlapping), then the value of the whole scope must not exceed the maximal value for its parts. This condition implicitly demands that the union of several opportunity scopes would not have created "essentially new" opportunities, which would yield a reward more than that for all opportunities contained in the initial scopes. In other words, more powerful opportunities cannot emerge as a result of merging some old ones.

Definition 14 Let $\mathcal{X} \subseteq \mathcal{U}$ be a family of admissible opportunity scopes $\mathcal{X} \subseteq U$. Introduce the *hypervalue of* $\mathcal{X}$ as

$$V(\mathcal{X}) = \max_{x \in \mathcal{X}} v(X). \tag{56}$$

Definition 15 We shall say that a scope value v on $\mathcal{U}$ satisfies the *nonemergence* condition, N, if for every $X \in \mathcal{U}$ and for every its decomposition $\mathcal{X}$ in $\mathcal{U}$, i.e., a family $\mathcal{X} \subseteq \mathcal{U}$ such that $\cup_{Y \in \mathcal{X}} Y = X$, holds

$$v(X) \leq V(\mathcal{X}). \tag{57}$$

The restrictive character of the condition N is obvious and can be illustrated by any counterexample in which joining several opportunities creates something essentially more powerful. To show it, consider

Game example 4. Take the simplest zero-sum game with the payoff matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{58}$$

and consider the standard maximin principle of behavior for player 1; admit the usage of mixed strategies. Then the optimal maximin strategy of player 1 is $(p_1, p_2) = (1/2, 1/2)$ with the gain (maximin expected reward) $1/2$, whereas for each of the two separate actions (pure strategies) 1 and 2 the maximin reward is 0. Thus, the creation of a new opportunity, mixed strategy, after joining two pure strategies, leads to a violation of the nonemergence condition.

Now we introduce a combined condition:

Definition 16 We shall say that a scope value v on $\mathcal{U}$ satisfies the *nonemergent monotonicity* condition, NM, if for every $X \in \mathcal{U}$ and for every its covering $\mathcal{X} \subseteq \mathcal{U}$ the inequality $v(X) \leq V(\mathcal{X})$ is fulfilled.

Proposition 10 For an arbitrary $\mathcal{U}$, NM implies both M and N. Conversely, if $\mathcal{U}$ is $\cup$-complete, then $M\&N$ implies NM.

Indeed, both M and N are special cases of MN: M - when $\mathcal{X} = \{X'\}$, one-member family; N - when the covering $\mathcal{X}$ is, in fact, the decomposition of X. Conversely, let both M and N be satisfied. Take an arbitrary $X \in \mathcal{U}$ and some of its covering $\mathcal{X}$, and consider $X^{\cup} = \bigcup\{X \mid X \in \mathcal{X}\}$. Then $X^{\cup} \supseteq X$, and $X^{\cup} \in \mathcal{U}$ by virtue of $\cup$-completeness of $\mathcal{U}$. Applying sequentially conditions M and N, we obtain

$$v(X) \leq v(X^{\cup}) \leq V(\mathcal{X}), \tag{59}$$

Q.E.D.

A small transformation of the condition M helps elucidate its role as a natural counterpart of the condition N:

Definition $13'$. We shall say that a scope value v on $\mathcal{U}$ satisfies the *monotonicity* condition, M, if for every $X \in \mathcal{U}$ and for every its covering $\mathcal{X} \subseteq \mathcal{U}$ holds

$$V(\mathcal{X}) \leq v(X). \tag{60}$$

It is easy to see that Definitions 13 and $13'$ are equivalent.

Definition 17 We shall say that a scope value v on $\mathcal{U}$ is *aggregable* if for any $X \in \mathcal{U}$ and $\{X^{\nu}\}_{\nu \in N} \subseteq \mathcal{U}$ such that $X = \cup_{\nu \in N} X^{\nu}$, holds

$$v(X) = \max_{\nu \in N} v(X^{\nu}). \tag{61}$$

From the very Definitions 17, $13'$ and 15 follows

Proposition 11 A scope value v on $\mathcal{U}$ is aggregable iff it satisfies both M and N.

We are now in the position to examine the inner structure of well-behaving scope value functions.

Definition 18 We shall say that a scope value v on $\mathcal{U}$ is *primitive rational*, if it satisfies the following *separability* condition: v is representable in the form

$$v(X) = \max_{x \in X} w(x) \tag{62}$$

for every $X \in \mathcal{U}$, where $w(x)$ is some *generating value function* $w : U \to L$.

Start with the simplest case when the inner structure of a scope value function v can be observed directly: it is the case of 1-complete families $\mathcal{U}$, when values v on all singletons are given.

Theorem 12 Let $\mathcal{U}$ be a 1-complete family. Then for a scope value v on $\mathcal{U}$ be primitive rational it is necessary and sufficient that v satisfies the conjunction of conditions $M\&N$.

Proof. Let $M\&N$ be satisfied. Then for every $X \in \mathcal{U}$ we can take as its decomposition the *primitive decomposition* $\mathcal{X}$ consisting of its singleton subsets: $\mathcal{X} = \{\{x\}\}_{x \in X}$. Since, owing to proposition 11, v is aggregable, holds

$$v(X) = \max_{x \in X} v(\{x\}). \tag{63}$$

We thus obtained the *autoseparable representation* (63) which is a special case of the separable representation (62), with $w(x) \equiv v(\{x\})$, $x \in U$. Conversely, if v is representable in a separable form (62), then it is easy to see that it obeys M and N. $\bigtriangledown$

For the general case, when $\mathcal{U}$ may or may not contain all singletons, we need a more complex analysis based on the above properties.

Definition 19 We shall say that a scope value v on $\mathcal{U}$ satisfies the *axiom of revealed value*, ARV, if for every $X \in \mathcal{U}$ holds

$$v(X) \le \max_{x \in X} w_v(x), \tag{64}$$

where $w_v : U \to L$ is the *revealed value function* defined by

$$w_v(x) = \min_{X:\, x \in X \in \mathcal{U}} v(X). \tag{65}$$

Lemma 5 Given any $X \in \mathcal{U}$, the fulfillment of (57) for every $\mathcal{X} \subseteq \mathcal{U}$ covering X is equivalent to the fulfillment of (64).

Proof. Take an arbitrary $\mathcal{X} \subseteq \mathcal{U}$ covering X. By definition of a covering, for every $x \in X$ there exists $X^x \in \mathcal{X}$ such that $X^x \ni x$. Then by definition of $w_v(x)$, holds $w_v \leq v(X^x)$, hence

$$\max_{x \in X} w_v(x) \leq \max_{x \in X} v(X^x) \leq \max_{Y \in \mathcal{X}} v(Y) = V(\mathcal{X}). \tag{66}$$

Therefore, (64) implies (57) for arbitrary $\mathcal{X}$. Conversely, denote X^{*x} a set in $\mathcal{X}$ such that $w_v(x) = v(X^{*x})$, and form the family $\mathcal{X}^* = \{X^{*x}\}_{x \in X}$. Then

$$V(\mathcal{X}^*) = \max_{x \in X} v(X^{*x}) = \max_{x \in X} w_v(x). \tag{67}$$

Therefore, the fulfillment of (57) for any $\mathcal{X}$, including $\mathcal{X}^*$, implies (64). ∇

Proposition 12 Conditions NM and ARV are equivalent.

This Proposition immediately follows from Lemma 5.

Theorem 13 With an arbitrary $\mathcal{U}$, for a scope value v on $\mathcal{U}$ be primitive rational it is necessary and sufficient that v satisfies ARV.

Proof. First, establish the following

Lemma 6 ARV in the form of the inequality (64) is equivalent to the equality

$$v(X) = \max_{x \in X} w_v(x). \tag{68}$$

Proof of Lemma 6. Due to the definition of the revealed value w_v, for each $x \in X$ we have $w_v(x) \leq v(X)$, hence

$$\max_{x \in X} w_v(x) \leq v(X). \tag{69}$$

Juxtaposing (69) and (64), we obtain Lemma. ∇

To prove sufficiency in Theorem, it is enough to note that the equation (68) in the modified ARV is exactly a separable representation of v, with w_v as the generating value function. To prove necessity, note that if v is represented in the separable form (62), then for any $x \in X \in \mathcal{U}$

$$w_v(x) = \min_{Y:\, x \in Y \in \mathcal{U}} v(Y) = \min_{Y:\, x \in Y \in \mathcal{U}} \max_{y \in Y} w(y) \geq$$

$$\geq \min_{Y:\, x \in Y \in \mathcal{U}} w(x) = w(x). \tag{70}$$

Therefore

$$\max_{x \in X} w_v(x) \geq \max_{x \in X} w(x) = v(\mathcal{X}), \tag{71}$$

which yields ARV in the original form (64). ∇

Note that for a separable v on an arbitrary $\mathcal{U}$ generally $w_v(x) \geq w(x)$ (**70**); but, if $\mathcal{U}$ is 1-complete, then we can assert from the definitions of w_v and of separability of v that, moreover, $w_v(x) = w(x) = v(\{x\})$ for all $x \in U$.

From Theorem 13 and Proposition 12 we obtain

Theorem 14 With an arbitrary $\mathcal{U}$, for a scope value v on $\mathcal{U}$ to be primitive rational it is necessary and sufficient that v satisfies NM.

Further, from Theorem 14 and Proposition 10 we obtain

Theorem 15 With an arbitrary $\mathcal{U}$, for a scope value v on $\mathcal{U}$ to be primitive rational it is necessary and, provided that $\mathcal{U}$ is $\cup$-complete, also sufficient that v satisfies M&N.

Thus, Theorems 12-15 present criteria of primitive rationality (separability) for scope value functions. Consider several examples of applications of these criteria. As an illustration, consider another deviation from the standard model of consumer, namely, the model of consumption under privileges, in particular characterizing the life of high officials in the recent Soviet history. The standard of living of such a person depended mainly on his access to privileged goods and services, such as special stores, dining halls, hotels and so on; the prices etc. played a minor role. The set X of privileges available depended on the person's position in the official hierarchy. If a person held several appointments, e.g., if a member of the Central Committee of the Communist Party was also a member of the Supreme Council of the USSR, then both sets of perks, X' and X'', might be united. However, since the privileges of the member of the Central Committee included more attractive perks than those of the member of the Supreme Council, he would not need to use the latter privileges. Hence the standard of living $v(X' \cup X'')$ determined by the union of both lists of privileges X' and X'' was not larger than the level $v(X')$ of the higher position. Therefore, the nonemergence condition N should be fulfilled for v. The fulfillment of the monotonicity condition M is even more so obvious. Since the $\cup$-completeness of the family $\mathcal{U}$ of lists of privileges here is naturally fulfilled, Theorem 15 guarantees the existence of the corresponding hierarchy of privileges x, measured according to some generating scale $w(x)$, such that the person's standard of living v is primitive rational. This means that $v(X)$ it is equal to the maximum level of the privileges $x \in X$ available to the person. Note that we have not considered explicitly alternative modes of person's behavior, so this example is actually within the framework of the model of decisions with hidden alternatives.

Formal examples. Now we shall consider more formal examples, which manifest the applicability of the above technique beyond the problem of estimation of "opportunity scopes" and even beyond the decision theory in the narrow sense. We shall confine ourselves to the simplest case when the reference ordered set L consists of just two elements, say $L = \{0, 1\}$ (where naturally $0 < 1$). Let U be a set of some experts, or voters, $\mathcal{U}$ be some set of groups $X \subseteq U$ of experts, and $v(X) \in L$ be the group attitude (estimate) of some issue. Let two possible values, 0 and 1, denote approval ("yea") and disapproval ("nay") respectively. Then the condition M means that enlarging the group can only enhance its negative, but not positive, attitude to the issue, or, more formally, it can switch the value v from 0 to 1 but not vice versa. The condition N means that if a group is decomposed into new smaller subgroups (with possible overlap), then the positive attitude (approval) by all subgroups implies the approval by the group as a whole. The condition MN means the same as M in the more general case, when new groups form an arbitrary covering (rather than decomposition) of the initial group; besides the initial participants, the new groups may include extra ones.

To describe this situation formally, it is convenient to use logical notations, taking as $v(X)$ a logical function $l(X)$ with the values 0 ("false") and 1 ("true"). The function $l(X)$ represents the group decision from the "negative" standpoint, i.e. $l(X) = 1$ iff the group X disapproves the issue, and $l(X) = 0$ iff X approves it. Then the condition M says that

$$\text{if} \quad X \subseteq X', \text{then} \quad l(X) \Rightarrow l(X'), \tag{72}$$

the condition N says that

$$\text{if} \quad X = \bigcup_{\nu \in N} X^\nu, \text{then} \quad \bigvee_{\nu \in N} l(X^\nu) \Rightarrow l(X), \tag{73}$$

and the condition NM is obtained from (73) by replacing the sign $=$ by $\subseteq$. (All X's above are meant to belong to the admissible family $\mathcal{U}$). Provided the corresponding conditions are fulfilled, the above primitive rationality criteria assert that the function l must be separable, which in logical terms means the representability of l in the disjunctive form

$$l(X) = \bigvee_{x \in X} a(x), \tag{74}$$

where $a : U \to \{0, 1\}$ is a logical function on the set of the experts. The function $a(x)$ can be interpreted as the personal attitude of the expert x to the issue considered: $a(x) = 0$ means the person's approval and $a(x) = 1$ - disapproval. Then (74) means that group X as a whole approves the issue ($l(X) = 0$) if and only if every member of the group does so. Consequently, each voter possesses "the right of veto": his single "nay" is enough to reject the issue. Since in the representation (74) the function a is fixed, we can explicitly list the subgroup $A \subseteq U$ of experts with the negative attitude:

$$A = \{x \in U \mid a(x) = 1\}. \tag{75}$$

Then the resulted attitude of each group $X \in \mathcal{U}$ can be also represented as

$$l(X) = 1 \Leftrightarrow X \cap A \neq \emptyset, \tag{76}$$

that is, the group X rejects the issue if and only if it contains at least one expert with the negative attitude.

To conclude, note that by Theorem 13 and Lemma 6 the function l, when primitive rational, can be represented by means of the *revealed attitude* function

$$a_l(x) = \bigwedge_{X : x \in X \in \mathcal{U}} l(X). \tag{77}$$

The function $a_l(x)$ describes the supposed attitude of the expert x, as if he votes in accordance with the prescribed attitude $a_l(x)$ and the group decision rule is the veto rule (74): the revealed attitude $a_l(x)$ assigned to the person x is "disapprove" if and only if each group that contains him rejects the issue, and "approve" if and only if at least one such group accepts the issue. Finally, if the family $\mathcal{U}$ is 1-complete, i.e., if every expert is questioned separately then we have the autoseparable representation (see (63)) of the group decision function l:

$$l(X) = \bigvee_{x \in X} l(\{x\}). \tag{78}$$

Here the revealed attitude $a_l(x)$ of expert x is the expert's genuine attitude $l(\{x\})$.

Now we shall apply the above model of group decisions to examine two problems which, though seemingly different from the collective estimation problem , turn out to be of the same type.

1. Group decision approach to individual choice.

Let us return to the conventional choice model described in Section 2, and treat it from the group decision standpoint. Fix an option $x \in U$ and consider the fact of x being chosen or rejected as a result of the influence of the choice context: the latter in the framework of the abstract choice model is the presented alternative set $X \ni x$. Then all alternatives $y \in X$ play the role of experts "examining" the alternative x. At the same time nothing prevents x from examining alternatives y. This occurs explicitly in tournaments; to follow this suit, let us consider the choice problem as a kind of an implicit tournament between alternatives presented. Within the model of rational choice of the form (18) or (19), the negative judgement of the "expert" y on the subject x is the strict preference yPx; in the form (16) or (17), the positive judgement of y on x is the nonstrict preference xRy.

In the general form, under fixed $x \in U$, let $\mathcal{U}^x = \{X \in \mathcal{U} \mid X \ni x\}$ be the family of feasible sets of experts evaluating x. We shall extract the group evaluation of x by the group X, observing the fact of choosing x from X (either in a single choice case or in a more general set-wise choice case). Let

$$l^x(X) = \begin{cases} 0, & \text{if } x = c(X) \quad (\text{or } x \in C(X)), \\ 1, & \text{otherwise.} \end{cases} \tag{79}$$

It is easy to verify, that, in particular, the monotonicity condition M for l is equivalent to the heredity condition H for the choice model, the nonemergence condition N is equivalent to the concordance condition C, the synthetic condition NM is equivalent to CH, and ARV is equivalent to RA. For example, in the single-choice case, the condition M in the form (72) yields: if $X, X' \in \mathcal{U}^x$ and $X \subseteq X'$, then $x = c(X') \Rightarrow x = c(X)$, i.e., we obtain the condition N (the extension onto the general set-wise choice is evident).

Thus, we have in fact embedded the standard (individual) choice model, together with its properties considered in Section 2, into the above group decision model. As a consequence, some important propositions and theorems of Section 2 can be shown to be particular cases of the corresponding propositions and theorems of the present section: viz., Propositions 3 and 5, Theorems 1,2,4 and 5 have become corollaries from Propositions 10 and 12, Theorems 13,14,15 and 12, respectively. To show this in relation to the theorems, which are, respectively, the criteria of rationality for choice functions c in Section 2 and the criteria of primitive rationality for logical group decision functions l^x (in the role of scope values v) in this section, we develop our construction. The primitive rationality of l^x means that there exists a function $a^x : U \longrightarrow \{0, 1\}$ such that for every $X \in \mathcal{U}^x$

$$l^x(X) = \bigvee_{y \in X} a^x(y). \tag{80}$$

Consider the collection $\{l^x\}_{x \in U}$ of functions l^x and introduce the binary relation R on U by

$$xRy \quad \text{iff} \quad a^x(y) = 0. \tag{81}$$

Then it is easy to see that the rationalization in the form (16) for c takes place if and only if the separability in the form (80) for each l^x, $x \in U$, defined by (79) takes place, and in such a case the correspondence (81) is valid. Therefore, the notion of the primitive rationality for scope values turns out to be a generalization of the notion of the commonly used rationality for choice functions.

2. Group decision approach to the problem of causal dependencies.

We now outline the problem of causality in terms of a simple model in the spirit of J.S.Mill (see Malishevski, 1989). Consider an observable "output" event. Let 1 denote the occurrence and 0 the absence of this event. Consider also a set U of some primary "input" events $x, y, \ldots$; let a function $l : U \to \{0, 1\}$ denote the logical dependence between the input and output events, where $\mathcal{U} \subseteq 2^U$ is a family of possible combinations (sets) of input events that occur simultaneously. Namely, $l(X) = 1$ if and only if the output event is observed when input events $x \in X$, but no event $y \in U \backslash X$, have occurred. We shall call the function l *primitive causal*, if there exists a subset $A \subseteq U$ called the set of *primitive causes* of the observed output event, such that:

$$l(X) = 1 \text{ if and only if } X \text{ contains at least one input event } x \in A. \tag{82}$$

Note that the requirement of primitive casualty (82) is exactly the condition (76). Thus, the primitive causality coincides with the separability, or the primitive rationality of the function l. Therefore, all the above criteria of primitive rationality can be used for examining primitive causality. The detailed investigation of the model of cause-effect relationships (with many output events as well as input events) can be found in Malishevski, 1989.

Problems 1 and 2 have been reduced above to a kind of separability problem which was then resolved by means of axiomatic conditions. The key role among those is played by the pair of conditions, Monotonicity (or Heritage, in choice problems), and Nonemergence (or Concordance, respectively). It should be noted that Sen's system of axioms α and γ was the first one which served as rationale for stating the corresponding kind of separability (rationality, or binariness) of choice functions.

Returning to the general problem of opportunity scope values, consider in conclusion a wider formulation of the problem — one that does not require the fulfillment of primitive rationality criteria. Namely, consider the case when only one of the conditions M and N is fulfilled, yielding two types of "semirational" scope functions.

Theorem 16 For an arbitrary $\mathcal{U}$, a scope value v satisfies the condition M or, respectively, the condition N, if and only if v can be represented in the following *primitive semirational* form: for any $X \in \mathcal{U}$

$$v(X) = \max_{x \in X} w(x; X), \tag{83}$$

with the generating value function (*pseudoscale*) $w(x; X)$ on $U \times \mathcal{U}$ monotonically increasing, resp., decreasing, in X on $\mathcal{U}$ (in the sense that $X \subseteq X' \Rightarrow w(x; X) \leq$ (resp. $\geq$) $w(x; X')$, with x fixed and $X, X' \in \mathcal{U}$).

Proof. Necessity. Define

$$w^-(x;X) = \min_{Y\in\mathcal{U}:\, x\in Y\subseteq X} v(Y), \tag{84}$$

$$w^+(x;X) = \max_{Y\in\mathcal{U}:\, x\in Y\subseteq X} v(Y) \tag{85}$$

for any $x \in X \in \mathcal{U}$. (Extension of the definition (84), (85) to $x \notin X$, if needed, can be done, e.g., by taking $w^-(x,X) \equiv \max_{Y\in\mathcal{U}} v(Y)$ and $w^+(x;X) \equiv \min_{Y\in\mathcal{U}} v(Y)$). It is obvious that for any $x \in X \in \mathcal{U}$

$$w^-(x;X) \le v(X) \le w^+(x;X), \tag{86}$$

and that w^-(resp.w^+) is monotonically decreasing (resp. increasing) in X. Let us show that a scope value satisfying N (resp., M) is representable in the form (83) with the generating value function w^- (resp., w^+). First, let v satisfy N. Take a set $Y^{X,x}$ yielding min in the definition of w^- (84), and collect the family $\{Y^{X,x}\}_{x\in X}$ that is obviously a decomposition of X. Therefore, by virtue of N,

$$\max_{x\in X} w^-(x;X) = \max_{x\in X} v(Y^{X,x}) \ge v(X). \tag{87}$$

Comparing (87) with (86), we obtain for w^- the equality needed. Now let v satisfy M. Then obviously $w^+(x;X) \equiv v(X)$, and so (83) with $w = w^+$ trivially holds.

Sufficiency. Consider that $w(x;X)$ in (83) is monotonic in X. First, let $w(x;X)$ be monotonically decreasing in X. Take some $X \in \mathcal{U}$ and an arbitrary decomposition $\mathcal{X}$ of X in $\mathcal{U}$. Then for v given by (83)

$$V(\mathcal{X}) = \max_{Y\in\mathcal{X}} v(Y) = \max_{Y\in\mathcal{X}} \max_{y\in Y} w(y,Y) \ge$$

$$\ge \max_{Y\in\mathcal{X}} \max_{y\in Y} w(y;X) = \max_{x\in X} w(x;X) = v(X), \tag{88}$$

which implies N. Finally, let $w(x;X)$ be monotonically increasing in X. Then, fulfilling M for v given by (83) is trivial. $\bigtriangledown$

Remark 8 It is easy to see that by using the logical technique developed and applied in this section for proving the statements of Section 2, one can also obtain Theorem 11 (about semirational choice functions) as a corollary Theorem 16 (about primitive semirational scope values).

As a comment for Theorem 16, note that the monotonic increase of the generating value function $w(x;X)$ in X reflects the increased importance of a fixed opportunity x when the whole opportunity scope X is enlarged. This implies a use of "combinations of opportunities". On the contrary, the decrease of $w(x;X)$ in X means that the use of a single opportunity x can be impeded by the presence of other opportunities. For example, the value of the opportunity x in the set X can have the form $w(x; X) = u(x) - f(X)$, where $f(X)$ is the expense increasing in X of realizing the opportunity x in the presence of other "distracting" opportunities $y \in X$.

5. Further generalizations

We have examined the possibility of judging the rationality of a decision without explicit consideration of alternatives. The simplest model studied in Section 4 is only the first step in this direction. This model is still closely tied to the source of the idea of rationality, viz., to the conventional choice model and the common notions of "alternatives" as points in the "space of possibilities", of values (or relative preferences) of these alternatives etc. A further refusal of such notions leads to a further development of models of rational behavior without an explicit notion of alternatives. Here I will sketch and comment on some directions of possible generalizations.

First, the model described in Section 4 demands too much. Indeed, we assumed the knowledge of (or an ability to observe or to obtain from the decision maker) the values of different opportunity scopes. It is more natural (and less restrictive) to suppose that the analyst possesses only "milder" information concerning the *relative* worths of different scopes. For example, we observe, in reality or mentally, the qualitative results of pairwise comparisons of different scopes from DM's standpoint. Formally this yields a binary relation between the scopes, which in terms of the measurement theory presents a structure of experimental relations on the object field. Then the purpose of theoretical investigation is to elaborate a measurement scale with an appropriate formal operation on corresponding scale values. What has been done in Section 4 is an example of such kind of problem resolution: the measurement scale $w(x)$ on U with the operation max has been introduced to represent the "experimental data" given in the form of pairs $(X, v(X))$, i.e., of the mapping $\mathcal{U} \to L$. It is easy to extend those results to the case when a weak order on $\mathcal{U}$ rather than an explicit primary scaling $\mathcal{U} \to L$ is given. Some other generalizations that can be obtained along this route have not been discussed due to lack of space.

A deeper, both technically and conceptually, is the direction of generalization connected with the refusal of "point-wise" representations of DM's opportunities. Indeed, in the above model we have still kept the representation of an opportunity scope as a set of some elements. Those elements were not necessarily alternatives, or options themselves; in the case of primitive rationality they turned out to be a kind of substitutes of alternatives, with their own revealed values as the rationale for optimal decisions. Such a situation takes place, for example, when the opportunity set is a legislation in the form of a "bill of rights". Then, the application of a set-theoretical operation, such as a union of several legislations seems to be justified. However, in general, a simple combining of several legislations does not lead to an intelligent text. It may be possible, though, to speak about some "creative" union of different legislations. To formalize this case, one needs to use algebraic systems with union (join) operations and covering relations that are more general than the set-theoretical ones. Such generalization (in semilattice-type terms) can be carried out as well, but this would require a separate exposition.

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